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Ulrich complexity

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A B S T R A C T

In this note we suggest a new measure of the complexity of polynomials, the *Ulrich complexity*. Valiant's conjecture on the exponential complexity of the permanent would imply exponential behavior of the Ulrich complexity as well, and this may be easier to prove. We compute some families of examples, one of which has provably exponential behavior.

0. Introduction

One measure of the complexity of a homogeneous polynomial $f(x_0, \dots, x_n)$ is the least size of a matrix of linear forms whose determinant is equal to $z^p f$, for some integer p, where z is an auxiliary variable. Called the *determinantal complexity* of f, this is interesting because determinants are relatively easy to compute. Valiant has famously conjectured that the $d \times d$ permanent polynomial is not easy to compute, and in particular that its determinantal complexity is exponential in d ([24, Section 13.4]).

Roots of polynomials can also be computed relatively easily (see Section 1), so one might also consider measuring the complexity of f by the size s of a matrix of linear forms whose determinant is a power f^r of f, or, adjusting for the degree d of f, by $r = s/d$. It turns out that if one adds a further natural restriction, then this measure is in the domain of a subject that has been intensively studied in algebraic geometry in recent years, the theory of Ulrich modules and sheaves [6,28]. To exploit this connection, we propose the following definition:

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Definition 0.1. The *Ulrich complexity* $\mathrm{uc}(f)$ of a homogeneous polynomial $f(x_1,\dots x_n)$ is the smallest number r such that there exists a square matrix M of homogeneous linear polynomials with

$$\det M = f^r$$

and such that there is a matrix N with $M \cdot N = f \cdot \mathrm{I}$, where I is the identity matrix.

In this paper we initiate the study of Ulrich complexity, and give some old and new examples where the complexity is exponential.

For example, the Ulrich complexity of the determinant polynomial is 1; in this case the matrix N is the matrix of cofactors (sometimes called the adjoint).

Recall that the *singular locus*, $\operatorname{sing} f$, of a polynomial f, or of the hypersurface $\{x \mid f(x) = 0\}$, is (in characteristic 0) the variety defined by the partial derivatives of f. As a special case of a conjecture of Buchweitz–Greuel–Schreyer [7] we have

Conjecture 0.2. *(Codimension Conjecture)*

$$\mathrm{uc}(f) \geq 2^{\lceil(\operatorname{codim} \operatorname{sing} f)/2\rceil - 2}$$

for all f.

The singular locus of the $n \times n$ generic determinantal hypersurface $\det(X) = 0$ is defined by the ideal of $(n-1) \times (n-1)$ minors of X, and it is well-known that this has codimension 4 ([14, Exercise 10.9]), so the conjecture predicts $\mathrm{uc}(\det X) \geq 2^{(4/2)-2} = 1$, as observed.

As a more interesting (and mysterious) example, the $n \times n$ permanental hypersurface $\mathrm{perm}_n = 0$ is singular along the codimension $2n$ locus where the variables in any pair of rows or columns vanish, and (though the singular locus has many other components) it seems natural to conjecture that the codimension is exactly $2n$. Combining this with Conjecture 0.2, we get:

Conjecture 0.3. *(Permanent Conjecture)*

$$\mathrm{uc}(\mathrm{perm}_n) \geq 2^{n-2}.$$

This is trivially sharp for $n = 2$, but we will show that it is also sharp for $n = 3$. The lower bound is proven in Section 3; for the upper bound, it suffices to exhibit an example:

Example 0.4. Let

$$M = \begin{pmatrix} 0 & 0 & 0 & -x_{1,1} & x_{2,1} & -x_{3,1} \\ 0 & 0 & -2x_{2,2} & -x_{1,2} & x_{2,2} & x_{3,2} \\ 0 & 2x_{2,2} & 0 & -x_{1,3} & -x_{2,3} & -x_{3,3} \\ x_{1,1} & x_{1,2} & x_{1,3} & 0 & 0 & 0 \\ -x_{2,1} & -x_{2,2} & x_{2,3} & 0 & 0 & x_{3,3} \\ x_{3,1} & -x_{3,2} & x_{3,3} & 0 & -x_{3,3} & 0 \end{pmatrix}.$$

Computation shows that $\det M = (\mathrm{perm}_3)^2$ and $MN = \mathrm{perm}_3 \cdot \mathrm{I}$, where N is a skew-symmetric matrix whose entries are the 4×4 signed Pfaffians of M. Thus $\mathrm{uc}(\mathrm{perm}_3) \leq 2$.

Valiant's conjecture implies (by the material in Section 1) that the lower bound for $\mathrm{uc}(\mathrm{perm}_n)$ should be exponential in n, and Conjecture 0.3 should be viewed in this light, but may be more accessible: Indeed, even some polynomials that are easy to compute have provably exponential Ulrich complexity, so to prove an exponential lower bound for Ulrich complexity might be easier than for the determinantal complexity.

For example,

$$\sum_{i=1}^{c} x_i y_i = \det \begin{pmatrix} 0 & x_1 & x_2 & \dots & x_c \\ y_1 & 1 & 0 & \dots & 0 \\ y_2 & 0 & 1 & & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ y_c & 0 & 0 & \dots & 1 \end{pmatrix}$$

shows that the determinantal complexity of the rank c quadric $\sum_{i=1}^{c} x_i y_i$ is at most $c+1$, and in fact this is sharp [25]. On the other hand, a result of Knörrer ([23]) gives the precise value of the Ulrich complexity for a rank c quadric over an algebraically closed field as

$$2^{\lceil c/2 \rceil - 2}$$

and this is a lower bound over any field (see [9] for the range of possible values).

To summarize: for quadrics, for $n \times n$ determinants and for the 3×3 permanent, Conjecture 0.2 is both true and sharp.

0.1. Ulrich modules and maximal Cohen–Macaulay modules

In this paper we will use the connection of these ideas with the theory of Cohen–Macaulay modules and sheaves, and we pause to review the definitions.

Suppose that $R = S/I$ where $S = k[x_0, \dots, x_n]$ is a graded polynomial ring over a field k with $\deg x_i = 1$ for all i, and I is a homogeneous ideal. If F is a finitely generated graded R-module, then F is said to be a *maximal Cohen–Macaulay* R-module if the projective dimension of F as an S-module (which is finite by Hilbert's Syzygy Theorem) is equal to the codimension of R, the smallest possible value. If, moreover F has a free resolution over S whose differentials are matrices of linear forms, then F is said to be an *Ulrich module* over R. The module F corresponds to a coherent sheaf $\tilde{F}$ on X, and $\tilde{F}$ is often said to be an Ulrich sheaf, and can be characterized by its cohomology. However, when we are interested in the projective scheme $X \subset \mathbb{P}^n$ defined by I we will still refer to F as an Ulrich module on X; we will not need the sheaf theoretic characterization.

If X is a hypersurface (that is, $R = S/(f)$), then to say that F is Ulrich means that its minimal free resolution has the form

$$0 \to S^n(-1) \xrightarrow{M} S^n \longrightarrow F \to 0\,,$$

where M is a matrix of linear forms. Since F is annihilated by f, the multiplication by f on S^n factors through M; that is, we can find a matrix of forms N such that $MN = f \cdot \mathrm{Id}_n$. In particular, $\det M \det N = f^n$, so if f is irreducible the determinant of M is some power f^r of f, as in the definition of Ulrich complexity. A result of Eisenbud [13, Proposition 5.6] shows that in this case the exponent r is equal to the rank of F as a module over $S/(f)$.

Thus the notion of Ulrich complexity, defined above for a single function, or equivalently a hypersurface, naturally extends to all projective algebraic varieties:

Definition 0.5. Let $X \subset \mathbb{P}^n$ be a projective variety of codimension k, let S_X be the homogeneous coordinate ring of X We define

$$\mathrm{uc}(X) = \inf\{\mathrm{rank}\, F \mid F \text{ is an Ulrich module on } X\}.$$

We have conjectured in [16] that there exist Ulrich sheaves on any variety X, that is, that $\mathrm{uc}(X) < \infty$ for every X. This is known in the case of hypersurfaces by [21], where it is proven using an idea of [12]; we give a direct exposition of the construction in Section 2.

0.2. Contents of this paper

We now describe the contents of this paper: In section 1, we explain in more detail why the Valiant conjecture would imply an exponential lower bound for the Ulrich complexity of the permanent.

Section 2 gives an upper bound on the Ulrich complexity of a hypersurface through an explicit construction of Ulrich modules depending on the decomposition of the equation as a sum of products of linear forms. The possibility of such a construction was discovered by Lindsay Childs [12], though his exposition claims much less than the method proves, and its relevance for the existence of Ulrich modules was recognized in [3,21]. The construction we give is slightly different than that of [21], where the proof given is a reference to the twisted tensor product construction described by Childs. In fact, the construction produces more than an Ulrich module: it produces a factorization of f times an identity matrix as a product of d matrices of linear forms. We give a direct proof of the correctness of the construction.

In Section 3 we give a simple lower bound for Ulrich complexity of a function or a hypersurface in terms of the singular locus of the hypersurface, much weaker than Conjecture 0.2, and we discuss some of the examples where the actual value is known. We also explain the Buchweitz–Greuel–Schreyer conjecture and its relation to Conjecture 0.2.

For example, in the case where f has degree d and the ground field contains a d-th root of unity, the construction produces an Ulrich module of rank d^{N-2}, and thus shows that $\mathrm{uc}(f) \leq d^{N-2}$, where N is the number of products of linear forms used in the decomposition of f.

It is clear from the definition that $uc(f) = 1$ if and only if f is the determinant of a $d \times d$ matrix of linear forms. More surprising is the case $\mathrm{uc}(f) = 2$: we prove, when the codimension of the singular locus of f is ≥ 5 that $\mathrm{uc}(f) = 2$ implies that f is the Pfaffian of a $2d \times 2d$ alternating matrix of linear forms, partly explaining Example 0.4 above. (This is true in many more cases, as well, as is explained in [4].) We use the result to show that the generic cubic trinomial

$$f = x_1x_2x_3 + y_1y_2y_3 + z_1z_2z_3$$

has $\mathrm{uc}(f) = 3 = 3^{3-2}$—that is, the upper bound given by Childs' construction is sharp. Note that Conjecture 0.2 only says $\mathrm{uc}(f) \geq 2$ in this case.

If Conjecture 0.2 is true, it would be natural to look for some special behavior of an Ulrich module along the singular locus of the hypersurface. We eliminate the simplest possibility by showing that every projective hypersurface in fact supports Ulrich sheaves that are vector bundles (Corollary 3.2). The ones we can construct are of substantially greater rank than the sheaves produced by Childs' formula, and thus it would be interesting to know more about the minimal rank of an Ulrich vector bundle—we don't even know this for the determinantal hypersurfaces!

In Section 5 we summarize some of the known facts about the Ulrich complexity of varieties of higher codimension. In particular, we compute the Ulrich complexity precisely for some Veronese embeddings.

In the final Section 6 we explain the close relation of Grenet's determinantal representation of the permanent to the Koszul complex and other sequences of composable matrices.

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1. Circuits for roots of polynomials, and the relation of Ulrich complexity to the Valiant conjecture

In this section we establish that a lower exponential bound for the Ulrich complexity is a consequence of the Valiant conjecture VP $\neq$ VNP, see [29,30] or [11] for more on the conjecture.

Assume we have a polynomial b of degree d in variables $x_1, \dots, x_n$ and we know that b is an rth power, that is, there is a polynomial a such that $a^r = b$. Consider the function $f(Z) = Z^r - b$ for some new indeterminate Z, so that a is a root of f. To approximate a, we can use classical Newton iteration as in [8]. Let $\mathfrak{m} = (x_1, \dots, x_n)$ be the maximal ideal.

We will consider a and b as formal power series and will compute approximations to a. Changing the coordinates by a translation, we can assume that the constant term of a is nonzero, so that a is invertible in the ring of formal power series. Assume that we are given an approximation $\hat{a}$ of a such that $\hat{a} \equiv a \mod \mathfrak{m}^t$. We claim that

$$\hat{a} - \frac{f(\hat{a})}{f'(\hat{a})} = \hat{a} - \frac{\hat{a}^r - b}{r\hat{a}^{r-1}}$$

is an approximation of order $2t$.

By Taylor expansion around a, we get

$$\begin{aligned} f(Z) &= \underbrace{a^r - b}_{=0} + r \cdot a^{r-1}(Z-a) + O((Z-a)^2), \\ f'(Z) &= ra^{r-1} + r(r-1)a^{r-2}(Z-a) + ((Z-a)^2) \\ &= ra^{r-1}\left(1 + \frac{r-1}{a}(Z-a) + O((Z-a)^2\right), \text{ and} \\ \frac{1}{f'(Z)} &= \frac{1}{ra^{r-1}}\left(1 - \frac{r-1}{a}(Z-a) + O((Z-a)^2\right). \end{aligned}$$

Therefore,

$$Z - \frac{f(Z)}{f'(Z)} = Z - (Z-a) + O((Z-a)^2) = a + O((Z-a)^2).$$

Thus, if $\hat{a} - a \in \mathfrak{m}^t$, then $\hat{a} - \frac{f(\hat{a})}{f'(\hat{a})}$ approximates a up to order $2t$.

Lemma 1.1. *Let a be a multivariate polynomial with nonzero constant term a_0, and let $\mathfrak{m}$ be the ideal of polynomials with no constant term. Let C be a circuit of size s computing $b = a^r$. There is an arithmetic circuit of size polynomial in s, $\log r$, and $\log t$ computing a modulo $\mathfrak{m}^t$.*

Proof. We will construct algebraic circuits C_i of size $O(s+i^2+i\log r)$ so that C_i computes an approximation to a modulo $\mathfrak{m}^{2^i}$. The construction is inductive. We take C_0 to be the constant circuit computing a_0. Assuming that we have constructed C_i, we construct the circuit C_{i+1} as

$$C_i - \frac{C_i^r - b}{r \cdot C_i^{r-1}}.$$

Given C_i and a circuit for b, this can be computed by a circuit of size $O(\log r)$ (using the square-and-multiply method) with one division. To remove this division, we need to compute a polynomial that approximates the inverse of C_i^{r-1} modulo $\mathfrak{m}^{2^i}$. Since C_i^{r-1} computes a polynomial $a_0^{r-1}(1+p)$, where p is a polynomial without constant term, the inverse is given by $a_0^{r-1}(1-p+p^2-p^3+\cdots)$. Given C_i^{r-1}, this gives immediately a circuit of size $O(2^i)$ using the Horner scheme. We can however do much better, namely size $O(i)$ by using another Newton iteration as above; see [10, Theorem 2.22] for more details.

Therefore, given the circuit C_i and a circuit for b, we need an additional $O(i + \log r)$ operations to build C_{i+1}. The claim follows. We get the statement of the lemma by choosing $i = \log_2 t$. □

Theorem 1.2. [27, Theorem 2.2] *Given a circuit C of size s computing a polynomial p, there is a circuit of size $O(s \cdot t^2)$ computing the first t homogeneous components of p.*

In Theorem 1.2 each homogeneous component is computed separately. Every addition gate is replaced by t addition gates and every multiplication gate by $O(t^2)$ multiplication and additions gates. (We can even achieve $O(t \log t \log\log t)$ by using fast polynomial multiplication.)

Theorem 1.3. *Let a be a polynomial of degree d. Assume there is a circuit of size s computing a^r for some r. Then there is a circuit of size polynomials in s, $\log r$, and d computing a.*

Proof. By Lemma 1.1, we can find a circuit of size s', which is polynomial in s, $\log d$, and $\log r$ that computes an approximation $\hat{a}$ of a up to order $d+1$. By the fact above, we can get a circuit computing the first $d+1$ homogeneous components of $\hat{a}$ by a circuit of size polynomial in s' and d. Since the a has degree d, the sum of these homogeneous components is a. □

For exponential Ulrich complexity, we need a determinant of exponential size. This determinant will in general have exponential complexity. Only in very special cases, like being a high power of a small degree polynomial, this determinant could have subexponential or even polynomial complexity. However, it could still be the case that the Ulrich complexity of the permanent is exponential but $\mathsf{VP} = \mathsf{VNP}$.

2. An upper bound: Childs' construction

The Clifford algebra of a quadratic form $q : V \to K$ on a vector space V over a field K is the tensor algebra $\bigotimes V$ modulo the relations $v \otimes v - q(v)$ for $v \in V$. If q is nonsingular, then this is a finite dimensional, $\mathbb{Z}/2$-graded semisimple algebra, and its $\mathbb{Z}/2$-graded representations correspond one-to-one with Ulrich modules (which, in this case, are the same as the maximal Cohen–Macaulay modules. The representation theory of such algebras is well-understood, and this gives rise to a classification of Ulrich modules summarized below in Example 4.1.

In the same way, if f is a form of any degree d, we can define a "generalized Clifford algebra" $C(f) := \bigotimes V/(\{v^{\otimes d} - f(v)\}$, and it was noted in [2] that finite dimensional $\mathbb{Z}/d$-graded representations of this algebra of dimension n are the same as factorizations

$$f \cdot \mathrm{id}_n = M_1 \cdots M_d$$

of an $n \times n$ diagonal matrix with entries equal to f into a product of d $n \times n$ matrices of linear forms – and thus each such representation gives rise to d Ulrich modules. The problem is that when $d > 2$ the algebra $C(f)$ is not finite-dimensional, and its representation theory is not known. However, Childs proved in [12] that the algebra does have finite dimensional representations, using the idea of twisted tensor products; this was noted in [2] and again in [21], where an explicit form of the representation is worked out, referring to Childs' paper for the proof. We give a variant of this construction, and a direct proof, not using the theory of twisted tensor products.

Theorem 2.1 *(Childs [12], Herzog–Ulrich–Backelin [21]). Let K be a field, over which the polynomial $z^d - 1 \in K[z]$ is a product of linear factors. If $f \in K[x_0, \ldots, x_n]$ is a homogeneous form of degree d that can be written* *as the sum of N products of linear forms, then there are square matrices of linear forms $B_1, \dots, B_d$ and B such that*

$$f \cdot \mathrm{id}_{d^{N-1}} = B_1 \cdot \dots \cdot B_d \, ,$$

and

$$f \cdot \mathrm{id}_{d^N} = B^d \, ,$$

where id_k for $k = d^{N-1}$ and $k = d^N$ respectively denotes the $k \times k$ identity matrix. If $z^d - 1$ does not split in $K[z]$ then there exist matrices B_i and B of slightly larger size d^N and d^{N+1} respectively whose product respectively d-th power equals f times an identity matrix.

The proof relies on the following identity among matrices.

Proposition 2.2. *Let $A_1, \dots, A_d$ and $B_d, \dots, B_1$ be two families of variables indexed by $\mathbb{Z}/(d)$ such that the A variables and the B variables commute cyclically*

$$A_1 \cdot A_2 \cdots A_d = A_2 \cdot A_3 \cdots A_{d+1} \quad \text{and so on,}$$

and

$$B_d \cdot B_{d-1} \cdots B_1 = B_{d+1} \cdot B_d \cdots B_2 \quad \text{and so on,}$$

while A_i commutes with B_j for all $i \neq j$. Suppose that z_j are elements of K such that $\prod_{j=1}^{d} (z - z_j) = z^d - 1 \in K[z]$.

The product of the matrices

$$X_j = \begin{pmatrix} A_j & 0 & 0 & 0 & \dots & z_j B_d \\ z_j B_1 & A_{j+1} & 0 & 0 & \dots & 0 \\ 0 & z_j B_2 & A_{j+2} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & & \vdots \\ 0 & 0 & 0 & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 0 & z_j B_{d-1} & A_{j+d} \end{pmatrix}$$

is

$$X_1 \cdots X_d = \big(A_1 \cdots A_d + (-1)^{d+1} B_d \cdots B_1\big) \cdot I$$

where I is a $d \times d$ identity matrix.

Proof. Let A, B be the diagonal matrices with entries A_i and B_i, respectively, and let C be the matrix

$$C = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}$$

of the cyclic permutation $(1, \dots, d)$. We have

$$X_j = C^{-j}AC^j + z_jCB.$$

Thus we may write the product of the X_j as

$$\prod_{1\le j\le d} C^{-j}AC^j + (\prod_{1\le j\le d} z_j)(CB)^d + D_1 + \cdots + D_{d-1},$$

where D_k is the sum of all the terms with k factors of CB and $d-k$ factors of conjugates of A.

• Since conjugation by C cyclically permutes the diagonal matrices, $\prod_j C^{-j}AC^j$ is a diagonal matrix with entries $A_j \cdot A_{j+1} \cdots A_{j+d}$ which, by hypothesis are all equal.

• $\prod_j z_jCB = \prod_j z_j \cdot (CB)^d$ is $\prod_j z_j = (-1)^{d+1}$ times a diagonal matrix with entries $B_d \cdot B_{d-1} \cdots B_1$. This is because

$$(CB)^d = (CB)^dC^{-d} = ((CB)^{d-1}C^{-d+1}) \cdot (C^dBC^{-d})$$

and proceeding by induction we see that this is

$$(CBC^{-1}) \cdot (C^2BC^{-2}) \cdot \cdots \cdot (C^dBC^{-d}),$$

which, as for the first product, is a diagonal matrix with entries all equal to $B_d \cdots B_1$.

• It remains to show that all the D_k are 0. We have

$$z_jCB \cdot C^{-i}AC^i = z_j(CBC^{-1})(C^{-i+1}AC^{i-1})C.$$

The matrix CBC^{-1} is diagonal with diagonal entries $B_d, B_1, \dots, B_{d-1}$, while the matrix $C^{-i+1}AC^{i-1}$ is diagonal, with diagonal entries $A_{i-1}, A_i, \dots A_{d+i-1}$. Thus if $1 \le i \le d$ the corresponding terms on the diagonal are B_j and A_i with $i \ne j$. Since these commute by hypothesis, we see that, when $1 \le i \le d$,

$$z_jCB \cdot C^{-i}AC^i = z_jC^{-i+1}AC^{i-1} \cdot CB.$$

It follows that all the terms in D_k can be written as the same matrix

$$A \cdot C^{-1}AC \cdots C^{-(d-k)}AC^{d-k}(CB)^k$$

times the term

$$\sum_{|K|=k} \prod_{m\in K} z_m.$$

But this factor is the coefficient of z^k in $\prod(z - z_j) = z^d - 1$, and is thus equal to 0, as required. □

Proof of Theorem 2.1. We first assume that the polynomial $z^d - 1$ splits over the ground field and apply induction on N, the number of summands in a presentation of $f = \sum_{k=1}^N p_k$ as a sum of products of linear forms $p_k = \prod_{j=1}^d \ell_{k,j}$. For example, we could write f as a linear combination of monomials. In case $N = 1$ we may take $B_j = \ell_j \mathrm{id}_1$ proving the first statement. In the induction step we write our polynomial in the form $f \pm g$, where g is a product of linear forms and f a sum of $N-1$ products. By the induction hypothesis there exist $d^{N-2} \times d^{N-2}$ matrices $A_1, \dots, A_d$ with

$$A_1 \cdot A_2 \cdots A_d = f \mathrm{id}_{2^{N-2}}.$$

Note that if $A_1 \cdots A_d = f \cdot \mathrm{id}$ is a factorization of a diagonal matrix, with f a nonzerodivisor, then the A_i automatically commute cyclically.

If $g = p_N = (-1)^{d+1} \prod_{j=1}^{d} \ell_j$ we choose $B_j = \ell_j \mathrm{id}_{2^{N-2}}$. Then B_j commute with A_i and we may choose $2^{N-1} \times 2^{N-1}$ matrices X_j as in Proposition 2.2.

For the second statement we take given $B_1, \dots, B_d$, the matrix B to be

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots & B_d \\ B_1 & 0 & 0 & 0 & \dots & 0 \\ 0 & B_2 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & & \vdots \\ 0 & 0 & 0 & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 0 & B_{d-1} & 0 \end{pmatrix}$$

If $z^d - 1$ does not split in linear forms, we may adjoin a d-th primitive root of unity ζ in case $\mathrm{char}\, K = 0$. In case $\mathrm{char}\, K = p > 0$ we may write $d = p^r e$ with $(p, e) = 1$ and write $z^d - 1 = (z^e - 1)^{p^r}$. The polynomial splits after adjoining an e-th primitive root ζ of unity. Hence we find matrices over $K[\zeta][x_0, \dots, x_n]$ which is a free $K[x_0, \dots, x_n]$-module of rank e. Since ed^{N-1} divides d^N we get matrices of desired size in all cases. □

Both the version of Childs' construction in [21] and the one given here lead to the following bounds on Ulrich complexity:

Corollary 2.3. *If $f \in K[x_0, \dots, x_n]$ can be written a sum of N products of linear forms, then*

$$\mathrm{uc}(f) \le d^{N-1},$$

and if moreover K contains all d-th roots of unity then $\mathrm{uc}(f) \le d^{N-2}$.

Note that if f has degree $d \ge 2$ and is a sum of N products of linear forms, then the singular locus of f has codimension at most $2N$. In case of equality, our Conjecture 0.2 implies the left side inequality in

$$2^{N-2} \le \mathrm{uc}(f) \le d^{N-1}$$

while the right side follows from Corollary 2.3. Sums of products of linear forms have been studied under the name $\Sigma\Pi\Sigma$-circuits in algebraic complexity theory. For some lower exponential bounds on $\Sigma\Pi\Sigma$-circuits, see [19,20].

The products of linear forms are the points of a projected Segre variety,

$$\mathbb{P}^n \times \dots \times \mathbb{P}^n \to \mathrm{Sym}_d(\mathbb{P}^n) \subset \mathbb{P}^{\binom{n+d}{n}-1}$$

Thus the general form of degree d in $n+1$ variables is NOT a sum of fewer than

$$N = \lceil \binom{n+d}{n} / (nd+1) \rceil$$

products of linear forms. Question: Which secant locus actually fills the space?

We note again that Childs' construction actually provides a "complete" factorization into linear matrices, which is more than needed for Ulrich complexity. For example, the Ulrich complexity of the generic determinant is 1, as already noted. Except in the trivial case $d = 2$, we do not know what size matrices are necessary to factor the generic $d \times d$ determinant into a product of d matrices of linear forms, as in Childs' Theorem above.

3. Singularities and a weak lower bound

If X is a variety and F is a module over the coordinate ring S_X, then the *singular locus* of F is by definition the set of points p of X such that F is not locally free at p.

Theorem 3.1. *If F is a maximal Cohen–Macaulay module on a variety $X \subset \mathbb{P}^n$, then the singular locus of F is contained in the singular locus of X. If X is a hypersurface and F is not free, then*

$$\operatorname{rank} F \geq \sqrt{c'} - 1 \geq \sqrt{c} - 1$$

where $c' = \operatorname{codim}_{\mathbb{P}^n} \operatorname{sing} F$ *and* $c = \operatorname{codim}_{\mathbb{P}^n} \operatorname{sing} X$*. Thus*

$$\operatorname{uc}(X) \geq \sqrt{c} - 1,$$

and, if X is smooth, then $\operatorname{uc}(X) \geq \sqrt{n+1} - 1$*.*

Proof. By the Auslander–Buchsbaum Formula [14, Theorem 19.9], a maximal Cohen–Macaulay module over a regular local ring is free, proving the first assertion.

If F is a module of rank r on X that is minimally generated by m elements, and F is not free, then $0 < r < m$, and F is not locally free on X exactly at the points of the subvariety of X defined by the $(m-r) \times (m-r)$ minors of a presentation matrix for F over S_X.

If X is a hypersurface, and F is a maximal Cohen–Macaulay module then, again by the Auslander–Buchsbaum Theorem, F has projective dimension 1 as an S-module. Since, as an S-module, F is torsion, it follows that F is the cokernel of the map given by an $m \times m$ matrix whose entries are forms of positive degrees. By a theorem of Eagon and Northcott (see for example [14, Exercise 10.9]) the ideal J of $(m-r) \times (m-r)$ minors of such a matrix has codimension $\leq (r+1)^2$, so $c' \leq (r+1)^2$, or equivalently $\sqrt{c'} - 1 \leq r$. Since the radical of the ideal J contains the hypersurface equation, the locus defined by J in $\mathbb{P}^n$ is equal to the singular locus of F in X. If X is singular then the desired inequalities follow from the first statement of the Theorem.

In the smooth case J must define the empty set, so $n+1 = \operatorname{codim} J \leq (r+1)^2$, and thus $r \geq \sqrt{n+1} - 1$ as required. □

Theorem 3.2. *Let $X \subset \mathbb{P}^n$ hypersurface. There exists an Ulrich module of the homogeneous coordinate ring of X that is locally free on the punctured spectrum.*

The idea of the proof is to restrict an Ulrich module from a smooth hypersurface in a larger projective space:

Lemma 3.3. *Let $f \in K[x_0, \ldots, x_n]$ in a homogeneous polynomial of degree d over infinite field K. Suppose the singular locus of f in $\mathbb{P}^n$ has dimension $d \geq 0$. Let $\ell \in K[x_0, \ldots, x_n]$ be a linear form whose zero locus does not contain any maximal dimensional component of* $\operatorname{sing} f$*. Let z be a additional variable and let $g \in K[\ell, z]$ be any homogeneous binary form of degree d that has no repeated factors over the algebraic closure of K. The polynomial $f + g \in K[x_0, \ldots, x_n, z]$ defines a hypersurface whose singular locus is* $\operatorname{sing} f \cap V(z, \ell)$*. In particular,* $\dim \operatorname{sing}(f+g) = \dim \operatorname{sing} f - 1$*.*

Proof. We may pass to an algebraically closed field and may assume that g factors $g = a\prod_{i=1}^{d}(z - \alpha_i \ell)$. If $p \in \operatorname{sing}(f+g) \subset \mathbb{P}^{n+1}$ is a point, then $f + g \in \mathfrak{m}^2$ where $\mathfrak{m}$ denotes the ideal of p. Since $f \not\equiv z - \alpha_i \ell$ mod $\mathfrak{m}^2$ because of the z, the point p is a singular point only if two of the factors of g vanish at p. Hence both z and ℓ vanish at p and thus $p \in V(z, \ell) \cap \operatorname{sing} f$. □

Proof of Theorem 3.2. Let Y be a nonsingular hypersurface such that X is a dimensionally transverse linear section of Y. By the Auslander–Buchsbaum formula, any maximal Cohen–Macaulay module for the homogeneous coordinate ring S_Y of Y is locally free on the punctured spectrum. By Corollary 2.3 the ring S_Y has an Ulrich module, which is thus locally free on the punctured spectrum, and these properties are preserved by restriction to X. □

Remark. If f is a sum of N products of linear forms and the singular locus of f has codimension s in $\mathbb{P}^n$ then the construction in the Theorem, used with Theorem 2.1, leads to locally free Ulrich sheaves of rank $d^{N-1+n-s}$. This suggests that some conjecture stronger than Conjecture 0.2 might be true for locally free Ulrich modules. For example, in the case of the generic determinant of rank n, the rank 1 Ulrich module is singular in codimension 4 in the ambient space. We don't know the minimal rank of a locally free Ulrich module even in this case!

4. Hypersurface examples

Example 4.1 *(Quadrics).* The following result is fundamental:

Theorem 4.2 *(Knörrer periodicity [23]). There is a one-to-one correspondence, given as in Theorem 2.1, between Ulrich modules for the hypersurface defined by a power series $f \in K[[z_1, \dots, z_n]]$ and the power series $f + xy \in K[[z_1, \dots, z_n, x, y]]$.*

It would be extremely interesting to have any analogue for higher degree additions, for example for $f(z) + wxy \in K[[z_1, \dots, z_n, w, x, y]]$.

For quadric hypersurfaces any nontrivial indecomposable matrix factorization must consist of two matrices of linear forms, so by [13] all maximal Cohen–Macaulay modules are Ulrich in this case. For a nonsingular quadric of rank n over an algebraically closed field of characteristic not 2, we can use Theorem 4.2 starting with z_1^2 in one variable and $z_1 z_2$ in two variables, and we see that there is a unique non-free indecomposable Ulrich module when n is odd, and exactly two (of equal rank) when n is even; the rank of the modules is $2^{\lceil n/2 \rceil - 2}$. In particular, we have:

Corollary 4.3. *Any non-free Ulrich module on a smooth quadric hypersurface in projective n-space has rank $2^{\lceil (n+1)/2 \rceil - 2}$.*

Using the theory of Clifford algebras (see for example [22, Volume 2]) [9] gives a picture of the Ulrich modules on a quadric hypersurface over an arbitrary field. It is always true that there is always just one or two indecomposable Ulrich modules but over a non-algebraically closed field, the rank of the modules may be as high as 2^{n-2}.

Example 4.4 *(Low Ulrich complexity).* Beauville's beautiful paper [4] develops techniques for deciding when a form of degree d can be written as the $d \times d$ determinant (possibly symmetric) or as the $2d \times 2d$ Pfaffian of a skew-symmetric matrix of linear forms, with special attention to the case of smooth hypersurfaces and general hypersurfaces. (In both these cases, the existence of the matrix N as in Definition 0.1 is automatic, so his results also decide the question of Ulrich complexity.) He reproves the classical results on plane curves and smooth cubics along the way. In many results the ground field is arbitrary. As the hypotheses vary, the paper is not easy to summarize (go and read it!) but here is a sample:

Theorem 4.5. ([4, Propositions 7.6, 8.5]) *If the ground field K is algebraically closed, then a general form of degree d on $\mathbb{P}^3_K$ can be written as the Pfaffian of a linear $2d \times 2d$ matrix if and only if $d \leq 15$. Moreover,* *this is true for every form of degree 3 on $\mathbb{P}^3_K$, and for every form of degree 3 on $\mathbb{P}^4_K$ that defines a smooth variety.*

Example 4.6 *(Permanents).* The cokernel of the matrix from the introduction is a rank 2 Ulrich module F on the 3×3 permanental hypersurface in $\mathbb{P}^8$. Since $\sqrt{6} - 1 > 1$, Theorem 3.1 proves that $\mathrm{uc}(\mathrm{perm}_3) = 2$.

The following result on larger permanents illustrates a method that is applicable more generally to give lower bounds for the rank of Cohen–Macaulay modules singular at a point p of a hypersurface X such that the intersection of X with the tangent plane at p is defined by a power series whose quadratic part has high rank.

Theorem 4.7. *Suppose that the ground field K has characteristic 0 or $\geq n$. Let Y be the linear subvariety of* perm_n *defined by the vanishing of some pair of rows or some pair of columns. Every maximal Cohen–Macaulay module on* perm_n *that is not generically free on Y has rank $\geq 2^{n-2}$.*

A computation shows that the $\mathrm{sing}(\mathrm{perm}_3)$ has 15 components, all of which have codimension 6. More precisely, there are 6 linear components defined by the vanishing of two rows or columns, and 9 quadratic components, defined by a row and a column and the complementary 2×2 permanent. The non-locally free locus of F above, consists of 4 of the linear components and 5 of the quadratic components. This shows that even Ulrich modules of minimal rank on a hypersurface X can be nonsingular on part of the singular locus of X.

Proof. Let $Y \subset \mathrm{perm}_n$ be the subvariety given by the vanishing of all the variables in two columns (or rows), say $Y = \{x_{1,j} = x_{2,j} = 0 \mid j = 1, \ldots, n\}$. Substituting $x_{i,j} = 1$ for all pairs (i, j) with $i \neq 1, 2$ the permanent specializes to the quadric

$$q = (n-2)! \sum_{i \neq j} (x_{1i}x_{2j} + x_{1j}x_{2i}).$$

The Hessian of this quadric is the matrix $(n-2)! \begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix}$ where $A = (a_{ij})$ is the $n \times n$ matrix with entries $a_{ij} = \begin{cases} 1 & \text{for } i \neq j \\ 0 & \text{for } i = j \end{cases}$. Subtracting the first row of A from all the other rows, and then replacing the first column with the sum of all the columns, we see that:

$$\det \begin{pmatrix} 0 & 1 & 1 & \ldots & 1 \\ 1 & 0 & 1 & \ldots & 1 \\ 1 & 1 & 0 & \ldots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \ldots & 0 \end{pmatrix} = \det \begin{pmatrix} 0 & 1 & 1 & \ldots & 1 \\ 1 & -1 & 0 & \ldots & 0 \\ 1 & 0 & -1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \ldots & -1 \end{pmatrix} = \det \begin{pmatrix} n-1 & 1 & 1 & \ldots & 1 \\ 0 & -1 & 0 & \ldots & 0 \\ 0 & 0 & -1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & -1 \end{pmatrix}.$$

Thus $\det A = (-1)^{n-1}(n-1) \neq 0$. This shows that the quadric q has full rank $2n$. A maximal Cohen–Macaulay on perm_n specializes to a maximal Cohen–Macaulay module on this quadric. By Corollary 4.3, a maximal Cohen–Macaulay module on perm_n that has rank $\geq 2^{n-2}$ is locally free along Y. □

Remark 4.8. We believe that the $\binom{n}{2}$ permanents of size $n-2$ of a generic $n \times (n-2)$ matrix are algebraically independent over a field of characteristic $\neq 2$. If this is true then we would obtain a proof of Theorem 4.7 valid over any field of characteristic $\neq 2$.

Example 4.9 *(Binomials and trinomials).* Note that an irreducible polynomial f of degree d cannot have a matrix factorization with a linear matrix of size less than d, since f^m has no factor of degree $< d$. Thus the upper bound d^{N-2} is sharp for any irreducible binomial. Since it is also sharp for quadratic forms over an algebraically closed field (Example 4.1), the first nontrivial case is a cubic trinomial:

Proposition 4.10. *If K is a field containing a cube root of 1, then the Ulrich complexity of the generic trinomial $f = abc + xyz + uvw \in K[a, b, c, x, y, z, u, v, w]$ is* $\mathrm{uc}(f) = 3$.

The proof uses the following general result:

Theorem 4.11. *Let $X \subset \mathbb{P}^n_K$ be a hypersurface whose singular locus has codimension ≥ 5 in $\mathbb{P}^n_K$, where K is a field of characteristic $\neq 2$. If F is an Urich module of rank 2 on X then the presentation matrix of F can be taken to be skew symmetric. In particular, the equation of X can be written as the Pfaffian of a matrix of linear forms.*

We note that [4] has a related result for smooth surfaces in 3-folds, strengthening Theorem 4.11 in this leading special case.

Proof. By the Lefschetz Theorem [18], the homogeneous coordinate ring S_X of X is factorial. Thus any rank 1 reflexive module on X is isomorphic to $S_X(m)$ for some integer m. In particular, we have a map

$$\wedge^2 F \to (\wedge^2 F)^{**} \cong S_X(m)$$

that is an isomorphism on the smooth locus of X. Equivalently, there is a skew-symmetric map

$$\phi : F \to \mathcal{H}om(F(-m), S_X) \cong \mathrm{Ext}^1_X(F, S)$$

that is an isomorphism on the smooth locus of X. Since both F and $\mathcal{H}om(F(-m), S_X)$ are maximal Cohen–Macaulay modules, and the singular locus of X has codimension > 1, ϕ is an isomorphism globally.

Since F is a maximal Cohen–Macaulay S_X-module, it has an S-free presentation

$$0 \to G_1 \to G_0 \to F \to 0\,.$$

Since F is Ulrich, we may assume, possibly after twisting, that G_0 is generated in degree 0 and G_1 is generated in degree 1. Thus $\mathrm{Ext}^1_X(F, S)$ is generated in degree 1, so $m = -1$. The skew-symmetric map ϕ induces a skew symmetric map on the elements of F of degree 0, and thus lifts to the unique map $\alpha : G_0 \to G_1^*(-1)$ making the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & G_1 & \xrightarrow{M} & G_0 & \longrightarrow & F & \longrightarrow & 0\,, \\
 & & \big\downarrow{\scriptstyle\beta} & \searrow^{\varphi} & \big\downarrow{\scriptstyle\alpha} & & \big\downarrow{\scriptstyle\phi} & & \\
0 & \longrightarrow & G_0^*(-1) & \xrightarrow{M^*} & G_1^*(-1) & \longrightarrow & \mathrm{Ext}^1_S(F, S(-1)) & \longrightarrow & 0
\end{array}$$

commute. The dual of the map β induces the map dual to ϕ:

$$\mathrm{Hom}(\phi, S(1)) : M(1) \cong \mathrm{Hom}(\mathrm{Ext}^1_{S_X}(F, S(-1)), S_X) \to \mathrm{Hom}(F, S_X) \cong \mathrm{Ext}^1_S(F, S)\,.$$

Since these modules are generated in a single degree, β is the unique map inducing the dual of ϕ. But the way that ϕ was constructed guarantees that the dual of ϕ is $-\phi$, and thus $\beta = -\alpha^*$. It follows that the diagonal map φ is equal to both $\alpha \circ M$ and to $-M^* \circ \alpha^*$; that is, φ is skew-symmetric. Since ϕ is an isomorphism, so is α, and thus φ is a skew-symmetric presentation matrix for F. □

Proof of Proposition 4.10. By Theorem 2.1, the trinomial f has a 9×9 matrix factorization, so $\mathrm{uc}(f) \leq 3$.

The singular locus of the hypersurface $t = 0$ consists of the 9 linear spaces of codimension 6 defined by the vanishing of a pair of variables from each monomial. The polynomial t is not equal to the Pfaffian of a 6×6 matrix of linear forms since the singular locus of the generic 6×6 Pfaffian is Gorenstein of codimension 6 and degree 14 (it is the cone over the Grassmannian $\mathbb{G}(2, 6)$ in its Plücker embedding) which is > 9. Thus, by Theorem 4.11 below, $\mathrm{uc}(t) > 2$. Hence the Ulrich complexity of the generic trinomial is $\mathrm{uc}(abc + xyz + uvw) = 3$. □

5. Higher codimension

Beyond the case of hypersurfaces, for which Theorem 2.1 gives a construction, complete intersections and linear determinantal varieties of maximal minors are known to carry Ulrich sheaves [9], as do all projective curves. There is a large literature giving other examples. Here we discuss the Veronese and Segre varieties, where we can sometimes compute the Ulrich complexity exactly.

5.1. Veronese and Segre varieties

If $X \subset \mathbb{P}^n$ has an Ulrich sheaf, then so do the Veronese embeddings $v_d : X \hookrightarrow \mathbb{P}^{\binom{n+d}{n}-1}$ and a similar statement is true for Segre products $X \times Y$ and dimensionally transversal intersections $X \cap Y$ of varieties $X, Y \subset \mathbb{P}^n$ with Ulrich sheaves, see [16, Prop. 2.6]. In particular the Ulrich complexity of $\mathbb{P}^m$ is finite in any of its embeddings.

The construction given in [16] produces Ulrich sheaves of minimal rank in some cases:

Theorem 5.1.

$$\mathrm{uc}(v_d(\mathbb{P}^m)) \leq m!,$$

and equality holds for a pair (m, d) if every prime $\leq m$ divides d. Moreover, if $\mathcal{F}$ is any Ulrich bundle on $v_d(\mathbb{P}^m)$ then $\operatorname{rank} \mathcal{F}$ *is divisible by the greatest common divisor* $\gcd(d^m, m!)$.

Proof. In [15, Theorem 6.1] we constructed a supernatural vector bundle $\mathcal{E}$ of rank $m!$ on $\mathbb{P}^m$ with Euler characteristic

$$\chi(\mathcal{E}(t)) = \prod_{k=1}^{m} (t + kd)$$

by pushing forward a line bundle on $(\mathbb{P}^1)^m$ via a linear projection from the Segre embedding of $(\mathbb{P}^1)^m$. In particular, $v_{d*}(\mathcal{E})$ is an Ulrich bundle on the $v_d(\mathbb{P}^m)$, establishing the upper bound.

Conversely, if $\mathcal{F}$ is an Ulrich bundle on $v_d(\mathbb{P}^m)$ then $v_d^*\mathcal{F}$ is a bundle with Euler characteristic

$$\chi(v_d^*\mathcal{F}(t)) = \frac{\operatorname{rank} \mathcal{F}}{m!} \prod_{k=1}^{m} (t + kd),$$

and it is supernatural by Boij–Söderberg theory ([15], Proposition 6.3). If p is a prime which divides d, then the residue class of $\prod_{k=1}^{m}(1 + kd) \equiv 1 \mod p$. Thus the p part of $m!$ divides $\operatorname{rank} \mathcal{F}$. This implies the second statement, and hence $\mathrm{uc}(v_d(\mathbb{P}^m)) = m!$ in case every prime $\leq m$ divides d. □

For Segre embeddings we have sub-multiplicativity:

Proposition 5.2. *If X_i are projective varieties then their product, in the Segre embedding, satisfies*

$$\mathrm{uc}(\prod_i X_i) \leq \prod_i \mathrm{uc}(X_i).$$

In particular, the Ulrich complexity of $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ in any of its embeddings is 1.

Proof. By Proposition 2.6 of [16], if $\mathcal{F}_i$ be an Ulrich sheaf on X_i, and $\dim X_i = n_i$, then

$$\mathcal{F} := \mathcal{F}_1 \boxtimes \mathcal{F}_2(n_1 H_2) \boxtimes \mathcal{F}_3((n_1 + n_2)H_3) \boxtimes \cdots$$

where H_i is the hyperplane class on X_i. □

5.2. Curves and Abelian surfaces

The Ulrich complexity of any curve $C \subset \mathbb{P}^n$ over an algebraically closed field is $\mathrm{uc}(C) = 1$, and the Ulrich line bundles of a smooth curve of genus g over an algebraically closed field all have the form $\mathcal{F} = \mathcal{L} \otimes \mathcal{O}_C(1)$ with $\mathcal{L} \in \mathrm{Pic}^{g-1}(C) \setminus \Theta$, where $\Theta = W^0_{g-1}(C)$ denotes the theta divisor.

Over a non-algebraically closed field the Ulrich complexity of a curve might be larger than 1 since the Ulrich line bundles defined over the algebraic closure may not be rationally defined. For example, the conic $C = V(x^2 + y^2 + z^2) \subset \mathbb{P}^2_{\mathbb{R}}$ has Ulrich complexity $uc_{\mathbb{R}}(C) = 2$ and

$$\mathcal{F} = \mathrm{coker}\,(\mathcal{O}^4(-1) \xrightarrow{M} \mathcal{O}^4)$$

with M the skew symmetric matrix

$$M = \begin{pmatrix} 0 & x & y & z \\ -x & 0 & z & -y \\ -y & -z & 0 & x \\ -z & y & -x & 0 \end{pmatrix}$$

is a rank 2 Ulrich bundle on the conic.

Beauville [5] proves the existence of Ulrich bundles on any Abelian surface X over an algebraically closed field, and shows $\mathrm{uc}(X) \leq 2$ in this case. But the existence of Ulrich sheaves on algebraic surfaces in general—even in $\mathbb{P}^4$—remains open.

6. Grenet's determinantal representation of the permanent

Let $S = k[x_{1,1}, \ldots, x_{n,n}, z]$ be a polynomial ring, and let $\mathrm{perm}_n = \mathrm{perm}_n(x)$ denote the $n \times n$ permanent of the matrix with entries $x_{i,j}$.

Proposition 6.1. *In case $n = 3$, and conjecturally in general: the singular locus of the $n \times n$ permanental hypersurface has codimension $2n$. The linear components of codimension $2n$ are given by pairs of rows or pairs of columns. The other components of codimension $2n$ are given by the vanishing of a row, a column, and the complementary permanent.*

Theorem 6.2. [17] *There is a matrix M of size $m = 2^n - 1$ whose determinant is $(-1)^{n-1} z^{m-n} \mathrm{perm}_n$.*

Proof. Let ϕ_i denote the i-th matrix of the Koszul complex on $x_{i,1}, \ldots, x_{i,n}$. Let $\tilde{\phi}_i$ be the matrix obtained from ϕ_i by replacing all minus signs by plus signs. Let Φ_n be the $m \times m$ matrix Φ_n that is the direct sum of the $\tilde{\phi}_i$, and let

$$M_n = \Phi_n + z \cdot J_n$$

where J_n is the $m \times m$ nilpotent matrix Jordan matrix with 1s on the sub diagonal and zeros elsewhere; for example, [24] p. 325,

$$M_3 = \left(\begin{array}{ccc|ccc|c} x_{1,1} & x_{1,2} & x_{1,3} & 0 & 0 & 0 & 0 \\ \hline z & 0 & 0 & 0 & x_{2,3} & x_{2,2} & 0 \\ 0 & z & 0 & x_{2,3} & 0 & x_{2,1} & 0 \\ 0 & 0 & z & x_{2,2} & x_{2,1} & 0 & 0 \\ \hline 0 & 0 & 0 & z & 0 & 0 & x_{3,1} \\ 0 & 0 & 0 & 0 & z & 0 & x_{3,2} \\ 0 & 0 & 0 & 0 & 0 & z & x_{3,3} \end{array}\right).$$

The following Lemma shows that after substituting $z = 1$ the determinant of M_n becomes equal to $(-1)^{n-1} \det(\tilde{\phi}_1 \cdot \ldots \cdot \tilde{\phi}_n) = (-1)^{n-1} \mathrm{perm}_n(x)$ as required. □

Lemma 6.3. *Let $\psi_1, \ldots, \psi_s$ be a composable sequence of matrices such that the number of rows of ψ_1 equals the number of columns of ψ_s. Setting*

$$M = \begin{pmatrix} \psi_1 & & & & \\ 1 & \psi_2 & & & \\ & 1 & \ddots & & \\ & & \ddots & \psi_{s-1} & \\ & & & 1 & \psi_s \end{pmatrix}$$

we have

$$\det M = (-1)^t \det(\psi_1 \cdot \ldots \cdot \psi_s)$$

where $t = (\sum_{i=1}^{s-1}$ number of columns of $\psi_i)($ number of columns of $\psi_s) + s - 1$.

Proof. Use row operations to clear out the first $s - 1$ blocks of the top rank of target of ψ_1 rows of M recursively. This leaves $(-1)^{s-1}$ times $\psi_1 \cdot \ldots \cdot \psi_s$ in the last block. Now permute the columns to bring this product into the upper left corner. This introduces an additional sign

$$(-1)^{(\sum_1^{s-1} \text{ number of columns of} \psi_i)(\text{ number of columns of } \psi_s)},$$

completing the argument. □

Given the central place of the Koszul complex, it is tempting to believe that this is best possible. This known in only one case:

Theorem 6.4. [1] *Let $x_{i,j}$ be a 3×3 matrix of variables, and let z be a further variable. There is no expression of $z^k \cdot \mathrm{perm}_3(x)$ as a determinant of linear homogeneous matrix of size $m := 3 + k < 7 = 2^3 - 1$.*

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