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Quadratic complete intersections [☆]

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ABSTRACT

We study Betti numbers of graded finitely generated modules over a quadratic complete intersection. In the case of codimension 1, we give a natural class of quadratic forms Q whose Clifford algebras are division rings, and deduce the possible Betti numbers of modules over the hypersurfaces $Q = 0$. Our approach leads to a new version of the Betti degree Conjecture. In higher codimensions, we obtain formulas for the Betti numbers in terms of the ranks of certain free modules in a higher matrix factorization.

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1. Introduction

We study Betti numbers—that is, the ranks of the free modules in minimal free resolutions—of graded finitely generated modules over a quadratic complete intersection $R := k[x_1, \dots, x_n]/(f_1, \dots, f_c)$ using the theory of higher matrix factorizations in [9].

In Section 5 we obtain a simplification and sharpening, in the case of quadratic complete intersections, of the formulas for Betti numbers given, in the more general case, in [9]. It follows from [9] that the graded Betti numbers are eventually given by the formula in Theorem 5.6 which involves the ranks of certain free $k[x_1, \dots, x_n]$-modules $B_0(p)$ and $B_1(p)$, with $1 \le p \le c$, associated to a higher matrix factorization of the quadratic regular sequence $f_1, \dots, f_c$. In Theorem 5.8 we sharpen this result by giving an explicit formula for the ranks of the modules $B_1(p)$ in terms of the ranks of the modules $B_0(p)$.

We begin in Section 2 by sharpening the known results in codimension 1. In the case of modules over a hypersurface ring, it is well known that free resolutions correspond to matrix factorizations, and in the case of a quadratic hypersurface $f = 0$ defined over a field K these correspond to $\mathbf{Z}/2$-graded modules over the Clifford algebra $\mathrm{Cliff}_K(f)$ of f. In Theorem 2.1 we prove that for a natural class of quadratic forms f, the Clifford algebra is a division ring. This implies such quadrics have only very large non-trivial matrix factorizations. In terms of Betti numbers, this yields a result for generic quadratic hypersurfaces:

Theorem 1.1. *Suppose that S is a regular local ring with maximal ideal $\mathbf{m}$ whose residue field characteristic is not 2, and that $f_1, \dots, f_m \in \mathbf{m}^2 \subset S$ are elements such that the quadratic forms f_i mod $(\mathbf{m}^3)$ generate an ideal of codimension c. Let $S' = S[z_1, \dots, z_m]$, where the z_i are indeterminates, and consider the local ring $S'' = S'_{\mathbf{m}S'}$. If M is any finitely generated module over the hypersurface ring $S'' \Big/ \sum_{i=1}^{m} z_i f_i$, then the Betti numbers of M are eventually given by a constant divisible by 2^{c-1}.*

To treat complete intersections of codimension > 1 we use the ideas of higher matrix factorizations in [9], which we briefly review in Section 3.

Section 4 is focused on the number rank $B_1(1) =$ rank $B_0(1)$. This number is equal to the Betti degree (if the complexity is maximal). In Theorem 4.3 we show for any finitely generated module over a (not-necessarily quadratic) complete intersection that its Betti degree is equal to the size of a minimal matrix factorization of a generic combination $\sum z_i f_i$, where z_i are new variables. This relates the Betti degree to the MCM Conjecture 4.1 of Buchweitz-Greuel-Schreyer [6], and leads to Conjecture 4.2. Matrix factorizations of a generic combination were considered in a different way by Burke, Orlov, Polishchuk, Vaintrob, Walker and others (Remark 4.5).

2. The Clifford algebra and the matrix factorizations of a generic combination of quadrics

A *matrix factorization* of an element f in a commutative ring S, in the sense of [8], is an ordered pair of square matrices (A, B) with entries in S such that AB and BA are both equal to f times an identity matrix, which we denote $f\,\mathrm{Id}$. Matrix factorizations are useful in many fields, from the study of maximal Cohen-Macaulay modules and singularity theory to knot theory and mathematical physics.

Recall that if $f : V \longrightarrow K$ is a quadratic function on a vector space V over a field K, then the Clifford algebra of f is

$$\mathrm{Cliff}_K(f) = \left(\bigotimes V\right) \Big/ \left(\{v \otimes v - f(v) \mid v \in V\}\right).$$

Matrix factorizations of a quadratic form f on a vector space V over a field K are related to modules over the Clifford algebra; one can use the structure of $\mathrm{Cliff}_K(f)$ to study their properties, and conversely. See [12, Section 4.8] for background on Clifford algebras.

If f is nonsingular then $\mathrm{Cliff}_K(f)$ is a semi-simple finite-dimensional algebra, but it is a difficult problem to determine when it is a division algebra; see [13] for a recent (negative) result. However, an obvious necessary condition is that f is anisotropic–that is, $f(v) = 0$ implies $v = 0$. (Proof: $V = \bigotimes^1 V$ is a subset of $\mathrm{Cliff}(f)$ because the relations defining $\mathrm{Cliff}_K(f)$ are $\mathbf{Z}/2$-homogeneous of even degree. Thus, if $f(v) = 0$ then v is a nilpotent element of $\mathrm{Cliff}_K(f)$.) We will show that for a natural class of quadratic forms, the converse is true as well:

Theorem 2.1. *Let k be a field of characteristic not 2, and let $K = k(z_1, \dots, z_m)$ be the field of rational functions in m variables over k. Suppose that $f_1, \dots, f_m \in k[x_1, \dots, x_c]$ are quadratic forms, and consider the quadratic form $f = \sum_i z_i f_i$ over K.*

(1) *If the ideal $(f_1, \dots, f_m)$ contains a power of $(x_1, \dots, x_c)$, then the Clifford algebra $\mathrm{Cliff}_K(f)$ of f over K is a division algebra.*
(2) *If k is algebraically closed, then $(f_1, \dots, f_m)$ contains a power of $(x_1, \dots, x_c)$ if and only if $f = \sum z_i f_i$ is anisotropic over K.*
(3) *If k is algebraically closed, then the converse of (1) is also true; that is, the Clifford algebra of f over K is a division algebra if and only if f is anisotropic over K.*

In the special case $f = \sum_i z_i x^i$, part (1) was proven by P.M. Cohn [7, Section 12.2] using different methods. A generalization is proven in [5].

Proof of Theorem 2.1. Let $C = \mathrm{Cliff}(f)$, the Clifford algebra of f over K, and set $\mathbf{m} = (x_1, \dots, x_c) \subset S$.

As already noted, if f is isotropic then C has non-zero nilpotent elements, so C is not a division ring. If k is algebraically closed, then the Nullstellensatz shows that $(f_1, \dots, f_m)$

contains a power of $(x_1, \dots, x_c)$ if and only if the f_i have no nontrivial common zero in k; and this is equivalent to the condition that $f = \sum z_i f_i$ is anisotropic over K. Thus it suffices to prove (1), for which we assume that $(f_1, \dots, f_m)$ contains a power of the maximal ideal.

The algebra C is finite dimensional over K. We will show that any finitely generated left C-module M has a finite free resolution. This implies that $\dim_K M$ is a multiple of $\dim_K C$. Consequently, C can have no proper ideals, whence C is a division ring.

We may harmlessly extend the ground field k, and assume that it is algebraically closed. We first consider the case $m = c$, so that $f_1, \dots, f_c$ is a regular sequence. We may replace the f_i by general k-linear combinations of the f_i. Since the finite map of projective spaces $\mathbb{P}^{c-1} \longrightarrow \mathbb{P}^{m-1}$ has degree 2^m, prime to the characteristic, the map is separable, and thus generically smooth. Each ideal of the form

$$I_j := \Big(\{f_i \mid i \neq j\}\Big)$$

defines the preimage of a general point of the image, and thus defines a set of reduced points in $\mathbb{P}^{c-1}$. Because $f_1, \dots, f_c$ is a regular sequence, each I_j is saturated, so the scheme it defines is not contained in a hyperplane. It follows that we may find linearly independent points $P_j \in k^c$ such that

$$f_i(P_j) = \delta_{i,j} := \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

Diagonalizing, we change variables so that $x_i(P_j) = \delta_{i,j}$ and

$$f_i = x_i^2 + \sum_{p<q} a^i_{p,q} x_p x_q.$$

The Clifford algebra C of f is by definition the free algebra $K\langle e_1, \dots, e_c\rangle$ modulo the ideal J generated by the quadratic relations

$$\sum_i u_i e_i \otimes \sum_i u_i e_i - f(u_1, \dots, u_c)$$

for all $(u_1, \dots, u_c) \in K^c$, or, more explicitly,

$$e_p e_q + e_q e_p = \sum_i a^i_{p,q} z_i \text{ for } p < q$$

$$e_p^2 = z_p \text{ for } p = 1, \dots, c.$$

Since $z_p = e_p^2$ is central in C we can eliminate the z_i from the description of C, and write C as an algebra generated by the e_i over the commutative polynomial ring $U := k[e_1^2, \dots, e_c^2]$. Further, C is obtained from the connected, positively graded U-algebra

$$C' = \frac{k\langle e_1, \dots, e_c\rangle}{(\{e_p e_q + e_q e_p = \sum_i a^i_{p,q} e_i^2 \mid p < q\})}$$

by inverting all non-zero elements of U.

Set $R := k[x_1, \dots, x_n]/(f_1, \dots, f_c)$. We next show that, as a quadratic algebra, C' is the algebra dual of R; that is, the vector space of non-commutative quadratic forms generated by the forms

$$e_p e_q + e_q e_p - \sum_i a^i_{p,q} e_i^2,$$

which defines C, is the annihilator of the vector space of non-commutative quadratic forms generated by

$$x_p x_q - x_q x_p = 0 \text{ for } p < q$$
$$x_i^2 + \sum_{p<q} a^i_{p,q} x_p x_q = 0 \text{ for } i = 1, \dots, c,$$

which defines R, under the pairing induced by taking $\{e_i\}$ to be the dual basis to $\{x_i\}$ (see [15] for background on duality for quadratic algebras and on Koszul algebras). One sees by inspection that each element of the first space annihilates each element of the second. Since the dimensions of these two vector spaces add up to c^2, this suffices.

As the quadratic algebra R is a commutative complete intersection of quadrics, the minimal R-free resolution of k is the *Tate resolution*, described in [17]. In particular, it is linear; that is, R is a Koszul algebra. The dual of the Tate resolution is the bi-graded algebra $C' \otimes_k R$, and after interchanging the gradings, this is also the dual of the free resolution of k over C'. Since R is finite-dimensional, this shows that k has a finite C'-free resolution. The algebra C' is Noetherian since it is generated as a U-module by the square-free monomials in the e_i. It can be positively graded by taking $\deg(e_p) = 1$ for all p, and, with this grading the degree 0 component C'_0 is equal to k. This shows that k has finite projective dimension as a C'-module, and it follows that C' has finite global dimension: every finitely generated C' module has a finite C'-free resolution.

If M is a finitely generated left C-module then, clearing denominators from a presentation matrix for M, we see that M is obtained from a finitely generated left C'-module M' by tensoring over k with K. Tensoring a finite free resolution of M' over C' with K, we see that M has a finite free resolution as required. The completes the proof in the case $m = c$.

To reduce to the case $m = c$, let $A = (a_{i,j})$ be an $(m \times c)$-matrix over k, and consider the map

$$\phi : k[z_1, \dots, z_m, x_1, \dots, x_c] \longrightarrow k[z'_1, \dots, z'_c, x_1, \dots, x_c]$$

sending x_i to x_i and z_i to $\sum_j z'_j a_{i,j}$. Consider

$$f' = \phi(f) = \sum_i \left(\sum_j z'_j a_{i,j} \right) f_i = \sum_j z'_j \left(\sum_i a_{i,j} f_i \right).$$

Setting $f'_j = \sum_i a_{i,j} f_i$, we may write this as $f' = \phi(f) = \sum_{j=1}^{c} z'_j f'_j$.

If $\mathrm{Cliff}(f)$ were not a division algebra, then it would contain non-zero elements u, v such that $uv = 0$. Write $u = u'/g(z)$, $v = v'/g(z)$ for some $0 \neq g(z) \in k[z_1, \dots z_m]$, and some u', v' in the image of the tensor algebra $T_{k[z]}(k[z] \otimes_k V)$ (the free algebra on generators V over $k[z]$). The set of matrices A as above such that $g(z') \neq 0$ is Zariski open and dense, and for any such A we have $(\widetilde{\phi}(u')/g(z'))(\widetilde{\phi}(v')/g(z')) = 0$ in $\mathrm{Cliff}(f')$, where we have written $\widetilde{\phi}$ for the obvious extension of ϕ to the free algebra $T_{k[z]}(k[z] \otimes_k V)$. For A in a smaller open dense set, both $\widetilde{\phi}(u')$ and $\widetilde{\phi}(v')$ are non-zero.

On the other hand, the set of matrices A such that the ideal $(f'_1, \dots, f'_j)$ contains a power of $\mathbf{m}$ is also Zariski open and dense. By what we have proven for the case $m = c$, $\mathrm{Cliff}(f')$ is a division ring for A in this set. This contradiction shows that $\mathrm{Cliff}(f)$ was a division ring to begin with. □

Remarks on the Proof. We are grateful to Michel van den Bergh and Michaela Vancliff for pointing out the connection with the work of Bøgvad [4] and Musson [14, Chapter 17], and the connection with finite global dimension. Avramov pointed out an alternative proof for Corollary 2.2, starting from the description of $\mathrm{Ext}_R(k, k)$ given by Sjödin [16]. The special case where $m = c$ and $f_i = x_i^2$ was proven by Buchweitz-Eisenbud-Herzog in [5], using a result of P.M. Cohn.

To apply Theorem 2.1 to matrix factorizations and other free resolutions, we first observe that if S is a local ring with maximal ideal $\mathbf{m}$, then any matrix factorization of an element $f \in S$ is equivalent, in a natural sense, to the direct sum of copies of the *trivial* matrix factorizations $\big(f\mathrm{Id}, \mathrm{Id}\big)$ and $\big(\mathrm{Id}, f\mathrm{Id}\big)$ and a *minimal matrix factorization* (A, B)—that is, one where the entries of the matrices A and B are in $\mathbf{m}$. We say that the factorization is *nontrivial* if the minimal component is present. We extend this definition in an obvious way to the case of homogeneous matrix factorizations over positively graded rings whose degree zero part is local. If S is a regular local ring then f admits nontrivial matrix factorizations if and only if $f \in \mathbf{m}^2$ since, by [8], a high truncation of any infinite minimal free resolution over $S/(f)$ is given by a nontrivial matrix factorization. Similar considerations hold in the graded case. It is interesting to ask about possible the sizes of matrices in minimal matrix factorizations of f. Theorem 1.1 follows from the next result, which is a consequence of Theorem 2.1.

Corollary 2.2. *Suppose that S is a regular local ring with maximal ideal $\mathbf{m}$ whose residue field characteristic is not 2, and let $f_1, \dots, f_m \subset \mathbf{m}^2 \subset S$. Let $S' = S[z_1, \dots, z_m]$, where the z_i are indeterminates, and consider the local ring $S'' = S'_{\mathbf{m}S'}$. Set $f = \sum_{i=1}^{m} z_i f_i$. Suppose the quadratic forms f_i mod $(\mathbf{m}^3)$ generate an ideal of codimension c.*

(1) *The size of the matrices in any minimal matrix factorization of f with entries in S'' is divisible by 2^{c-1}.*
(2) *If Q is a minimal matrix with entries in S'' such that $Q^2 = f \cdot \mathrm{Id}$ then the size of Q is divisible by 2^c.*

Proof. First, we will prove (2). Factoring out a system of $\dim S - c$ regular parameters, we may assume from the outset that $\dim S = \dim S'' = c$. Let K be the field of rational functions $K := k(z_1, \dots, z_c)$. Since the leading forms $\mathrm{in}(f_i)$ of the f_i form a maximal regular sequence of quadrics in the associated graded ring $k[x_1, \dots, x_c]$ of S, the same is true in the associated graded ring $K[x_1, \dots, x_c]$ of S''. The initial form $\mathrm{in}(f) = \sum z_i f_i$ of f is thus a generic linear combination of a maximal regular sequence of quadrics; in particular, it does not vanish non-trivially over the algebraic closure of k.

Since the entries of Q are in $\mathbf{m}'' := \mathbf{m}S''$ we may define a matrix P of linear forms over S'' by taking the entries of P to be the classes of the entries of Q modulo $\mathbf{m}''^2$, and it follows that

$$P^2 = \mathrm{in}(f)\,\mathrm{Id}$$

in $K[x_1, \dots, x_c]$ since $\mathrm{in}(f)$ is a quadric.

Let r be the size of the matrix P. Set $F = K^r$, and let X be the k-vector space spanned by $x_1, \dots, x_c$. The matrix P defines a map of K-vector spaces

$$F \longrightarrow (K \otimes_k X) \otimes_K F = X \otimes_k F\,.$$

Giving such a map is equivalent to giving a map

$$X^* \otimes_k F \longrightarrow F\,.$$

Since $P^2 = \mathrm{in}(f)\,\mathrm{Id}$, the latter map makes F into a module over the Clifford algebra of $\mathrm{in}(f)$ over K. By Theorem 2.1, the Clifford algebra is a division algebra over K. Since its dimension is 2^c, the dimension of any module over it is an integral multiple of 2^c.

To deduce the statement about matrix factorizations in (1) we observe that if (A, B) is a matrix factorization of f, so that $AB = BA = f \cdot \mathrm{Id}$, then

$$Q = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}$$

satisfies $Q^2 = f \cdot \mathrm{Id}$. □

3. Review of higher matrix factorizations

We are going to use the concept of a higher matrix factorization (d, h) for a regular sequence $f_1, \dots, f_c$, introduced in [9, Definition 1.2.2]:

Definition 3.1. A *higher matrix factorization* (d, h) *with respect to a regular sequence* $f_1, \dots, f_c$ in a commutative ring S consists of the data in (1) and (2) subject to conditions (a) and (b):

(1) A pair of free finitely generated S-modules A_0, A_1 with filtrations

$$0 \subseteq A_s(1) \subseteq \cdots \subseteq A_s(c) = A_s, \quad \text{for } s = 0, 1,$$

such that each $A_s(p-1)$ is a free summand of $A_s(p)$;

(2) A pair of maps d, h preserving filtrations,

$$\bigoplus_{q=1}^{c} A_0(q) \xrightarrow{h} A_1 \xrightarrow{d} A_0,$$

where we regard $\oplus_q A_0(q)$ as filtered by the submodules $\oplus_{q \le p} A_0(q)$;

such that, writing $A_0(p) \xrightarrow{h_p} A_1(p) \xrightarrow{d_p} A_0(p)$ for the induced maps, we have

(a) $d_p h_p \equiv f_p \operatorname{Id}_{A_0(p)} \bmod (f_1, \dots, f_{p-1}) A_0(p)$;

(b) $\pi_p h_p d_p \equiv f_p \pi_p \bmod (f_1, \dots, f_{p-1}) \big(A_1(p)/A_1(p-1)\big)$, where π_p denotes the projection $A_1(p) \longrightarrow A_1(p)/A_1(p-1)$.

For our constructions we choose splittings

$$A_s(p) = A_s(p-1) \oplus B_s(p)$$

so $A_s(p) = \oplus_{1 \le q \le p} B_s(q)$.

Set $R := S/(f_1, \dots, f_c)$. We define *the module of the higher matrix factorization* (d, h) to be $M := \operatorname{Coker}(R \otimes d)$. We refer to modules of this form as *higher matrix factorization modules* or *HMF modules.*

We call the higher matrix factorization *minimal* if d and h are minimal (that is, the image of each map is contained in the maximal ideal times the target).

The next theorem follows immediately from [9, Theorem 1.3.1].

Theorem 3.2. *Let k be an infinite field, $S = k[x_1, \dots, x_n]$ be standard graded with* $\deg(x_i) = 1$ *for each i, and I be an ideal generated by a regular sequence of c forms of degree r. Let N be a finitely generated graded S/I-module, and $f_1, \dots, f_c$ be a generic for N regular sequence of r-forms minimally generating I. If M is a sufficiently high graded syzygy of N over $R := S/I$, then M is the module of a minimal graded higher matrix factorization.*

4. Betti degrees and matrix factorizations

Let $f_1, \dots, f_c$ be a regular sequence in a local ring S. Consider the quotient $R = S/(f_1, \dots, f_c)$, and suppose that N is a finitely generated R-module of finite projective dimension over S, and let $\mathcal{R} = k[\chi_1, \dots, \chi_c]$ be the ring of CI-operators. Here each χ_i operates with degree 2 on $\operatorname{Ext}^*_R(N, k)$, which thus splits into even and odd parts

$$\operatorname{Ext}^{even}_R(N,k) := \oplus_{0\leq i} \operatorname{Ext}^{2i}_R(N,k)$$
$$\operatorname{Ext}^{odd}_R(N,k) := \oplus_{0\leq i} \operatorname{Ext}^{2i+1}_R(N,k)\,.$$

By a theorem of Gulliksen as interpreted by Avramov-Sun, each of these modules is finitely generated over $\mathcal{R}$; a short proof is given in [9, Theorem 4.5].

The *complexity* $\operatorname{cx}_R(N)$ of N is defined to be $\dim_{\mathcal{R}}(\operatorname{Ext}^{even}_R(N,k))$ and the *Betti degree* of N to be the multiplicity, $\operatorname{mult}(\operatorname{Ext}^{even}_R(N,k))$ computed with respect to the standard grading of $k[\chi_1, \dots, \chi_c]$ with $\deg(\chi_i) = 1$ for each i. It is shown in [2] that these have the same values as the dimension and multiplicity of $\operatorname{Ext}^{odd}_R(N,k)$. The Betti degree and complexity are defined similarly in the graded situation.

As N has finite projective dimension over S, we have $0 \leq \operatorname{cx}_R(N) \leq c$, which follows immediately from the Eisenbud-Shamash (possibly non-minimal) standard resolution. The complexity is 0 if and only if N has finite projective dimension over R.

Suppose that the module N has infinite projective dimension over R, so $\operatorname{cx}_R(N) \geq 1$. Avramov and Buchweitz [3, Conjecture 7.5] had made the interesting conjecture that the Betti degree of N should be at least $2^{\operatorname{cx}_R(N)-1}$, but recently Iyengar-Walker [11] gave a counterexample. In this section, we demonstrate that the following related conjecture of Buchweitz-Greuel-Schreyer provides a hint that 4.2 might be the right substitute.

MCM Conjecture 4.1. (Buchweitz-Greuel-Schreyer [6, Conjecture A]) *Let T be an irreducible hypersurface whose singular locus has codimension r. Any Maximal Cohen-Macaulay module L without free summands over T satisfies*

$$\operatorname{rank} L \geq 2^{\lfloor \frac{r}{2} - 1 \rfloor}\,,$$

and $\lfloor q \rfloor$ stands for the largest integer $\leq q$.

By results of [8], the size of the minimal matrix factorization of L is the sum of the ranks of L and of its first syzygy. Thus the Buchweitz-Greuel-Schreyer conjecture would imply that the size of the minimal matrix factorization associated to L is at least $2^{\lfloor \frac{r}{2} \rfloor}$.

In Theorem 4.3 we will show that the Betti degree of a module N over a complete intersection is equal to the size of a minimal matrix factorization for a hypersurface whose singular locus has codimension at least the complexity of N. Combining Theorem 4.3 and Conjecture 4.1 leads to:

Betti Degree Conjecture 4.2. *Let $f_1, \dots, f_c$ be a regular sequence in a regular local ring S, and set $R = S/(f_1, \dots, f_c)$. Let N be a finitely generated R-module of infinite projective dimension over R. The Betti degree of N is at least $2^{\lfloor \frac{\mathrm{cx}_R(N)}{2} \rfloor}$.*

Similarly, one can make the corresponding conjecture in the case where S is a standard graded polynomial ring over an infinite field, $f_1, \dots, f_c$ is a regular sequence of forms of the same degree, and N is a finitely generated graded $S/(f_1, \dots, f_c)$-module.

It remains to prove:

Theorem 4.3. *Let I be an ideal generated by a regular sequence in a local ring S with infinite residue field k and maximal ideal $\mathbf{m}$, and set $R = S/I$. Suppose that N is a finitely generated R-module of finite projective dimension over S. Let $f_1, \dots, f_c$ be a regular sequence minimally generating I that is generic for N in the sense of [9, Theorem 1.3.1]. Set $\gamma := c - \mathrm{cx}_R\, N + 1$.*

Consider $S[z_\gamma, \dots, z_c]_{\mathbf{n}}$, where the z_i are indeterminates and $\mathbf{n} = \mathbf{m} + (\mathbf{z}_\gamma, \dots, \mathbf{z_c})$. Let V be the local complete intersection

$$V := S[z_\gamma, \dots, z_c]_{\mathbf{n}}/(f_1, \dots, f_{\gamma-1}).$$

The Betti degree of N is equal to the size of a minimal matrix factorization over V of the element

$$f := \sum_{i=\gamma}^{c} z_i f_i \,.$$

If N has maximal complexity $\mathrm{cx}_R N = c$, then $\gamma = 1$ and the singular locus of $f \in V = S[z_\gamma, \dots, z_c]_{\mathbf{n}}$ has codimension $\geq \mathrm{cx}_R N$.

Proof. The estimate on the singular locus of $f = \sum_{i=1}^{c} z_i f_i$ follows because the derivative with respect to z_i is f_i.

We may harmlessly extend the ground field k, and assume that it is algebraically closed. Let M be a high R-syzygy of N in the sense of [9, Theorem 1.3.1]. The Betti degree and the complexity of the module M are the same as those of N. By [9, Theorems 7.6 and 9.2], M is the module of a minimal higher matrix factorization (d, h) involving modules $B_0(1), \dots, B_0(c), B_1(1), \dots, B_1(c)$, where the notation follows that of Section 3. By [9, Corollary 5.2.3], we have

$$\gamma = \min\{\, p \mid B_0(p) \neq 0 \,\} \,,$$

and the Betti degree of M is the size of the square matrix $b_\gamma : B_1(\gamma) \longrightarrow B_0(\gamma)$, which is a minimal matrix factorization for f_γ over the ring $W := S/(f_1, \dots, f_{\gamma-1})$. By [9, Theorem 3.1.4], the module M has finite projective dimension over the ring W.

The V-module $V \otimes M$ has the same projective dimension over V, Betti degree and complexity as a module over $V/(f_\gamma, \dots, f_c)$ as M has, and is the V-module of the minimal higher matrix factorization $(V \otimes d,\ V \otimes h)$ with respect to $f_\gamma, \dots, f_c$. Writing $V \otimes h$ as a tuple $(h_\gamma, \dots, h_c)$ we set $h' := (z_\gamma h_\gamma, \dots, z_c h_c)$. Let K be the V-module of the minimal higher matrix factorization $(V \otimes d,\ V \otimes h')$ with respect to $z_\gamma f_\gamma, \dots, z_c f_c$. By [9, Theorem 5.1.2], the V-module K has the same Betti degree and the same complexity over the ring $Q := V/(z_\gamma f_\gamma, \dots, z_c f_c)$ as does $V \otimes M$ over the ring $V/(f_\gamma, \dots, f_c)$.

By [9, Proposition 6.1.11 and Theorem 6.1.2] we can add scalar combinations of later elements of the regular sequence $z_\gamma f_\gamma, \dots, z_c f_c$ to earlier ones to make the sequence generic for K. After making the corresponding change of the variables z_i, the ideal $I' := (z_\gamma f_\gamma, \dots, z_c f_c)$ may be written in the form

$$I' = \left(\sum_{i=\gamma}^{c} z_i f_i,\ \sum_{i=\gamma+1}^{c} a_{\gamma+1,i} f_i,\ \sum_{i=\gamma+2}^{c} a_{\gamma+2,i} f_i,\ \dots \right),$$

where the $a_{j,i}$ are linear forms in the variables z_q. Let L be a sufficiently high Q-syzygy of K. By [9, Theorem 1.3.1], L is the module of a minimal higher matrix factorization $(\bar{d}, \bar{h})$ with respect to the regular sequence

$$f = \sum_{i=\gamma}^{c} z_i f_i,\quad \sum_{i=\gamma+1}^{c} a_{\gamma+1,i} f_i,\quad \sum_{i=\gamma+3}^{c} a_{\gamma+2,i} f_i,\ \dots\ ,$$

involving modules $\overline{B_0}(\gamma), \dots, \overline{B_0}(c), \overline{B_1}(\gamma), \dots, \overline{B_1}(c)$. The complexity and the Betti degree of L are equal to those of K, so equal to those of N.

By [9, Corollary 5.2.3], we have

$$\gamma = \min\{\, p \mid \overline{B_0}(p) \neq 0 \,\},$$

and the Betti degree of L is the size of the square matrix $\overline{b_\gamma} : \overline{B_1}(\gamma) \longrightarrow \overline{B_0}(\gamma)$, which is a minimal matrix factorization for f over the ring V. $\square$

Remark 4.4. The same conclusion as in Theorem 4.3 holds, with essentially the same proof, in case S is a standard graded polynomial ring over an infinite field, I is generated by a regular sequence of forms of the same degree, and N is a finitely generated graded module over S/I.

Remark 4.5. Matrix factorizations of a generic combination were considered in a different way by Burke, Orlov, Polishchuk, Vaintrob, Walker and others. They regard a complete intersection as a family of hypersurfaces parametrized by a projective space: if $S = k[x_1, \dots, x_n]$ is the coordinate ring of the affine n-space $\mathbf{A}^n_k$ over a field k, and R is the complete intersection $R = S/(f_1, \dots, f_c)$, then one may consider the element $f = \sum z_i f_i \in S[z_1, \dots, z_c]$ as defining a hypersurface in the product of $\mathbf{A}^n$ and the projective space $\mathbf{P}^{c-1}$, and consider the category of matrix factorizations of f.

5. Linear higher matrix factorizations and Betti numbers

Under the assumptions of Theorem 3.2, we have graded Betti numbers

$$b^R_{i,j}(M) = \dim_k \mathrm{Tor}^R_i(M,k)_j\ .$$

In this section we study these Betti numbers.

The (graded) Betti numbers of an HMF module over a quadratic complete intersection are given by the formulas in the following Proposition 5.1, which follows from [9, Corollary 5.2.1].

Proposition 5.1. *Let $S = k[x_1,\dots,x_n]$ be a standard graded polynomial ring over a field, with $\deg(x_i) = 1$ for each i, and I be an ideal generated by a regular sequence of quadrics. Let M be the module of a higher graded matrix factorization (d,h) over $R = S/I$. The Poincaré series $\mathcal{P}^R_M(x) = \sum_{i\geq 0} b^R_i(M)x^i$ of M over R is*

$$\mathcal{P}^R_M(x) = \sum_{1\leq p\leq c} \frac{1}{(1-x^2)^{c-p+1}}\Big(x\,\mathrm{rank}\ B_1(p) + \mathrm{rank}\ B_0(p)\Big)\,.$$

The graded Poincaré series $\mathcal{P}^R_M(x,z) = \sum_{i\geq 0} b^R_{i,j}(M)x^iz^j$ of M over R is

$$\mathcal{P}^R_M(x,z) = \sum_{1\leq p\leq c} \frac{z^j}{(1-x^2z^2)^{c-p+1}}\Big(xz\,m_{p;1}(z) + m_{p;0}(z)\Big)\,, \tag{5.2}$$

where, for each $s = 0,1$ and $1 \leq p \leq c$,

$$m_{p;s}(z) := \sum_{j\geq 0} b_{p;s,j}\,z^j$$

is the polynomial whose j-th coefficient $b_{p;s,j}$ is equal to the number of minimal generators of degree $j+s$ of the S-free module $B_s(p)$.

We call a minimal R-free resolution *piecewise linear* if the entries in the differential matrices are linear forms. A free resolution is piecewise linear if and only if it is a direct sum of linear free resolutions:

Lemma 5.3. *Let $\mathbf{U}$ be a piecewise linear R-free resolution. If*

$$U_0 = R(-a_1)^{b_{0,a_1}} \oplus \cdots \oplus R(-a_m)^{b_{0,a_m}}\ ,$$

then $\mathbf{U} = \mathbf{U}(1) \oplus \cdots \oplus \mathbf{U}(m)$, where each $\mathbf{U}(i)$ is a linear R-free resolution of a module generated in degree a_i.

Proof. For every $j \geq 0$ we have $U_j = \oplus_q R(-q)^{b_{j,q}}$. For each $i \leq m$, set $\mathbf{U}(i)$ to be the subcomplex of $\mathbf{U}$ such that $\mathbf{U}(i)_j = R(-(a_i + j))^{b_{j,a_i+j}}$. This is a subcomplex since the resolution $\mathbf{U}$ is piecewise linear. $\square$

In the notation of Definition 3.1, we say that a higher matrix factorization (d, h) is *linear* (or more precisely, *q-linear*) if d and h have linear entries and A_0 is generated in degree q. In this case, we say that $\mathrm{Coker}(d) \otimes R$ is a *linear* (or more precisely, *q-linear*) higher matrix factorization module. We have:

Corollary 5.4. *The module of a q-linear higher matrix factorization (d, h), with respect to a quadratic regular sequence, has a q-linear minimal R-free resolution.*

Proof. Because the higher matrix factorization is linear, the homotopies used in the construction of the minimal R-free resolution as given in [9, Chapter 5] may be taken to be linear, and thus M has a q-linear free resolution.

The result also follows directly from formula (5.2) in Theorem 5.1. $\square$

Theorem 5.5 shows that the linearization of a sufficiently high truncation of any minimal resolution is a finite sum of resolutions of linear higher matrix factorization modules. Herzog and Iyengar [10] proved that the linearization of a sufficiently high truncation splits as a sum of linear resolutions; thus our result gives a more precise description of the linear resolutions that occur (after a slight further truncation).

Theorem 5.5. *Let $S = k[x_1, \ldots, x_n]$ be standard graded with $\deg(x_i) = 1$ for each i, and I be an ideal generated by a regular sequence of quadrics. Let M be a higher graded matrix factorization module over $R = S/I$. The Betti table of M is a finite sum of Betti tables of linear higher matrix factorization modules.*

Proof. Suppose that M is the module of a minimal higher matrix factorization (d, h) with respect to $f_1, \ldots, f_c$. We adopt the notation of 3.1. Let d'_p and h'_p be the linearizations of d and h respectively, that is, d'_p and h'_p are obtained by erasing all the terms of degree greater than 1 in the matrices d_p and h_p respectively. Then d'_p and h'_p form a new matrix factorization (d', h') with the same underlying modules $A_s(p)$. Moreover, (d', h') is a direct sum of linear higher matrix factorizations. By Proposition 5.1, the Betti table of M is the sum of the Betti tables of the higher matrix factorization modules corresponding to these linear summands. $\square$

Combining Theorem 3.2, Proposition 5.1, and Theorem 5.5, we get:

Theorem 5.6. *Let k be an infinite field, $S = k[x_1, \ldots, x_n]$ be standard graded with $\deg(x_i) = 1$ for each i, and I be an ideal generated by a regular sequence of c quadrics. Let N be a finitely generated graded S/I-module. If M is a sufficiently high graded syzygy of N over $R := S/I$, then the graded Poincaré series of M over R is*

$$\mathcal{P}_M^R(x,z) = \sum_q \sum_{1\le p\le c} \frac{z^q}{(1-x^2z^2)^{c-p+1}}\Big(xz\,\mathrm{rank}\,B_1^q(p) + \mathrm{rank}\,B_0^q(p)\Big)\,, \tag{5.7}$$

where $B_1^q(p), B_0^q(p)$ are the free modules in a q-linear higher matrix factorization and q ranges over the degrees in a minimal set of generators of M.

The graded Betti numbers of a graded finitely generated module over a quadratic complete intersection are eventually given by the formula in Theorem 5.6. They are expressed in terms of the ranks of the free modules $B_0(p)$ and $B_1(p)$ (where $1 \le p \le c$) appearing in a linear higher matrix factorization in the notation of 3.1. Next, we will show how to obtain the ranks of the modules $B_1(p)$ from the ranks of the modules $B_0(p)$.

Theorem 5.8. *Let $S = k[x_1,\dots,x_n]$ be standard graded with $\deg(x_i) = 1$ for each i, and $f_1,\dots,f_c$ be a regular sequence of quadrics. Let M be the module of a linear higher matrix factorization (d,h) over $R = S/(f_1,\dots,f_c)$. The ranks of the modules $B_1(p)$ are determined by the ranks of the modules $B_0(p)$; in fact,*

$$\mathrm{rank}\,B_1(p) = \sum_{j=0}^{p-1} 2^{-2j}\binom{2j}{j}\mathrm{rank}\,B_0(p-j)$$

for every p.

For example, when $c = 6$, the formula above gives:

$$\begin{pmatrix}\mathrm{rank}\,B_1(1)\\ \vdots\\ \vdots\\ \vdots\\ \mathrm{rank}\,B_1(6)\end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0\\ 1/2 & 1 & 0 & 0 & 0 & 0\\ 3/8 & 1/2 & 1 & 0 & 0 & 0\\ 5/16 & 3/8 & 1/2 & 1 & 0 & 0\\ 35/128 & 5/16 & 3/8 & 1/2 & 1 & 0\\ 63/256 & 35/128 & 5/16 & 3/8 & 1/2 & 1\end{pmatrix}\begin{pmatrix}\mathrm{rank}\,B_0(1)\\ \vdots\\ \vdots\\ \vdots\\ \mathrm{rank}\,B_0(6)\end{pmatrix}.$$

Proof. By [9, Corollary 5.6] the Betti numbers of M over R are given by the following two polynomials in z:

$$b_{2z}^R(M) = \sum_{1\le p\le c}\binom{c-p+z}{c-p}\mathrm{rank}\,B_0(p)$$

$$b_{2z+1}^R(M) = \sum_{1\le p\le c}\binom{c-p+z}{c-p}\mathrm{rank}\,B_1(p)\,.$$

To simplify the notation we set $r(p) = \mathrm{rank}\,B_0(p)$. Since a quadratic complete intersection is a Koszul algebra, it follows that the Betti numbers of M over R are given by one polynomial $E(z)$ of degree $\le c$, see [1]. Since

$$E(2z) = b_{2z} = \sum_{1\le p\le c} \binom{c-p+z}{c-p} r(p)$$

we must have

$$E(2z+1) = \sum_{1\le p\le c} \binom{c-p+z+1/2}{c-p} r(p);$$

that is,

$$\sum_{1\le p\le c} \binom{c-p+z+1/2}{c-p} r(p) = \sum_{1\le p\le c} \binom{c-p+z}{c-p} \operatorname{rank} B_1(p)\,.$$

For $1 \le c' \le c$ the restrictions of d and h to the $A_i(c')$ form a higher matrix factorization of codimension c', so in fact

$$\sum_{1\le p\le c'} \binom{c'-p+z+1/2}{c'-p} r(p) = \sum_{1\le p\le c'} \binom{c'-p+z}{c'-p} \operatorname{rank} B_1(p)$$

for all $1 \le c' \le c$.

For $c' = 1$ this gives $\operatorname{rank} B_1(1) = r(1)$ as required. Moreover, the coefficient of $\operatorname{rank} B_1(c')$ in the equation is 1, so the c equations above inductively determine $\operatorname{rank} B_1(p)$ in terms of the numbers $\operatorname{rank} B_1(q)$ for $q < p$ and all the $r(q')$ for $q' \le p$. By induction, it suffices to show that the values for $\operatorname{rank} B_1(p)$ given in the Theorem satisfy the c-th equation above; that is,

$$\sum_{1\le p\le c} \binom{c-p+z+1/2}{c-p} r(p) = \sum_{1\le p\le c} \binom{c-p+z}{c-p} \sum_{j=0}^{p-1} 2^{-2j} \binom{2j}{j} r(p-j).$$

Since these equations are linear functions of the vector $(r(1),\dots,r(p))$, it suffices to check, for each $1 \le q \le c$, that the equation holds when $r(q) = 1$ while $r(p) = 0$ for $p \ne q$. In this case the summand on the left vanishes except when $p = q$, while the summand on the right vanishes except when $j = p - q$, so it suffices to show that

$$\binom{c-q+z+1/2}{c-q} = \sum_{q\le p\le c} 2^{-2(p-q)} \binom{2(p-q)}{p-q} \binom{c-p+z}{c-p}.$$

Equivalently, setting $m = c - q$ and $i = p - q$ we have to prove that

$$\binom{m+z+1/2}{m} = \sum_{i=0}^{m} 2^{-2i} \binom{2i}{i} \binom{m-i+z}{m-i}.$$

The following proof of this identity was generously communicated to us by Joe Buhler: Define the “lower difference operator” Δ on a function F by

$$\Delta F(z) := F(z) - F(z-1).$$

If $F(z)$ is a polynomial of degree at most e, then

$$F(z) = \sum_{v=0}^{e} \Delta^v F(-1) \binom{z+v}{v}.$$

It thus suffices to show that, if $F(z) = \binom{z+1/2+m}{m}$ then

$$\Delta^v F(-1) = 2^{-2(m-v)} \binom{2(m-v)}{m-v}. \tag{5.9}$$

We will prove (5.9). It is easy to see by induction that

$$\Delta^v F(z) = \binom{z+1/2+m-v}{m-v},$$

and hence it suffices to consider (5.9) in the case $v = 0$, that is, to show that $F(-1) = 2^{-2m}\binom{2m}{m}$. It is immediate that

$$2^m F(-1) = \frac{1 \cdot 3 \cdot 5 \cdots (2m-1)}{m!}.$$

Furthermore,

$$\begin{aligned} 2^m \frac{1 \cdot 3 \cdot 5 \cdots (2m-1)}{m!} &= 2\binom{2m-1}{m} \\ &= \binom{2m-1}{m} + \binom{2m-1}{m-1} = \binom{2m}{m}, \end{aligned}$$

yielding the desired formula. □

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