Contemporary Mathematics
Volume **773**, 2021
https://doi.org/10.1090/conm/773/15531

Tate resolutions and MCM approximations

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Abstract. Let M be a finitely generated Cohen-Macaulay module of codimension m over a Gorenstein Ring $R = S/I$, where S is a regular ring. We show how to form a quasi-isomorphism ϕ from an R-free resolution of M to the dual of an R-free resolution of $M^\vee := \mathrm{Ext}^m_R(M, R)$ using the S-free resolutions of R and M. The mapping cone of ϕ is then a Tate resolution of M, allowing us to compute the maximal Cohen-Macaulay approximation of M.

In the case when R is 0-dimensional local, and M is the residue field, the formula for ϕ becomes a formula for the socle of R generalizing a well-known formula for the socle of a zero-dimensional complete intersection.

When $I \subset J \subset S$ are ideals generated by regular sequences, the R-module $M = S/J$ is called a *quasi-complete intersection*, and ϕ was studied in detail by Kustin and Şega. We relate their construction to the sequence of "Eagon-Northcott"-like complexes originally introduced by Buchsbaum and Eisenbud.

Introduction

Tate resolutions. Let R be a Gorenstein local ring, and let M be a finitely generated R-module. A *Tate resolution* or *complete resolution* $\mathbb{T}$ of M is a free, doubly infinite complex that coincides with a free resolution of M in sufficiently high homological degree (it suffices to truncate the resolution at the first syzygy module that is a maximal Cohen-Macaulay module). If $\mathbb{T}$ is minimal, in the sense that the differential of $R/\mathfrak{m}_R \otimes \mathbb{T}$ is 0, then we speak of a minimal Tate resolution. Such minimal Tate resolutions exist for any finitely generated R-module M, and are unique up to (typically non-unique) isomorphism. As explained below, the Tate resolution of M is the same as the Tate resolution of the *essential maximal Cohen-Macaulay (MCM) approximation* M' of M, so information about the Tate resolution gives information about the MCM approximation.

In Section 1 we study the construction of Tate resolutions of a (not necessarily maximal) Cohen-Macaulay module M over a Gorenstein ring R: we show that the Tate resolution of M is the mapping cone of a quasi-isomorphism ϕ from the free resolution of M to the dual of the resolution of the dual, $M^\vee := \mathrm{Ext}^{\mathrm{codim}_R M}(M, R)$, appropriately shifted. If $R = S/I$, with S regular, then we show how to construct ϕ from the S-free resolutions of R and M.

2020 *Mathematics Subject Classification.* Primary 14H99; Secondary 13D02, 14H51.

The first author is grateful to the National Science Foundation for partial support. This work is a contribution to Project I.6 of the second author within the SFB-TRR 195 "Symbolic Tools in Mathematics and their Application" of the German Research Foundation (DFG)..

Quasi-complete intersections. If R and $M := R/J$ are both complete intersections in S, then R/J is called a quasi-complete intersection in R, and the map $R \to R/J$ is a special case of what is called a *quasi-complete intersection morphism*, studied in [**AHS**]. In this case the module M is self-dual, and its minimal free resolution $\mathbb{G}$ was constructed by Tate [**Tate**]. The quasi-isomorphism $\phi : \mathbb{G} \to \mathbb{G}^*[-m]$, where m is the codimension of J in R and $(-)^* := \mathrm{Hom}_R(-, R)$, was constructed by Kustin and Şega [**KS**] as part of a much larger study of quasi-complete intersections. In Section 2 we give a basis-free exposition of this construction, adding the observation that the Tate resolution is naturally a double complex in which the "vertical" strands are the complexes generalizing the Eagon-Northcott complex that were described in [**BE**]. This is explained in Theorem 2.3.

Maximal Cohen-Macaulay approximations. Auslander and Buchweitz [**AB**] defined the *maximal Cohen-Macaulay (MCM) approximation* of a finitely generated module M over a local Gorenstein ring R to be the unique minimal surjection from an MCM R-module N such that the kernel of $N \to M$ has finite projective dimension (see [**EP**] for a recent summary and application of the theory). The module N may always be decomposed as the direct sum of a free module and an MCM R-module M' with no free summand. The module M', with its induced map to M, is called the *essential MCM approximation* of M. If k is any integer $\geq \max(2, \operatorname{depth} R - \operatorname{depth} M)$ then M' is isomorphic to the minimal k-th syzygy of the dual into R of the minimal k-th syzygy of M.

Thus the essential MCM approximation M' of M is the cokernel of the first differential in the minimal Tate resolution of M. The original motivation for our work was to compute the minimal number of generators of the essential MCM approximation of a complete intersection $M = S/J$ as a module over a complete intersection $R = S/I$. This is done in Corollary 2.4.

1. Duality in the Tate resolution of a Cohen-Macaulay module

Tate resolutions associated with Cohen-Macaulay modules always exhibit a sort of duality. The result is well-known in the case $\dim R = 0$ (see for example [**LW**]), and appears in an unpublished manuscript of Buchweitz in the general case [**Bu**, Subsetion 4.5 and Section 5]

PROPOSITION 1.1. *Let R be a Gorenstein ring, and let M be a Cohen-Macaulay R-module whose annihilator has codimension m in R. Let $(\mathbb{F}, \delta)$ and $(\mathbb{G}, \partial)$ be R-free resolutions of M and of $M^\vee := \mathrm{Ext}^m_R(M, R)$, with terms F_i and G_i, respectively. There is a quasi-isomorphism $\phi : \mathbb{F} \to \mathbb{G}^*[-m]$:*

$$\begin{tikzcd}
\cdots & G_m^* \arrow[l, "\partial^*"'] & G_{m-1}^* \arrow[l, "\partial^*"'] & \cdots \arrow[l, "\partial^*"'] & G_0^* \arrow[l, "\partial^*"'] & 0 \arrow[l] \\
 & F_0 \arrow[u, "\phi_0"] & F_1 \arrow[l, "\delta"'] \arrow[u, "\phi_1"] & \cdots \arrow[l, "\delta"'] & F_m \arrow[l, "\delta"'] \arrow[u, "\phi_m"] & \cdots \arrow[l, "\delta"']
\end{tikzcd} .$$

The mapping cone $\mathbb{M}(\phi)$, the total complex of the double complex above, is a Tate resolution for M, and $\mathbb{M}(\phi^*)$ is a Tate resolution of $M^\vee$.

Thus in the local case (possibly after factoring out a free summand), the essential MCM approximation of M over R has a presentation

$$F_0 \oplus G^*_{m-1} \xleftarrow{\begin{pmatrix}\delta & 0\\ \phi_1 & \partial^*\end{pmatrix}} F_1 \oplus G^*_{m-2}.$$

Remark. If we drop the Gorenstein hypothesis but still assume that R is Cohen-Macaulay, and replace $(-)^*$ with $(-)^\vee = \mathrm{Hom}_R(-,\omega_R)$, then similar statements still hold.

Proof. Because M and $M^\vee$ are Cohen-Macaulay modules of codimension m we have $M \cong \mathrm{Ext}^m_R(M^\vee, R) = \mathrm{H}^m(\mathbb{G}^*)$, while $\mathrm{Ext}^i_R(M^\vee, R) = 0$ for $i \neq m$.

The isomorphism $M = \mathrm{H}_0(\mathbb{F}) \cong \mathrm{H}^m(\mathbb{G}^*)$ lifts to a map $F_0 \to G^*_m$ which induces a map $F_1 \to G^*_{m-1}$, and thus to a quasi-isomorphism $\phi : \mathbb{F} \to \mathbb{G}^*$ of cohomological degree m. It follows that the mapping cone $\mathbb{M}(\phi)$ of ϕ has no homology. Since it coincides with $\mathbb{F}$ in large homological degree, it is a Tate resolution of M.

Since the image of each map in $\mathbb{M}(\phi)$ is a maximal Cohen-Macaulay module, every truncation of $\mathbb{M}$ is a resolution of such a module, and thus the dual $\mathbb{M}^*(\phi) = \mathbb{M}(\phi)^*$ has no homology. It follows that ϕ^* is also a quasi-isomorphism. ☐

From Proposition 1.1 we see that, beyond the free resolutions of M and $M^\vee$, the new information in the Tate resolution lies in the description of the map of complexes ϕ. The rest of this paper is devoted to further description of this map.

In the situation of Proposition 1.1, suppose in addition that $R = S/I$. To construct a (generally non-minimal) R-free resolution $\mathbb{F}$ of M one might take an S-free resolution $\mathbb{K}$ of M, tensor with R, and then extend it to an R-free resolution. The next result applies, in particular, to the case when S is regular local and $R = S/I$ is Gorenstein, and also to the case where S is arbitrary and I is generated by a regular sequence, and gives a way of constructing the map of complexes in Proposition 1.1. In the special case of Proposition 1.4 the maps σ^M are made more explicit.

Theorem 1.2. *Suppose that S and $R = S/I$ are Noetherian rings and that R has an S-free resolution $\mathbb{E}$*

$$\mathbb{E} : S \xleftarrow{\partial_1} E_1 \longleftarrow \cdots \longleftarrow E_{c-1} \xleftarrow{\partial_c} E_c \longleftarrow 0.$$

with $\partial_c \cong \partial_1^*$. Let M be an R-module, and let $(\mathbb{K}, \delta)$ be an S-free resolution of M. Let σ^M be a map of complexes $\mathbb{E} \otimes \mathbb{K} \to \mathbb{K}$ that induces the multiplication map $R \otimes M \to M$, and let

$$\sigma^M_{i,j} : E_i \otimes K_j \to K_{i+j}$$

be the components of σ^M. The diagram

$$\begin{tikzcd}
\cdots & R \otimes K_{c-1} \arrow[l, "R\otimes\delta"'] & R \otimes K_c \arrow[l, "R\otimes\delta"'] & R \otimes K_{c+1} \arrow[l, "R\otimes\delta"'] & \cdots \arrow[l, "R\otimes\delta"'] \\
 & & & & \cdots . \\
 & 0 & R \otimes K_0 \arrow[l] \arrow[uu, "R\otimes\sigma^M_{c,0}"] & R \otimes K_1 \arrow[l, "R\otimes\delta"'] \arrow[uu, "R\otimes\sigma^M_{c,1}"] & \cdots \arrow[l, "R\otimes\delta"']
\end{tikzcd}$$

is a map of complexes inducing an isomorphism $M = H_0(R \otimes \mathbb{K}) \to H_c(R \otimes \mathbb{K}) = \mathrm{Tor}^S_c(R, M)$.

LEMMA 1.3. *If S and R are as in Theorem 1.2, then the functor* $\mathrm{Tor}_c^S(R,-)$*, restricted to the category of R-modules, is equivalent to the identity functor.*

PROOF OF LEMMA 1.3. We compute $\mathrm{Tor}_c^S(R,M)$ using the given resolution of R. Because $\partial_1^* \otimes M = 0$ we see that

$$\mathrm{Tor}_c^S(R,M) = \ker(E_c \otimes M \xrightarrow{\partial_1^* \otimes M} E_{c-1} \otimes M) \cong M.$$

Any choice of an isomorphism $E_c \cong S$ gives an equivalence between this functor and the identity functor. ☐

PROOF OF THEOREM 1.2. We first prove that the maps $\sigma^M_{c,*}$ form a map of complexes $\sigma^M_c : R \otimes \mathbb{K} \to R \otimes \mathbb{K}[-c]$. From the definition of the $\sigma^M_{i,j}$ we see that there are commutative diagrams

$$\begin{tikzcd}
R \otimes K_{i-1+c} & & R \otimes K_{i+c} \arrow[ll] \\
R \otimes \big((E_c \otimes K_{i-1}) \oplus (E_{c-1} \otimes K_i)\big) \arrow[u, "R \otimes \left(\sigma^M_{c,i-1} \quad \sigma^M_{c-1,i}\right)"] & & R \otimes E_c \otimes K_i \arrow[ll, "R \otimes \begin{pmatrix} \delta \otimes 1 \\ \pm 1 \otimes \partial \end{pmatrix}"'] \arrow[u, "R \otimes \sigma^M_{c,i}"']
\end{tikzcd}$$

However, $R \otimes (1 \otimes \partial) : E_c \to E_{c-1}$ is 0, so the diagrams

$$\begin{tikzcd}
R \otimes K_{i-1+c} & R \otimes K_{i+c} \arrow[l] \\
R \otimes E_c \otimes K_{i-1} \arrow[u, "R \otimes \sigma^M_{c,i-1}"] & R \otimes E_c \otimes K_i \arrow[l, "\delta \otimes 1"'] \arrow[u, "R \otimes \sigma^M_{c,i}"']
\end{tikzcd}$$

also commute, as required.

We next show that for any R-module M the map σ^M_c induces a functorial isomorphism $M = \mathrm{Tor}_0^S(R,M) \to \mathrm{Tor}_c^S(R,M)$. We first prove functoriality. Let $\alpha : M \to N$ be a homomorphism of R-modules, let $\mathbb{L}$ be the S-free resolution of N, and let $\tilde{\alpha} : \mathbb{K} \to \mathbb{L}$ be a map extending α. Choose maps $\sigma^M : \mathbb{E} \otimes \mathbb{K} \to \mathbb{K}$ and $\sigma^N : \mathbb{E} \otimes \mathbb{L} \to \mathbb{L}$ extending the multiplication maps as above.

There is a homotopy τ between the two compositions

$$\mathbb{E} \otimes \mathbb{K} \xrightarrow{\sigma^N \circ (1 \otimes \tilde{\alpha})} \mathbb{L}$$

and

$$\mathbb{E} \otimes \mathbb{K} \xrightarrow{\tilde{\alpha} \circ \sigma^M} \mathbb{L}.$$

because they cover the same map $R \otimes M \to N$ and $\mathbb{L}$ is acyclic. Because $R \otimes \partial_c = 0$, this homotopy restricts to a homotopy between the induced maps

$$R \otimes E_c \otimes \mathbb{K} \xrightarrow{R \otimes \left(\sigma^N \circ (1 \otimes \tilde{\alpha})\right)} R \otimes \mathbb{L}$$

and

$$R \otimes E_c \otimes \mathbb{K} \xrightarrow{R \otimes (\tilde{\alpha} \circ \sigma^M)} R \otimes \mathbb{L}.$$

In particular, the diagrams

$$\begin{tikzcd}
\operatorname{Tor}^S_{i+c}(R,M) \arrow[r, "\operatorname{Tor}^S_{i+c}(R,\alpha)"] & \operatorname{Tor}^S_{i+c}(R,N) \\
\operatorname{Tor}^S_{i}(R,M) \arrow[r, "\operatorname{Tor}^S_{i}(R,\alpha)"] \arrow[u, "\sigma^M_{c,i*}"] & \operatorname{Tor}^S_{i}(R,N) \arrow[u, "\sigma^N_{c,i*}"']
\end{tikzcd}$$

commute. This proves the functoriality.

We next observe that $\sigma^R_{c,0}$ is an isomorphism. This follows because we may choose $\sigma^R : \mathbb{E}\otimes\mathbb{E} \to \mathbb{E}$ to restrict to the identity map on the subcomplex $\mathbb{E}\otimes E_0 = \mathbb{E}$. It follows that $\sigma^{R^s}_{c,0}$ is an isomorphism for any s.

From the right exact sequence

$$R\otimes K_1 \to R\otimes K_0 \to M \to 0$$

we get a commutative diagram

$$\begin{tikzcd}
\operatorname{Tor}^S_c(R, R\otimes K_1) \arrow[r] & \operatorname{Tor}^S_c(R, R\otimes K_0) \arrow[r] & \operatorname{Tor}^S_c(R, M) \arrow[r] & 0 \\
\operatorname{Tor}^S_0(R, R\otimes K_1) \arrow[r] \arrow[u, "\sigma^{K_1}_{c,0}"] & \operatorname{Tor}^S_0(R, R\otimes K_0) \arrow[r] \arrow[u, "\sigma^{K_0}_{c,0}"] & \operatorname{Tor}^S_0(R, M) \arrow[r] \arrow[u, "\sigma^{M}_{c,0}"] & 0
\end{tikzcd}.$$

The bottom row is the R-free presentation of M, and the top row is also right exact because $\operatorname{Tor}^S_c(R,-)$ is an equivalence on the category of R-modules. The two left-hand vertical maps are isomorphisms because $R\otimes K_1$ and $R\otimes K_0$ are free. It follows by a diagram chase that σ^M_* is an isomorphism as well, completing the proof. ☐

If R is a complete intersection in S the maps $\sigma_{i,j}$ of Theorem 1.2 have a simpler description:

PROPOSITION 1.4. *Suppose that S is a Noetherian ring, that $g_1,\dots,g_c$ is a regular sequence in S, and that $R = S/(g_1,\dots,g_c)$, so that the S-free resolution of R is the Koszul complex*

$$S \xleftarrow{\partial} S^c \xleftarrow{\partial} \bigwedge^2 S^c \xleftarrow{\partial} \cdots \xleftarrow{\partial} \bigwedge^c S^c \longleftarrow 0.$$

Suppose that $\mathbb{K}$ is an S-free resolution of an R-module M, and, for $1 \le j \le c$, let $\tau_j : \mathbb{K} \to \mathbb{K}[1]$ be a homotopy for multiplication by g_j on $\mathbb{K}$. Let $e_1,\dots,e_c$ be a basis for S^c such that $\partial(e_i) = g_i$. The map

$$\sigma_{i,j} : \bigwedge^i S^c \otimes K_j \to K_{i+j}$$

that takes an element $e_{i_1}\wedge\cdots\wedge e_{i_s}\otimes a$ with $i_1 < \cdots < i_s$ to $\tau_{i_1}\circ\cdots\circ\tau_{i_s}(a)$ satisfies the properties of the maps σ of Theorem 1.2.

In particular, the map $\sigma_{c,i} : K_i \to K_{c+i}$ of Theorem 1.2 may be chosen to be $\tau_1\circ\cdots\circ\tau_c$.

Note that if $(\mathbb{F},\delta)$ is any complex and τ is a homotopy for multiplication by an element g, then δ anti-commutes with τ modulo g. Thus the order of the τ_i in the formula does not matter modulo I.

Proof. Since the elements $e_{i_1} \wedge \cdots \wedge e_{i_s}$ with $i_1 < \cdots < i_s$ form a basis for $\bigwedge^s S^c$, the maps $\sigma_{i,j}$ are well-defined.

Write δ for the differential of $\mathbb{K}$. The differential of $\mathbb{E} \otimes \mathbb{K}$ acts on $E_s \otimes K_j$ as $\partial \otimes 1 + 1 \otimes (-1)^s \delta$. From the defining property of the homotopies τ_i we have

$$\begin{aligned}
&\delta\sigma_{s,j}(e_{i_1} \wedge \cdots \wedge e_{i_s} \otimes a)\\
&= \delta \circ \tau_{i_1} \circ \cdots \circ \tau_{i_s}(a) = (g_{i_1} - \tau_{i_1}) \circ \delta \circ \tau_{i_2} \circ \cdots \circ \tau_{i_s}(a) = \cdots\\
&= \Big(\sum_{j=0}^{i-1} (-1)^j g_{i_j} \tau_{i_1} \circ \cdots \circ \widehat{\tau_{i_{i_j}}} \circ \cdots \circ \tau_{i_s}(a)\Big) + (-1)^s \tau_{i_1} \circ \cdots \circ \tau_{i_s} \circ \delta(a)\\
&= \sigma_{s-1,j}(\partial(e_{i_1} \wedge \cdots \wedge e_{i_s}) \otimes a) + \sigma_{i,j-1}(e_{i_1} \wedge \cdots \wedge e_{i_s} \otimes (-1)^s \delta(a))
\end{aligned}$$

as required. ☐

To apply Proposition 1.1 using Theorem 1.2, we will use the following result in the case $N = M^\vee, N' = R$.

Lemma 1.5. *Suppose that S is a Noetherian ring, let $R = S/I$, and let N, N' be a finitely generated R-modules. Let $\mathbb{L}$ be an S-free resolution of N and let $\mathbb{G}$ be an R-free resolution of N, and let $\alpha : \mathbb{L} \to \mathbb{G}$ be a map of complexes extending the identity map of N. If the depth of $J := \operatorname{ann} N$ on N' is m, then the map $H^m(\operatorname{Hom}_R(\mathbb{G}, N')) \xrightarrow{\alpha_*} H^m(\operatorname{Hom}_S(\mathbb{L}, N'))$ is an isomorphism.*

Remark. It is well-known from duality theory that

$$\begin{aligned}
H^m(\operatorname{Hom}_R(\mathbb{G}, N')) &= \operatorname{Ext}^m_R(N, N')\\
&\cong \operatorname{Ext}^m_S(N, N') = H^m(\operatorname{Hom}_S(\mathbb{L}, N'));
\end{aligned}$$

the point of the Lemma is that the comparison map α induces the isomorphism, which doesn't seem to follow immediately from the standard proofs.

Proof. Suppose first that $m = 0$. Since α induces the identity on N it induces the identity on $\operatorname{Hom}_R(N, N') = \operatorname{Hom}_S(N, N')$.

We now induct on m, and we may suppose $m > 0$. We may choose an element $x \in J$ that is a non-zerodivisor on N', and consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Hom}_R(\mathbb{G}, N') \arrow[r, "x"] \arrow[d] & \operatorname{Hom}_R(\mathbb{G}, N') \arrow[r] \arrow[d] & \operatorname{Hom}_R(\mathbb{G}, N'/xN') \arrow[r] \arrow[d] & 0\\
0 \arrow[r] & \operatorname{Hom}_L(\mathbb{L}, N') \arrow[r, "x"] & \operatorname{Hom}_S(\mathbb{L}, N') \arrow[r] & \operatorname{Hom}_S(\mathbb{G}, N'/xN') \arrow[r] & 0
\end{tikzcd}$$

Since $\operatorname{Ext}^{m-1}_S(N, N') = 0 = \operatorname{Ext}^{m-1}_R(N, N')$ while x annihilates $\operatorname{Ext}^m_S(N, N')$ and $\operatorname{Ext}^m_R(N, N')$. Thus we get a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & H^{m-1} \operatorname{Hom}_R(\mathbb{G}, N') \arrow[r] \arrow[d, "H^{m-1} \operatorname{Hom}(\alpha, N')"'] & H^m \operatorname{Hom}_R(\mathbb{G}, N'/xN') \arrow[r] \arrow[d, "H^m \operatorname{Hom}(\alpha, N')"'] & 0\\
0 \arrow[r] & H^{m-1} \operatorname{Hom}_S(\mathbb{L}, N') \arrow[r] & H^m \operatorname{Hom}_S(\mathbb{L}, N'/xN') \arrow[r] & 0
\end{tikzcd}$$

and the left-hand vertical map is an isomorphism by induction. ☐

2. The case of two regular sequences

In this section we give an exposition from a double-complex point of view of the complete resolution of a quasi-complete intersection that was described in [**KS**]. We fix the following notation: S will denote a Gorenstein ring. Let $M = S/I$, where I is generated by a regular sequence $f_1, \dots, f_n$. Let $R = S/J$ where $J \subset I$ is also generated by a regular sequence $(g_1, \dots, g_c)$. We set $m = n - c$, and we write $\mathbb{K}$ for the Koszul complex of $f_1, \dots, f_n$ over S.

Let $A : S^c \to S^n$ be a map making the diagram

(1)
$$\begin{tikzcd}
S & S^n \arrow[l, "{(f_1, \dots, f_n)}"'] \\
S \arrow[u, "1"] & S^c \arrow[l, "{(g_1, \dots, g_c)}"'] \arrow[u, "A"']
\end{tikzcd}$$

that is, $g_j = \sum_{i=1}^{n} a_{i,j} f_i$. We choose an identification of $\bigwedge^c S^c$ with S and write $\alpha \in \bigwedge^c S^n$ for the image of 1 under $\bigwedge^c A$.

Before doing the general case, it may be helpful to see the case $m = 0$, which was worked out by Tate himself, in the form that we will generalize:

Example 2.1. With notation as above, suppose $c = n$ and that M is the residue field of R.

Tate's paper [**Tate**] provides an explicit minimal free resolution that may be written as the total complex of a double complex beginning

$$\begin{tikzcd}
\mathbb{F}: & R & R^c \arrow[l, "{(f_1 \ \ \dots \ \ f_c)}"'] & \bigwedge^2 R^c \arrow[l] & \cdots \\
 & & & & \cdots \\
 & & R^c \otimes R \arrow[uu] & R^c \otimes R^c \arrow[l] \arrow[uu] & \cdots \\
 & & & {} \arrow[u] & \cdots
\end{tikzcd}$$

Here we have written $R^c \otimes R$ instead of R^c to emphasize that the second row is the tensor product of R^c with the first row.

Because $M = S/(f_1, \dots f_c)$ is a maximal Cohen-Macaulay module over R the dual $\mathbb{F}^* = \operatorname{Hom}_R(\mathbb{F}, R)$ is exact except at F_0^*. Furthermore,

$$H^0(\mathbb{F}^*) = \ker\Big(F_0^* \cong S \xrightarrow{\begin{pmatrix} f_1 \\ \vdots \\ f_c \end{pmatrix}} S^{n*} \cong F_1^*\Big) \cong M$$

Thus the Tate resolution of M is obtained by "splicing" together $\mathbb{F}$ and $\mathbb{F}^*$ via a map $\alpha : S \to S \cong S^*$ which may be taken to be multiplication by any generator of $(g_1, \dots, g_c) : (f_1, \dots f_c)$. One well-known expression for such a generator is $\det A$, where A, as in diagram (1) is a matrix expressing the g_i as linear combinations of the f_i. Thus the Tate resolution of M has the form:

$$\begin{tikzcd}
 & \cdots & & & & & \\
\cdots & R^{c*} \otimes R^{c*} \arrow[u] & R^{c*} \arrow[l] & & & & \\
\cdots & & & & & & \\
\cdots & \bigwedge^2 R^{c*} \arrow[uu] & R^{c*} \arrow[l] \arrow[uu] & R \arrow[l, "\begin{pmatrix} f_1 \\ \vdots \\ f_c \end{pmatrix}"'] & & & \\
 & & & R \arrow[u, "\det A"'] & R^c \arrow[l, "(f_1\ \dots\ f_c)"'] & \bigwedge^2 R^c \arrow[l] & \cdots \\
 & & & & & & \cdots \\
 & & & & R^c \otimes R \arrow[uu] & R^c \otimes R^c \arrow[l] \arrow[uu] & \cdots \\
 & & & & & {} \arrow[u] & \cdots
\end{tikzcd}$$

We now turn to the general case, $m = n - c \geq 0$.

We will give an explicit description the Tate resolution of M over R in terms of $\mathbb{K}$ and α, following the outline of Proposition 1.1. What makes this case simpler than the more general case treated in Section 1 is the that we can choose the map $\sigma_{c,*}$ to kill certain differentials in the resolution of M.

We recall the construction of the minimal R-free resolution $\mathbb{F}$ of M. Following Tate, it is usually written as a complex whose underlying graded free module is the tensor product of $R \otimes \mathbb{K}$ with $\mathcal{D}(R^c)$, the divided power algebra on the free module R^c. For our purposes it will be useful to write it as the total complex of the double complex:

$$\begin{tikzcd}
R \arrow[dr, dashed, no head] & \bigwedge^1 \arrow[l] & \bigwedge^2 \arrow[l] & \cdots \arrow[l] & \bigwedge^k \arrow[l] & \cdots \arrow[l] & \bigwedge^n \arrow[l] & 0 \arrow[l] & {} \arrow[l] \\
 & \mathcal{D}_1 \arrow[u] \arrow[dr, dashed, no head] & \mathcal{D}_1 \bigwedge^2 \arrow[l] \arrow[u] & \cdots \arrow[l] & \mathcal{D}_1 \bigwedge^{k-1} \arrow[l] \arrow[u] & \cdots \arrow[l] & \mathcal{D}_1 \bigwedge^{n-1} \arrow[l] \arrow[u] & \mathcal{D}_1 \bigwedge^n \arrow[l] & {} \arrow[l] \\
 & & \mathcal{D}_2 \arrow[u] \arrow[rrddd, dashed, no head] & \cdots \arrow[l] & \mathcal{D}_2 \bigwedge^{k-2} \arrow[l] \arrow[u] & \cdots \arrow[l] & \mathcal{D}_2 \bigwedge^{n-2} \arrow[l] \arrow[u] & \mathcal{D}_2 \bigwedge^{n-1} \arrow[l] & {} \arrow[l] \\
 & \mathbb{F}: & & & & & & & \\
 & & & & \vdots & \cdots & \vdots & \cdots & \\
 & & & & \mathcal{D}_k \bigwedge^{k-1} \arrow[dr, dashed, no head] & \cdots \arrow[l] & \mathcal{D}_k \bigwedge^{n-k} \arrow[l] & \mathcal{D}_2 \bigwedge^{n-k+1} \arrow[l] & {} \arrow[l] \\
 & & & & & \cdots & \vdots & \cdots &
\end{tikzcd}$$

where for compactness we have written $\bigwedge^i$ for $R \otimes \bigwedge^i S^n$ and $\mathcal{D}_i$ for the i-th divided power of R^c and suppressed the tensor product signs so that, for example, we write $\mathcal{D}_2 \bigwedge^k$ in place of $\mathcal{D}_2 R^c \otimes_S \bigwedge^k R^n$.

Because M is a complete intersection in S, we have $M^\vee = \operatorname{Ext}^m_R(M, R) \cong \operatorname{Ext}^c_S(M, S) \cong M$, and thus the R-free resolution of $M^\vee$ is again $\mathbb{F}$. By Proposition 1.1 the Tate resolution of M is the mapping cone of any map of complexes

$\phi : \mathbb{F} \to \mathbb{F}^*[-m]$ that induces an isomorphism

$$M = H_0(\mathbb{F}) \to H_0(\mathbb{F}^*[-m]) = H_0(\mathbb{F}^*[-m]) \cong M.$$

We will see that such a map of complexes can be constructed from the maps $\sigma_{c,i} : K_i \to K_{i+c}$ of Theorem 1.2. By virtue of Proposition 1.4, these take a particularly simple form. (A similar result holds for all the $\sigma_{i,j}$, but we do not need this.)

Proposition 2.2. *Let $e'_1, \dots e'_n$ be a basis of S^n, and let $\delta_1 : S^n \to S$ send $e'_i \to f_i$. Let $(\mathbb{K}, \delta)$ be the Koszul complex*

$$\begin{tikzcd} \mathbb{K} : S & S^n \arrow[l, "\delta_1"'] & \bigwedge^2 S^n \arrow[l, "\delta_2"'] & \cdots \arrow[l] \end{tikzcd}$$

The homotopy for an element $g = \sum_j a_j f_j$ on $\mathbb{K}$ is exterior multiplication by $\sum_j a_j e'_j$. Thus if $A : S^c \to S^n$ is a map as in diagram (1), and α is the image of the generator of $\bigwedge^c S^c$ in $\bigwedge^c S^n$, under $\bigwedge^c A$ then $\sigma_{c,0} : \mathbb{K} \to \mathbb{K}[-c]$ may be taken to be exterior multiplication by α.

Proof. It is easy to check directly that a homotopy for f_i on $\mathbb{K}$ is exterior multiplication by e'_i. The given formula for a homotopy for g follows by linearity.

By Proposition 1.4 the maps $\sigma_{i,j}$ are defined by compositions of the homotopies τ_j for the g_j on $\mathbb{K}$, and the composition $\tau_1 \circ \cdots \circ \tau_c$ is thus exterior multiplication by $\alpha = \bigwedge^c A(e_1 \wedge \cdots \wedge e_c)$, as claimed. ☐

Theorem 2.3. *With notation as above, let $\mathbb{K}$ be the Koszul complex resolving M over S and let $\mathbb{F}$ be the resolution of M over R as in the diagram above. Let $\phi' : R \otimes \mathbb{K} \to R \otimes \mathbb{K}[-c] \cong R \otimes \mathbb{K}^*[-m]$ be the composition of the map defined in Proposition 2.2 with the isomorphism induced by a choice of isomorphism $\beta : \bigwedge^n S^n \to S$. Let $\phi : \mathbb{F} \to \mathbb{F}^*[-m]$ be the composition*

$$\begin{tikzcd} \mathbb{F} \arrow[r, "\pi"] & R \otimes \mathbb{K} \arrow[r, "\phi'"] & R \otimes \mathbb{K}^*[-m] \arrow[r, "\pi^*"] & \mathbb{F}^*[-m]. \end{tikzcd}$$

where π is the projection with kernel $\oplus_{i \geq 1} \mathcal{D}_i(R^c) \otimes \mathbb{K}[-i]$. The map ϕ is a homomorphism of complexes and maps $M = H_0(\mathbb{F})$ isomorphically to $H_0(\mathbb{F}^*[-m])$. Thus the mapping cone $\mathbb{M}(\phi)$ of ϕ is a Tate resolution of M over R. If $I \subset \mathfrak{m}J$, then this Tate resolution is minimal.

Note that π and π^* are *not* maps of complexes (the complex $R \otimes \mathbb{K}$ is a subcomplex of $\mathbb{F}$, *not* a quotient complex.). Nevertheless, the composition ϕ is a map of complexes.

Proof of Theorem 2.3. Let $\mathcal{S}(R^c)$ denote the symmetric algebra of R^c as an R-module. Consider the doubly infinite diagram whose i-th and $i+1$-st columns are:

$$\begin{tikzcd}
 & {} & {} & \\
\cdots & \mathcal{S}_1(R^c) \otimes \bigwedge^{i+c+1} R^n \arrow[l] \arrow[u] & \mathcal{S}_1(R^{c*}) \otimes \bigwedge^{i+c+2} R^n \arrow[l] \arrow[u] & \cdots \arrow[l] \\
\cdots & & & \cdots \\
\cdots & \bigwedge^{i+c} R^n \arrow[l] \arrow[uu] & \bigwedge^{i+c+1} R^n \arrow[l] \arrow[uu] & \cdots \arrow[l] \\
\cdots & \bigwedge^{i} R^n \arrow[l] \arrow[u, "(-1)^i\phi'_i"] & \bigwedge^{i+1} R^n \arrow[l] \arrow[u, "(-1)^{i+1}\phi'_{i+1}"] & \cdots \arrow[l] \\
\cdots & & & \cdots \\
\cdots & \mathcal{D}_1(R^c) \otimes \bigwedge^{i-1} R^n \arrow[l] \arrow[uu] & \mathcal{D}_1(R^c) \otimes \bigwedge^{i} R^n \arrow[l] \arrow[uu] & \cdots \arrow[l]
\end{tikzcd}$$

with the term $\bigwedge^i R^n$ in position $(i,0)$, where the maps in the bottom two rows of the diagram are those of the minimal R-free resolution of $S/(f_1,\dots,f_n)$. Using the isomorphism β we may identify $\bigwedge^j R^n$ with $\bigwedge^{n-j}(R^{n*})$, and with this identification, taking into account that $(\mathcal{D}_i R^c)^*$ is is naturally isomorphic to $\mathcal{S}_i(R^{c*})$, the upper two rows of the diagram are isomorphic to the dual of the lower two rows, shifted m steps to the left. Thus each row is itself a complex and the squares in the lower two rows, and dually in the upper two rows, commute up to sign.

We claim that, with the map ϕ' between the two middle rows, the diagram is a double complex: that is, the squares in the middle two rows commute up to sign, and the vertical maps as well as the horizontal ones compose to zero.

The lower of the middle two rows is the Koszul complex of $f_1,\dots,f_n$, and the upper of the middle two rows is the same Koszul complex, shifted c steps to the left. We have already shown in Proposition 2.2 that the maps ϕ_i commute with the differentials of the these Koszul complexes.

We must still show that the composition of consecutive vertical maps is 0. But the columns of the diagram are exactly the complexes first described in [**BE**, Section 2] and [**Ki**], and given an exposition in [**E97**, Appendix A.2.10]. (See also the more conceptual construction in [**W**], which follows ideas of [**Ke**].)

However, as this is the only fact about the vertical columns that we need, it seems worth pointing out that the result is elementary, a direct extension of "Cramer's rule" for solving linear equations: since the whole diagram is self-dual up to shifts, it may be reduced to showing that the composition

$$\mathcal{D}_1(R^c) \otimes \bigwedge^{i-1} R^n = R^c \otimes \bigwedge^{i-1} R^n \to \bigwedge^{i} R^n \xrightarrow{\phi'} \bigwedge^{i+c} R^n$$

is zero, and direct computation shows that the components of this map are the $(c+1)\times(c+1)$ minors of the matrix derived from A by repeating a row.

Finally, we must show that the composed map of complexes $\pi^* \circ \phi' \circ \pi$ induces an isomorphism $H_0(\mathbb{F}) \to H_0(\mathbb{F}^*[-m]) = H_{-m}(\mathbb{F}^*)$. To this end, consider the maps of complexes

$$R \otimes \mathbb{K} \xhookrightarrow{\iota} \mathbb{F} \xrightarrow{\pi^* \circ \phi' \circ \pi} \mathbb{F}^*[-m] \overset{\iota^*}{\twoheadrightarrow} R \otimes \mathbb{K}^*[-m]$$

where ι is the natural inclusion of complexes. It is obvious that ι induces an isomorphism $M = H_0(R \otimes \mathbb{K}) \to H_0(\mathbb{F})$. Lemma 1.5 shows that ι^* induces an isomorphism

$$H_0(\mathbb{F}^*[-m]) = H_{-m}(\mathbb{F}^*) \to H_{-m}(R \otimes \mathbb{K}^*) = H_c(R \otimes \mathbb{K}).$$

Finally, the composition $\iota^* \circ (\pi^* \circ \phi' \circ \pi) \circ \iota$ is just ϕ' composed with the isomorphism $R \otimes \mathbb{K}[-c] \cong R \otimes \mathbb{K}^*[-m]$ induced by β. This induces an isomorphism $H_0(R \otimes \mathbb{K}) \to H_0(R \otimes \mathbb{K}^*[m])$ by Proposition 2.2. Thus

$$\pi \circ \phi' \circ \pi : H_0\mathbb{F} \to H_0(\mathbb{F}^*[-m])$$

is an isomorphism as well, completing the proof. □

From the description in the proof we see that the mapping cone $\mathbb{M}(\phi)$ is the total complex of the double complex below, where we have illustrated the case when $m = n - c = 2k + 1$ is odd, and the dotted line below runs through the terms of homological degree 0. The bold arrows are given by wedge product with α. The columns are the complexes $\mathcal{C}_i$ that appear in Figure A2.6 of [**E97**]. The dashed line passes through the terms of homological degree 0.

$$\begin{tikzcd}
\ddots & \ddots & \ddots & & & & \ddots & & & & & \\
\mathcal{S}_{n-c-k} \wedge^{n-k-1} & \mathcal{S}_{n-c-k} \wedge^{n-k} \arrow[l] & \mathcal{S}_{n-c-k} \wedge^{n-k+1} \arrow[l] & \mathcal{S}_{n-c-k} \wedge^{n-k+2} \arrow[l] & {} \arrow[l] & \cdots & \mathcal{S}_{n-c-k} \wedge^{n} & 0 \arrow[l] & & & & \\
\vdots & \vdots & \vdots & \vdots & & & \vdots & & & & & \\
\mathcal{S}_1 \wedge^{c} & \mathcal{S}_1 \wedge^{c+1} \arrow[l] & \mathcal{S}_1 \wedge^{c+2} \arrow[l] & \mathcal{S}_1 \wedge^{c+3} \arrow[l] \arrow[uurrr, dash, dashed] & {} \arrow[l] & \cdots & \mathcal{S}_1 \wedge^{c+k+1} \arrow[l] & \cdots \arrow[l] & \mathcal{S}_1 \wedge^{n} & 0 \arrow[l] & & \\
\wedge^{c-1} \arrow[u] & \wedge^{c} \arrow[l] \arrow[u] & \wedge^{c+1} \arrow[l] \arrow[u] \arrow[urr, dash, dashed] & \wedge^{c+2} \arrow[l] \arrow[u] & {} \arrow[l] & \cdots & \wedge^{c+k} \arrow[l] \arrow[u] & \cdots \arrow[l] & \wedge^{n-1} \arrow[u] & \wedge^{n} \arrow[l] & 0 \arrow[l] & \\
0 & R \arrow[l] \arrow[u, line width=1.5pt] \arrow[ur, dash, dashed] & \wedge^{1} \arrow[l] \arrow[u, line width=1.5pt] & \wedge^{2} \arrow[l] \arrow[u, line width=1.5pt] & {} \arrow[l] & \cdots & \wedge^{k} \arrow[l] \arrow[u, line width=1.5pt] & \cdots \arrow[l] & \cdots & \wedge^{n-c} \arrow[l] \arrow[u, line width=1.5pt] & \wedge^{n-c+1} \arrow[l] & {} \arrow[l] \\
& 0 & \mathcal{D}_1 \arrow[l] \arrow[u] & \mathcal{D}_1 \wedge^{1} \arrow[l] \arrow[u] & {} \arrow[l] & \cdots & \mathcal{D}_1 \wedge^{k-1} \arrow[l] \arrow[u] & \cdots \arrow[l] & \cdots & \mathcal{D}_1 \wedge^{n-c-1} \arrow[l] \arrow[u] & \mathcal{D}_1 \wedge^{n-c} \arrow[l] & {} \arrow[l] \\
& & 0 & \mathcal{D}_2 \arrow[l] \arrow[u] & {} \arrow[l] & \cdots & \mathcal{D}_2 \wedge^{k-2} \arrow[l] \arrow[u] & \cdots \arrow[l] & \cdots & \mathcal{D}_2 \wedge^{n-c-2} \arrow[l] \arrow[u] & \mathcal{D}_2 \wedge^{n-c-1} \arrow[l] & {} \arrow[l] \\
& & & & & & \vdots & \cdots & & \vdots & \cdots & \\
& & & & & 0 & \mathcal{D}_k \arrow[l] & \cdots \arrow[l] & \cdots & \mathcal{D}_k \wedge^{n-c-k} \arrow[l] & \mathcal{D}_2 \wedge^{n-c-k+1} \arrow[l] & {} \arrow[l] \\
& & & & & & & \ddots & \ddots & \ddots & \ddots &
\end{tikzcd}$$

Tate Resolution of M; case when m is odd.

Here the entries of the matrices represented by horizontal arrows are the f_i, and the entries of the matrices represented by the vertical arrows are entries of a matrix representing A, except for the bold arrows, whose entries are the $c \times c$ minors of A.

The part of the complex represented by the lower half of the diagram is infinite. Each row is a tensor product of a $\mathcal{D}_k$ or an $\mathcal{S}_k$ with the Koszul complex on $f_1, \dots, f_n$. The columns, on the other hand, are the complexes $\mathcal{C}_i$ that appear in Figure A2.6 of [**E97**].

Corollary 2.4. *The minimal free resolution of the essential maximal Cohen-Macaulay approximation M' of M has the form shown in Figure* 2. *Thus M'*

requires

$$1+\sum_{1\le i\le (n-c-1)/2}\binom{n}{c+1+2i}\binom{c-1+i}{i}$$

generators.

Proof. Add the ranks of the free modules appearing along the 0-th diagonal of the Tate resolution of M (this is marked with a dashed line in the diagram above.) □

$$\begin{tikzcd}
 & & & & \mathcal{S}_k\wedge^n & 0 \arrow[l] & & & & \\
 & & & & \vdots & & & & & \\
 & & \mathcal{S}_1\wedge^{c+3} \arrow[uurr, dashed, no head] & \cdots \arrow[l] & \mathcal{S}_1\wedge^{c+k+1} \arrow[l] & \cdots \arrow[l] & \mathcal{S}_1\wedge^n & 0 \arrow[l] & & \\
 & \wedge^{c+1} \arrow[ur, dashed, no head] & \wedge^{c+2} \arrow[l] \arrow[u] & \cdots \arrow[l] & \wedge^{c+k} \arrow[l] \arrow[u] & \cdots \arrow[l] & \wedge^{n-1} \arrow[u] & \wedge^n \arrow[l] & 0 \arrow[l] & \\
R \arrow[ur, dashed, no head] & \wedge^1 \arrow[l] \arrow[u, thick] & \wedge^2 \arrow[l] \arrow[u, thick] & \cdots \arrow[l] & \wedge^k \arrow[l] \arrow[u, thick] & \cdots \arrow[l] & \cdots & \wedge^{n-c} \arrow[l] \arrow[u, thick] & \wedge^{n-c+1} \arrow[l] & \arrow[l] \\
 & \mathcal{D}_1 \arrow[ul, dashed, no head] \arrow[u] & \mathcal{D}_1\wedge^1 \arrow[l] \arrow[u] & \cdots \arrow[l] & \mathcal{D}_1\wedge^{k-1} \arrow[l] \arrow[u] & \cdots \arrow[l] & \cdots & \mathcal{D}_1\wedge^{n-c-1} \arrow[l] \arrow[u] & \mathcal{D}_1\wedge^{n-c} \arrow[l] & \arrow[l] \\
 & & \mathcal{D}_2 \arrow[ul, dashed, no head] \arrow[u] & \cdots \arrow[l] & \mathcal{D}_2\wedge^{k-2} \arrow[l] \arrow[u] & \cdots \arrow[l] & \cdots & \mathcal{D}_2\wedge^{n-c-2} \arrow[l] \arrow[u] & \mathcal{D}_2\wedge^{n-c-1} \arrow[l] & \arrow[l] \\
 & & & & \vdots & \cdots & & \vdots & \cdots & \\
 & & & & \mathcal{D}_k \arrow[uull, dashed, no head] & \cdots \arrow[l] & \cdots & \mathcal{D}_k\wedge^{n-c-k} \arrow[l] & \mathcal{D}_2\wedge^{n-c-k+1} \arrow[l] & \arrow[l] \\
 & & & & & \cdots \arrow[ul, dashed, no head] & \cdots & \vdots & \cdots &
\end{tikzcd}$$

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