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Ideals and their Fitting ideals

BY DAVID EISENBUD
Department of Mathematics, University of California at Berkeley, and the Mathematical Science Research Institute, Berkeley, CA, 94720*, U.S.A.*
e-mail: de@msri.org

ANTONINO FICARRA
Department of Mathematics and Computer Sciences, Physics and Earth Sciences, University of Messina, Viale Ferdinando Stagno d'Alcontres 31, 98166 *Messina, Italy.*
e-mail: antficarra@unime.it

JÜRGEN HERZOG
Fakultät für Mathematik, Universität Duisburg-Essen, 45117 *Essen, Germany.*
e-mail: juergen.herzog@uni-essen.de

AND SOMAYEH MORADI
Department of Mathematics, Faculty of Science, Ilam University, P.O.Box 69315-516*, Ilam, Iran.*
e-mail: so.moradi@ilam.ac.ir

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Abstract

For an ideal I in a Noetherian ring R, the Fitting ideals $\mathrm{Fitt}_j(I)$ are studied. We discuss the question of when $\mathrm{Fitt}_j(I) = I$ or $\sqrt{\mathrm{Fitt}_j(I)} = \sqrt{I}$ for some j. A classical case is the Hilbert–Burch theorem when $j = 1$ and I is a perfect ideal of grade 2 in a local ring.

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1. *Introduction*

Fitting ideals are defined for finitely generated modules over a Noetherian ring. They convey some information about the complexity of a module, as we will describe below. Fitting ideals have many applications in commutative algebra as well as in number theory and arithmetic. In 1936, H. Fitting introduced these module invariants in [**5**].

Fitting ideals are defined as follows: for a finitely generated R-module M one chooses a finite free presentation $G \xrightarrow{\varphi} F \to M \to 0$. Choosing bases of F and G, the R-module homomorphism φ can be described by a matrix A. If m is the rank of F, then the jth Fitting ideal of M is defined to be the ideal $\mathrm{Fitt}_j(M) = I_{m-j}(A)$, where for an integer t, $I_t(A)$ denotes the ideal of t-minors of A. Here we use the convention that $I_t(A) = R$, for $t \leq 0$. Fitting showed that the ideals $\mathrm{Fitt}_j(M)$ are invariants of M, that is, they only depend on the module M and do not depend on the free presentation of M nor on the choice of the bases of the corresponding free modules. Obviously one has $\mathrm{Fitt}_0(M) \subseteq \mathrm{Fitt}_1(M) \subseteq \cdots \subseteq \mathrm{Fitt}_m(M) = R$. Fitting ideals commute with the change of base rings, and in particular with localisation. In other words, if $R \to S$ is a ring homomorphism, then $\mathrm{Fitt}_j(M \otimes_R S) = \mathrm{Fitt}_j(M)S$. An important property of Fitting ideals is the following: the closed subset of R defined by $\mathrm{Fitt}_j(M)$ is the set of prime ideals $P \in \mathrm{Spec}(R)$ such that M_P cannot be generated by j elements. For details and the proofs of these statements we refer the reader to the book of Eisenbud [**4**, section 20.2].

In this paper we are interested in the Fitting ideals $\mathrm{Fitt}_j(M)$, when M is an ideal. The reason for this arises from the Hilbert-Burch theorem. Let R be a local ring, and let $I \subset R$ be a perfect ideal of grade 2. Then the Hilbert-Burch theorem implies that $\mathrm{Fitt}_1(I) = I$. A similar statement holds for graded ideals in a graded K-algebra, where K is a field. Considering this theorem one may ask more generally for which ideals I we have $\mathrm{Fitt}_j(I) = I$ for some j. Dealing with this problem will be one of our main concerns in this paper.

In the first section of the paper we study the relationship of an ideal to its Fitting ideals and prove in theorem 2·1 that for an ideal I in a Noetherian ring which is generated by m elements, $I^{m-j} \subseteq \mathrm{Fitt}_j(I)$ for all $1 \le j \le m$. Equality holds, if I is generated by a regular sequence. Moreover, if $\mathrm{grade}\, I = j+1$, then $\mathrm{Fitt}_j(I) \subseteq I$. As a result, in corollary 2·3 we obtain that $\sqrt{\mathrm{Fitt}_i(I)} = \sqrt{I}$ for $i = 1, \ldots, \mathrm{grade}\, I - 1$. Later in the paper, with a different argument, we show in theorem 5·1 that this statement remains true if one replaces grade I by height I, which in general is bigger than grade I. Also from theorem 2·1 we obtain in corollary 2·4 a first simple example of an ideal I for which $\mathrm{Fitt}_j(I) = I$ for some j. This is indeed the case when $(R, \mathfrak{m})$ is local, $I = \mathfrak{m}$ and $j = \mu(\mathfrak{m}) - 1$. Proposition 2·5 describes another case under which a Fitting ideal of I coincides with I. This case is for example given, when R is a regular ring and $\mathfrak{m}$ is a maximal ideal of R of height at least two.

The trace of a module M is the ideal $\mathrm{tr}(M) = \sum_{\varphi} \varphi(M)$, where the sum is taken over all $\varphi \in \mathrm{tr}_R(M, R)$. If R is a domain and $M = I$ is an ideal, then $\mathrm{tr}(I) = I^{-1}I$, see [**9**]. This allows us to compare $\mathrm{tr}(I)$ with $\mathrm{Fitt}_1(I)$. Indeed, we obtain in Proposition 2·7 that $\mathrm{Fitt}_1(I) = \mathrm{tr}(I)$, if I is generated by 2 elements.

In section 3 we consider the trace of the canonical module ω_R, and observe in corollary 3·2 that the radicals of the ideals $\mathrm{Fitt}_1(\omega_R)$ and $\mathrm{tr}(\omega_R)$ coincide. If R is a local Cohen-Macaulay ring, which is generically a complete intersection, then ω_R can be identified with an ideal of R, and one may ask when it happens that $\mathrm{Fitt}_1(\omega_R) = \omega_R$. Trivially, this equality holds if R is Gorenstein, and we expect that equality holds if and only if R is Gorenstein. So far we have not found any counterexample to this expectation, but also could not prove it yet. However, we show in Proposition 3·4 that if $\mathrm{Fitt}_1(\omega_R) \cong \omega_R$ and the Cohen-Macaulay type of R is at most two, then R is Gorenstein.

Section 4 focusses on one of our main questions, namely, when $\mathrm{Fitt}_1(I) = I$. In theorem 4·1 a certain converse of the Hilbert-Burch theorem is presented. Indeed, it is shown that if R is a local ring and $I \subset R$ is an ideal with $\mathrm{grade}\, I \ge 2$ and $\mathrm{Fitt}_1(I) = I$, then I is a perfect ideal of grade 2.

Suppose that I is an ideal with $\mathrm{grade}\, I \ge j > 2$. For such ideals one may presume, like in the case $j = 2$, that I is a perfect ideal of grade j, if $\mathrm{Fitt}_{j-1}(I) = I$. But it turns out that this is not the case. Indeed, in theorem 4·3 it is shown that if S is a polynomial ring over a field, and $I \subset S$ is a squarefree monomial ideal with $\mathrm{grade}\, I \ge j \ge 2$, then following conditions are equivalent: (i) $\mathrm{Fitt}_{j-1}(I) = I$, (ii) $\mathrm{Fitt}_{j-1}(I)$ is squarefree, and (iii) If $j = 2$, then I is a perfect ideal with $\mathrm{grade}\, I = 2$, and if $j > 2$, then I is generated by a regular sequence of length j. For the proof of this result combinatorial arguments are used. But we expect that the equivalence of (i) and (iii) is true for any ideal in I in a local ring with grade $I \geq j \geq 2$. What we only can say in larger generality is phrased in Proposition 4·2 which says that if $I \subset R$ is a radical ideal of grade $\geq j \geq 2$ with $\mathrm{Fitt}_{j-1}(I) = I$, then I is an unmixed ideal of grade j and R_P is regular for all $P \in \mathrm{Ass} I$.

For an ideal I in a one-dimensional local ring, it may very well happen that $\mathrm{Fitt}_1(I) = I$. In corollary 4·4 we show that if $(R, \mathfrak{m})$ is a one-dimensional local domain with infinite residue class field and multiplicity $e(R) = 2$, then $\mathrm{Fitt}_1(I) = I$ if and only if $\mathrm{tr}(I) = I$. In Examples 4·5(a), this result is applied to numerical semigroup rings of multiplicity 2, where the monomial ideals with $\mathrm{Fitt}_1(I) = I$ are classified. In part (b) an example of a monomial ideal I in a numerical semigroup ring of multiplicity 4 is given which also satisfies $\mathrm{Fitt}_1(I) = I$, and which is not generated by 2 elements. In view of this one may ask whether each one-dimensional local Cohen-Macaulay ring admits a proper ideal I with $\mathrm{Fitt}_1(I) = I$.

In the last section of this paper we compare the radical of $\mathrm{Fitt}_j(I)$ with the radical of I, when I is an ideal in a Noetherian ring. As mentioned before, these two radicals coincide for $j = 1, \ldots,$ height $I - 1$. As a consequence we observe in corollary 5·2 that in the same range for j, the jth Fitting ideal of I and that of $\sqrt{I}$ coincide. Finally, in theorem 5·3 we succeed to compute $\sqrt{\mathrm{Fitt}_j(I)}$ for all j, when $I = I(G)$ is the edge ideal of a finite simple graph G. The result shows that it in general it may be hard to find a nice description of $\sqrt{\mathrm{Fitt}_j(I)}$ for $j \geq$ height I.

2. *The relationship of an ideal to its Fitting ideals*

In this section we compare the Fitting ideals of an ideal I with powers of I, the radical and the trace of I. As the first result we have

THEOREM 2·1. *Let R be a Noetherian ring and $I \subset R$ be an ideal generated by m elements. Then the following hold*:

(a) $I^{m-j} \subseteq \mathrm{Fitt}_j(I)$ *for all* $1 \leq j \leq m$. *Equality holds, if I is generated by a regular sequence, or R is local with maximal ideal $\mathfrak{m}$ and $I = \mathfrak{m}$.*

(b) *if* grade $I = j + 1$, *then* $\mathrm{Fitt}_j(I) \subseteq I$.

Proof. (*a*) By definition, $\mathrm{Fitt}_j(I) = I_{m-j}(A)$, where A is a relation matrix of I. Let $\mathbf{f} = f_1, \ldots, f_m$ be system of generators of I. Consider the Koszul complex $K_. = (K_.(\mathbf{f};R), \partial_.)$ attached to the sequence $\mathbf{f}$. The complex $K_.$ is a differential graded algebra, whose algebra structure is the exterior algebra of K_1. Let $Z_1 = \mathrm{Ker}\,\partial_1$ be the R-module of 1-cycles of the Koszul complex. Then $Z_1^{m-j} \subset K_{m-j}$. We may assume that A is the matrix whose columns correspond to a generating set of Z_1. After fixing a basis of K_{m-j} each element of Z_1^{m-j} can be uniquely written as a linear combination of the elements of this basis. Then $I_{m-j}(A)$ is generated by the coefficients of this linear combinations as z runs through a system of generators of Z_1^{m-j}. To see this, let $z_1, \ldots, z_r$ be the generating set of Z_1 so that z_i presents the ith column of A for all i. Then a $(m-j)$-minor of A corresponding to the rows $i_1, \ldots, i_{m-j}$ and the columns $k_1, \ldots, k_{m-j}$ is the coefficient of $e_{i_1} \wedge \cdots \wedge e_{i_{m-j}}$ in $z_{k_1} \cdots z_{k_{m-j}}$.

For K_1 we choose the basis $e_1, \ldots, e_m$ with $\partial_1(e_i) = f_i$ for $i = 1, \ldots, m$. For any subset $A \subseteq [m]$, $A = \{a_1 < a_2 < \cdots < a_\ell\}$ we set $e_A = e_{a_1} \wedge \cdots \wedge e_{a_\ell}$. Then the elements e_A with $|A| = \ell$ form a basis of K_ℓ.

Let $1 \leq j < m$, and let $f = f_{i_1} f_{i_2} \cdots f_{i_{m-j}}$ with $1 \leq i_1 \leq i_2 \leq \cdots \leq i_{m-j} < m$ be a generator of I^{m-j}. By induction on $m-j$ one shows that there exist integers $1 \leq k_1 < \cdots < k_{m-j} \leq m$ such that $k_\ell \neq i_\ell$ for $\ell = 1, \ldots, m-j$. Let $z = z_1 \wedge \cdots \wedge z_{m-j}$, where $z_\ell = f_{i_\ell} e_{k_\ell} - f_{k_\ell} e_{i_\ell}$ is an element of Z_1. Then $z \in Z_1^{m-j}$, and the coefficient of $e_{k_1} \wedge \cdots \wedge e_{k_{m-j}}$ in z is equal to $\pm f$. This proves that $I^{m-j} \subseteq I_{m-j}(A) = \mathrm{Fitt}_j(I)$.

Now, suppose that $f_1, \ldots, f_m$ is a regular sequence. Then Z_1 is generated by the 1-boundaries $z_{i,j} = f_i e_j - f_j e_i$ with $1 \leq i < j \leq m$. This implies that the coefficients of Z_1^{m-j} belong to I^{m-j}.

Finally, assume that R is local with maximal ideal $\mathfrak{m}$ and $I = \mathfrak{m}$. By the previous part $\mathfrak{m}^{m-j} \subseteq \mathrm{Fitt}_j(\mathfrak{m})$ for all $1 \leq j \leq m$. On the other hand, any nonzero entry of a minimal presentation matrix A of $\mathfrak{m}$ belongs to $\mathfrak{m}$. Hence $\mathrm{Fitt}_j(\mathfrak{m}) = I_{m-j}(A) \subseteq \mathfrak{m}^{m-j}$ for all $1 \leq j \leq m$, and this completes the proof.

(b) Let $f = f_1, \ldots, f_m$ be a system of generators of I, and let as before Z_1 be the module of 1-cycles of the Koszul complex $K_.(\mathbf{f}; R)$. As observed in *(a)*, the coefficients of the elements $z \in Z_1^{m-j}$ (with respect the canonical basis of K_{m-j}) generate $\mathrm{Fitt}_j(I)$. Since we assume that grade $I = j+1$, it follows that $H_{m-j}(f; R) = 0$, see [1, theorem 1·6·17(b)]. Therefore, any $z \in Z_1^{m-j}$ is a boundary, since $Z_1^{m-j} \subseteq Z_{m-j} = B_{m-j}$, and so all coefficients of z belong to I. This shows that $\mathrm{Fitt}_j(I) \subseteq I$.

Remark 2·2. With the assumptions and notation given in theorem 2·1, part (b) of its proof shows that if $Z_1^{m-j} = Z_{m-j}$ for $j = \mathrm{grade}\, I - 1$, then $\mathrm{Fitt}_j(I) = I$.

COROLLARY 2·3. *Let R be a Noetherian ring, and $I \subset R$ be an ideal with* grade $I = j$. *Then*

$$\sqrt{\mathrm{Fitt}_i(I)} = \sqrt{I} \quad \text{for} \quad i = 1, \ldots, j-1.$$

Proof. By the previous theorem, we have $I^{m-1} \subseteq \mathrm{Fitt}_1(I) \subseteq \cdots \subseteq \mathrm{Fitt}_{j-1}(I) \subseteq I$, where $m = \mu(I)$. Taking the radicals, the statement follows.

In theorem 5·1 it is shown that for the equality in corollary 2·3, grade I can be replaced by height I.

On the other hand, when grade $I = j$, we cannot expect that $\sqrt{\mathrm{Fitt}_j(I)} = \sqrt{I}$. Indeed, consider the ideal $I = (x_1x_2, x_1x_3) \subset S = K[x_1, x_2, x_3]$ and K a field. Then grade $I = 1$, $\sqrt{\mathrm{Fitt}_1(I)} = (x_2, x_3)$ and $\sqrt{I} = I$. More examples follow later.

When I is the maximal ideal in a local ring, theorem 2·1 gives the answer to our general question of when $\mathrm{Fitt}_j(I) = I$.

COROLLARY 2·4. *Let R be a local ring with maximal ideal $\mathfrak{m}$. Then* $\mathrm{Fitt}_j(\mathfrak{m}) = \mathfrak{m}$ *if and only if $j = \mu(\mathfrak{m}) - 1$.*

The next result describes another situation for which a Fitting ideal of an ideal I coincides with I.

PROPOSITION 2·5. *Let R be a Noetherian Cohen-Macaulay ring, and let $I \subset R$ be an ideal with* $\mu(IR_P) = \mathrm{grade}\, I = j \geq 2$ *for any $P \in V(I)$. Then* $\mathrm{Fitt}_{j-1}(I) = I$.

Proof. From the inequalities

$$\mathrm{grade}\, I \leq \mathrm{grade}\, IR_P \leq \mathrm{height}\, IR_P \leq \mu(IR_P) = \mathrm{grade}\, I,$$

we obtain that IR_P is a complete intersection for any $P \in V(I)$. Hence, by theorem 2·1*(a)*, we conclude that $\mathrm{Fitt}_{j-1}(I)R_P = \mathrm{Fitt}_{j-1}(IR_P) = IR_P$ for any $P \in V(I)$. This together with theorem 2·1*(b)* implies that $\mathrm{Fitt}_{j-1}(I) = I$.

Using Proposition 2·5 and under the additional assumption that R is a regular ring, corollary 2·4 can be extended as follows.

COROLLARY 2·6. *Let R be a regular ring (not necessarily local), and let $\mathfrak{m}$ be maximal ideal of R such that* $\dim R_{\mathfrak{m}} = d \geq 2$. *Then* $\mathrm{Fitt}_{d-1}(\mathfrak{m}) = \mathfrak{m}$.

The next result shows that if I is a 2-generated ideal in a domain, then $\mathrm{Fitt}_1(I)$ is the trace of I.

PROPOSITION 2·7. *Let R be a domain, and let $I \subset R$ be an ideal generated by 2 elements. Then* $\mathrm{Fitt}_1(I) = \mathrm{tr}(I)$.

Proof. Let $h \in I^{-1}$ be nonzero. Then by theorem 2·1, $hI \subseteq \mathrm{Fitt}_1(hI) = \mathrm{Fitt}_1(I)$. This implies that $\mathrm{tr}(I) \subseteq \mathrm{Fitt}_1(I)$. Now, we show that $\mathrm{Fitt}_1(I) \subseteq \mathrm{tr}(I)$. Let $I = (f_1, f_2)$. Let $g_1 f_1 + g_2 f_2 = 0$, and let $h = g_1/f_2$. Then $hf_2 = g_1$ and $hf_2 f_1 = g_1 f_1 = -g_2 f_2$. Therefore, $hf_1 = -g_2$. This implies that $h \in I^{-1}$. Hence $g_1 \in I^{-1}I = \mathrm{tr}(I)$. Similarly, $g_2 \in \mathrm{tr}(I)$. Thus, the desired conclusion follows.

3. *Fitting ideals of the canonical module*

Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring with canonical module ω_R. The Cohen-Macaulay type of a Cohen-Macaulay module M will be denoted by $r(M)$. We have

LEMMA 3·1. *Let $j \geq 1$ be an integer. Then the set of prime ideals P of R for which $r(R_P) \geq j + 1$ is a closed set in* $\mathrm{Spec}(R)$.

Proof. We note that $\mu(\omega_{R_P}) = r(R_P)$. Hence the assertion follows from the fact that $V(\mathrm{Fitt}_j(\omega_R))$ is the set of prime ideals P for which $(\omega_R)_P = \omega_{R_P}$ cannot be generated by j elements.

COROLLARY 3·2. *Let R be a Cohen-Macaulay local ring. Then* $\sqrt{\mathrm{Fitt}_1(\omega_R)} = \sqrt{\mathrm{tr}(\omega_R)}$.

Proof. For $P \in \mathrm{Spec}(R)$, R_P is Gorenstein if and only if ω_{R_P} is generated by one element. Thus, $\mathrm{Fitt}_1(\omega_R)$ describes the non-Gorenstein locus of R. Since the non-Gorenstein locus of R is also given by $\mathrm{tr}(\omega_R)$, as noted in [**9**], the statement follows.

In the next two results we assume that $(R, \mathfrak{m})$ is a local Cohen–Macaulay ring which is generically Gorenstein. We furthermore assume that R admits a canonical module ω_R. By [**1**, Proposition 3·3·18], ω_R can be identified with a Cohen-Macaulay ideal of grade 1. Assume that $\mathrm{Fitt}_j(\omega_R) \cong \omega_R$ for some j. Then there exists a non-zerodivisor $t \in R$ such that $\mathrm{Fitt}_j(\omega_R) = t\omega_R$. Then $\mathrm{Fitt}_j(t\omega_R) = \mathrm{Fitt}_j(\omega_R) = t\omega_R$. Since the ideal $t\omega_R$ is also a canonical ideal of R, the condition $\mathrm{Fitt}_j(\omega_R) \cong \omega_R$ can always be replaced by $\mathrm{Fitt}_j(\omega_R) = \omega_R$ for a suitable choice of ω_R.

COROLLARY 3·3. *Assume that* $\mathrm{Fitt}_1(\omega_R) = \omega_R$. *Then R_P is Gorenstein if and only if $\omega_R \not\subseteq P$. Moreover, if R is not Gorenstein, then the non-Gorenstein locus of R is of dimension* $\dim R - 1$.

Proof. Let $P \in \mathrm{Spec}(R)$. Our assumptions and lemma 3·1 imply that R_P is not Gorenstein if and only if $\omega_R \subseteq P$.

If R is not Gorenstein, then it follows from our assumption that ω_R is a proper ideal of R. The ideal ω_R is of height one, since it is a Cohen-Macaulay module. This together with the first part of the proof completes the proof.

Assume that R is not Gorenstein, and let P be a minimal prime ideal of the canonical ideal ω_R. The proof of corollary 3·3 shows that if $\mathrm{Fitt}_1(\omega_R) = \omega_R$, then R_P is a one-dimensional Cohen-Macaulay ring but not Gorenstein. So far we could not find a one-dimensional non-Gorenstein ring with canonical ideal ω_R for which $\mathrm{Fitt}_1(\omega_R) = \omega_R$. However, we have

PROPOSITION 3·4. *Let R be a local Cohen-Macaulay domain with canonical module ω_R. Suppose that $\mathrm{Fitt}_1(\omega_R) = \omega_R$ and the Cohen-Macaulay type of R is at most 2. Then R is Gorenstein.*

Proof. Suppose that R is not Gorenstein. Then $\mu(\omega_R) = 2$ and it follows from Proposition 2·7 and our hypothesis that $\omega_R = \omega_R^{-1}\omega_R$. This implies that

$$R = \omega_R{:}\omega_R = \omega_R{:}(\omega_R^{-1}\omega_R) = (\omega_R{:}\omega_R){:}\omega_R^{-1} = R{:}\omega_R^{-1} = (\omega_R^{-1})^{-1}.$$

Therefore, $R = ((\omega_R^{-1})^{-1})^{-1} = \omega_R^{-1}$.

Since $\mathrm{tr}_R(R/\omega_R, R) = 0$, the exact sequence $0 \to \omega_R \to R \to R/\omega_R \to 0$, induces the exact sequence

$$0 \to R \to \omega_R^{-1} \to \mathrm{Ext}^1_R(R/\omega_R, R) \to 0.$$

It follows that $\mathrm{Ext}^1_R(R/\omega_R, R) = 0$. Since R is not Gorenstein, $R/\omega_R \neq 0$, and since $\mathrm{grade}\, \omega_R = 1$, the theorem of Rees (cf. [**1**, theorem 1·2·5]) implies that $\mathrm{Ext}^1_R(R/\omega_R, R) \neq 0$, a contradiction.

4. *When is* $\mathrm{Fitt}_j(I) = I$?

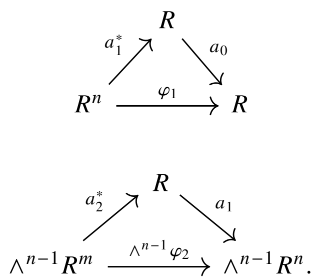
Let R be a Noetherian local ring or a finitely generated graded K-algebra. The Hilbert-Burch theorem implies that if $I \subset R$ is a perfect ideal of grade 2, then $\mathrm{Fitt}_1(I) = I$. In this section we are interested in a converse of this theorem, and ask more generally that if $\mathrm{grade}\, I \geq j$ and $\mathrm{Fitt}_{j-1}(I) = I$, then what can be said about I. The following theorem gives an answer for $j = 2$.

THEOREM 4·1. *Let R be a local ring, and let $I \subset R$ be an ideal. If* $\mathrm{grade}\, I \geq 2$ *and* $\mathrm{Fitt}_1(I) = I$, *then I is a perfect ideal of grade 2.*

Proof. We show that the projective dimension of I is 1. Let

$$\cdots \longrightarrow R^m \xrightarrow{\varphi_2} R^n \xrightarrow{\varphi_1} R \to R/I \longrightarrow 0$$

be a minimal free resolution of R/I. Since grade $I \geq 2$, the first two steps of the dual of this resolution is a resolution of $\mathrm{Coker}(\varphi_2^*)$. By [**3**, theorem 3·1], there are maps a_0, a_1 and a_2 making the following diagrams commute:



For a map φ of free modules, let $I_j(\varphi)$ be the ideal of j-minors of φ. Since grade $I \geq 2$, the map a_0 must be a unit, so the ideal $I_1(a_1) = I_1(a_1^*)$ is equal to I. Since, by assumption, $I_{n-1}(\varphi_2) = I = I_1(a_1)$, it follows that a_2^* is surjective. Thus one of the usual basis elements of $\wedge^{n-1}R^m$ must map by a_2^* to a unit, and it follows that if φ_2' is the restriction to an appropriate summand of rank $n-1$ in R^m, then $I_{n-1}(\varphi_2')$ has grade 2. Thus by [**2**, corollary 1],

$$0 \longrightarrow R^{n-1} \xrightarrow{\varphi_2'} R^n \xrightarrow{\varphi_1} R \to R/I \longrightarrow 0$$

is a resolution.

Now we consider radical ideals I of grade $\geq j \geq 2$ which satisfy the condition $\mathrm{Fitt}_{j-1}(I) = I$. Recall that an ideal I is called *unmixed*, if all the associated prime ideals of I have the same height, equal to height I.

PROPOSITION 4·2. *Let $j \geq 2$ be an integer, and let $I \subset R$ a radical ideal of grade $\geq j$. If* $\mathrm{Fitt}_{j-1}(I) = I$, *then I is an unmixed ideal of grade j and R_P is a regular ring for all $P \in \mathrm{Ass} I$.*

Proof. Let $P \in \mathrm{Ass} I$. Then $PR_P = IR_P$, since I is reduced. Let $m = \mu(PR_P)$. theorem 2·1 and our assumptions imply that

$$PR_P = IR_P = \mathrm{Fitt}_{j-1}(I)R_P = \mathrm{Fitt}_{j-1}(IR_P) = \mathrm{Fitt}_{j-1}(PR_P) = P^{m-j+1}R_P.$$

It follows that $\mu(PR_P) = j$. Krull's generalized principal ideal theorem implies that height $P =$ height$PR_P \leq j$. On the other hand, $j \leq$ grade $I \leq$ height P. Therefore, grade $I = j$ and height $P = j$ for all $P \in \mathrm{Ass} I$. This shows that I is an unmixed ideal of grade j.

We also have seen that $\mu(PR_P) =$ height $PR_P = \dim R_P$ for all $P \in \mathrm{Ass} I$. It follows that R_P is regular for all $P \in \mathrm{Ass} I$.

In the following theorem we answer the main question of this section for squarefree monomial ideals. Before stating this result we need to recall some concepts and notation.

For a finite simple graph G, we denote the vertex set and the edge set of G by $V(G)$ and $E(G)$, respectively. The *edge ideal* of G is the squarefree monomial ideal in the polynomial ring $S = K[x_i : i \in V(G)]$ generated by all monomials $x_i x_j$ such that $\{i, j\} \in E(G)$. The *complementary graph* of G is the graph G^c on vertex set $V(G)$ whose edges are the non-edges of G. A cycle C in G is called an *induced cycle* if no non-adjacent vertices of C form an edge in G. The graph G is called *chordal* if it has no induced cycles of length bigger than 3.

For a monomial ideal I, the relation matrix of I arising from the Taylor relations is quite useful in the study of the Fitting ideals of I. Let $\{u_1, \ldots, u_m\}$ be the minimal set of monomial generators of I. Recall that the Taylor relations of I are of the form $(u_j/\gcd(u_i,u_j))e_i - (u_i/\gcd(u_i,u_j))e_j$ for any distinct indices i and j. So any nonzero entry of the relation matrix is of the form $u_j/\gcd(u_i,u_j)$ for some i and j.

THEOREM 4·3. *Let $I \subset S$ be a squarefree monomial ideal with* grade $I \geq j \geq 2$. *The following conditions are equivalent*:

(i) $\mathrm{Fitt}_{j-1}(I) = I$;

(ii) $\mathrm{Fitt}_{j-1}(I)$ *is squarefree;*

(iii) *if $j = 2$, then I is a perfect ideal with* grade $I = 2$, *and if $j > 2$, then I is generated by a regular sequence of length j.*

Proof.

(i) $\Longrightarrow$ (ii) is obvious.

(ii) $\Longrightarrow$ (i). Let $\mathrm{Fitt}_{j-1}(I)$ be squarefree. Then by corollary 2·3, we have

$$\mathrm{Fitt}_{j-1}(I) = \sqrt{\mathrm{Fitt}_{j-1}(I)} = \sqrt{I} = I.$$

(i) $\Longrightarrow$ (iii). It follows from Proposition 4·2 that I is an unmixed ideal of grade j. First assume that $j = 2$. Then I is unmixed of height 2. It is enough to show that S/I is a Cohen-Macaulay ring. Let G be the graph on $[n]$ such that $\{i, j\} \in E(G)$ if and only if (x_i, x_j) is a minimal prime ideal of I. Then $I = \bigcap_{\{i,j\}\in E(G)} (x_i, x_j)$. Since $I^\vee = I(G)$, by Eagon-Reiner [**8**, theorem 8·1·9], S/I is a Cohen-Macaulay ring if and only if $I(G)$ has a linear resolution. Thus by [**6**], (see, also, [**8**, theorem 9·2·3]), we need to show that G^c is a chordal graph. By contradiction assume that G^c has an induced cycle C_k with $k \geq 4$. Then C_k^c is an induced subgraph of G. Consider the prime ideal $P = (x_i \colon i \in V(C_k))$ in S and set $J = \bigcap_{\{i,j\}\in E(C_k^c)} (x_i, x_j)$. Then $\mathrm{Fitt}_1(JS_P) = \mathrm{Fitt}_1(IS_P) = IS_P = JS_P$. The minimal monomial generators of J are of the form x_A, where A is a minimal vertex cover of C_k^c. One can see that each minimal vertex cover A of C_k^c is of the form $A = V(C_k) \setminus e$, where $e \in E(C_k)$. Therefore JS_P is generated in degree $k-2$ and $\mu(JS_P) = k$. On the other hand, $\mathrm{Fitt}_1(JS_P)$ is the ideal which is generated by $(k-1)$-minors of a relation matrix of JS_P. So each monomial generator of $\mathrm{Fitt}_1(JS_P)$ is of degree at least $k-1$. This contradicts to the equality $\mathrm{Fitt}_1(JS_P) = JS_P$. Hence G^c is chordal, as claimed.

Let $j > 2$. Let $\mathcal{G}(I) = \{u_1, \ldots, u_m\}$ be the set of minimal monomial generators of I. Without loss of generality we may assume that $\bigcup_{i=1}^m \mathrm{supp}(u_i) = [n]$. Let k_I be the number of minimal prime ideals of I. By induction on k_I we show that if $\mathrm{Fitt}_{j-1}(I) = I$ and grade $I = j$, then $u_1, \ldots, u_m$ form a regular sequence. If $k_I = 1$, then I is a prime ideal generated by variables and there is nothing to prove. So we may assume that $k_I \geq 2$. We claim that $m = j$. By contradiction assume that $m > j$. Any monomial generator of $\mathrm{Fitt}_{j-1}(I)$ has degree at least $m - j + 1 \geq 2$. So $\deg(u_i) \geq 2$ for all i. For each $1 \leq i \leq n$, let J_i be the monomial localisation of I at x_i, that is the monomial ideal obtained from I by substituting the variable x_i by 1. Then J_i is a monomial ideal of grade j with $k_{J_i} < k_I$ and $\mathrm{Fitt}_{j-1}(J_i) = J_i$. So by induction hypothesis we may assume that J_i is generated by a regular sequence of monomials of length j. This implies that for distinct p and q if $\gcd(u_p, u_q) \neq 1$, then it has degree at most one. Hence any nonzero entry of a relation matrix A of I has degree at least $d-1$, where $d=\min\{\deg(u_p)\colon 1\leq p\leq m\}$. Therefore any $(m-j+1)$-minor of A has degree at least $(m-j+1)(d-1)$. Since $\mathrm{Fitt}_{j-1}(I)=I$, we obtain $(m-j+1)(d-1)\leq d$. Since $d\geq 2$, we get $m-j+1=2$ and $d=2$. So $m=j+1$. Without loss of generality let $u_1=x_1x_2$ be a monomial of degree 2 in I. Since $x_1x_2\in\mathrm{Fitt}_{j-1}(I)$, there exists $u_2,u_3\in\mathcal{G}(I)$ such that $x_1|u_2$ and $x_2|u_3$. The equality $m=j+1$ together with height $I=j$ implies that any variable x_i appears in at most two of the u_i's. Note that $m\geq 4$. Since $m=j+1$ and $(u_1,u_2,u_3)\subset(x_1,x_2)$ we conclude that $u_4,\ldots,u_m$ form a regular sequence. Therefore, from $u_4\in\mathrm{Fitt}_{j-1}(I)$, we have $u_4=(u_2/x_1)(u_3/x_2)$. Since $\gcd(u_2,u_4)$ has degree at most one, we conclude that $u_2=x_1x_r$ for some r. Similarly, $u_3=x_2x_s$ for some s. Hence $u_4=x_rx_s$. If $m>4$, then $u_5=x_rx_s$ as well, which is not possible. Hence $m=4$ and we may write $I=(x_1x_2,x_2x_3,x_1x_4,x_3x_4)$. Hence height $I=2$, a contradiction.

(iii) $\Longrightarrow$ (i) follows from the Hilbert-Burch theorem together with theorem 2·1(a).

For an ideal I in a one-dimensional local ring, it may very well happen that $\mathrm{Fitt}_1(I)=I$, as we will see below.

An ideal I in a ring R is called a *trace ideal*, if $I=\mathrm{tr}(M)$ for some R-module M. By [**10**, Proposition 2·8], I is a trace ideal if and only if $I=\mathrm{tr}(I)$.

COROLLARY 4·4. *Let $(R,\mathfrak{m})$ be a one-dimensional local domain with infinite residue class field and multiplicity $e(R)=2$. Then $\mathrm{Fitt}_1(I)=I$ if and only if I is a trace ideal.*

Proof. Since the residue class field is infinite, there exists an element $x\in R$ such that $e(R)=\ell(R/(x))$. Let $I\subset R$ be an arbitrary ideal. Then $\mu(I)\leq\ell(I/xI)=\ell(R/(x))=e(R)=2$, see for example [**1**, corollary 4·7·11]. So by Proposition 2·7 we have $\mathrm{Fitt}_1(I)=\mathrm{tr}(I)$ for all ideals I in R. Now, by [**10**, Proposition 2·8] the assertion follows.

EXAMPLES 4·5.

(a) *Let $R=Kt^2,t^{2k+1}$ be a numerical semigroup ring, and let $I\subset R$ be a monomial ideal which is not principal. Then there exists an integer $i=1,\ldots,k$ such that I is isomorphic to the monomial fractionary ideal $J=(1,t^{2i-1})$. The integer i is uniquely determined by I. By Proposition 2·7,* $\mathrm{Fitt}_1(I)=I^{-1}I=J^{-1}J$. *We have* $J^{-1}=(t^{2(k-i+1)},t^{2k+1})$. *Therefore,* $\mathrm{Fitt}_1(I)=(t^{2(k-i+1)},t^{2k+1})(1,t^{2i-1})=(t^{2(k-i+1)},t^{2k+1})$. *By corollary 4·4,* $\mathrm{Fitt}_1(I)=I$ *if and only if* $I=(t^{2(k-i+1)},t^{2k+1})$ *for some* $i\in[k]$.

(b) *Let $R=Kt^4,t^5$, and let $I=(t^{12},t^{13},t^{14},t^{15})$. Then* $\mathrm{Fitt}_1(I)=I$.

Even when the multiplicity of a one-dimensional local domain R is not 2, there may exist a proper ideal $I\subset R$ with $\mathrm{Fitt}_1(I)=I$, as the above example shows. We expect that for any one-dimensional local Cohen-Macaulay ring there always exists an ideal I with $\mathrm{Fitt}_1(I)=I$.

5. *On the radical of Fitting ideals*

In this section we compare the ideal I with the radical of $\mathrm{Fitt}_j(I)$.

THEOREM 5·1 *Let R be a Noetherian ring and $I\subset R$ be an ideal of height $I=h$. Then for any $1\leq j\leq h-1$ we have*

$$\sqrt{\mathrm{Fitt}_j(I)}=\sqrt{I}.$$

Proof. Fix an integer $1 \leq j \leq h-1$. It follows from theorem 2·1 that $\sqrt{I} \subseteq \sqrt{\mathrm{Fitt}_j(I)}$. Since $\sqrt{I} = \bigcap_{P \in V(I)} P$, we only need to show that $\mathrm{Fitt}_j(I) \subseteq P$ for any prime ideal P containing I. Let P be such a prime ideal. Since $j+1 \leq h = \text{height } I \leq \text{height } IR_P \leq \mu(IR_P)$, it follows from [4, Proposition 20·6] that $\mathrm{Fitt}_j(IR_P) \subseteq PR_P$. Hence $\mathrm{Fitt}_j(I) \subseteq \mathrm{Fitt}_j(I)R_P \cap R \subseteq PR_P \cap R = P$, as desired.

COROLLARY 5·2. *Let R be a Noetherian ring and let $I \subset R$ be an ideal of height $I = h$. Then for any $1 \leq j \leq h-1$ we have*

$$\sqrt{\mathrm{Fitt}_j(I)} = \sqrt{\mathrm{Fitt}_j(\sqrt{I})}.$$

Proof. Since $\text{height} I = \text{height } \sqrt{I}$ and $\sqrt{\sqrt{I}} = \sqrt{I}$, the claim follows from the previous theorem.

Next, we determine radical of Fitting ideals of I, when I is the edge ideal of a graph. Let G be a finite simple graph with the vertex set $V(G) = [n]$. The *open neighbourhood* of $i \in V(G)$ is the set $N(i) = \{j : \{i,j\} \in E(G)\}$. Whereas, the *closed neighbourhood* of i is the set $N[i] = N(i) \cup \{i\}$. For a subset A of $V(G)$, we denote by $G \setminus A$, the subgraph of G with the vertices of A and their incident edges deleted. An edge e of G is called a *neighbour* of i if $i \notin e$ and i is adjacent to an endpoint of e. The *edge neighbourhood* of i is the set of all edges in G which are neighbours of i and is denoted by $E(i)$. Let $F = \{i_1, \ldots, i_k\} \subseteq V(G)$. We call a subset $A = \{e_1, \ldots, e_t\}$ of $E(G)$ an *admissible cover* of F of size t if:

(a) $A = \bigsqcup_{s=1}^{k} A_s$, where $\emptyset \neq A_s \subseteq E(i_s)$ for all s.

(b) $\{i_s\} \cup e \neq \{i_t\} \cup e'$ for any distinct integers $s, t \in [k]$, $e \in A_s$ and $e' \in A_t$.

The minimum cardinality of a vertex cover of G is denoted by c_G.

THEOREM 5·3. *Let G be a graph with m edges. Then*

$$\sqrt{\mathrm{Fitt}_j(I(G))} = \begin{cases} I(G) & \text{if } j < c_G, \\ I(G) + J & \text{if } j \geq c_G, \end{cases}$$

where J is the squarefree monomial ideal minimally generated by those monomials $x_{i_1} \cdots x_{i_k}$ for which $F = \{i_1, \ldots, i_k\}$ is an independent set of G, F has an admissible cover of size $m-j$ and no proper subset of F has such an admissible cover.

Proof. If $j < c_G$, the result follows from theorem 5·1, since height $I(G) = c_G$. Let $j \geq c_G$, let $V(G) = [n]$, $E(G) = \{e_1, \ldots, e_m\}$ and $I = I(G)$. For any edge $e_t = \{i,j\}$, we set $f_{e_t} = x_i x_j$.

Note that any Fitting ideal of a monomial ideal is a monomial ideal. To see this, let $u_1, \ldots, u_m$ be the minimal set of monomial generators of I and A be a relation matrix of I whose rows correspond to u_i's and whose columns correspond to generating relations of I, say $g_1, \ldots, g_s$. Each g_i can be chosen to be homogeneous with respect to the multigrading on the polynomial ring. Then any nonzero term in the $(m-j)$-minor of A corresponding to the rows $r_1, \ldots, r_{m-j}$ and the columns $c_1, \ldots, c_{m-j}$ has multidegree equal to $\sum_{i=1}^{m-j} \mathrm{multideg}(g_{c_i}) - \sum_{i=1}^{m-j} \mathrm{multideg}(u_{r_i})$, where $\mathrm{multideg}(g)$ denotes the multidegree of a multigraded element g.

First we show that for any minimal generator $x_{i_1}\cdots x_{i_k}\in\sqrt{\mathrm{Fitt}_j(I)}\setminus I$ of $\sqrt{\mathrm{Fitt}_j(I)}$, $F=\{i_1,\ldots,i_k\}$ is an independent set of G and F has an admissible cover of size $m-j$. The fact that F is an independent set of G follows from the assumption that $x_{i_1}\cdots x_{i_k}\notin I$. Moreover, since $x_{i_1}\cdots x_{i_k}$ is a minimal generator of $\sqrt{\mathrm{Fitt}_j(I)}$, we have $x_{i_1}^{r_1}\cdots x_{i_k}^{r_k}\in\mathrm{Fitt}_j(I)$ for some positive integers $r_1,\ldots,r_k$ with $r_1+\cdots+r_k\geq m-j$. Considering the relation matrix B of I arising from the Taylor relations, one can see that the nonzero entries of B are either variables or monomials x_ix_j for some edges $\{i,j\}$ of G. If $r_1+\cdots+r_k>m-j$, then this means that the minor $x_{i_1}^{r_1}\cdots x_{i_k}^{r_k}$ is the product of $m-j$ entries of B such that at least one of these entries is of the form x_ix_j with $\{i,j\}\in E(G)$. So x_ix_j divides $x_{i_1}\cdots x_{i_k}$, contradicting the assumption that $x_{i_1}\cdots x_{i_k}\notin I$. Hence $r_1+\cdots+r_k=m-j$ and all the entries of B whose product form the minor $x_{i_1}^{r_1}\cdots x_{i_k}^{r_k}$ are variables. Up to relabelling, there exists a subset $A=\{e_1,\ldots,e_{m-j}\}$ of $E(G)$ with $A=\bigsqcup_{s=1}^k A_s$ such that for any $1\leq s\leq k$, $|A_s|=r_s$ and for any $e_\ell\in A_s$, $x_{i_s}f_{e_\ell}=x_pf_{e'}$ for some $p\in V(G)$ with $p\neq i_s$ and some $e'\in E(G)$. Hence if $e_\ell=\{p,q\}$, then $\{i_s,q\}=e'\in E(G)$. Therefore $e_\ell\in E(i_s)$ and hence $A_s\subseteq E(i_s)$.

To show property (b) of A to be an admissible cover of F, by contradiction assume that there exist distinct integers $s,t\in[k]$, $e\in A_s$ and $e'\in A_t$ such that $\{i_s\}\cup e=\{i_t\}\cup e'$. Note that $\{i_s\}\cup e=\{p\}\cup e''$ for some $p\in[n]$ and $e''\in A$ and $\{i_t\}\cup e'=\{q\}\cup e_0$ for some $q\in[n]$ and $e_0\in A$. Thus $e''=e_0$ and $p=q$. This means that $e''=\{i_s,i_t\}$ and hence $x_{i_1}\cdots x_{i_k}\in I$, which is a contradiction. This shows that F has the admissible cover A of size $m-t$.

Next we show that if $F=\{i_1,\ldots,i_k\}$ is an independent set of G and $A=\bigsqcup_{s=1}^k A_s=\{e_1,\ldots,e_{m-j}\}$ is an admissible cover of F, then $x_{i_1}\cdots x_{i_k}\in\sqrt{\mathrm{Fitt}_j(I)}\setminus I$. To prove this, first by relabelling the e_i's we may assume that if $s<t$, then for any $e_i\in A_s$ and $e_\ell\in A_t$ one has $i<\ell$. For any $e_t\in A_s$, there exists a relation $x_{i_s}f_{e_t}=x_pf_{e_r}$ for some $p\neq i_s$ and some $e_r\in E(G)$. Let B_t be the $m\times 1$-matrix corresponding to this relation, i.e., whose tth entry is x_{i_s}, its r-th entry is $-x_p$ and its other entries are zero. Let B be the $m\times(m-j)$-matrix whose i-th column is B_i, and let B' be the $(m-j)$-minor of B obtained by deleting the last j rows of B. Then, we see that B' is the determinant of the following matrix:

$$\begin{array}{r l}
A_1\left\{\begin{array}{l} f_{e_1}\\ \\ \\ \end{array}\right. & \\
\vdots & \\
A_k\left\{\begin{array}{l} \\ \\ f_{e_{m-j}}\end{array}\right. &
\end{array}
\begin{bmatrix}
x_{i_1} & & & & & \\
& \ddots & & & & \\
& & x_{i_1} & & & \\
& & & \ddots & & \\
& & & & x_{i_k} & \\
& & & & & \ddots \\
& & & & & & x_{i_k}
\end{bmatrix}.$$

If $\det(B')\neq 0$, then it is equal to $x_{i_1}^{r_1}\cdots x_{i_k}^{r_k}$, where $r_i=|A_i|$ for all i. For any nonzero entry x_t of B with $t\notin F$ replacing x_t by zero in B' does not change $\det(B')$. Indeed, if $\det(B')\neq 0$, since x_t does not divide $\det(B')$ this is the case, and if $\det(B')=0$ this is obviously true, as well. Hence without loss of generality we may assume that each column of B' has at most two nonzero entries which belong to $\{x_{i_1},\ldots,x_{i_k}\}$. We claim that the columns of B' are linearly independent. If this is not the case, then one can find two columns B'_i and B'_j of B' whose r-th rows are equal up to sign to x_{i_ℓ} for some r. Then from the shape of B' one can see that at least one of B'_i and B'_j has another nonzero entry, say x_{i_t}. This means that we have the relation $x_{i_t}f_{e_d}=x_{i_\ell}f_{e_s}$ for some $\ell,s\in[m-j]$. This contradicts to property (b) of being an admissible cover. Hence, the columns of B' are linearly independent and so $\det(B') \neq 0$ is the required monomial in $\mathrm{Fitt}_j(I)$. Therefore $x_{i_1}\cdots x_{i_k} \in \sqrt{\mathrm{Fitt}_j(I)} \setminus I$. Now, in addition assume that no proper subset of F has an admissible cover of size $m-j$. Then $x_{i_1}\cdots x_{i_k}$ is a minimal generator of $\sqrt{\mathrm{Fitt}_j(I)}$. Otherwise, there exists $F' \subsetneq F$ such that $\prod_{t\in F'} x_t$ is a minimal generator of $\sqrt{\mathrm{Fitt}_j(I)}$. Then it follows from the first part of the proof that F' has an admissible cover of size $m-j$, a contradiction.

We complete the proof by showing that if $x_{i_1}\cdots x_{i_k} \in \sqrt{\mathrm{Fitt}_j(I)} \setminus I$ is a minimal generator of $\sqrt{\mathrm{Fitt}_j(I)}$ and $F=\{i_1,\ldots,i_k\}$, then no proper subset of F has an admissible cover of size $m-j$. On the contrary suppose that $F' \subsetneq F$ is a set which has an admissible cover of size $m-j$. Then F' is an independent set of G as well. As was shown above these imply that $\prod_{t\in F'} x_t \in \sqrt{\mathrm{Fitt}_j(I)} \setminus I$, which contradicts to the minimality of $x_{i_1}\cdots x_{i_k}$.

As a corollary, we characterise when the radical of Fitting ideal of an edge ideal $I(G)$ is the graded maximal ideal, in terms of the combinatorics of G.

COROLLARY 5·4. *Let G be a graph on [n] and let* $m=|E(G)|$. *Then* $x_i \in \sqrt{\mathrm{Fitt}_j(I)}$ *if and only if* $|E(i)| \geq m-j$. *In particular,* $\sqrt{\mathrm{Fitt}_j(I)} = (x_1,\ldots,x_n)$ *if and only if* $m-\min\{|E(i)| : i \in [n]\} \leq j < m$.

Let d be a positive integer. A graph G is called *d-regular* if $|N(i)| = d$ for all $i \in V(G)$. For instance, a complete graph on n vertices is $(n-1)$-regular, whereas any cycle is 2-regular.

Remark 5·5. The second statement of corollary 5·4 can be rephrased as follows, as well. We have $\sqrt{\mathrm{Fitt}_j(I)} = (x_1,\ldots,x_n)$ if and only if

$$\max\{|N(i)| + |E(G_i)| : i \in V(G)\} \leq j < m,$$

where $G_i = G \setminus N[i]$. In particular, when G is a d-regular graph, then $\sqrt{\mathrm{Fitt}_j(I)} = (x_1,\ldots,x_n)$ if and only if $j \geq d + |E(G_i)|$ for each i.

Example 5·6. Let $G=K_n$ be the complete graph on $[n]$ and $I=I(G)$. Then,

$$\sqrt{\mathrm{Fitt}_j(I)} = \begin{cases} I & \text{if } j \leq n-1, \\ (x_1,\ldots,x_n) & \text{if } n-1 \leq j < \binom{n}{2}, \\ S & \text{otherwise.} \end{cases}$$

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