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NOTES ON REGULARITY STABILIZATION

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Abstract. When M is a finitely generated graded module over a standard graded algebra S and I is an ideal of S, it is known from work of Cutkosky, Herzog, Kodiyalam, Römer, Trung and Wang that the Castelnuovo-Mumford regularity of I^mM has the form $dm+e$ when $m \gg 0$. We give an explicit bound on the m for which this is true, under the hypotheses that I is generated in a single degree and M/IM has finite length, and we explore the phenomena that occur when these hypotheses are not satisfied. Finally, we prove a regularity bound for a reduced, equidimensional projective scheme of codimension 2 that is similar to the bound in the Eisenbud-Goto conjecture, under the additional hypotheses that the scheme lies on a quadric and has nice singularities.

Introduction

Let S be a standard graded algebra over a field k, that is, an algebra generated by finitely many forms of degree one, let M be a finitely generated graded S-module, and let I be a homogeneous ideal not contained in the radical of $\operatorname{ann} M$. If H is an Artinian S-module, we set $\operatorname{reg} H = \max\{d \mid H_d \neq 0\}$ and we write $\operatorname{reg} M$ for the Castelnuovo-Mumford regularity

$$\operatorname{reg} M = \operatorname{reg}_{S_+} M := \max\{\operatorname{reg} H^i_{S_+}(M) + i\}.$$

Combining results of Cutkosky-Herzog-Trung [C-H-T], Kodiyalam [Kod], Römer [R] and Trung-Wang [T-W], we have:

Theorem 0.1. *There exist integers $m_0 = m_0(I, M)$, $d = d(I, M)$ and $e = e(I, M)$ such that for all $m \geq m_0$,*

$$\operatorname{reg} I^m M = dm + e\,.$$

Furthermore, d is the asymptotic generator degree of I on M, i.e., the minimal number such that if $J \subset I$ is the ideal generated by the elements of I of degree $\leq d$, then $I + \operatorname{ann} M$ is integral over $J + \operatorname{ann} M$.

This beautiful result begs for an answer to several questions: What is the significance of the number e? What is a reasonable bound m_0? What is the nature of the function $m \mapsto \operatorname{reg} I^m M$ for $m < m_0$ …? In general very little is known. But the result of the first section of this paper gives a value for m_0 in case

(∗) I is generated in a single degree and M/IM has finite length.

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Here is a summary of our knowledge in this case. Under the hypothesis $(*)$ one has:

- The number d in Theorem 0.1 is equal to the common degree of the generators of I.
- The differences $e_m := \operatorname{reg} I^m - dm$ form a weakly decreasing sequence of nonnegative integers.
- The asymptotic value e of the e_m can be identified with the regularity of the restriction of the sheaf associated to S to the fibers of the morphism defined by I.
- If S is a polynomial ring and I is not a complete intersection, then the numbers e_m are equal to the asymptotic value e for all $m \geq m_0$, where m_0 is the $(0, 1)$-regularity (defined below) of the Rees algebra $\mathcal{R}(I)$.

The first item in this list is immediate from Theorem 0.1. The next two are proved in Eisenbud-Harris [E-H]. The last is the subject of the first section of this paper, where we also derive a sharper but more technical bound that is often optimal. We note that a different (somewhat larger) value for m_0 was proposed in Cutkosky-Herzog-Trung [C-H-T], but the proof given was incomplete, as the authors of that paper have pointed out. Also, a bound similar to ours has been shown by Marc Chardin (private communication) with a spectral sequence proof.

In connection with the second item of the list, we observed in many cases that the sequence of first differences of the $e_m - e_{m+1}$ is also weakly decreasing. Is this always the case, under the assumption of $(*)$?

A key definition in this development is the $(0, 1)$ (Castelnuovo-Mumford) regularity of the Rees module $\mathcal{R}(I.M)$. To define it, we recall that the *Rees ring* of I is

$$\mathcal{R}(I) := \bigoplus_{j\geq 0} I^j \cong \bigoplus_{j\geq 0} I^j t^j = S[It] \subset S[t]\,.$$

This ring is an epimorphic image of the polynomial ring $T := S[y_1, \ldots, y_r]$ via the map of S-algebras sending the y_i to t times the homogeneous minimal generators of I. In fact, this becomes a map of bigraded k-algebras if we set $\deg x_i = (1, 0)$ and $\deg y_i = (0, 1)$ (note that this is only possible because the generator degrees of I are assumed to be equal). Next, if M is a finitely generated graded S-module, we define

$$\mathcal{R}(I, M) := S[It]M \subset M \otimes_S S[t]\,,$$

which is a finitely generated bigraded module over $\mathcal{R}(I)$ and hence over T. Thus we consider a bigraded minimal free resolution

$$\cdots \quad F_1 \longrightarrow F_0 \longrightarrow \mathcal{R}(I, M) \to 0$$

of $\mathcal{R}(I, M)$ as a T-module, and we define the $(0, 1)$-*regularity* $\operatorname{reg}_{(y_1,\ldots,y_r)} \mathcal{R}(I, M)$ to be the maximum integer j such that F_i has a free summand of the form $T(-a, -i-j)$ for some i and a. As with the usual Castelnuovo-Mumford regularity, there is also a definition in terms of local cohomology, which we will use freely; see Römer [R] for a detailed treatment.

In the second section of this paper, we turn to the question of what happens if we weaken the hypothesis $(*)$ to allow ideals that are not necessarily generated in a single degree. We found it surprisingly hard to give formulas for the numbers $e_m(I, M) := \operatorname{reg} I^m M - d(I, M)m$, even in very special cases; but we are able to provide such a formula when $M = S = k[x_1, \ldots, x_n]$ is a polynomial ring and

$I = J + (x_1, \ldots, x_n)^D$ for some D, with J an $(x_1, \ldots, x_n)$-primary ideal generated in a single degree, in terms of the numbers $e_m(J, M)$. In particular, we find that in this situation the numbers $e_m(I, M) - e_{m+1}(I, M)$ need not be weakly decreasing.

Section 3 of the paper uses some of the same ideas to prove a result close in spirit to the Eisenbud-Goto conjecture. Let $I \subset (x_1, \ldots, x_n)^2$ be a reduced, equidimensional homogeneous ideal in S, and suppose that k is algebraically closed. The Eisenbud-Goto conjecture then asserts the following: *if the projective variety X associated to I is connected in codimension* 1*, then* $\operatorname{reg} I \leq \deg X - \operatorname{codim} X + 1$. This conjecture is wide open, even for smooth varieties X, when the dimension of X is large.

In the conjecture the hypothesis "connected in codimension 1" is necessary, as an example of Giaimo (included in Section 3) shows; without the hypothesis, one must expect exponentially large regularity in general. But we are able to prove a bound that is only slightly weaker than that of the Eisenbud-Goto conjecture *without any connectedness hypothesis,* assuming instead that X has codimension 2, lies on a quadric, and has only isolated "bad" singularities.

1. $\mathfrak{m}$-primary ideals generated in one degree

In this section, S denotes a standard graded algebra over a field k. We write $\mathfrak{m}$ for the homogeneous maximal ideal of S. Let $I \subset S$ be a homogeneous ideal generated in a single degree d.

We consider the Rees ring $\mathcal{R}(I) = S[It]$ of I, a standard bigraded k-algebra as described above. Let A be the ring

$$A := k[I_d t] = \bigoplus_j \mathcal{R}(I)_{(0,j)} \subset \mathcal{R}(I).$$

It is a bigraded subalgebra of $\mathcal{R}(I)$, generated in degree $(0, 1)$, which is a direct summand as an A-module. We regard A as a standard graded algebra, generated in degree 1 over k. We write $\mathfrak{n}$ for the homogeneous maximal ideal of A. Since I is generated in one degree, A is isomorphic to the *special fiber ring* $\mathcal{F}(I) := \mathcal{R}(I) \otimes_S k$.

For M a finitely generated graded S-module we consider the *Rees module* $\mathcal{R}(I, M) = S[It]M$, which is a finitely generated bigraded $\mathcal{R}(I)$-module. We define

$$N_i(I, M) := k[I_d t]M_i \subset \mathcal{R}(I, M).$$

With the $(0, 1)$-grading, the A-module $N_i(I, M)$ is generated in degree 0 and has degrees determined by the powers of t. As an A-module, $\mathcal{R}(I, M)$ is isomorphic to the direct sum of the $N_i(I, M)$. In particular,

$$\operatorname{reg}_{(y_1, \ldots, y_r)} \mathcal{R}(I, M),$$

the $(0, 1)$-regularity of $\mathcal{R}(I, M)$, is the maximum of the regularities of the $N_i(I, M)$ (as A-modules). We shall see later how to restrict the range of i required.

Theorem 1.1. *Suppose that $I \subset S$ is an ideal generated by forms of a single degree d, and M is a finitely generated graded S-module, generated in a single degree, such that M/IM has finite length but M does not. Let e be the number such that*

$$\operatorname{reg} I^m M = md + e$$

for $m \gg 0$. Let $N_e := N_e(I, M)$.

(1) *The equality* $\operatorname{reg} I^m M = md + e$ *holds if*

$$m \geq \max\{\operatorname{reg} H^1_{\mathfrak{n}}(N_e) + 1,\ \frac{\operatorname{reg} M - e + 1}{d}\}.$$

(2) *In case* $\operatorname{reg} H^1_{\mathfrak{n}}(N_e) \geq (\operatorname{reg} M - e + 1)/d$ *and* $m \geq 1$, *the equality* $\operatorname{reg} I^m M = md + e$ *holds if and only if*

$$m \geq \operatorname{reg} H^1_{\mathfrak{n}}(N_e) + 1.$$

Corollary 1.2. *Let* I, S, M, d, e *be as in Theorem* 1.1, *and assume that* M *is generated in degree* 0. *The equality* $\operatorname{reg} I^m M = md + e$ *holds for all* $m \geq \max\{\operatorname{reg}_{(y_1,\ldots,y_r)} \mathcal{R}(I, M), \frac{\operatorname{reg} M+1}{d}\}$.

Proof of the Corollary. Since N_e is an A-direct summand of $\mathcal{R}(I, M)$,

$$\operatorname{reg} H^1_{\mathfrak{n}}(N_e) + 1 \leq \operatorname{reg} N_e \leq \operatorname{reg}_{(y_1,\ldots,y_r)} \mathcal{R}(I, M). \qquad \square$$

Proof of the Theorem. After a shift of degree we may assume that M is generated in degree 0. Consider first part (1), and assume that

$$m \geq \max\{\operatorname{reg} H^1_{\mathfrak{n}}(N_e) + 1,\ \frac{\operatorname{reg} M - e + 1}{d}\}.$$

By Eisenbud-Harris [E-H], Proposition 1.1, $\{e_n\}$ is a nonincreasing sequence of nonnegative integers. Thus it suffices to show that $\operatorname{reg} I^m M \leq md + e$. Our assumption on m implies that $\operatorname{reg} M \leq md + e - 1$. Because of the exact sequence

$$0 \to I^m M \to M \to M/I^m M \to 0 \tag{1}$$

we only need to show that $\operatorname{reg} M/I^m M \leq md + e - 1$. Since $M/I^m M$ has finite length, this is equivalent to the statement that

$$(I^m M)_{md+e} = M_{md+e}.$$

The definition of e implies, by the same argument, that this equality at least holds for sufficiently large m.

Let $N'_e := N_e(\mathfrak{m}^d, M) = \bigoplus_{j\geq 0} M_{jd+e} t^j$, where the last equality holds because M is generated in degree $0 \leq e$. Note that N'_e is naturally a graded A-module (with j-th graded piece $M_{jd+e} t^j$) and that N_e is a submodule. Let

$$E := N'_e/N_e = \frac{\bigoplus_{j\geq 0} M_{jd+e} t^j}{\bigoplus_{j\geq 0} (I^j)_{jd} M_e t^j}.$$

By the preceding remark, the module E has finite length.

We wish to show that $E_m = 0$. Since $m \geq \operatorname{reg} H^1_{\mathfrak{n}}(N_e) + 1$ we see from the exact sequence

$$\cdots \to H^0_{\mathfrak{n}}(N'_e) \to E \to H^1_{\mathfrak{n}}(N_e) \to H^1_{\mathfrak{n}}(N'_e) \to \cdots \tag{2}$$

that it suffices to prove $H^0_{\mathfrak{n}}(N'_e)_m = 0$.

We may identify the A-module N'_e with the $k[I_d]$-module $\bigoplus M_{dj+e}$, which is a $k[I_d]$-direct summand of M. Note that this identification sends the degree j part of N'_e to the degree $dj + e$ part of M. Moreover, since $I_d S = I$ contains a power of $\mathfrak{m}$, the module $H^0_{\mathfrak{n}}(N'_e)$ is a summand of $H^0_{\mathfrak{m}}(M)$ (with the same degree shift). On the other hand, $H^0_{\mathfrak{m}}(M)_{dj+e} = 0$ when $dj + e \geq \operatorname{reg} M + 1$. Thus $H^0_{\mathfrak{n}}(N'_e)_j = 0$ when $j \geq (\operatorname{reg} M - e + 1)/d$, concluding the proof of part (1).

We now consider part (2). Given part (1) and Eisenbud-Harris [E-H], Proposition 1.1, it suffices to show that if $m = \operatorname{reg} H^1_{\mathfrak{n}}(N_e)$, then $\operatorname{reg} I^m M \geq md + e + 1$. It follows from the hypothesis of part (2) that $\operatorname{reg} M \leq md + e - 1$. Because of the exact sequence (1) we only need to show that $\operatorname{reg}(M/I^m M) \geq md + e$. Let N'_e and E be as in the proof of part (1). We want to show that $E_m \neq 0$.

Using the exact sequence (2) and the fact that $H^1_{\mathfrak{n}}(N_e)_m \neq 0$, we see that it suffices to show that $H^1_{\mathfrak{n}}(N'_e)_m = 0$. Since N'_e is a summand of M (with a shift of degree) it suffices to show that $H^1_{\mathfrak{m}}(M)_{md+e} = 0$. This holds because, by hypothesis, $\operatorname{reg} M \leq md + e - 1$. □

Conjecture. *If I, S, M are as in Theorem 1.1 and M is generated in degree 0, then the regularity of N_i is nonincreasing from $i = 0$. In particular, the $(0,1)$-regularity of $\mathcal{R}(I)$ is equal to the regularity of $k[I_d]$.*

We can prove the conjecture in the case where I is a power of the maximal ideal.

Proposition 1.3. *Let M be a finitely generated graded S-module, generated in degree 0.*

(1) *If $i \geq 0$, then*

$$\operatorname{reg} N_i(\mathfrak{m}^d, M) \leq \max\left\{0, \frac{\operatorname{reg} M - i + (d-1)\dim M}{d}\right\}.$$

In particular $\operatorname{reg} N_i(\mathfrak{m}^d, M) = 0$ *for* $i \geq \operatorname{reg} M + (d-1)(\dim M - 1)$.

(2) *If $H^0_{\mathfrak{m}}(M) = 0$, then the sequence of numbers $\{\operatorname{reg} N_i(\mathfrak{m}^d, M) \mid i \geq 0\}$ is weakly decreasing.*

Proof. In the previous proof we have seen that there is a homogeneous isomorphism of $k[S_d]$-modules

$$N_i := N_i(\mathfrak{m}^d, M) \cong M_i K[S_d](i) = \bigoplus_{j \geq 0} M_{dj+i} = (M(i)_{\geq 0})^{(d)},$$

where we consider N_i as a $k[S_d]$-module via the identification $k[S_d t] \cong k[S_d]$; here $-^{(d)}$ denotes the Veronese functor.

The exact sequence

$$0 \to M(i)_{\geq 0} \to M(i) \to M(i)/M(i)_{\geq 0} \to 0$$

gives rise to an exact sequence

$$\begin{aligned} 0 \to & H^0_{\mathfrak{m}}(M(i)_{\geq 0}) \to H^0_{\mathfrak{m}}(M(i)) \to M(i)/M(i)_{\geq 0} \\ & \to H^1_{\mathfrak{m}}(M(i)_{\geq 0}) \to H^1_{\mathfrak{m}}(M(i)) \to 0 \end{aligned}$$

and isomorphisms $H^\ell_{\mathfrak{m}}(M(i)_{\geq 0}) \cong H^\ell_{\mathfrak{m}}(M(i))$ for $2 \leq \ell$.

Since the d-th Veronese functor commutes with taking local cohomology, it follows that

(3)
$$
\begin{aligned}
&\operatorname{reg}(M(i)_{\geq 0})^{(d)}\\
&\leq \max\{1+\operatorname{reg}(M(i)/M(i)_{\geq 0})^{(d)}, \max\{\operatorname{reg}(H^{\ell}_{\mathfrak{m}}(M(i)))^{(d)}+\ell \mid 0\leq \ell\leq \dim M\}\}\\
&\leq \max\left\{0, \max\left\{\left\lfloor \frac{\operatorname{reg} H^{\ell}_{\mathfrak{m}}(M)-i}{d}\right\rfloor+\ell \mid 0\leq \ell\leq \dim M\right\}\right\}\\
&\leq \max\left\{0, \max\left\{\left\lfloor \frac{\operatorname{reg} M-i-\ell}{d}\right\rfloor+\ell \mid 0\leq \ell\leq \dim M\right\}\right\}\\
&\leq \max\left\{0, \left\lfloor \frac{\operatorname{reg} M-i+(d-1)\dim M}{d}\right\rfloor\right\},
\end{aligned}
$$

which gives the desired formula. If $H^0_{\mathfrak{m}}(M)=0$ and $M\neq 0$, then the first two inequalities are equalities, which implies part (2). □

We can also prove the above conjecture for $i\geq e$, at least when $H^0_{\mathfrak{m}}(M)=0$.

Proposition 1.4. *Suppose that $I\subset S$ is an ideal generated by forms of a single degree d, and $M\neq 0$ is a finitely generated graded S-module, generated in a single degree, such that M/IM has finite length and $H^0_{\mathfrak{m}}(M)=0$. For each m, let e_m be the number such that $\operatorname{reg} I^mM = md+e_m$, and let $e:=e_m$ for $m\gg 0$. Let $N_i := N_i(I,M)$ be the module defined above.*

(1) *$e_m\geq e_{m+1}\geq e_m-d$.*
(2) *If $i\geq e$, then $\operatorname{reg} N_{i+1}\leq \operatorname{reg} N_i$.*

Proof. Again, after a shift of degree we may assume that M is generated in degree 0. The inequality $e_m\geq e_{m+1}$ of part (1) is proven in Eisenbud-Harris [E-H], Proposition 1.1.

For the second inequality it suffices to prove that $\operatorname{reg} I^mM\leq \operatorname{reg} I^{m+1}M$, for then $dm+e_m\leq d(m+1)+e_{m+1}$, that is, $e_m\leq d+e_{m+1}$.

Recall that $M/I^{m+1}M$ has finite length and $H^0_{\mathfrak{m}}(M)=0$. The exact sequence

$$0\to I^{m+1}M\to M\to M/I^{m+1}M\to 0$$

shows that $\operatorname{reg} H^1_{\mathfrak{m}}(I^{m+1}M) = \max\{\operatorname{reg} M/I^{m+1}M, \operatorname{reg} H^1_{\mathfrak{m}}(M)\}$ and moreover $H^{\ell}_{\mathfrak{m}}(I^{m+1}M)=\operatorname{reg} H^{\ell}_{\mathfrak{m}}(M)$ for $\ell\geq 2$. The same equalities hold for I^mM in place of $I^{m+1}M$. The epimorphism of finite length modules $M/I^{m+1}M\twoheadrightarrow M/I^mM$ implies that $\operatorname{reg} M/I^{m+1}M\geq \operatorname{reg} M/I^mM$, and the desired inequality follows.

For part (2), we note that for $i\geq e$ we can embed N_i into $N'_i := N_i(\mathfrak{m}^d, M)$ with finite length cokernel and that N_i is a submodule of M (with a shift of degree); see the proof of Theorem 1.1(1). From $H^0_{\mathfrak{m}}(M)=0$ we deduce $H^0_{\mathfrak{m}}(N'_i)=0$ and thus $H^0_{\mathfrak{m}}(N_i)=0$. Therefore $\operatorname{reg} N_i=\max\{\operatorname{reg} N'_i, \operatorname{reg}(N'_i/N_i)+1\}$.

Since $H^0_{\mathfrak{m}}(M)=0$, part (2) of Proposition 1.3 shows that the numbers $\operatorname{reg} N'_i$ are weakly decreasing. On the other hand, the generators of $\mathfrak{m}$ provide a homogeneous epimorphism $\bigoplus N'_i\to N'_{i+1}$ that induces an epimorphism $\bigoplus N'_i/N_i\to N'_{i+1}/N_{i+1}$. Thus the $(0,1)$-regularity of the finite length module N'_i/N_i is also weakly decreasing when $i\geq e$. □

Corollary 1.5. *Let $S = k[x_1, \dots, x_n]$ be a polynomial ring and let I, d, e be as in Theorem* 1.1*. If $e = 0$ and $m \geq \operatorname{reg} k[I_d]$, then $\operatorname{reg} I^m = md + e$.*

Proof. One uses Theorem 1.1(1). □

Example 1.6. The regularity of $\mathcal{R}(I)$ is often much larger than the regularity of the module N_e. For the ideal $I = (x^{20}, x^3y^{17}, x^{12}y^8, y^{20}) \subset k[x, y]$ we have $\operatorname{reg} I^m \geq 20m + 7$, with equality if and only if $m \geq 2$. Here the $(0, 1)$-regularity of the Rees algebra, and also the regularity of $k[I_d]$, are equal to 7. By Theorem 1.1, $\operatorname{reg} H^1_{\mathfrak{n}}(N_e) \leq 1$ (and in fact equality holds). Now Proposition 1.3(1) shows that $\operatorname{reg} N_e = 2$.

For the ideal $I = (x^{20}, x^3y^{17}, x^{15}y^5, y^{20}) \subset k[x, y]$ we have $\operatorname{reg} I^m \geq 20m + 4$, with equality if and only if $m \geq 4$. Here again the $(0, 1)$-regularity of the Rees algebra, and also the regularity of $k[I_d]$, are equal to 7. By Theorem 1.1, $\operatorname{reg} H^1_{\mathfrak{n}}(N_e) \leq 3$ (and again, in fact, equality holds), and then $\operatorname{reg} N_e = 4$ according to Proposition 1.3(1).

2. Ideals with generators in more than one degree

As a first example, we have:

Proposition 2.1. *Let $I \subset S = k[x_1, \dots, x_n]$ be a homogeneous ideal and let M be a finitely generated graded S-module. If $I \subset S$ is generated by an M-regular sequence of degrees $d = d_1 \geq \cdots \geq d_t$ and $m \geq 1$, then $\operatorname{reg} I^m M = dm + e$, where $e = \operatorname{reg} M + \sum_{i=2}^t (d_i - 1)$.*

Proof. Since I is generated by a regular sequence on M, we may tensor M with the Eagon-Northcott resolution of I^m and get a resolution of $I^m \otimes M = I^m M$ by shifted copies of M. Analyzing the shifts, we see that $\operatorname{reg} I^m M = dm + e$. □

Corollary 2.2. *Let $I \subset S = k[x_1, \dots, x_n]$ be a homogeneous ideal, and M a finitely generated graded S-module. Let d be the asymptotic generator degree of I on M, and write $\operatorname{reg} I^m M = dm + e_m$. If I contains an M-regular sequence of degrees $d = d_1 \geq \cdots \geq d_t$ with $t = \dim M$, then $e_m \leq \operatorname{reg} M + \sum_{i=2}^t (d_i - 1)$ for every $m \geq 1$.*

In general, we can analyze only special cases.

Theorem 2.3. *Let $J \subset S = k[x_1, \dots, x_n]$ be an $\mathfrak{m}$-primary ideal generated by forms of a single degree d. Write $I := J + \mathfrak{m}^{d+k}$ for some $k \geq 0$. Let $f_m(p) := (d+k)m - kp$ and*

$$p_m := \min\{p \geq 0 \mid \operatorname{reg} J^p \geq f_m(p)\}.$$

For $m \geq 1$ we have

$$\operatorname{reg} I^m = \min\{\operatorname{reg} J^{p_m}, f_m(p_m - 1)\}.$$

Proof. Define e_p by the formula $\operatorname{reg} J^p = dp + e_p$. Note that p_m is finite, and in fact $p_m \leq m$ since $\operatorname{reg} J^m \geq dm$.

We have

$$I^m = \sum_{p=0}^{m} J^p (\mathfrak{m}^{d+k})^{m-p}.$$

Thus, $\operatorname{reg} I^m \leq \min\{\operatorname{reg} J^p(\mathfrak{m}^{d+k})^{m-p} \mid 0 \leq p \leq m\}$. Moreover, $J^p(\mathfrak{m}^{d+k})^{m-p} = (J^p)_{\geq dp+(d+k)(m-p)} = (J^p)_{\geq f_m(p)}$, so

$$\operatorname{reg} J^p(\mathfrak{m}^{d+k})^{m-p} = \max\{\operatorname{reg} J^p, f_m(p)\}.$$

We claim that the minimum value of $\operatorname{reg} J^p(\mathfrak{m}^{d+k})^{m-p}$ is taken on either for $p = p_m$ or $p = p_m - 1$, and that in either case it is

$$\min_{0\leq p\leq m}\{\operatorname{reg} J^p(\mathfrak{m}^{d+k})^{m-p}\} = \min\{\operatorname{reg} J^{p_m}, f_m(p_m-1)\}.$$

This follows because, as p increases, the function $\operatorname{reg} J^p$ is weakly increasing (see the proof of Proposition 1.4(1)) while $f_m(p)$ is decreasing, and for $p = m$ the first is at least as large as the second, and $p_m \geq 1$ (except when $I = S$); see Figure 1. Note that the minimum value is the value claimed in the theorem for $\operatorname{reg} I^m$.

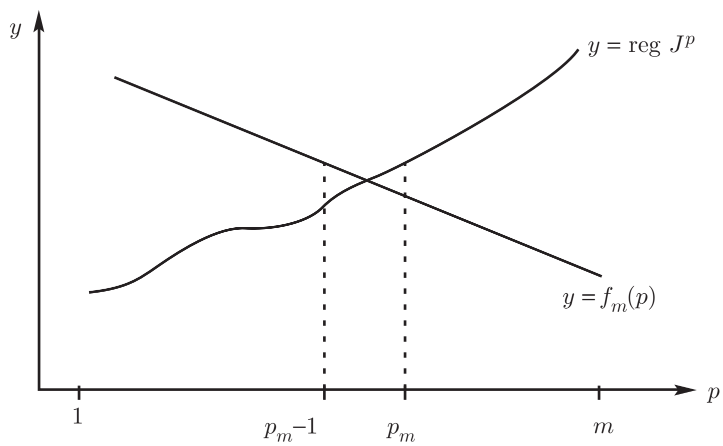

FIGURE 1. Where the graphs of $f_m(p)$ and $\operatorname{reg} J^p$ cross

Thus it is enough to show that

$$\operatorname{reg} I^m \geq \min\{\operatorname{reg} J^{p_m}, f_m(p_m-1)\}.$$

Write $a := \min\{\operatorname{reg} J^{p_m}, f_m(p_m-1)\}$. Note that $I^m \subset J^{p_m} + \mathfrak{m}^{f_m(p_m-1)}$. Thus it suffices to prove that

$$\mathfrak{m}^{a-1} \not\subset J^{p_m} + \mathfrak{m}^{f_m(p_m-1)}.$$

Since $a - 1 < f_m(p_m - 1)$, this is equivalent to $\mathfrak{m}^{a-1} \not\subset J^{p_m}$. But the latter holds because $a - 1 < \operatorname{reg} J^{p_m}$. $\square$

Example 2.4. If I is not generated in a single degree, then in the formula $\operatorname{reg} I^m = md + e_m$ the e_m may not be weakly decreasing. They can even go up and then down. For example, using Theorem 2.3 one can easily compute that if

$$I = (x_1^4, \ldots, x_4^4)(x_1, \ldots, x_4) + (x_1, \ldots, x_4)^6 \subset S = k[x_1, \ldots, x_4],$$

then $\operatorname{reg} I^m = 5m + e_m$, where the successive values of e_m for $m = 1, 2, \ldots$ are $1, 2, 2, 1, 1, 1, 1, 1, 0, 0, 0, \ldots$.

Proposition 2.5. *Let $I \subset S = k[x_1, \dots, x_n]$ be a homogeneous ideal and let M be a finitely generated graded S-module, concentrated in nonnegative degrees, such that M/IM has finite length but M does not. Let d be the asymptotic generator degree of I on M, and write* $\operatorname{reg} I^m M = dm + e_m$.

(1) *If I is generated in degrees $\leq d$, then the sequence of integers $\{e_m \mid m \geq (\operatorname{reg} M + 1)/d\}$ is weakly decreasing.*
(2) *If the associated graded module $\mathfrak{gr}_I(M)$ has positive depth, then the sequence $\{e_m \mid m \geq (\operatorname{reg} M + 1)/d\}$ is weakly increasing.*

Proof. We first prove part (1). If I is generated by homogeneous elements of degrees d_i, then multiplication by these elements gives a homogeneous surjection

$$\bigoplus_i \left(\frac{I^{m-1}M}{I^m M}(-d_i)\right) \to \frac{I^m M}{I^{m+1}M}$$

of modules of finite length. Thus

$$\operatorname{reg} I^m M/I^{m+1}M \leq \operatorname{reg} I^{m-1}M/I^m M + d \leq \operatorname{reg} M/I^m M + d.$$

Now the exact sequence

$$0 \to I^m M/I^{m+1}M \to M/I^{m+1}M \to M/I^m M \to 0$$

shows that $\operatorname{reg} M/I^{m+1}M \leq \operatorname{reg} M/I^m M + d$.

Since $\operatorname{reg}(I^m)^p M = (dm)p + e_{mp}$ for $p \gg 0$, we conclude that the asymptotic generator degree of I^m on M is dm. Thus the highest generator degree of $I^m M$ is at least dm because M is concentrated in nonnegative degrees. It follows that $\operatorname{reg} I^m M \geq dm$. Thus, if $m \geq (\operatorname{reg} M + 1)/d$, then $\operatorname{reg} M \leq dm - 1 \leq \operatorname{reg} I^m M - 1$. Now the inequality $\operatorname{reg} M/I^{m+1}M \leq \operatorname{reg} M/I^m M + d$ implies that $\operatorname{reg} I^{m+1}M \leq \operatorname{reg} I^m M + d$.

For part (2) we may assume that k is infinite. The definition of d shows that for some integer p we have

$$(I/I^2)^p \mathfrak{gr}_I(M) \subset ((I_{\leq d} + I^2)/I^2)\mathfrak{gr}_I(M).$$

It follows that there exists an element $a \in I_d$ whose leading form $a + I^2 \in \mathfrak{gr}_I(S)$ is not a zero-divisor on $\mathfrak{gr}_I(M)$. Hence $I^{m+1}M :_M a = I^m M$. Thus multiplication by a induces an embedding

$$\frac{M}{I^m M}(-d) \hookrightarrow \frac{M}{I^{m+1}M}.$$

This implies that $\operatorname{reg} M/I^{m+1}M \geq \operatorname{reg} M/I^m M + d$, and hence $\operatorname{reg} I^{m+1}M \geq \operatorname{reg} I^m M + d$ whenever $m \geq \operatorname{reg} M/d$. □

Corollary 2.6. *Let $I \subset S = k[x_1, \dots, x_n]$ be a homogeneous $\mathfrak{m}$-primary ideal with asymptotic generator degree d. If I is generated in degrees $\leq d$ and $\mathfrak{gr}_I(S)$ has positive depth, then $\operatorname{reg} I^m = dm + e$ for some e and every $m \geq 1$.* □

Example 2.7. One cannot drop the assumption of generation in degree $\leq d$ from Corollary 2.6. If

$$I = (x^4, y^4, z^4) + (x, y, z)^5 \subset S = k[x, y, z],$$

then $\operatorname{reg} I^m = 4m + e_m$, where the successive values of e_m for $m = 1, 2, \dots$ are $1, 2, 2, 2, 2, \dots$. Computation with Macaulay2 shows that the depth of the associated graded ring of I is positive.

3. A case of the (almost) Eisenbud-Goto conjecture

Eisenbud and Goto [E-G] conjecture that the regularity of a nondegenerate, geometrically reduced irreducible subscheme $X \subset \mathbb{P}^n$ has regularity at most $\deg X - \operatorname{codim} X + 1$. They further conjecture that the hypothesis can be weakened to say that the nondegenerate scheme is geometrically reduced and connected in codimension 1, and this has been proved by Giaimo [G] for curves. The bound can fail for disconnected schemes. For example, if X is the union of two skew lines in $\mathbb{P}^3$, then the degree of X is 2 but the regularity (that is, the regularity of the ideal of X) is 2 rather than 1. Derksen and Sidman [D-S] have shown that in general a union of linear subspaces of projective space has regularity at most the number of subspaces.

One might guess from this that the regularity of a reduced equidimensional scheme would be bounded by the degree of the scheme, but this is not the case.

Example 3.1 (Giaimo, unpublished)**.** Here is a reduced equidimensional union of two irreducible complete intersections whose regularity is much larger than its degree:

By Mayr-Meyer [M-M] there is a homogeneous ideal $I \subset S = \mathbb{C}[x_1, \ldots, x_n]$ generated by $10n$ forms of degrees two and three, having regularity of the order of 2^{2^n}. In the ring $R = S[z_1, \ldots]$ we build an ideal I' whose generators correspond to those of I by replacing the monomials in the generators of I with products of new variables z_j in such a way that each z_j occurs only linearly, and no z_j occurs twice. Clearly the generators of this new ideal are a regular sequence. If any of the generators are monomials, we add further new variables w_j and make each a binomial that will be a prime. Since the variables are all distinct, the resulting complete intersection will also be prime, and modulo an ideal of the form $L = (\{z_j - x_{p(j)}\}) + (\{w_j\})$ the ideal I' becomes equal to the ideal I. The codimension of L is clearly at least as big as the codimension of the complete intersection. We add further variables to the ambient ring and to the complete intersection I' to make the codimensions the same.

The ideal $I' \cap L$ now defines the union of two reduced, irreducible complete intersections, while the ideal $I' + L$ defines the same factor ring as the original Mayr-Meyer example. From the short exact sequence

$$0 \to I' \cap L \to I' \oplus L \to I' + L \to 0,$$

we see that the regularity of $I' \cap L$ is of the order of 2^{2^n}. On the other hand, the degree of the subscheme defined by $I' \cap L$ is at most of the order of 3^{10n}.

We state our result in terms of the regularity of the homogeneous coordinate ring S_X of X, which is one less than $\operatorname{reg} X$, to emphasize the parallel between the two parts of the theorem. The first part of the theorem deals with the Eisenbud-Goto conjecture, whereas the second part is motivated by the estimate of Corollary 1.2. Recall that a local algebra essentially of finite type over a field of characteristic zero is said to have a rational singularity if it is normal and Cohen-Macaulay and, if $\pi : \tilde{X} \to \operatorname{Spec} R$ is a resolution of singularities, then $\pi_*(\omega_{\tilde{X}}) = \omega_{\operatorname{Spec} R}$.

Theorem 3.2. *Let X be a reduced equidimensional subscheme of codimension 2 in $\mathbb{P}^n_k$, where k is a field of characteristic zero. Assume that X lies on a quadric hypersurface and that the locus of nonrational singularities of X has dimension at most zero. Let S_X be the homogeneous coordinate ring of X.*

(1) $\operatorname{reg} S_X \leq \deg X$.
(2) *If* $x_1, \dots, x_n$ *are general linear forms in* S_X*, and* I *is the ideal they generate, then* $\operatorname{reg}_{(y_1,\dots,y_r)} \mathcal{R}(I, S_X) \leq \deg X - \operatorname{codim} X + 1$.

Note that the Eisenbud-Goto conjecture would say, under the additional hypothesis that X is nondegenerate and connected in codimension 1, that $\operatorname{reg} S_X \leq \deg X - \operatorname{codim} X = \deg X - 2$.

Proof. We make use of the notation introduced in part (2) of the theorem, and we write $\mathfrak{m}$ for the homogeneous maximal ideal of S_X. Let $\mathcal{F} := k[I_1] \subset S_X$ and note that $\mathcal{F}$ is isomorphic to the special fiber ring $\mathcal{F} \cong \mathcal{R}(I, S_X)/\mathfrak{m}\mathcal{R}(I, S_X)$. Let x be a linear form such that $\mathfrak{m} = (I, x)$. Because the $x_1, \dots, x_n$ are general and the ideal defining X contains a quadric, $S_X = \mathcal{F} + \mathcal{F}x$. Thus $S_X/\mathcal{F} \cong (\mathcal{F}/(\mathcal{F} :_{\mathcal{F}} S_X))(-1)$. The extension $\mathcal{F} \subset S_X$ is finite and birational. Hence $\mathcal{F}$ is the ring of a hypersurface whose degree is $\deg S_X$ in $\mathbb{P}^{n-1}$. It follows that $\operatorname{reg} \mathcal{F} = \deg S_X - 1$.

As $\omega_{\mathcal{F}} = \mathcal{F}(-n + \deg S_X)$ we have $\mathcal{F} :_{\mathcal{F}} S_X = \operatorname{Hom}_{\mathcal{F}}(S_X, \mathcal{F}) = \omega_{S_X}(n - \deg S_X)$. The hypothesis that the characteristic is zero and that the equidimensional scheme X has at most isolated nonrational singularities implies that the regularity of ω_{S_X} is at most $\dim S_X = n-1$ (see Chardin-Ulrich [C-U], Theorem 1.3, which is based on results of Ohsawa [O] and Kollár [Kol], Theorem 2.1(iii)). It follows that $\operatorname{reg}(\mathcal{F} :_{\mathcal{F}} S_X) \leq n - 1 - (n - \deg S_X) = \deg S_X - 1$. Thus $\operatorname{reg} S_X/\mathcal{F} \leq \deg S_X$, and therefore $\operatorname{reg} S_X \leq \deg S_X$, proving the first statement.

For the second statement, let $G := \mathfrak{gr}_I(S_X)$ be the associated graded ring of S_X with respect to I, which is an S_X-module via the map $S_X \to S_X/I = G_0$. By Johnson and Ulrich [J-U], Proposition 4.1, one has $\operatorname{reg}_{(y_1,\dots,y_r)} \mathcal{R}(I, S_X) = \operatorname{reg}_{(y_1,\dots,y_r)} G$, so it suffices to bound the latter.

Note that $\mathcal{F} = G/\mathfrak{m}G = G/xG$. Because the ideal defining X contains a quadric we have $x^2 \in I$. It follows that $x^2 G = 0$. Of course $xG \cong G/(0 :_G x)$. We will show that $G/(0 :_G x) \cong \mathcal{F}/(\mathcal{F} :_{\mathcal{F}} S_X)$. Indeed, the embedding $\mathcal{F} \cong k[I_1 t] \subset \mathcal{R}(I, S_X)$ induces a map $\mathcal{F} \to G/(0 :_G x)$, which is surjective because $xG \subset 0 :_G x$. To compute the kernel, let $f \in \mathcal{F}$ be a form of degree i. The image of fx in G is 0 if and only if, as elements of S_X, we have $fx \in I^{i+1}$. But the degree (in S_X) of fx is $i+1$, so this happens if and only if $fx \in \mathcal{F}_{i+1}$. This in turn means that $f \in \mathcal{F} :_{\mathcal{F}} x = \mathcal{F} :_{\mathcal{F}} S_X$.

From the computation of the regularity of $\mathcal{F} :_{\mathcal{F}} S_X$ above, we get $\operatorname{reg} G \leq \max\{\operatorname{reg} \mathcal{F}/(\mathcal{F} :_{\mathcal{F}} S_X), \operatorname{reg} \mathcal{F}\} = \deg S_X - 1$. □

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