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Tor as a module over an exterior algebra

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Abstract. Let S be a regular local ring with residue field k and let M be a finitely generated S-module. Suppose that $f_1, \dots, f_c \in S$ is a regular sequence that annihilates M, and let E be an exterior algebra over k generated by c elements. The homotopies for the f_i on a free resolution of M induce a natural structure of graded E-module on $\mathrm{Tor}^S(M, k)$. In the case where M is a high syzygy over the complete intersection $R := S/(f_1, \dots, f_c)$ we describe this E-module structure in detail, including its minimal free resolution over E.

Turning to $\mathrm{Ext}_R(M, k)$ we show that, when M is a high syzygy over R, the minimal free resolution of $\mathrm{Ext}_R(M, k)$ as a module over the ring of CI operators is the Bernstein–Gel'fand–Gel'fand dual of the E-module $\mathrm{Tor}^S(M, k)$.

For the proof we introduce *higher CI operators*, and give a construction of a (generally non-minimal) resolution of M over S starting from a resolution of M over R and its higher CI operators.

Keywords. Free resolutions, exterior algebras, Tor, Eisenbud operators

1. Introduction

Throughout this paper we write S for a regular local ring with maximal ideal $\mathfrak{m}$ and residue field k, and we let $f_1, \dots, f_c \in S$ be a regular sequence. Set $I := (f_1, \dots, f_c) \subset S$ and consider the complete intersection $R := S/I$. Let M be a finitely generated S-module annihilated by I. We denote by E the exterior algebra

$$E := \bigwedge\nolimits_k (I/\mathfrak{m}I) =: k\langle e_1, \dots, e_c\rangle.$$

The finite-dimensional graded vector space $\mathrm{Tor}^S(M, k)$ has a natural E-module structure induced by the action of homotopies for the f_i on the minimal S-free resolution of M (Section 2). For some modules M, the action of E on $\mathrm{Tor}_S(M, k)$ is trivial, but in the case where M is a high R-syzygy in the sense of [EP1] (explicit bounds are given in [EP1] and [EP2]) we prove that it is highly nontrivial:

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(i) We prove that the E-module $\operatorname{Tor}^S(M,k)$ is generated by $\operatorname{Tor}_0^S(M,k)$ and $\operatorname{Tor}_1^S(M,k)$, and its (Castelnuovo–Mumford) regularity is 1 (Corollary 5.1 and Theorem 5.3).

(ii) Let

$$T' := E\cdot \operatorname{Tor}_0^S(M,k) \subset \operatorname{Tor}^S(M,k) \quad \text{and} \quad T'' := \operatorname{Tor}^S(M,k)/T'.$$

Assuming that k is infinite and the generators of $(f_1,\dots,f_c)$ are chosen generally, we compute vector space bases of T' and T'', and show that, as E-modules, T' and T'' have Gröbner deformations to direct sums of copies of $E/(e_p,\dots,e_c)$ for $p=1,\dots,c$ (Theorem 4.6). It follows that even when k is finite, T' and T'' have linear E-free resolutions, given explicitly in (iv) below.

(iii) We prove that the Betti numbers of the 0-linear strand of the minimal E-free graded resolution of $\operatorname{Tor}^S(M,k)$ are given by the even Betti numbers of M over R, and the Betti numbers of the 1-linear strand are given by the odd Betti numbers of M over R:

$$\begin{aligned}\beta^E_{i,i}(\operatorname{Tor}^S(M,k)) &= \beta^R_{2i}(M),\\ \beta^E_{i,i+1}(\operatorname{Tor}^S(M,k)) &= \beta^R_{2i+1}(M)\end{aligned}$$

(Theorem 4.8).

(iv) We show that the numerical statement in (iii) is a consequence of the structure of the minimal E-free resolution of $\operatorname{Tor}^S(M,k)$ by proving that the resolution is the mapping cone:

$$\begin{tikzcd}
\cdots \arrow[r, "t_2"] & \operatorname{Tor}_4^R(M,k)\otimes_R E \arrow[r, "t_2"] & \operatorname{Tor}_2^R(M,k)\otimes_R E \arrow[r, "t_2"] & \operatorname{Tor}_0^R(M,k)\otimes_R E \\
\cdots & \bigoplus & \bigoplus & \bigoplus \\
\cdots \arrow[r, "t_2"] & \operatorname{Tor}_5^R(M,k)\otimes_R E \arrow[r, "t_2"] \arrow[uur, "t_3"] & \operatorname{Tor}_3^R(M,k)\otimes_R E \arrow[r, "t_2"] \arrow[uur, "t_3"] & \operatorname{Tor}_1^R(M,k)\otimes_R E
\end{tikzcd}$$

(Theorem 9.2, see also Theorem 4.6(iii)) where the two rows are themselves minimal linear free resolutions of the E-submodule T' and the quotient T''. The maps labeled t_2 are the CI (= Complete Intersection) operators (also called Eisenbud operators), while the maps labeled t_3 between the two strands are some of the higher CI operators, introduced in Section 7.

A curious consequence of (iv) is that if M is a high syzygy over R, then the *first* syzygy over E of the E-module $\operatorname{Tor}^S(M,k)$ is $\operatorname{Tor}^S(L,k)$, where L is the *second* syzygy of M as an R-module. For a slightly sharper statement see Corollary 9.3.

Next we focus on $\operatorname{Ext}_R(M,k)$. The action of the CI operators makes the graded vector space $\operatorname{Ext}_R(M,k)$ into a finitely generated module over the ring

$$\mathcal{R} := \operatorname{Sym}_k((I/\mathfrak{m}I)^\vee) =: k[\chi_1,\dots,\chi_c].$$

In Theorem 9.4 we prove that when M is a high R-syzygy, the minimal $\mathcal{R}$-free resolution of $\operatorname{Ext}_R^{\text{even}}(M,k)$ is obtained by the Bernstein–Gel'fand–Gel'fand (BGG) correspondence from the E-module structure of $T'^\vee$, and similarly for $\operatorname{Ext}_R^{\text{odd}}(M,k)$ and $T''^\vee$.

Corollary 9.5 does not even require the definition of T'. Write

$$\mu : E_1 \otimes_k \operatorname{Tor}_0^R(M,k) \to \operatorname{Tor}_1^R(M,k)$$

for the multiplication map and

$$\mu^\vee : \operatorname{Ext}_S^1(M,k) \to \operatorname{Ext}_S^1(M,k) \otimes \mathcal{R}_1$$

for its vector space dual. The $\mathcal{R}$-module $\operatorname{Ext}_R^{\mathrm{even}}(M,k)$ then has the (nonminimal) linear free presentation

$$\operatorname{Ext}_S^1(M,k) \otimes \mathcal{R}(-1) \xrightarrow{\tau} \operatorname{Hom}(M,k) \otimes \mathcal{R} \to \operatorname{Ext}_R^{\mathrm{even}}(M,k) \to 0$$

where τ is the map of free modules whose linear part is $\mu^\vee$. This follows from Theorem 9.4 because $\mu^\vee$ is 0 on the submodule $T''^\vee$.

An essential ingredient in the proofs in Section 9 is a new theory of *higher CI operators*, introduced in Section 7. Just as the Eisenbud–Shamash construction allows one to describe an R-free resolution of any R-module from the higher homotopies on an S-free resolution, one can describe an S-free resolution from the higher CI operators on an R-free resolution. This construction was discovered independently by Jessie Burke [Bu]. The differentials in the E-free resolution of $\operatorname{Tor}_S(M,k)$ are related, as above, to the higher CI operators.

We also use the “layered” structures of the minimal S-free and R-free resolutions of M [EP2], which come from the higher matrix factorizations of [EP1]. We review the necessary definitions and results about layered resolutions in Section 3.

Remark. One could often replace the hypothesis that S is regular with a hypothesis that S is Gorenstein and M has finite projective dimension over S, or that M is the module of a minimal higher matrix factorization. Moreover, the hypothesis that M is the module of a minimal higher matrix factorization could be replaced by the (possibly) more general hypothesis that the layered resolutions described in Section 3 are minimal. We leave these refinements to the interested reader.

The following example is computed using Macaulay2 code, which may be found in the documentation for the function “exteriorTorModule” in the Macaulay2 package CompleteIntersectionResolutions.m2 [M2, Version 1.9.1 and higher].

Example 1.1. Let $S = k[\![x_1, x_2, x_3]\!]$ and $R = S/(x_1^3, x_2^3, x_3^3)$. Denote by N_i the i-th syzygy of k as an R-module. The minimal S-free resolution of $N_0 = k$ is the Koszul complex on x_1, x_2, x_3, and x_i^2 times the homotopy for x_i is a homotopy σ_i for $f_i := x_i^3$. Thus the action of E on $\operatorname{Tor}^S(k,k)$ is trivial.

By contrast, the action of E on $\operatorname{Tor}^S(N_i,k)$ is nontrivial for $i \geq 1$. The beginnings of the Betti tables of the minimal E-free resolutions of $\operatorname{Tor}^S(N_i,k)$ for $i = 1, 2, 3$ are:

Betti table of $\operatorname{Tor}^S(N_1,k)$:

total:	10	27	52	85	126	175	⋯
0:	3	9	18	30	45	63	⋯
1:	6	15	28	45	66	91	⋯
2:	1	3	6	10	15	21	⋯

Betti table of $\operatorname{Tor}^S(N_2,k)$:

total:	16	36	64	100	144	196	⋯
0:	6	15	28	45	66	91	⋯
1:	10	21	36	55	78	105	⋯

Betti table of $\operatorname{Tor}^S(N_3,k)$:

total:	25	49	81	121	169	225	⋯
0:	10	21	36	55	78	105	⋯
1:	15	28	45	66	91	120	⋯

From the first table we see that, as an E-module, $\operatorname{Tor}^S(N_1,k)$ is generated in degrees 0, 1, 2. Since N_1 is artinian, we have $\operatorname{Tor}^S_3(N_1,k) \neq 0$. Hence, the E-module structure of $\operatorname{Tor}^S(N_1,k)$ is nontrivial.

The smallest i for which N_i is a high syzygy (in the sense of [EP1]) is $i = 3$, but in fact the *layered resolution* of N_2 with respect to f_1, f_2, f_3 described in Section 3 is also minimal. The E-module $\operatorname{Tor}^S(N_i,k)$ has a free resolution with just two linear strands for $i = 2, 3$, illustrating assertion (i) above.

Further, Macaulay2 computes the Betti table of N_2 as an R-module as

total:	6	10	15	21	28	36	45	55	66	78	91	105	⋯
0:	6	10	15	21	28	36	45	55	66	78	91	105	⋯

and we see that, for $s = 0, 1$,

$$\beta^E_{i,i+s}(\operatorname{Tor}^S(N_2,k)) = \beta^R_{2i+s}(N_2),$$

which illustrates (iii).

We can also illustrate Theorem 9.4 in this context. It turns out that the homogeneous components of $T' = E \cdot \operatorname{Tor}^S_0(N_2,k)$ are $T'_0 = E^6,\ T'_1 = E^3,\ T'_2 = E^1$, and it follows that the minimal $\mathcal{R}$-free resolution of $\operatorname{Ext}^{\text{even}}_R(N_2,k)$ has the form

$$0 \to \mathcal{R}^1(-2) \xrightarrow{d_2} \mathcal{R}^3(-1) \xrightarrow{d_1} \mathcal{R}^6 \to \operatorname{Ext}^{\text{even}}_R(N_2,k) \to 0.$$

The differentials are easily computed from the action of E on $\operatorname{Tor}^S(N_2,k)$. The map

$$\langle f_1,\dots,f_3\rangle \otimes \operatorname{Tor}^S_0(N_2,k) \to \operatorname{Tor}^S_1(N_2,k)$$

given by the homotopies induces the map $E_1 \otimes T'_0 \to T'_1$, whose dual induces the map $d_1 : \mathcal{R}^3(-1) \to \mathcal{R}^6$. Computation shows that, in suitable bases, this map is given by the matrix

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \chi_1 & \chi_2 \\ -\chi_1 & 0 & \chi_3 \\ -\chi_2 & -\chi_3 & 0 \end{pmatrix}$$

where the χ_i form a dual basis to f_1, f_2, f_3. From this presentation matrix we see that $\mathrm{Ext}_R^{\mathrm{even}}(N_2, k)$ is the direct sum of $\mathcal{R}^3$ and one copy of the maximal ideal $(\chi_1, \chi_2, \chi_3) \subset \mathcal{R}$, shifted so that it is generated in degree 0. Similar conclusions hold for $\mathrm{Ext}^{\mathrm{odd}}(N_2, k)$.

Related work. Avramov and Buchweitz [AB] made use of the simple classification of modules over an exterior algebra on two generators to study free resolutions of modules over complete intersections of codimension 2, and this study is carried further in [AY]. For other points of view on the module structure of Tor see [Da, HW]. For further results on resolutions over exterior algebras, see for example [AI, Ei2, Fl].

2. Homotopies and the action of the exterior algebra on Tor

In this section we review the action of the exterior algebra on Tor. We will use the notation at the beginning of the Introduction.

For each i we choose a homotopy σ_i for f_i on a free resolution $\mathbf{F}$ of the module M. Up to homotopy, the homotopies σ_i anticommute and square to 0—see for example [EP1, Proposition 3.4.2]. Though the σ_i are not maps of complexes, they become maps of complexes when tensored with an S-module N annihilated by $f_1, \dots, f_c$, so $\sigma_i \otimes 1$ takes cycles in the complex $\mathbf{F} \otimes_S N$ to cycles, while raising the homological degree by 1. Thus the action of the σ_i gives $H_*(\mathbf{F} \otimes N) = \mathrm{Tor}^S(M, N)$ the structure of a graded module over the exterior algebra $\bigwedge_S(I)$. The action factors through an action of $\bigwedge_S(I/I^2)$ because $I \cdot \mathrm{Tor}^S(M, N) = 0$.

As an example, consider the Koszul complex $\mathbf{K} = \mathbf{K}(f_1, \dots, f_c)$. Denote by e_i the basis element of $\mathbf{K}_1$ that maps to f_i. An immediate computation shows that multiplication by e_i is a homotopy for f_i. These homotopies anticommute and square to 0, making $\mathbf{K}$ a free module over the exterior algebra $S \otimes_k \bigwedge_k k^c = S\langle e_1, \dots, e_c\rangle$.

The action of $\bigwedge(I/I^2)$ on $\mathrm{Tor}^S(M, N)$ is functorial by Lemma 2.1 below.

Lemma 2.1. *Let* $(\mathbf{F}, \partial) : \cdots \to F_1 \to F_0$ *and* $(\mathbf{G}, d) : \cdots \to G_1 \to G_0$ *be complexes of S modules, and suppose that* $\mathbf{F}$ *and* $\mathbf{G}$ *admit homotopies* σ *for some element* $f \in S$, *respectively. Let* $\varphi : \mathbf{F} \to \mathbf{G}$ *be a map of complexes. If the* F_i *are free and the complex* $\mathbf{G}$ *is acyclic, then there are maps* $\alpha_i : F_i \to G_{i+2}$ *such that* $\varphi\sigma - \tau\varphi = d\alpha - \alpha\partial$.

Thus, if N is a module annihilated by f, the maps $\varphi\sigma$ and $\tau\varphi$ are homotopic maps of complexes; in particular, the maps

$$\mathrm{H}(\mathbf{F} \otimes N) \to \mathrm{H}(\mathbf{G} \otimes N) \quad \textit{and} \quad \mathrm{H}(\mathrm{Hom}(\mathbf{G}, N)) \to \mathrm{H}(\mathrm{Hom}(\mathbf{F}, N))$$

induced by φ commute with the action of the homotopies.

The proof is an immediate computation.

For example, taking φ to be the identity on a free resolution of M, we see that the induced action of $\bigwedge_S(I/I^2)$ on $\mathrm{Tor}_S(M, N)$ is independent of the choice of homotopies.

We note that $\bigwedge_R(I/I^2) = \bigwedge \mathrm{Tor}_1^S(R, R)$, and one can see the above action on Tor as being, up to sign, induced by the action of the algebra $\mathrm{Tor}^S(R, R)$ on $\mathrm{Tor}^S(M, N)$. This is a special case of the natural product $\mathrm{Tor}^S(A, B) \otimes \mathrm{Tor}^S(M, N) \to \mathrm{Tor}^S(A \otimes B, M \otimes N)$ defined in the book of Cartan and Eilenberg [CE, Chapter XI, Section 1], as one can prove from the fact that homotopies on a tensor product complex can be defined from homotopies on one factor. In particular, the E-module structure on $\mathrm{Tor}^S(M, N)$ computed from homotopies on a resolution of M is, up to sign, the same as that computed from a resolution of N. We will not use these facts.

In this paper we focus on the structure of $\mathrm{Tor}^S(M, k)$, where S is a regular local ring with residue field k, and M is annihilated by $f_1, \dots, f_c \in S$. Since $\mathrm{Tor}^S(M, k)$ is annihilated by the maximal ideal, it may be regarded as a graded module over the exterior algebra $E := \bigwedge_k(I/\mathfrak{m}I) =: k\langle e_1, \dots, e_c\rangle$.

3. Layered resolutions

Continuing with the notation of the introduction, we consider a regular local ring S and a complete intersection $R = S/(f_1, \dots, f_c)$ of codimension c. Throughout the paper all the modules are assumed finitely generated.

Let M be a Cohen–Macaulay S-module of codimension c and let $\mathbf{f} := f_1, \dots, f_c$ be a regular sequence in the annihilator of M. In [EP2] we construct an S-free resolution $\mathbf{L}{\uparrow}^S(M, \mathbf{f})$ and an R-free resolution $\mathbf{L}{\downarrow}_R(M, \mathbf{f})$, called *layered resolutions* of M. In this section we recall the features of the construction that will play a role in this paper.

The importance of the layered resolutions comes from the following result:

Theorem 3.1. (i) *If M is the module of a minimal higher matrix factorization with respect to $f_1, \dots, f_c$ in the sense of* [EP1, *Definition* 1.2.2]*, then $\mathbf{L}{\uparrow}^S(M, \mathbf{f})$ and $\mathbf{L}{\downarrow}_R(M, \mathbf{f})$ are minimal resolutions.*

(ii) *Suppose that the ground field k is infinite. If M is a sufficiently high R-syzygy of an R-module N and the elements $f_1', \dots, f_c'$ are sufficiently general among generators of $(f_1, \dots, f_c)$, then M is the module of a minimal higher matrix factorization with respect to $f_1', \dots, f_c'$.*

Proof. See [EP1, Theorems 1.3.1, 3.1.4, 5.1.2] and [EP2]. ◻

The S-free layered resolution. First we will review that $\mathbf{L}{\uparrow}^S(M, \mathbf{f})$ has a filtration by acyclic free subcomplexes that are resolutions of maximal Cohen–Macaulay modules over the intermediate rings $R(p) := S/(f_1, \dots, f_p)$. Let $M'(p) \to M$ be the maximal Cohen–Macaulay $R(p)$-approximation of M in the sense of [AB]. We may write $M'(p) = M(p) \oplus R(p)^{m_p}$, where $M(p)$ has no free summand. Following [EP1, Definition 7.3.1] we call $M(p)$ the *essential MCM approximation* of M over $R(p)$. Let $\mathbf{L}(p) := \mathbf{L}{\uparrow}^S(M(p), f_1, \dots, f_p)$ be the layered S-free resolution of $M(p)$. By [EP1, Corollary 7.3.4] the essential Cohen–Macaulay approximation of $M(p)$ over $R(p-1)$ is

$M(p-1)$, and thus we have maps

$$0 = M(0) \to M(1) \to \cdots \to M(c) = M. \tag{3.2}$$

They induce inclusions of complexes

$$0 = \mathbf{L}(0) \subset \mathbf{L}(1) \subset \cdots \subset \mathbf{L}(c) =: \mathbf{L},$$

with quotients

$$\mathbf{L}(p)/\mathbf{L}(p-1) = \mathbf{K}(f_1, \ldots, f_{p-1}) \otimes_S \mathbf{B}(p), \tag{3.3}$$

where $\mathbf{K}(f_1, \ldots, f_{p-1})$ is the Koszul complex on $f_1, \ldots, f_{p-1}$, and $\mathbf{B}(p)$ is a two-term free complex of the form

$$\mathbf{B}(p) : B_1(p) \xrightarrow{b_p} B_0(p). \tag{3.4}$$

Thus, as free S-modules,

$$\mathbf{L}(p) = \mathbf{L}(p-1) \oplus S\langle e_1, \ldots, e_{p-1}\rangle \otimes_S \mathbf{B}(p). \tag{3.5}$$

In particular

$$A_0(p) := \bigoplus_{q=1}^{p} B_0(q) = \mathbf{L}(p)_0,$$

while

$$A_1(p) := \bigoplus_{q=1}^{p} B_1(q) \subset \mathbf{L}(p)_1$$

is a summand of $\mathbf{L}(p)_1$.

Let

$$E(p) := E/(e_{p+1}, \ldots, e_c) = k\langle e_1, \ldots, e_p\rangle.$$

From Lemma 2.1 we deduce

Lemma 3.6. *For $1 \le p \le c-1$, the inclusion $\mathbf{L}(p) \subset \mathbf{L}$ induces an inclusion*

$$\operatorname{Tor}_S(M(p), k) \subset \operatorname{Tor}_S(M, k)$$

of $E(p)$-modules. □

We will make use of the following property:

Proposition 3.7. *If M is the module of a minimal higher matrix factorization with respect to $f_1, \ldots, f_c$, then the homotopy σ_p for f_p on $\mathbf{L}(p)$ can be chosen so that its component*

$$h_p : \mathbf{L}(p)_0 = \bigoplus_{q=1}^{p} B_0(q) \to \bigoplus_{q=1}^{p} B_1(q) \subset \mathbf{L}(p)_1$$

is minimal.

Proof. By [EP1, Theorem 5.3.1], the higher matrix factorization (d, h) for M and the homotopy σ_p for f_p can be chosen so that h_p is a component of h. Furthermore, [EP1, Theorem 5.1.2] shows that $R \otimes h$ is the second differential in the minimal R-free resolution of M over R. Hence, h is minimal. □

As a complex, $\mathbf{L}(p)$ is a *Koszul extension* of $\mathbf{L}(p-1)$ in the sense of [EP1, Definition 3.1.1]. By definition, this is the mapping cone of a map of complexes

$$\Psi : \mathbf{K}(f_1, \dots, f_{p-1}) \otimes_S \mathbf{B}(p)[-1] \to \mathbf{L}(p-1)$$

that is zero on $\mathbf{K}(f_1, \dots, f_{p-1}) \otimes_S B_0(p)$.

Example 3.8. We illustrate the constructions above in the codimension 2 case. When $c = 2$, the resolution $\mathbf{L}{\uparrow}^S(M, f_1, f_2)$ may be represented by the diagram

$$\begin{tikzcd}
& B_1(1) \arrow[r, "b_1"] & B_0(1) \\
& \oplus & \oplus \\
& B_1(2) \arrow[r, "b_2"] \arrow[uur, "\psi_2"] & B_0(2) \\
& \oplus & \\
e_1B_1(2) \arrow[r, "b_2"'] \arrow[uuuur] \arrow[uur, "f_1"] & e_1B_0(2) \arrow[uur, "-f_1"'] &
\end{tikzcd}$$

In the notation above, $\mathbf{L}(1)$ is the two-term complex

$$B_1(1) \xrightarrow{b_1} B_0(1)$$

and $\mathbf{K}(f_1) \otimes \mathbf{B}(2)$ is the complex

$$\begin{tikzcd}
& B_1(2) \arrow[r, "b_2"] & B_0(2) \\
& \oplus & \\
e_1B_1(2) \arrow[r, "b_2"'] \arrow[uur, "f_1"] & e_1B_0(2) \arrow[uur, "-f_1"'] &
\end{tikzcd}$$

where e_1 denotes the basis element of $I/\mathfrak{m}I$ corresponding to f_1 and we have written three underlying free modules of $\mathbf{K}(f_1) \otimes \mathbf{B}(2)$, in homological degrees 2, 1, 0, as

$$e_1B_1(2) \to B_1(2) \oplus e_1B_0(2) \to B_0(2).$$

The R-free layered resolution. The maps (3.2) induce inclusions of complexes

$$0 = \mathbf{T}(0) \subset R \otimes \mathbf{T}(1) \subset \cdots \subset R \otimes \mathbf{T}(c) =: \mathbf{T},$$

where $\mathbf{T}(p)$ is the layered resolution of $M(p)$ over $R(p)$. By [EP2], $\mathbf{T}(p+1)$ is obtained from $\mathbf{T}(p)$ by the Shamash construction applied to the box complex

$$\begin{tikzcd}
& \cdots \arrow[r] & T(p)_2 \arrow[r, "\partial_2"] & T(p)_1 \arrow[r, "\partial_1"] & T(p)_0 \\
T'(p): & & & \oplus & \oplus \\
& & & R(p) \otimes B_1(p) \arrow[r, "b_p"'] \arrow[uur, "\psi_0"] & R(p) \otimes B_0(p)
\end{tikzcd}$$

where $\mathbf{B}(p)$ is the two-term complex from (3.4). In particular, the following property holds.

Proposition 3.9. *The CI operator* $t_j : \mathbf{T} \to \mathbf{T}$ *for* f_j *on the layered resolution* $\mathbf{T}$ *can be chosen so that, for* $j \le p$, *it preserves the box complex* $\mathbf{T}'(p)$ *as a subcomplex of* $\mathbf{T}$, *and its components*

$$\begin{aligned} R \otimes T(p)_2 &\to R \otimes B_0(p), \\ R \otimes T(p)_3 &\to R \otimes B_1(p) \end{aligned}$$

are zero. The dual maps $\chi_j : \mathrm{Hom}(\mathbf{T}, R) \to \mathrm{Hom}(\mathbf{T}, R)$ *then vanish on* $\mathrm{Hom}_R(B_0(p), R)$ *and* $\mathrm{Hom}_R(B_1(p), R)$. ☐

4. The structure of Tor

Throughout this section, as in the introduction, we assume that S is a regular local ring, $R = S/(f_1, \ldots, f_c)$ is a complete intersection of codimension c, and modules are finitely generated. Write $\mathbf{f} := f_1, \ldots, f_c$. We consider a module M that is the module of a minimal higher matrix factorization with respect to $\mathbf{f}$. We write $T' := E \cdot \mathrm{Tor}_0^S(M, k)$ for the E-submodule of $\mathrm{Tor}^S(M, k)$ generated by $\mathrm{Tor}_0^S(M, k)$ and set $T'' := \mathrm{Tor}^S(M, k)/T'$. For each module in the short exact sequence

$$0 \to T' \to \mathrm{Tor}^S(M, k) \to T'' \to 0,$$

we will identify a vector space decomposition, the minimal generators as an E-module, a Gröbner basis for the relations, and the ranks of the free modules in a minimal E-free resolution. In Section 9 we will determine the structure of the resolutions themselves.

Notation 4.1. In addition to the notation and hypotheses above, we adopt the notations $R(p)$, $M(p)$, $\mathbf{L}(p)$, $B_0(p)$, $B_1(p)$, $E(p)$ of Section 3. We set

$$A_s(p) = \bigoplus_{q=1}^{p} B_s(q)$$

for $s = 0, 1$, as in Section 3, and $A_s := A_s(c)$. We write $\overline{}$ for $k \otimes -$.

Since the differential $B_1(p) \to B_0(p)$ is minimal, we may regard $\overline{\mathbf{B}(p)}$ as the direct sum $\overline{B_1(p)} \oplus \overline{B_0(p)}$.

The next result is the key to the E-module structure of Tor:

Theorem 4.2. *Let the notation and hypotheses be as in 4.1. For* $0 \le p \le c - 1$ *there is an isomorphism of* $E(p)$*-modules*

$$\mathrm{Tor}^S(M(p+1), k) \cong \mathrm{Tor}^S(M(p), k) \oplus \big(E(p) \otimes_k \overline{B_0(p+1)}\big) \oplus \big(E(p) \otimes_k \overline{B_1(p+1)}\big),$$

where the action of $E(p)$ *on the second and third summands is by multiplication on the tensor factor* $E(p)$.

Proof. The minimal free resolution $\mathbf{L}(p+1)$ of $M(p+1)$ is a Koszul extension of $\mathbf{L}(p)$ by $\mathbf{K}(f_1,\dots,f_p)\otimes_S \mathbf{B}(p+1)$, that is, the mapping cone of a map

$$\psi : \mathbf{K}(f_1,\dots,f_p)\otimes_S \mathbf{B}(p+1)[-1] \to \mathbf{L}(p)$$

such that the induced map $\mathbf{K}(f_1,\dots,f_p)\otimes_S B_0(p+1) \to \mathbf{L}(p)$ is zero, as in [EP1, 3.1.1].

It follows that we may also regard $\mathbf{L}(p+1)$ as the mapping cone of a map

$$\psi' : \mathbf{K}(f_1,\dots,f_p)\otimes_S B_1(p+1)[-1] \to \mathbf{L}(p)\oplus \mathbf{K}(f_1,\dots,f_p)\otimes_S B_0(p+1)[-1].$$

where the target complex is a direct sum, as complexes. We can define homotopies for $f_1,\dots,f_p$ on $\mathbf{K}(f_1,\dots,f_p)\otimes_S B_0(p+1)$ and on $\mathbf{K}(f_1,\dots,f_p)\otimes_S B_1(p+1)$ by simple multiplication on the tensor factor $\mathbf{K}(f_1,\dots,f_p)$.

We can apply Lemma 2.1 to each of the maps in the exact sequence of the mapping cone

$$0 \to \mathbf{L}(p)\oplus (\mathbf{K}(f_1,\dots,f_p)\otimes_S B_0(p+1)) \to \mathbf{L}(p+1) \\ \to \mathbf{K}(f_1,\dots,f_p)\otimes_S B_1(p+1) \to 0.$$

Thus, tensoring over S with the residue field k, we get an exact sequence of $E(p)$-modules

$$0 \to \overline{\mathbf{L}(p)} \oplus \big(E(p)\otimes_k \overline{B_0(p+1)}\big) \to \overline{\mathbf{L}(p+1)} \to E(p)\otimes_k \overline{B_1(p+1)} \to 0.$$

Since $E(p)\otimes_k \overline{B_1(p+1)}$ is a free $E(p)$-module, the sequence splits, as claimed. □

Note that Theorem 4.2 does not assert that the "obvious" copy of $E(p)\otimes_k \overline{B_1(p+1)}$ in $\mathrm{Tor}^S(M,k)$ is a submodule, but only that there is a submodule isomorphic to it.

Taking $p=c$, we get

Corollary 4.3.

$$\mathrm{Tor}^S(M,k) = \bigoplus_{p=1}^{c} E(p-1)\otimes_k \overline{\mathbf{B}(p)}$$

as vector spaces. The subspace $E(p-1)\otimes_k \overline{\mathbf{B}(p)}$ *is an* $E(p-1)$*-submodule and the action of* $E(p-1)$ *is via the left tensor factor. In particular,* $\mathrm{Tor}^S(M,k)$ *is generated as an* E*-module in degrees* 0 *and* 1*, by*

$$\overline{A_0\oplus A_1} = \bigoplus_{p=1}^{c} \overline{\mathbf{B}(p)}.$$ □

We can now give a Gröbner basis of relations for $\mathrm{Tor}^S(M,k)$:

Theorem 4.4. *Suppose that* M *is the module of a minimal higher matrix factorization for a regular sequence* $f_1,\dots,f_c$*. Let* $1\le p\le c$*. For every*

$$a \in \overline{\mathbf{B}(p)}$$

and every $r\ge p$ *there is a homogeneous relation on* $\mathrm{Tor}^S(M,k)$ *of the form*

$$e_r a - b \tag{4.5}$$

with $b \in (e_1, \dots, e_{r-1})\big(\overline{A_0 \oplus A_1}\big)$. *These relations form a Gröbner basis for the relations on* $\operatorname{Tor}^S(M, k)$ *as an* E*-module with respect to any term order that refines the lexicographic order on the monomials of* E *with* $e_c \succ \cdots \succ e_1$. *The* E*-module defined by the leading terms of these relations is*

$$\bigoplus_{p=1}^{c} E/(e_p, \dots, e_c) \otimes_k \overline{\mathbf{B}(p)}.$$

Proof. By Lemma 3.6 it suffices to consider the $E(r)$-module $H\big(\overline{\mathbf{L}(r)}\big)$. Corollary 4.3 shows that this module can be written as

$$\operatorname{Tor}^S(M(r), k) \cong H\big(\overline{\mathbf{L}(r)}\big) = E(r-1) \cdot \big(\overline{A_1(r) \oplus A_0(r)}\big).$$

It follows that there exists a relation on $\operatorname{Tor}^S(M, k)$ of the form $e_r a - b$ with b in $E(r-1)\big(\overline{A_0(r) \oplus A_1(r)}\big)$.

If b had a nonzero component in $\overline{A_0 \oplus A_1}$, then we would have $a \in \overline{B_0(p)}$, which contradicts Proposition 3.7. Thus $e_r a$ is the leading term of the relation in the monomial order $\succ$.

If we factor out these leading terms from the free E-module generated by $\overline{A_0 \oplus A_1}$, we obtain the module

$$\bigoplus_{1 \le p \le c} E(p-1) \otimes_k \overline{\mathbf{B}(p)}.$$

By Corollary 4.3, this has the same vector space dimension as the E-module $\operatorname{Tor}^S(M, k)$. Therefore, the given relations form a Gröbner basis for the module of relations. ☐

We can now prove assertions (i) and (ii) of the Introduction:

Theorem 4.6. *Suppose that* M *is the module of a minimal higher matrix factorization for a regular sequence* $f_1, \dots, f_c$.

(i) *The submodule* $T' := E \cdot \operatorname{Tor}_0^S(M, k)$ *has underlying vector space*

$$T' = \bigoplus_{p=1}^{c} E(p-1) \otimes_k \overline{B_0(p)},$$

and thus the quotient $T'' := \operatorname{Tor}^S(M, k)/T'$ *has underlying vector space*

$$T'' \cong \bigoplus_{p=1}^{c} E(p-1) \otimes_k \overline{B_1(p)},$$

where, for $s = 0, 1$, *the action of* $E(p-1)$ *on the summand* $E(p-1) \otimes_k \overline{B_s(p)}$ *is by multiplication on the left tensor factor.*

(ii) *The module* T' *is generated by* $\overline{A_0}$, *in degree* 0, *while* T'' *is generated by* $\overline{A_1}$, *in degree* 1. *Both* T' *and* T'' *have linear* E*-free resolutions.*

(iii) *The minimal free resolution of* $\operatorname{Tor}^S(M,k)$ *as an* E*-module is the mapping cone of a map from the minimal free resolution of* T'' *(shifted by* -1*) to the minimal free resolution of* T'.

(iv) *The relations given in* (4.5) *with* $a \in \overline{A_0}$ *form a Gröbner basis of the relations on the* E*-module* T'*, and those with* $a \in \overline{A_1}$ *form a Gröbner basis of the relations on the* E*-module* T''.

We will make use of the following well-known lemma:

Lemma 4.7. *The minimal* E*-free resolution of* $E(p) = E/(e_{p+1},\dots,e_c)$ *has underlying free module*

$$E \otimes_k \operatorname{Hom}_{\mathrm{gr}}(k[x_{p+1},\dots,x_c],k),$$

where $k[x_{p+1},\dots,x_c]$ *denotes the polynomial ring on* $c-p$ *variables of homological and internal degree* 1 *generating a vector space that is dual to* $\operatorname{span}\langle e_{p+1},\dots,e_c\rangle$.

Proof. The case $p = 0$ is the resolution of the residue field k. This resolution is the "generalized Koszul complex" of Priddy and others—see for example [Ei3, Exercise 17.22].

The minimal resolution of $E(p)$ as an E-module is easily seen to be the tensor product, over k, of $E(p)$ with the minimal resolution of k as a module over the exterior algebra $k\langle e_{p+1},\dots,e_c\rangle$. □

Proof of Theorem 4.6. By Theorem 4.3,

$$U := \bigoplus_{p=1}^{c} E(p-1) \otimes_k \overline{B_0(p)} \subseteq T'.$$

The homogeneous relations (4.5) with $a \in \overline{A_0}$ must have $b \in \overline{A_0}$ as well, so they are relations on the free E-module $E \otimes_k \overline{A_0}$. If we factor out their leading terms $e_r a$, we obtain the E-module

$$\widehat{T'} := \bigoplus_{p=1}^{c} E/(e_p,\dots,e_c) \otimes_k \overline{B_0(p)}.$$

It has the same dimension as the vector space U, proving both that $U = T'$ and that we have a Gröbner basis for T'.

As for T'', if we factor out the leading terms of the relations (4.5) with $a \in \overline{A_1}$ from the free E-module $E \otimes_k \overline{A_1}$, we obtain the E-module

$$\widehat{T''} := \bigoplus_{p=1}^{c} E/(e_p,\dots,e_c) \otimes_k \overline{B_1(p)},$$

which has the same vector space dimension as T'', proving that these relations form a Gröbner basis for the relations on T'' as claimed. This concludes the proofs of parts (i) and (iv) of the theorem.

To prove part (ii), observe first that, by Lemma 4.7, the minimal free resolutions of the E-modules $E/(e_p,\dots,e_c)$ are linear. It follows that the minimal E-free resolutions of $\widehat{T'}$ and $\widehat{T''}$ are linear, and thus the miminal free resolutions $\mathbf{F}'$ of T' and $\mathbf{F}''$ of T'' are also linear.

It remains to prove part (iii). From the short exact sequence

$$0 \to T' \to \operatorname{Tor}^S(M,k) \to T'' \to 0$$

we see that $\operatorname{Tor}^S(M,k)$ has a free resolution that is the mapping cone of some map of complexes $\alpha : \mathbf{F}''[-1] \to \mathbf{F}'$. The j-th term F'_j of $\mathbf{F}'$ is generated in degree j, while the j-th term F''_j of $\mathbf{F}''$ is generated in degree $j+1$ since the generators $\overline{A_1}$ of T'' have degree 1. Hence, the matrices in the map α have entries of degree 2. In particular, the mapping cone is a minimal resolution of the form

$$\cdots \to F'_j \oplus F''_j \to \cdots \to F'_0 \oplus F''_0 = E \otimes_k \left(\overline{A_0 \oplus A_1}\right). \qquad \square$$

In Section 9 we will identify the resolutions $\mathbf{F}'$ and $\mathbf{F}''$ and the map α in terms of the minimal free resolution of M as an R-module. We already have enough information to interpret the Betti numbers:

Theorem 4.8. *Suppose that M is the module of a minimal higher matrix factorization for a regular sequence $f_1, \ldots, f_c$ and for $s = 1, 2$ and $p = 1, \ldots, c$, let $b_s(p) = \operatorname{rank} B_s(p)$. With notation as above, the graded Betti numbers of $\operatorname{Tor}^S(M,k)$ as an E-module are*

$$\beta^E_{i,i}(\operatorname{Tor}^S(M,k)) = \sum_{p=1}^{c} \binom{c-p+i}{c-p} b_0(p) = \dim_k \operatorname{Ext}^{2i}_R(M,k),$$

$$\beta^E_{i,i+1}(\operatorname{Tor}^S(M,k)) = \sum_{p=1}^{c} \binom{c-p+i}{c-p} b_1(p) = \dim_k \operatorname{Ext}^{2i+1}_R(M,k),$$

and these two formulas give the graded Betti numbers of T' and T'' individually.

Proof. The minimal graded E-free resolutions of T' and T'' are linear, and so their Betti numbers are equal to the Betti numbers of the modules $\widehat{T}'$ and $\widehat{T}''$ used in the proof of Theorem 4.6. These Betti numbers can be obtained from Lemma 4.7. Furthermore, the minimality of the layered resolution $\mathbf{L}{\downarrow}_R(M,\mathbf{f})$ implies the identical formula for $\dim \operatorname{Ext}^{2i}_R(M,k) = \beta^R_{2i}(M)$ and $\dim \operatorname{Ext}^{2i+1}_R(M,k) = \beta^R_{2i+1}(M)$ (see [EP1, Corollary 1.3.3]). $\square$

In experiments, we have observed that the sequence of E-modules

$$0 \to T' \to \operatorname{Tor}^S(M,k) \to T'' \to 0$$

often splits. Here is the simplest example we know where this is not the case:

Example 4.9. Let $S = k[a,b,c]$, $R = S/(a^4, b^4, c^4)$,

$$N = R \otimes_S \operatorname{Coker} \begin{pmatrix} a & b & c \\ b & c & a \end{pmatrix},$$

and let M be a sufficiently high syzygy of the R-module N. Computation using Macaulay2 shows that the dual E-module, which up to a shift in grading is $\operatorname{Ext}^S(M,k)$, has smaller Betti numbers than does the direct sum of the duals of T' and T''. In particular, the E-submodule $T' \subset \operatorname{Tor}_S(M,k)$ is not a direct summand.

5. Regularity

If V is a finite-dimensional $\mathbb{Z}$-graded vector space, we set $\max V = \max\{j \mid V_j \neq 0\}$. We define the *regularity* of a graded E-module L to be

$$\operatorname{reg}_E L := \sup_i \{\max \operatorname{Tor}_i^E(L, k) - i\}.$$

The minimal E-free resolution $\mathbf{U}$ of k is linear, so $\operatorname{Tor}^E(L, k) \cong \mathrm{H}(L \otimes \mathbf{U})$ gives $\max \operatorname{Tor}_i^E(L, k) \leq i + \max L$. Thus

$$\operatorname{reg}_E L \leq \max L.$$

From Theorem 4.6 we get the following result.

Corollary 5.1. *If M is the module of a minimal higher matrix factorization for a regular sequence $f_1, \ldots, f_c$, then*

$$\operatorname{reg}_E \operatorname{Tor}^S(M, k) = 1.$$

We will provide a short alternative proof of this result. To do this, we compare the regularity of an E-module L with the regularity of L regarded as a module over $E(p) := k\langle e_1, \ldots, e_p\rangle$, regarded as a subalgebra of E.

Theorem 5.2. *If L is a finitely generated graded E-module, then*

$$\operatorname{reg}_E L \leq \operatorname{reg}_{E(p)} L \leq \operatorname{reg}_E L + c - p.$$

Proof. First, we will prove the left inequality. Take $\mathbf{F}$ to be the tensor product over $E(p)$ of a minimal free resolution $\mathbf{G}$ of L as an $E(p)$-module with the minimal free resolution $\mathbf{D}$ of $E(p)$ as an E-module. Since the latter is split exact as a sequence of $E(p)$-modules, $\mathbf{F}$ is a (possibly nonminimal) E-free resolution of L as an E-module. By Lemma 4.7, the resolution $\mathbf{D}$ is linear. Therefore, for each i,

$$\max F_i \otimes k \leq \max_{q \leq i} \{\max G_q \otimes k\} + (i - q) \leq \operatorname{reg}_{E(p)} L + i.$$

For the second inequality, note that $E = E(p) \otimes_k k\langle e_{p+1}, \ldots, e_c\rangle$ is a free $E(p)$-module with generators in degrees $\leq c - p$, so a minimal E-free resolution is a (possibly nonminimal) $E(p)$-free resolution with regularity $\operatorname{reg}_E L + c - p$. ◻

We now return to the situation of Notation 4.1.

Theorem 5.3. *Suppose that M is the module of a minimal higher matrix factorization for a regular sequence $f_1, \ldots, f_c$ with $c \geq 1$. The regularity of* $\operatorname{Tor}^S(M, k)$ *as an E-module is* 1.

Proof. By [EP1, Theorem 3.1.4] the projective dimension of the S-module M is c. The description of its minimal resoution $\mathbf{L} := \mathbf{L}{\uparrow}^S(M, \mathbf{f})$ given in Section 3 shows that the c-th free module in $\mathbf{L}$ is $E(c-1) \otimes B_1(c)$. Hence, $B_1(c) \neq 0$.

By Proposition 3.7 it follows that the E-module $\operatorname{Tor}^S(M,k)$ requires generators of degree 1 from $B_1(c)$. Thus its regularity cannot be < 1, and we need only prove that it is ≤ 1. We will prove this by induction on p.

If $p = 1$ then $M(1)$ is a maximal Cohen–Macaulay module over the hypersurface $S/(f_1)$ and the resolution $\mathbf{L}(1)$ has projective dimension 1. Consequently, we have $\operatorname{reg}_{E(1)} \operatorname{Tor}^S(M(1),k) = 1$.

By induction on p, the direct sum in Corollary 4.3 shows that

$$\operatorname{reg}_{E(p-1)} \operatorname{Tor}^S(M(p),k) \leq 1.$$

Applying the left inequality in Theorem 5.2, we conclude

$$\operatorname{reg}_{E(p)} \operatorname{Tor}^S(M(p),k) \leq 1. \qquad \square$$

6. A Gröbner basis for the relations on $\operatorname{Ext}_R(M,k)$

As in the Introduction, we write $\mathcal{R} := k[\chi_1,\dots,\chi_c]$ for the ring of CI operators acting on $\operatorname{Ext}_R(M,k)$. Note that the χ_i have degree 2.

Throughout this section we will suppose that M is the module of a minimal higher matrix factorization for the regular sequence $f_1,\dots,f_c$. We will provide results for $\operatorname{Ext}_R(M,k)$ as an $\mathcal{R}$-module that are analogous to results proved above for $\operatorname{Tor}^S(M,k)$ as an E-module.

We use the notation and hypotheses of 4.1. Furthermore, we write $-^\vee$ for $\operatorname{Hom}(-,k)$. Since the differential $B_1(p) \to B_0(p)$ is minimal, we may think of $\mathbf{B}(p)^\vee$ as the direct sum $B_1(p)^\vee \oplus B_0(p)^\vee$. We set

$$\mathcal{R}(p) := k[\chi_p,\dots,\chi_c] \subset \mathcal{R}.$$

The following result is the analogue of Corollary 4.3.

Corollary 6.1 ([EP1, Corollary 5.1.6]). *There is an isomorphism*

$$\operatorname{Ext}_R(M,k) \cong \bigoplus_{p=1}^{c} k[\chi_p,\dots,\chi_c] \otimes_k \mathbf{B}(p)^\vee = \bigoplus_{p=1}^{c} \mathcal{R}(p) \otimes_k \mathbf{B}(p)^\vee$$

of graded vector spaces. The subspace $\mathcal{R}(p) \otimes \mathbf{B}(p)^\vee$ is an $\mathcal{R}(p)$-submodule and $\mathcal{R}(p)$ acts on it via the action on the first factor. $\square$

The result above can be used to prove an analogue to Theorem 4.4:

Theorem 6.2. *Suppose that M is the module of a minimal higher matrix factorization for a regular sequence $f_1,\dots,f_c$. Let $1 \leq p \leq c$. For every*

$$a \in \mathbf{B}(p)^\vee$$

and every $r < p$ there is a homogeneous relation on $\operatorname{Ext}_R(M,k)$ of the form

$$\chi_r a - b \tag{6.3}$$

with $b \in (\chi_{r+1}, \dots, \chi_c)(A_0 \oplus A_1)^\vee$*. These relations form a Gröbner basis for the relations on* $\operatorname{Ext}_R(M, k)$ *as an* $\mathcal{R}$*-module with respect to any term order that refines the lexicographic order on the monomials of* $\mathcal{R}$ *with* $\chi_1 \succ \cdots \succ \chi_c$*. The module defined by the leading terms of these relations is*

$$\bigoplus_{p=1}^{c} \mathcal{R}/(\chi_1, \dots, \chi_{p-1})\, \mathbf{B}(p)^\vee.$$

Proof. The existence of the desired relations follows from Proposition 3.9. The leading term of the relation $\chi_r a - b$ in the monomial order $\succ$, is $\chi_r a$. If we factor out these leading terms from the free $\mathcal{R}$-module generated by $(A_0 \oplus A_1)^\vee$, we obtain the module

$$\bigoplus_{1 \le p \le c} \mathcal{R}(p) \otimes_k \mathbf{B}(p)^\vee.$$

By Corollary 6.1, this has the same Hilbert function as the $\mathcal{R}$-module $\operatorname{Ext}_R(M, k)$. Therefore, the given relations form a Gröbner basis, and in particular they generate the module of all relations. □

Finally, we provide an analogue to Corollary 5.1.

Corollary 6.4. *Suppose that* M *is the module of a minimal higher matrix factorization for a regular sequence* $f_1, \dots, f_c$*. The* $\mathcal{R}$*-module* $\operatorname{Ext}_R^{\text{even}}(M, k)$ *has regularity* 0*, and the* $\mathcal{R}$*-module* $\operatorname{Ext}_R^{\text{odd}}(M, k)$ *has regularity* 1. □

7. Higher CI operators and an inverse Eisenbud–Shamash construction

The Eisenbud–Shamash construction (see [Sh] for the codimension 1 case and [Ei1, Section 7] for the general case) allows one to construct a (generally nonminimal) R-free resolution of an R-module from an S-free resolution together with a system of higher homotopies on the S-free resolution. In this section we will explain a construction that goes the other way: from an R-free resolution of an R-module together with a *system of higher CI operators* $\{t_i\}$ as defined below, we will construct a (generally nonminimal) S-free resolution.

The classic CI operators were first defined on $\operatorname{Ext}_R(M, k)$ by Gulliksen [Gu], and then in the form used here by Eisenbud [Ei1]. The material in this section was discovered independently by Jesse Burke, and a more general version will appear in his paper [Bu].

Proposition 7.1. *Let* S *be a commutative ring, let* $f_1, \dots, f_c$ *be a regular sequence, and let* $R = S/(f_1, \dots, f_c)$*. Let*

$$\mathbf{K} := \mathbf{K}(f_1, \dots, f_c) : \quad \cdots \xrightarrow{t_0'^3} \textstyle\bigwedge^2 S^c \xrightarrow{t_0'^2} S^c \xrightarrow{t_0'^1} S$$

be the Koszul complex resolving R*. Let* $\overline{\mathbf{G}}$ *be a complex of free* R*-modules, and suppose that*

$$\mathbf{G} : \quad \cdots \xrightarrow{t_1'} G_p \xrightarrow{t_1'} G_{p-1} \xrightarrow{t_1'} \cdots \xrightarrow{t_1'} G_0$$

is a lifting of $\overline{\mathbf{G}}$ *to a sequence of maps of free* S*-modules. There exist operators*

$$t_i = \sum_{p,q} t_i^{p,q} : \mathbf{G} \otimes \mathbf{K} \to (\mathbf{G} \otimes \mathbf{K})[-1]$$

that commute with the natural action of $\bigwedge S^c$ *on* $\mathbf{K}$*, having components*

$$t_i^{p,q} : G_p \otimes K_q \to G_{p-i} \otimes K_{q+i-1}$$

for $i, q \geq 0,\ p \geq i$ *and satisfying the conditions*

$$\begin{aligned} t_0^{p,q} &= 1 \otimes (-1)^p t'^{q}_0, \\ t_1 &= t'_1 \otimes 1, \end{aligned}$$

and

$$\sum_{i+j=n} t_i t_j = 0 \quad \textit{for all } n.$$

The maps $R \otimes t_i$ *are determined uniquely by these conditions.*

The positions of the maps $t_0, \ldots, t_3$, for example, are shown in the following figure where, for clarity, the upper indices are not shown and not all the maps have been labeled:

$$\begin{tikzcd}
\cdots G_4 \otimes K_0 \arrow[r, "t_1"] \arrow[drr, "t_2"] \arrow[ddrrr, "t_3"] & G_3 \otimes K_0 \arrow[r, "t_1"] \arrow[drr, "t_2"] \arrow[ddrrr, "t_3"] & G_2 \otimes K_0 \arrow[r, "t_1"] \arrow[drr, "t_2"] & G_1 \otimes K_0 \arrow[r, "t_1"] & G_0 \otimes K_0 \\
\cdots G_4 \otimes K_1 \arrow[r] \arrow[u] \arrow[drr] & G_3 \otimes K_1 \arrow[r] \arrow[u] \arrow[drr] & G_2 \otimes K_1 \arrow[r] \arrow[u] \arrow[drr] & G_1 \otimes K_1 \arrow[r] \arrow[u] & G_0 \otimes K_1 \arrow[u, "t_0"'] \\
\cdots G_4 \otimes K_2 \arrow[r] \arrow[u] & G_3 \otimes K_2 \arrow[r] \arrow[u] & G_2 \otimes K_2 \arrow[r] \arrow[u] & G_1 \otimes K_2 \arrow[r] \arrow[u] & G_0 \otimes K_2 \arrow[u, "t_0"'] \\
\vdots & \vdots & \vdots & \vdots & \vdots
\end{tikzcd}$$

Proof of Proposition 7.1. We construct the $t_n^{p,q}$ by induction on n. The condition $\sum_{i+j=n} t_i t_j = 0$ holds for $n = 0$ because $\mathbf{K}$ is a complex, and for $n = 1$ by our choice of signs.

Thus we assume that $t_j^{p,q}$ has been defined for all $j < n$. We next construct the maps $t_n^{p,0}$, and we then define $t_n^{p,q}$ for $q > 0$ to be the unique maps that make

$$\sum_q t_n^{p,q} : G_p \otimes \bigwedge S^c \to G_{p-n} \otimes \bigwedge S^c$$

into a map of free $\bigwedge S^c$-modules.

Because $t_0^{p,0} = 0$, the desired condition for $t_n^{p,0}$ is

$$\sum_{\substack{i+j=n \\ j>0}} t_i^{p-j,j-1} t_j^{p,0} = 0.$$

To simplify the notation, we drop the upper indices, which are functions of n, p and j, and write the condition as

$$t_0 t_n = -\sum_{\substack{i+j=n\\ i,j>0}} t_i t_j.$$

Since **K** is acyclic, both existence of t_n and the uniqueness of $R \otimes t_n$ will follow if we show that

$$t_0 \sum_{\substack{i+j=n\\ i,j>0}} t_i t_j = 0.$$

Using the induction hypothesis

$$t_0 t_i = -\sum_{\substack{\ell+m=i\\ \ell>0}} t_\ell t_m$$

for $i < n$, we get

$$t_0 \sum_{\substack{i+j=n\\ i,j>0}} t_i t_j = -\sum_{\substack{\ell+m+j=n\\ j,\ell>0}} t_\ell t_m t_j = -\sum_{\ell>0} t_\ell \sum_{\substack{m+j=n-\ell\\ j>0}} t_m t_j.$$

Since $\ell > 0$, we can use the induction hypothesis again, and we see that each sum

$$\sum_{\substack{m+j=n-\ell\\ j>0}} t_m t_j$$

is 0, yielding the desired vanishing. □

Corollary 7.2. *With hypotheses as in Proposition 7.1, the sequence*

$$\mathbf{GK}: \quad \cdots \to \sum_{i+j=n} G_i \otimes_S K_j \xrightarrow{T_n} \sum_{i+j=n-1} G_i \otimes_S K_j \to \cdots \to G_0 \otimes_S K_0$$

with

$$T_n = \begin{pmatrix} t_1^{n,0} & t_0^{n-1,1} & 0 & \dots & 0 \\ t_2^{n,0} & t_1^{n-1,1} & t_0^{n-2,2} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ t_n^{n,0} & t_{n-1}^{n-1,1} & t_{n-2}^{n-2,2} & \dots & t_0^{0,n} \end{pmatrix}$$

is a complex. □

Theorem 7.3. *Let S be a commutative ring, let $f_1, \dots, f_c$ be a regular sequence, and let $R = S/(f_1, \dots, f_c)$. Let $\mathbf{K} = \mathbf{K}(f_1, \dots, f_c)$ be the Koszul complex. If $\mathbf{G}$ is a sequence of maps of free S-modules*

$$\mathbf{G}: \quad \cdots \xrightarrow{t_1'} G_p \xrightarrow{t_1'} G_{p-1} \xrightarrow{t_1'} \cdots \xrightarrow{t_1'} G_0$$

such that $R \otimes_S \mathbf{G}$ is an R-free resolution of an R-module M, then the complex **GK** *of Corollary 7.2 is an S-free resolution of M.*

Proof. To see that the complex is a resolution, we note that it is filtered by the subcomplexes involving just the $G_i \otimes K_j$ with $i \leq m$: all the t_i except t_0 decrease the index of G_i, while t_0 keeps the index of G_i the same. The associated graded complex is thus the direct sum of the $G_i \otimes \mathbf{K}$, with differentials t_0; that is, the direct sum of copies of the resolution $\mathbf{K}$ of R. The homology is thus $\mathrm{H}_i(\mathrm{gr}(\mathbf{GK})) = R \otimes G_i$.

In the spectral sequence converging from the homology of the associated graded complex $\mathrm{gr}(\mathbf{GK})$ to the homology of $\mathbf{GK}$, the E_1 page thus has nonzero terms $E_1^{(i,0)} = R \otimes G_i$ in position $(i, 0)$, and differentials induced by the differential of $\mathbf{GK}$. But the only differential that reduces the first index by only 1 is t_1; thus the E_1 differential is the differential of the complex $R \otimes \mathbf{G}$, which is a resolution of M. ◻

For any R-modules M, N there is a spectral sequence

$$\mathrm{Tor}_i^R(\mathrm{Tor}_j^S(M, R), N) \Rightarrow \mathrm{Tor}_{i+j}^S(M, N)$$

that comes from the double complex $\mathbf{G} \otimes_S (\mathbf{K} \otimes_S N)$, and that allows the computation of a certain associated graded module of $\mathrm{Tor}^S(M, N)$. It is natural to expect that the t_i are special liftings of the differentials in this spectral sequence; Burke [Bu] shows that this is indeed the case. The complex $\mathbf{GK}$ allows the computation of $\mathrm{Tor}^S(M, N)$ itself.

8. The Bernstein–Gel'fand–Gel'fand (BGG) correspondence

The results in the next section depend on properties of the Bernstein–Gel'fand–Gel'fand (BGG) correspondence from [EFS], and in this short section we review what is necessary.

Let W be the vector space generated by the regular sequence $f_1, \ldots, f_c$ so that $E = \bigwedge W$. We set $W = V^\vee$ and $\mathcal{R} := \mathrm{Sym}(W)$. In this section, for simplicity, we regard both V and W as having degree 1, though in the next section we will need to adjust to the situation where W has degree 2.

The BGG correspondence establishes equivalences between the category of $\mathbb{Z}$-graded $\mathcal{R}$-modules and linear free complexes over E, and also between $\mathbb{Z}$-graded E-modules and linear free complexes over $\mathcal{R}$.

For example, giving a graded vector space $U = \bigoplus_i U_i$ the structure of a graded $\mathcal{R}$-module is the same as giving multiplication maps $\mu_i : W \otimes_k U_i \to U_{i+1}$ that satisfy the commutativity and associativity conditions. But giving the map μ_i is equivalent to giving a map $\delta_i : U_i \to \mathrm{Hom}_k(V, U_{i+1})$, and this is equivalent, in turn, to giving a linear map of free E-modules

$$\mathrm{Hom}_k(E, U_i)(-1) \to \mathrm{Hom}_k(E, U_{i+1}).$$

It turns out that the associative and commutative conditions on the μ_i are equivalent to the conditions $\delta_{i+1}\delta_i = 0$ for all i. We write $\mathbb{R}(U)$ for the resulting linear E-free complex with i-th term $\mathrm{Hom}(E, U_i)(i)$.

Similarly, given a graded E-module $T = \bigoplus T_i$ we construct a linear $\mathcal{R}$-free complex $\mathbb{L}(T)$ having i-th term $(\mathcal{R} \otimes T_i)(i)$. Here are the results we need:

Theorem 8.1 ([EFS, Theorem 3.7 (Reciprocity)]). *Let U and T be finitely generated graded modules over $\mathcal{R}$ and E, respectively. The complex $\mathbb{L}(T)$ is a free resolution of U if and only if the complex $\mathbb{R}(U)$ is an injective resolution of T.* □

Since the complex $\mathbb{L}(T)$ is linear, the equivalent conditions of the theorem can only be satisfied if the Castelnuovo–Mumford regularity of U is 0. In fact, this is sufficient:

Corollary 8.2 ([EFS, Corollary 2.4]). *Suppose that U is a finitely generated graded $\mathcal{R}$-module. The complex $\mathbb{R}(U)$ is acyclic—that is, the only homology of $\mathbb{R}(U)$ is H^0—if and only if U has Castelnuovo–Mumford regularity* 0. □

9. Free resolutions of $\mathrm{Tor}^S(M, k)$ and $\mathrm{Ext}_R(M, k)$

We will make use of the BGG correspondence in two ways: first, if $\mathbf{F}$ is any free complex of $R = S/(f_1, \ldots, f_c)$-modules, where $f_1, \ldots, f_c$ is a regular sequence, then the CI operators on $\mathbf{F}$ define an $\mathcal{R}$-module structure on $H_*(\mathbf{F} \otimes_R k)$, and thus, since the CI operators have degree 2, we get a linear free complex of E-modules

$$\cdots \xrightarrow{t_2} \mathrm{H}_{2i+s}(\mathbf{F} \otimes_R k) \otimes_k E \xrightarrow{t_2} \mathrm{H}_{2i+s-2}(\mathbf{F} \otimes_R k) \otimes_k E \xrightarrow{t_2} \cdots$$

for $s = 0$ and for $s = 1$. When M is a high R-syzygy and $\mathbf{F}$ is its minimal free resolution, we shall see that this is a resolution of $\mathrm{Tor}^S(M, k)$.

Second, given an S-module M, the action of E on the sub- and quotient modules T' and T'' of $\mathrm{Tor}^S(M, k)$ gives us two linear complexes of $\mathcal{R}$-modules; when M is a high R-syzygy, we shall see that these are minimal $\mathcal{R}$-free resolutions of $\mathrm{Ext}_R^{\mathrm{even}}(M, k)$ and $\mathrm{Ext}_R^{\mathrm{odd}}(M, k)$, respectively.

To prove these results, we will use the complexes constructed in Corollary 7.2. With notation and hypotheses as in Proposition 7.1, we may regard $t_2^{0,*}$ as a map $\mathbf{G} \to \mathbf{G} \otimes S^c$ whose components $t_{2,i}$ satisfy $\sum_i f_i t_{2,i} = t_1^2$; that is, the $R \otimes t_{2,i}$ are the same as the CI operators defined in [Ei1].

Corollary 9.1. *With hypotheses as in Corollary 7.2, suppose that $R \otimes \mathbf{G}$ is a minimal complex. The induced maps*

$$t_2 : G_{i+2} \to G_i \otimes k^c, \qquad t_3 : G_{i+3} \to G_i \otimes \bigwedge^2 k^c$$

yield a complex of the form

$$\begin{tikzcd}
\cdots \arrow[r, "t_2"] & G_4 \otimes E \arrow[r, "t_2"] & G_2 \otimes E \arrow[r, "t_2"] & G_0 \otimes E \\
\cdots & \bigoplus & \bigoplus & \bigoplus \\
\cdots \arrow[r, "t_2"] & G_5 \otimes E \arrow[r, "t_2"] \arrow[uur, "t_3"] & G_3 \otimes E \arrow[r, "t_2"] \arrow[uur, "t_3"] & G_1 \otimes E
\end{tikzcd}$$

Proof. By minimality, $t_1 \otimes k = 0$, so $\mathrm{H}_i\mathbf{G} = G_i \otimes k$. Thus by Corollary 9.1 each row of the diagram is a complex. Further, Proposition 7.1 gives the identity $\sum_{i=0}^{5} t_i t_{5-i} = 0$, and tensoring with k we get

$$(t_2 t_3 + t_3 t_2) \otimes k = 0$$

as required. □

Note that if $R \otimes \mathbf{G}$ is the minimal resolution of an R-module M, then $G_i \otimes E = \mathrm{Tor}_i^R(M, k)$.

Theorem 9.2. *Let $f_1, \dots, f_c \subset S$ be a regular sequence in a regular local ring with maximal ideal $\mathfrak{m}$ and residue field k. Let $I = (f_1, \dots, f_c)$ and let $R = S/I$. Let $\mathcal{R} := k[\chi_1, \dots, \chi_c]$ be the ring of CI operators. If* $\operatorname{reg} \mathrm{Ext}_R^{\mathrm{even}}(M, k) = 0$ *and* $\operatorname{reg} \mathrm{Ext}_R^{\mathrm{odd}}(M, k) = 1$ *as $\mathcal{R}$-modules, where the operator χ_i acts on* $\mathrm{Ext}_R(M, k) = \mathrm{Hom}_k(\mathrm{Tor}^R(M, k), k)$ *via the action of $t_{2,i}$ on* $\mathrm{Tor}^R(M, k)$*, then the complex*

$$\begin{tikzcd}
 & \cdots \arrow[r, "t_2"] & \mathrm{Tor}_4^R(M,k) \otimes E \arrow[r, "t_2"] & \mathrm{Tor}_2^R(M,k) \otimes E \arrow[r, "t_2"] & \mathrm{Tor}_0^R(M,k) \otimes E \\
\mathbf{T}(M): & & \bigoplus & \bigoplus & \bigoplus \\
 & \cdots \arrow[r, "t_2"] & \mathrm{Tor}_5^R(M,k) \otimes E \arrow[r, "t_2"] \arrow[uur, "t_3"] & \mathrm{Tor}_3^R(M,k) \otimes E \arrow[r, "t_2"] \arrow[uur, "t_3"] & \mathrm{Tor}_1^R(M,k) \otimes E
\end{tikzcd}$$

is a minimal free resolution of $\mathrm{Tor}^S(M, k)$ *as a module over* $E = \mathrm{Tor}^S(R, k)$*. Moreover, the upper row is a minimal free resolution of the submodule* $T' := E \cdot \mathrm{Tor}_0^S(M, k) \subset \mathrm{Tor}^S(M, k)$*, and the lower row is a minimal free resolution of the quotient* $T'' = \mathrm{Tor}^S(M, k)/T'$.

Note that if M is a minimal higher matrix factorization module then, by Corollary 6.4, the $\mathcal{R}$-modules $\mathrm{Ext}_R^{\mathrm{even}}(M, k)$ and $\mathrm{Ext}_R^{\mathrm{odd}}(M, k)$ satisfy the regularity hypothesis.

We think of the minimal E-free resolution of $\mathrm{Tor}^S(M, k)$ as having two "strands": the resolution of T' and the resolution of T''.

Proof of Theorem 9.2. We first show that $\mathbf{T}(M)$ is acyclic. By Corollary 8.2, the complexes corresponding to $\mathrm{Ext}^{\mathrm{even}}(M, k)$ and $\mathrm{Ext}^{\mathrm{odd}}(M, k)$ are acyclic. Since $\mathrm{Tor}^R(M, k)$ is the graded dual of $\mathrm{Ext}_R(M, k)$ and E is injective as an E-module, the rows of the complex in the theorem are acyclic. The total complex $\mathbf{T}(M)$ is the mapping cone of the map t_3 between these complexes, so it is acyclic as well.

Now let $\mathbf{G}$ be a sequence of maps of free S-modules such that $\mathbf{G} \otimes R$ is a minimal R-free resolution of M. By Theorem 7.3 the homology of the complex $\mathbf{GK} \otimes k$ is $\mathrm{Tor}^S(M, k)$. In particular, $\mathrm{Tor}_0^S(M, k)$, $\mathrm{Tor}_1^S(M, k)$ and $\mathrm{Tor}_2^S(M, k)$ are the homology of the following complexes at the middle position:

$$0 \longrightarrow \mathrm{Tor}_0^R(M, k) \otimes E_0 \longrightarrow 0,$$

$$\begin{tikzcd}
\mathrm{Tor}_2^R(M,k) \otimes E_0 \arrow[r, "t_2"] & \mathrm{Tor}_0^R(M,k) \otimes E_1 & \\
 & \bigoplus & \\
 & \mathrm{Tor}_1^R(M,k) \otimes E_0 \arrow[r] & 0,
\end{tikzcd}$$

$$\begin{tikzcd}
\operatorname{Tor}_2^R(M,k)\otimes E_1 \arrow[r, "t_2"] & \operatorname{Tor}_0^R(M,k)\otimes E_2 & \\
\bigoplus & \bigoplus & \\
\operatorname{Tor}_3^R(M,k)\otimes E_0 \arrow[r, "t_2"'] \arrow[ruu, "t_3"] & \operatorname{Tor}_1^R(M,k)\otimes E_1 & \\
 & \bigoplus & \\
 & \operatorname{Tor}_2^R(M,k)\otimes E_0 \arrow[r, "t_2"] & \operatorname{Tor}_0^R(M,k)\otimes E_1.
\end{tikzcd}$$

Under the regularity hypothesis of the theorem the rows of the diagram $\mathbf{T}(M)$ are exact. In particular, the map

$$\operatorname{Tor}_2^R(M,k)\otimes E_0 \xrightarrow{t_2} \operatorname{Tor}_0^R(M,k)\otimes E_1$$

in the sequence for $\operatorname{Tor}_2^S(M,k)$ above is injective. Thus $\mathrm{H}_0(\mathbf{T}(M))$ coincides with $\operatorname{Tor}^S(M,k)$ in degrees ≤ 2. Since $\operatorname{Tor}^S(M,k)$ is 1-regular by Theorem 4.6, this implies that $\mathrm{H}_0(\mathbf{T}(M))$ coincides with $\operatorname{Tor}^S(M,k)$ in all degrees. Together with the exactness of the two strands, this proves the theorem. □

Corollary 9.3. *With hypotheses and notation as in Theorem 4.6, let M_1 be the first syzygy of M over R. The degree 0 strand of the minimal resolution of $\operatorname{Tor}^S(M_1,k)$ is equal to the degree 1 strand of the minimal resolution of $\operatorname{Tor}^S(M,k)$; and the degree 1 strand of the minimal resolution of $\operatorname{Tor}^S(M_1,k)$ is equal to the degree 0 strand of the minimal resolution of $\operatorname{Tor}^S(M,k)$, truncated at homological degree 1.* □

We now turn to the free resolution of $\operatorname{Ext}_R(M,k)$. As the generators of $\mathcal{R}$ have degree 2, we have

$$\operatorname{Ext}_R(M,k) = \operatorname{Ext}_R^{\text{even}}(M,k) \oplus \operatorname{Ext}_R^{\text{odd}}(M,k)$$

as $\mathcal{R}$-modules. We treat only the even part in detail, as the odd part is analogous.

Theorem 9.4. *With the hypotheses of Theorem 4.6, let*

$$\sigma_i' : E_1 \otimes T_{i-1}' \to T_i'$$

be the multiplication maps. The i-th differential in the minimal $\mathcal{R}$-free resolution of $\operatorname{Ext}_R^{\text{even}}(M,k)$ is the map

$$\tau_i' : T_i' \otimes_k \mathcal{R}(-i) \to T_{i-1}' \otimes_k \mathcal{R}(-i+1)$$

whose linear part

$$T_i' \otimes_k \mathcal{R}_1 \to T_{i-1}'$$

is the vector space dual of σ_i'. The corresponding statement holds for $\operatorname{Ext}_R^{\text{odd}}(M,k)$ and T'' as well.

Proof. By Theorem 9.2, the minimal E-free resolutions of T' is given by the $\mathcal{R}$-module structure of the even part of $\operatorname{Tor}^R(M,k)$. Since $\omega_E := \operatorname{Hom}(E,k) \cong E(c)$ is an injective E-module, the vector space dual of this resolution is the injective resolution of the E-modules $\operatorname{Hom}(T',k)$. Furthermore, the differentials in this injective resolution come, via the BGG correspondence, from the module structure of the even part of the graded vector space dual of $\operatorname{Tor}_{\text{even}}^R(M,k)$, which is the $\mathcal{R}$-module $\operatorname{Ext}_R^{\text{even}}(M,k)$.

By Theorem 8.1, the resolution of $\mathrm{Ext}_R^{\mathrm{even}}(M,k)$ is the BGG dual of $\mathrm{Hom}(T',k)$. If the module structure of T' is given by maps $\mu_i : E_1 \otimes T'_i \to T'_{i+1}$, then the module structure of $\mathrm{Hom}(T',k)$ is given by maps

$$\mu'_i : E_1 \otimes \mathrm{Hom}(T'_{i+1},k) \to \mathrm{Hom}(T'_i,k)$$

and the BGG dual complex

$$\cdots \to E \otimes \mathrm{Hom}(T'_{i+1},k) \to \cdots \to E \otimes \mathrm{Hom}(T'_1,k) \to E \otimes \mathrm{Hom}(T'_0,k)$$

is induced by maps $\mu''_i : \mathrm{Hom}(T'_{i+1},k) \to \mathrm{Hom}(E_1,k) \otimes \mathrm{Hom}(T'_i,k)$. Identifying $\mathrm{Hom}(E_1,k) \otimes \mathrm{Hom}(T'_i,k)$ with $\mathrm{Hom}(E_1 \otimes T'_i,k)$, we see that μ''_i is, up to change of basis, the same as $\mathrm{Hom}(\mu_i,k)$, proving the theorem. □

Since $T'_1 = \bigoplus_p E_1(p-1) \otimes B_0(p)$, the minimal $\mathcal{R}$-free presentation of $\mathrm{Ext}_R^{\mathrm{even}}(M,k)$, with the hypotheses in Theorem 9.4, can be written as

$$\mathcal{R}(-1) \otimes \Big(\sum_{p=1}^{c} \mathrm{span}_k\langle e_1,\dots,e_{p-1}\rangle \otimes B_0(p)\Big) \to \mathcal{R} \otimes \Big(\sum_{p=1}^{c} B_0(p)\Big) \\ \to \mathrm{Ext}_R^{\mathrm{even}}(M,k) \to 0,$$

where the map is induced by the appropriate components of the homotopies.

There is an even more direct way of getting a free presentation for the even part of Ext:

Corollary 9.5. *With the hypotheses of Theorem 4.6, the module* $\mathrm{Ext}_R^{\mathrm{even}}(M,k)$ *has an* $\mathcal{R}$*-free presentation as the cokernel of the map*

$$\tau : \mathrm{Ext}^1_S(M,k) \otimes_k \mathcal{R}(-1) \to \mathrm{Hom}(M,k) \otimes_k \mathcal{R}$$

whose linear part

$$\mu^\vee : \mathrm{Ext}^1_S(M,k) \to \mathrm{Hom}(M,k) \otimes_k \mathcal{R}_1$$

is the vector space dual of the multiplication map

$$\mu : E_1 \otimes_k \mathrm{Tor}^S_0(M,k) \to \mathrm{Tor}^S_1(M,k).$$

Proof. With notation as above, $\mathrm{Tor}^S_0(M,k) = T'_0$, and by Theorem 9.4 the even Ext module has minimal $\mathcal{R}$-free presentation as the cokernel of the map

$$(T'_1)^\vee \otimes_k \mathcal{R}(-1) \to (\mathrm{Tor}^S_0(M,k))^\vee \otimes_k \mathcal{R}.$$

We have

$$\mathrm{Tor}^S_1(M,k) = T'_1 \oplus T''_1,$$

so it suffices to show that the image of μ is contained in T'_1. This follows from Proposition 3.7. □

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