Boij–Söderberg Theory

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Abstract In this article we start with a survey of the recent breakthroughs concerning Betti table of graded modules over the polynomial ring and cohomology tables coherent sheaves on projective space. We then ask how this theory can be extended to a more general setting. Our first new result concerns cohomology tables of very ample polarized varieties. We prove a necessary and sufficient condition that the Boij–Söderberg cone of cohomology tables of coherent sheaves on a variety X of dimension d with polarization $\mathscr{O}_X(1)$ coincides with the corresponding cone of $(\mathbb{P}^d, \mathscr{O}(1))$, and conjecture that our condition is always satisfied. The last section concerns cohomology tables of vector bundles with respect to more than one line bundle, where we start with the simplest case $\mathbb{P}^1 \times \mathbb{P}^1$. We identify some extremal rays in the Boij–Söderberg cone of vector bundles on $\mathbb{P}^1 \times \mathbb{P}^1$, and conjecture that these are all.

1 Introduction

The Hilbert polynomial of a graded module over a polynomial ring is an important invariant that is refined by the *graded Betti numbers* of the module, which are usually encoded together as the *Betti Table* of the module (see below for precise definitions). If we think of the module as representing a sheaf on projective space, then the Hilbert polynomial appears as the Euler characteristic of twists of the sheaf, and a more natural refinement is, perhaps, the table of dimensions of the cohomology spaces of different twists, the *Cohomology Table* of the sheaf.

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In late 2006 Mats Boij and Jonas Söderberg, made a remarkable conjecture specifying the possible Betti Tables of modules of finite length, up to a rational multiple [2]. It is easy to see that the Betti tables form a semigroup—the direct sum of modules corresponds to the addition of Betti tables. Allowing multiplication by positive rational numbers instead of just positive integers, we get a rational convex cone. What Boij and Söderberg did was to conjecture that the Betti tables of what are called *pure resolutions* define the *extremal rays* of this cone. They also conjectured the existence of such pure resolutions—very few were known—and thus that the cone was closed, as well.

We proved these conjectures by showing, on the one hand, the existence of the necessary pure resolutions and, on the other, identifying the facets of the cone as coming from the extremal rays in the cone of cohomology tables of vector bundles on projective space. We also identified the extremal rays in the corresponding cone of cohomology tables of vector bundles, and showed that vector bundles with such extremal cohomology tables actually exist.

A flurry of other papers and preprints including [3, 6, 9, 10, 11] and [12] have added to the basic picture and its applications; in particular, the whole picture now extends in some form to arbitrary finitely generated graded modules over a polynomial ring, and to coherent sheaves on a projective space. The first two sections of this note survey some of what we now know about these cases.

One can imagine many extensions of the basic ideas in this theory. For example, one might ask about the cohomology tables of vector bundles or sheaves whose support is restricted to a subvariety of projective space. Somewhat surprisingly, the possibilities are often identical to those for a projective space of the same dimension. We will show in Sect. 4 that this depends on the existence of a single "Ulrich" sheaf on X. We have conjectured elsewhere that Ulrich sheaves exist on every projective variety.

A different sort of extension would be to pack more data into the cones. For example we could look at multigraded modules, and ask about the cone of multigraded Betti numbers. More generally, we could look at modules equivariant for the action of a reductive group, and ask about the representations in the resolution instead of the degrees. This direction has been pursued by Sam and Weyman [10]. As a first step, one might try to determine—or at least guess!—the extremal tables. We can ask similar questions about the cohomology of vector bundles or coherent sheaves.

The last section of this paper takes up one aspect of this. We study vector bundles $\mathscr{F}$ on $\mathbb{P}^1 \times \mathbb{P}^1$ and ask about the rational cone of *bigraded* cohomology tables $\{h^i(\mathscr{F}(a,b)\}$. We give a conjectural description of the cone in terms of extremal rays.

2 Betti Tables

Let $S = k[x_1,\dots,x_n]$ be a polynomial ring over a field k, graded with each x_i of degree 1, and let M be a finitely generated graded S-module. As usual, we write $S(-j)$ for the graded free module of rank 1 with generator in degree j. By the

Hilbert Syzygy theorem there exist *finite free resolutions* of M; that is, sequences of graded free modules and degree 0 homomorphisms

$$\mathbf{F}: \quad F_0 \xleftarrow{\varphi_1} \cdots \xleftarrow{\varphi_m} F_m \longleftarrow 0$$

that are exact at F_i for $i > 0$, while coker $\varphi_1 \cong M$. Such a resolution is said to be *minimal* if no proper summand of F_i maps onto the kernel of φ_{i-1}. Minimal resolutions are unique up to isomorphism, and have *length* $m \leq n$. In particular, if $\mathbf{F}$ is a minimal free resolution, and we write $F_i = \oplus_j S(-j)^{\beta_{i,j}(M)}$, then the *graded Betti numbers* $\beta_{i,j}(M) = dim((F_i \otimes_S k)_j)$ are invariants of M alone. We define the *Betti table* of M to be this collection of numbers $\{\beta_{i,j}(M)\}$.

We may regard the Betti table of M as an element of an (infinite-dimensional) rational vector space,

$$B := \bigoplus_{-\infty}^{\infty} \mathbb{Q}^{n+1}$$

with coordinates $\beta_{i,j}(M)$. Since $\beta_{i,j}(M \oplus N) = \beta_{i,j}(M) + \beta_{i,j}(N)$ the Betti tables of finitely generated modules form a sub-semigroup of this vector space. The following Theorem, conjectured by Boij and Söderberg, specifies the cone of positive rational linear combinations of Betti tables of finitely generated modules precisely.

One way of specifying a cone is to give it's extremal rays—the half-lines in the cone that are not in the convex hull of the remaining elements of the cone. In the case of the cone of Betti tables, the extremal rays will turn out to be the *pure* modules:

Definition 2.1. A finitely generated graded S-module M is called *pure* of type $d := (d_0, \dots, d_m)$ if

(a) In a minimal free resolution of M as above, the free module F_i generated by elements of degree d_i; that is, $\beta_{i,j} = 0$ when $j \neq d_i$.
(b) M is Cohen–Macaulay of codimension m; that is, $F_i = 0$ for $i > m$ and the annihilator of M is an ideal of codimension m.

It is easy to see that if there is a pure module of type d, then $d_0 < \cdots < d_m$. Much more is true: if M is a pure module, then a result of Herzog and Kühl [15] shows that the Betti table of M is determined by d up to a rational multiple: that is, there is a constant $r = r(M)$ depending on M such that

$$\beta_{i,d_i}(M) = \frac{r}{\prod_{t \neq i} |d_t - d_i|}.$$

Thus the pure modules of type d define a single ray in the cone of Betti tables.

One more preparation is necessary: we order the strictly increasing sequences d: we say that

$$d = (d_0, \dots, d_m) \leq d' = (d'_0, \dots, d'_{m'})$$

if $m \geq m'$ and $d_i \leq d'_i$ for $i = 1, \dots, m'$. (One can think of this as the termwise order if one simply extends each sequence $d = (d_0, \dots, d_m)$ to $d = (d_0, \dots, d_m, \infty, \infty, \dots)$.)

We can now state the main result of the theory concerning the cone of Betti tables:

Theorem 2.2 ([3, 8]). *Let $S = k[x_1,\dots,x_n]$ be as above.*

(a) For every strictly increasing sequence of integers $d = (d_0,\dots,d_m)$ with $m \le n$, there exist pure finitely generated graded S-modules of type d.
(b) The Betti table of any finitely generated graded S-module may be written uniquely as a positive rational linear combination of the Betti tables of a set of pure finitely generated modules whose types form a totally ordered sequence.

The second statement of the Theorem has two nice interpretations, which may help clarify its meaning. First, geometrically, it really says that the cone of Betti tables is a *simplicial fan*, that is, it is the union of simplicial cones, meeting along facets, with each simplicial cone spanned by the rays corresponding to a set of pure Betti tables whose types form a totally ordered set. These simplices and cones are of course infinite dimensional; but one can easily reduce to the finite-dimensional case by specifying that one wants to work with resolutions where the free modules are generated in a given bounded range of degrees.

Second, algorithmically, the Theorem implies that there is a greedy algorithm that gives the decomposition. Rather than trying to specify this formally, we give an Example. For this purpose, we write the Betti table of a module M as an array whose entries in the i-th column are the $\beta_{i,j}$—that is, the i-th column corresponds to the free module F_i for reasons of efficiency and tradition, we put $\beta_{i,j}$ in the $(j-i)$-th row.

For our example we take $n = 3$, and let $M = S/(x^2, xy, xz^2)$. The minimal free resolution of M has the form

$$S \longleftarrow S(-2)^2 \oplus S(-3) \longleftarrow S(-3) \oplus S(-4)^2 \longleftarrow S(-5) \longleftarrow 0$$

and is represented by an array

$$\beta(M) = \begin{pmatrix} 1 & & & \\ & 2 & 1 & \\ & 1 & 2 & 1 \end{pmatrix}$$

where all the entries not shown are equal to zero.

To write this as a positive rational linear combination of pure diagrams, we first consider the “top row”, corresponding to the generators of lowest degree in the free modules of the resolution. These are in positions

$$\begin{pmatrix} * & & & \\ & * & * & \\ & & & * \end{pmatrix}$$

corresponding to the degree sequence $(0,2,3,5)$. There is in fact a pure module $M_1 = S/I_1$ with resolution

$$\beta(M_1) = \begin{pmatrix} 1 & & & \\ & 5 & 5 & \\ & & & 1 \end{pmatrix}.$$

The greedy algorithm now instructs us to subtract *the largest possible multiple* q_1 *of* $\beta(M_1)$ *that will leave the resulting table* $\beta(M) - q_1\beta(M_1)$ *having only non-negative terms.* We see at once that $q_1 = 1/5$.

We now repeat this process starting from $\beta(M) - q_1\beta(M_1)$; the Theorem guarantees that there will always be a pure resolution whose degree sequence matches the top row of the successive remainders. In this case we arrive at the expression

$$\beta(M) = \begin{pmatrix} 1 & & & \\ & 2 & 1 & \\ & 1 & 2 & 1 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 1 & & & \\ & 5 & 5 & \\ & & & 1 \end{pmatrix} + \frac{1}{10}\begin{pmatrix} 3 & & & \\ & 10 & & \\ & & 15 & 8 \end{pmatrix} + \frac{1}{6}\begin{pmatrix} 1 & & & \\ & & & \\ & 4 & 3 & \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 1 & & & \\ & & & \\ & 1 & & \end{pmatrix}.$$

All the fractions and diagrams that occur are of course invariants—apparently new invariants—of M.

3 Facets of the Cone and Cohomology Tables

We next focus on the facets of the cone of Betti tables. According to the simplicial structure of the cone [2], an outer facet corresponds to a sequence of three degree sequences which differ in at most two consecutive positions. For example the degree sequences to the following Betti tables form such a chain.

$$\begin{pmatrix} 3 & 8 & 6 & \\ & & & 1 \end{pmatrix} < \begin{pmatrix} 2 & 4 & & \\ & & 4 & 2 \end{pmatrix} < \begin{pmatrix} 1 & & & \\ & 6 & 8 & 3 \end{pmatrix}$$

The facet equation is defined by the vanishing on all Betti tables of pure resolutions corresponding to degree sequences that are $< (0,1,3,4)$ and all Betti table of pure resolutions corresponding to degree sequences $> (0,1,3,4)$. This allows to compute the coefficients of the facet equation recursively using zero coefficients on the support of the right hand table as start values.

$$\begin{pmatrix} \vdots & \vdots & \vdots & \vdots \\ 21 & -12 & 5 & 0 \\ 12 & -5 & 0 & 3 \\ 5 & 0 & -3 & 4 \\ \mathbf{0} & 3 & -4 & 3 \\ 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

What we have to prove is that this linear form is non-negative on the Betti table of any minimal free resolution. Our key observation is that the numbers appearing are dimensions of cohomology groups of what we call supernatural vector bundles.

Definition 3.1. A vector bundle $\mathscr{E}$ on $\mathbb{P}^m$ has natural cohomology [13], if for each k at most one of the groups

$$H^i(\mathscr{E}(k)) \neq 0.$$

It has supernatural cohomology, if in addition the Hilbert polynomial

$$\chi(\mathscr{E}(k)) = \frac{\operatorname{rank} \mathscr{E}}{m!} \prod_{j=1}^{m} (k - z_j)$$

has m distinct integral roots $z_1 < z_2 < \ldots < z_m$.

For a coherent sheaf $\mathscr{E}$ on $\mathbb{P}^m$ we denote by

$$\gamma(\mathscr{E}) = (\gamma_{j,k}) \in \prod_{k=-\infty}^{\infty} \mathbb{Q}^{m+1} \text{ with } \gamma_{j,k} = h^j(\mathscr{E}(k))$$

its cohomology table. Analogous to the Theorem on free resolutions we have

Theorem 3.2 ([8])**.** *The extremal rays of the rational cone of cohomology tables of vector bundles are generated by cohomology tables of supernatural vector bundles.*

The crucial new concept is the following pairing between Betti tables of modules and cohomology table of coherent sheaves. We define $\langle \beta, \gamma \rangle$ for a Betti table $\beta = (\beta_{i,k})$ and a cohomology table $\gamma = (\gamma_{j,k})$ by

$$\langle \beta, \gamma \rangle = \sum_{i \geq j} (-1)^{i-j} \sum_k \beta_{i,k} \gamma_{j,-k}$$

Theorem 3.3 (Positivity 1, [8, 9])**.** *For F any free resolution of a finitely generated graded $K[x_0, \ldots, x_m]$-module and $\mathscr{E}$ any coherent sheaf on $\mathbb{P}^m$ we have*

$$\langle \beta(F), \gamma(\mathscr{E}) \rangle \geq 0.$$

For example the facet equation above, is obtained from the vector bundles $\mathscr{E}$ on $\mathbb{P}^2$, which is the kernel of a general map $\mathscr{O}^5(-1) \to \mathscr{O}^3$. The coefficients of the functional $\langle -, \gamma(\mathscr{E}) \rangle$ are

$$\begin{pmatrix} \vdots & \vdots & \vdots & \vdots \\ 21 & -12 & 5 & 0 \\ 12 & -5 & 0 & 3 \\ 5 & 0 & -3 & 4 \\ 0 & 3 & -4 & 3 \\ 0 & 4 & -3 & 0 \\ 0 & 3 & 0 & -5 \\ 0 & 0 & 5 & -12 \\ 0 & 0 & 12 & -21 \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

This is not quite what we wanted. We define truncate functionals $\langle -, \gamma \rangle_{\tau,c}$ by putting zero coefficients in the appropriate spots.

Theorem 3.4 (Positivity 2, [8, 9])**.** *For F any minimal free resolution of a finitely generated graded $K[x_0,\dots,x_m]$-module and $\mathscr{E}$ any coherent sheaf on $\mathbb{P}^m$ we have*

$$\langle \beta(F), \gamma(\mathscr{E}) \rangle_{\tau,c} \geq 0.$$

To prove the positivity theorems we consider the tensor product of F with the Čech resolution

$$C : \ldots \to C^p(E) = \sum_{0 \leq i_0 < i_1 < \ldots < i_p \leq m} E[(x_{i_0} \cdot \ldots \cdot x_{i_p})^{-1}] \to \ldots$$

of $\mathscr{E}$, where E denotes any graded module whose associated sheaf is $\mathscr{E}$.

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\uparrow & & \uparrow & & \uparrow & & \\
F_0 \otimes C^2 & \leftarrow & F_1 \otimes C^2 & \leftarrow & F_2 \otimes C^2 & \leftarrow & \ldots \\
\uparrow & & \uparrow & & \uparrow & & \\
F_0 \otimes C^1 & \leftarrow & F_1 \otimes C^1 & \leftarrow & F_2 \otimes C^1 & \leftarrow & \ldots \\
\uparrow & & \uparrow & & \uparrow & & \\
F_0 \otimes C^0 & \leftarrow & F_1 \otimes C^0 & \leftarrow & F_2 \otimes C^0 & \leftarrow & \ldots
\end{array}$$

Since we want to prove a purely numerical statement, we can replace E with its translate under a general element of $PGL(m+1,K)$ to achieve that E and F are cohomologically transverse [19, 20]. The horizontal homology is then concentrated in the first column and the total complex has cohomology only in positive degrees. On the other hand the lower diagonal part of the vertical cohomology of internal degree zero is

$$\begin{array}{cccc}
 & & H^2(F_2 \otimes \mathscr{E}) & \ldots \\
 & H^1(F_1 \otimes \mathscr{E}) & H^1(F_2 \otimes \mathscr{E}) & \ldots \\
H^0(F_0 \otimes \mathscr{E}) & H^0(F_1 \otimes \mathscr{E}) & H^0(F_2 \otimes \mathscr{E}) & \ldots
\end{array}$$

and the Euler characteristic of this diagram is the desired value $\langle \beta(F), \gamma(\mathscr{E}) \rangle$. We can split the spectral sequence which starts with the vertical cohomology and converge to the total cohomology as a sequence of K-vector spaces. The part displayed above has then no cohomology except the cokernel in total cohomological degree 0. So $\langle \beta(F), \gamma(\mathscr{E}) \rangle$ is the dimension of a vector space. Using the minimality one sees that the truncated functionals are even more positive.

The main remaining part of the proof of both Boij–Söderberg decompositions is now to establish the existence of supernatural vector bundles and pure resolutions for arbitrary zero or degree sequences. There are two methods for both cases known. For equivariant resolution or homogeneous vector bundles one can use appropriate explicit Schur functors [6, 7, 10] in characteristic 0. For arbitrary fields, one can use a push down method [8]. For bundles this is a simple application of the Künneth formula applied to $\mathscr{E} = \pi_* \mathscr{O}(a_1, \dots, a_m)$, where π is a finite linear projection $\pi : \mathbb{P}^1 \times \dots \times \mathbb{P}^1 \to \mathbb{P}^m$ and $\mathscr{O}(a_1, \dots, a_m)$ is a suitable line bundle on the product. For resolutions this consists of an iteration of the Lascoux method [18] to get the Buchsbaum–Eisenbud family of complexes associated to generic matrices [4]: We start with $\mathscr{K}$, a Koszul complex on $\mathbb{P}^m \times \mathbb{P}^{m_1} \times \dots \times \mathbb{P}^{m_s}$ of $1 + m + \sum_{i=1}^{s} m_i$ forms of multidegree $(1, \dots, 1)$ tensored with $\mathscr{O}(d_0, a_1, \dots, a_s)$. Here s is the number of desired non linear maps and $m_j + 1$ their degrees. The spectral sequence for $R\pi_* \mathscr{K}$ of the projection $\pi : \mathbb{P}^m \times \mathbb{P}^{m_1} \times \dots \times \mathbb{P}^{m_s} \to \mathbb{P}^m$ give rise to the desired complex, if we choose $a_1, \dots, a_s$ suitably.

4 Sheaves on a Subvariety

If X is a scheme then we can define the cohomology table of a sheaf on X with respect to a line bundle $\mathscr{L}$ to the be function

$$\mathbb{Z} \times \{0, 1, \dots, \dim X\} \to \mathbb{Z}, \ (m, i) \mapsto h^i(\mathscr{F} \otimes \mathscr{L}^m).$$

The cohomology table of the direct sum of two sheaves is the sum of the two cohomology tables, so the set of cohomology tables forms a semigroup inside the vector space of functions

$$\mathbb{Z} \times \{0, 1, \dots, \dim X\} \to \mathbb{Q},$$

and if we include linear combinations with non-negative rational coefficients, we obtain a rational cone $C(X, \mathscr{L})$, generalizing the one defined in Boij–Söderberg theory (the case $X = \mathbb{P}^n, \mathscr{L} = \mathscr{O}_{\mathbb{P}^n}(1)$.) Given the success of that theory, it seems natural to ask about the cones $C(X, \mathscr{L})$ more generally.

As a first step, we consider the case where $\mathscr{L}$ is very ample. That is, we take $X \subset \mathbb{P}^n$ to be a subscheme, say of dimension d, and consider the cone $C := C(X, \mathscr{O}_X(1))$ of cohomology tables of coherent sheaves on X with respect to $\mathscr{O}_X(1)$. Thinking of X as embedded in $\mathbb{P}^n$, we will simply refer to this as the cone of cohomology tables of coherent sheaves on X (and similarly with vector bundles.)

Let $\pi : X \to \mathbb{P}^d$ be a general linear projection, so that π is a finite mapping and $\pi^* \mathscr{O}_{\mathbb{P}^d}(1) = \mathscr{O}_X(1)$. It follows that the cohomology table of a sheaf $\mathscr{F}$ on X is the same as the cohomology table of $\pi_* \mathscr{F}$ with respect to $\mathscr{O}_{\mathbb{P}^d}(1)$. Hence the cone C is naturally contained in the cone $C(\mathbb{P}^d, \mathscr{O}_{\mathbb{P}^d}(1))$. We conjecture that they are equal:

Conjecture 4.1. $C(X, \mathscr{O}_X(1)) = C(\mathbb{P}^d, \mathscr{O}_{\mathbb{P}^d}(1))$ *via the inclusion induced by a general linear projection* π.

We will show that this conjecture is equivalent to a conjecture made in our paper [7], and known to be true for a class of schemes including complete intersections, arbitrary smooth curves and many others. Recall that an *Ulrich Sheaf* on a d-dimensional scheme $X \subset \mathbb{P}^n$ may be defined as a coherent sheaf $\mathscr{U}$ on X such that $\pi_* \mathscr{U} \cong \mathscr{O}^r_{\mathbb{P}^d}$ for some $r > 0$. It is clear from the definition that a d-dimensional scheme X possesses an Ulrich sheaf if some d-dimensional component of X_{red} does.

Theorem 4.2. *The cone of cohomology tables of coherent sheaves (respectively, vector bundles) on a d-dimensional scheme $X \subset \mathbb{P}^n$ is the same as the cone of cohomology tables of coherent sheaves (respectively, vector bundles) on $\mathbb{P}^d$ if and only if X has an Ulrich sheaf (respectively, an Ulrich sheaf that is a vector bundle).*

If X is smooth, then any Ulrich sheaf on X is a vector bundle: the vanishing of the intermediate cohomology shows that it can be represented by a maximal Cohen–Macaulay module on the homogeneous coordinate ring of X, and by the Auslander–Buchsbaum formula such a module is locally free at any nonsingular point of X. This proves the following corollary. Is there a direct proof?

Corollary 4.3. *If $X \subset \mathbb{P}^n$ is a smooth variety, then the cone of cohomology tables of coherent sheaves on X is the same as the cone of cohomology tables of coherent sheaves on $\mathbb{P}^d$ if and only if the corresponding result is true for vector bundles.*

Proof of Theorem 4.2. If $C(X, \mathscr{O}_X(1)) = C(\mathbb{P}^d, \mathscr{O}_{\mathbb{P}^d}(1))$, then X possesses a sheaf $\mathscr{U}$ whose cohomology table is a multiple of that of $\mathscr{O}_{\mathbb{P}^d}$, and this is by definition an Ulrich sheaf.

Now suppose that X possesses an Ulrich sheaf $\mathscr{U}$. By Horrocks Theorem (or the Auslander–Buchsbaum formula) any coherent sheaf on $\mathbb{P}^d$ with the same cohomology as $\mathscr{O}^r_{\mathbb{P}^d}$ is actually isomorphic to $\mathscr{O}^r_{\mathbb{P}^d}$. Thus $\pi_* \mathscr{U} = \mathscr{O}^r_{\mathbb{P}^d}$.

For any coherent sheaf $\mathscr{G}$ on $\mathbb{P}^d$,

$$\begin{aligned} h^i(\mathscr{U} \otimes \pi^* \mathscr{G} \otimes \mathscr{O}_X(m)) &= h^i(\pi_*(\mathscr{U} \otimes \pi^*(\mathscr{G} \otimes \mathscr{O}_{\mathbb{P}^d}(m))) \\ &= h^i(\pi_*(\mathscr{U}) \otimes \mathscr{G}(m)) = r\, h^i \mathscr{G}(m) \end{aligned}$$

so the cohomology table of $\mathscr{U} \otimes \pi^* \mathscr{G}$ is r times the cohomology table of $\mathscr{G}$, proving that the cones are equal. □

The results above address only one side of Boij–Söderberg theory. A different kind of extension to the case of a subscheme X would be to consider Betti tables of the free resolutions of modules over $S_X := K[x_1, \dots, x_n]/I(X)$, either as modules over $S = K[x_0, \dots, x_n]$ or over S_X. The second case may involve more radically new phenomena, since the modules will generally not have finite projective dimension.

5 Several Line Bundles

One can also consider the cone of cohomology tables with respect to several line bundles $\mathscr{L}_1, \dots, \mathscr{L}_g$. We define this "cohomology table" to be the function

$$h\mathscr{F} : \mathbb{Z}^g \times \{0, 1, \dots, \dim X\} \to \mathbb{Z},\ (n_1 \dots, n_g, i) \mapsto h^i(\mathscr{F} \otimes \mathscr{L}_1^{n_1} \otimes \cdots \otimes \mathscr{L}_g^{n_g}).$$

The first interesting case is the cone C of cohomology tables of vector bundles on $\mathbb{P}^1 \times \mathbb{P}^1$ with respect to the line bundles $\mathscr{O}(1,0)$ and $\mathscr{O}(0,1)$. We regard the cohomology table as a refined version of Hilbert polynomial $(a,b) \mapsto \chi\mathscr{F}(a,b)$. Since the bundles $\mathscr{O}(1,0)$ and $\mathscr{O}(0,1)$ are pulled back from copies of $\mathbb{P}^1$, this polynomial is linear in each of its arguments. In addition, since $\mathscr{O}(1,1)$ is very ample, $\chi\mathscr{F}(a,a)$ is asymptotically equal to $r \cdot a^2$ where r is the rank of $\mathscr{F}$. Thus $\chi\mathscr{F}(a,b)$ has the form

$$(*) \quad \chi\mathscr{F}(a,b) = \textstyle\sum_i (-1)^i h^i \mathscr{F}(a,b) = r \cdot ab + r_1 \cdot a + r_2 \cdot b + \chi_0.$$

It is easy to see that

$$r = \text{rank } \mathscr{F} = \chi(\mathscr{F} \otimes \mathscr{O}_p), r_1 = \chi(\mathscr{F} \otimes \mathscr{O}_{L_a}), r_2 = \chi(\mathscr{F} \otimes \mathscr{O}_{L_b}) \text{ and } \chi_0 = \chi\mathscr{F}.$$

Here L_a and L_b denote lines of class $(1,0)$ and $(0,1)$ respectively, and p denotes a point.

Following the notation on projective space, we say that a vector bundle $\mathscr{F}$ has natural cohomology if, for each pair of integers (a,b), the group $H^i\mathscr{F}(a,b)$ is nonzero for at most one value of i. For example, any line bundle on $\mathbb{P}^1 \times \mathbb{P}^1$ has natural cohomology.

For a bundle $\mathscr{F}$ with natural cohomology, the vanishing locus of the polynomial $\chi\mathscr{F}(x,y)$ divides the plane into regions where H^0, H^1 or H^2 cohomology is nonzero. We will exploit the relation of $\mathscr{F}$ to the geometry of this hyperbola. The following turns out to be a useful step.

Lemma 5.1. *Let $\mathscr{F}$ be a coherent sheaf on $\mathbb{P}^1 \times \mathbb{P}^1$ having no zero-dimensional components. If, for some pair of integers (a,b), the middle cohomology groups $H^1\mathscr{F}(a+k,b+k)$ vanish for all $k \in \mathbb{Z}$, then $\mathscr{F}$ is a direct sum of copies of the line bundles $\mathscr{O}(d,d)$, $\mathscr{O}(d,d+1)$ and $\mathscr{O}(d+1,d)$.*

Proof. Consider $M = \sum_k H^0(\mathscr{F}(a+k,b+k)$ as a module over the homogeneous coordinate ring S_X of the quadric $X = \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$. The associated sheaf on $\mathbb{P}^3$ is $\mathscr{F}(a,b)$. Since none of its twists has H^1 cohomology, the module M has depth 2; that is, it is a maximal Cohen–Macaulay module over S_X. By the classification of such modules [16] and [1],

$$\mathscr{F}(a,b) \cong \bigoplus_d [\mathscr{O}(d,d)^{\alpha_d} \oplus \mathscr{O}(d+1,d)^{\beta_d} \oplus \mathscr{O}(d,d+1)^{\gamma_d}].$$

□

From the Lemma we see that if $\mathscr{F}$ is not a sum of line bundles, then on each diagonal $\{(x,y) \mid x - y = const\}$ at least one H^1-group is nonzero. By Serre vanishing, the group $H^0\mathscr{F}(a+k,b+k)$ must be nonzero for $k \gg 0$; and by duality, $H^2\mathscr{F}(a+k,b+k)$ must be nonzero for $k \ll 0$. Thus when $\mathscr{F}$ is not a sum of line bundles, the hyperbola defined by the vanishing of the Euler characteristic,

$$\chi\mathscr{F}(x,y) := r \cdot xy + r_1\mathscr{F} \cdot x + r_2\mathscr{F} \cdot y + \chi_0 = 0,$$

will divide the (x,y) plane in three connected regions; if $\mathscr{F}$ has natural cohomology, these are the regions where $\mathscr{F}(x,y)$ has nonvanishing H^0, H^1 or H^2.

We can show that the cohomology tables of certain vector bundles with natural cohomology lie in extremal rays:

Theorem 5.2. *The cohomology table of a vector bundle $\mathscr{F}$ with natural cohomology generates an extremal ray in the Boij–Söderberg cone of cohomology tables, if either*

(a) $\mathscr{F}$ is a line bundle, or
(b) $\mathscr{F}$ is not a line bundle, and for the hyperbola $\chi\mathscr{F}(x,y)$ the number of integral asymptotes plus the number of integral points not on an integral asymptote is at least 3.

By analogy with the case of vector bundles on projective spaces, we call these bundles supernatural.

Note that the (possibly degenerate) hyperbolas of the form $(*)$ are those whose points at infinity are $(1,0)$ and $(0,1)$, and the asymptotes are the slopes of the tangent lines at infinity. Thus when we fix 3 data, each either an integral point or an integral asymptote, we are fixing n points of the projective closure of the hyperbola, with $3 \le n \le 5$ and $5-n$ tangent lines at those points. Among the points, exactly two lie on the line at infinity. It follows that knowing these data determines the hyperbola uniquely.

Proof. We will say that a sheaf $\mathscr{F}_1$ is a *numerical summand* of $\mathscr{F}$ if the cohomology table of $\mathscr{F}$ bounds a positive scalar multiple of the cohomology table $\mathscr{F}_1$ from above. This condition is slightly stronger than the requirement that the cohomology table of $\mathscr{F}_1$ has a zero wherever the cohomology table of $\mathscr{F}$ does.

Now let $\mathscr{F}$ be as in the hypothesis of the Theorem, and let $\mathscr{F}_1$ be a numerical summand. We will prove that the cohomology table of $\mathscr{F}_1$ is a scalar multiple of that of $\mathscr{F}$. Since $\mathscr{F}$ has natural cohomology, so does $\mathscr{F}_1$. Thus it will suffice to prove that $\chi\mathscr{F}_1(x,y)$ is a scalar multiple of $\chi\mathscr{F}(x,y)$.

Consider first the case where $\mathscr{F}$ is a line bundle; without loss of generality we may take $\mathscr{F} = \mathscr{O}(-1,-1)$. Since $\mathscr{F}(a,b)$ has no cohomology at all when $a=0$ or $b=0$, the same is true of $\mathscr{F}_1$. Hence $\chi\mathscr{F}_1(x,y) = \operatorname{rank} \mathscr{F}_1 xy$ as desired. Note that Lemma 5.1 even implies $\mathscr{F}_1 \cong \mathscr{O}(-1,-1)^{\operatorname{rank} \mathscr{F}_1}$ in this case.

For the second case we first note that if (a,b) is an integral zero of $\chi\mathscr{F}(x,y)$ then, by natural cohomology, $H^i\mathscr{F}(a,b) = 0$ for all i, and it follows that $\chi\mathscr{F}_1(a,b) = 0$ as well. On the other hand, if $\chi\mathscr{F}(x,y)$ has an integral asymptote, say the line $x = a$, then writing $\chi\mathscr{F}(x,y) = r(x + r_2/r)(y + r_1/r) + c$ we see that $a = -r_2/r$. Thus $\chi\mathscr{F}(a,y) = c$ is constant as a function of y. By the form of the polynomial $\chi\mathscr{F}_1(x,y)$ we see that $\chi\mathscr{F}_1(a,y)$ is a linear function of y, and it follows that this must be constant as well; that is, the hyperbola $\chi\mathscr{F}_1(x,y)$ has the same integral asymptotes as $\chi\mathscr{F}(x,y)$. Since a total of 3 points and asymptotes determine a hyperbola of the form $(*)$, this shows that the vanishing loci of $\chi\mathscr{F}(x,y)$ is the same as that of $\chi\mathscr{F}_1(x,y)$. Thus these polynomials are scalar multiples of each other as required. $\square$

Example 5.3. Consider the polynomials

$$3ab-a-b-1, 2ab+b-1, 2ab+a-1, ab-1$$

whose values we indicate for $-2 \le a,b \le 2$ in the following tables:

$$\begin{vmatrix} -13 & -6 & 1 & 8 & 15 \\ -8 & -4 & 0 & 4 & 8 \\ -3 & -2 & -1 & 0 & 1 \\ 2 & 0 & -2 & -4 & -6 \\ 7 & 2 & -3 & -8 & -13 \end{vmatrix}, \begin{vmatrix} -7 & -3 & 1 & 5 & 9 \\ -4 & -2 & 0 & 2 & 4 \\ -1 & -1 & -1 & -1 & -1 \\ 2 & 0 & -2 & -4 & -6 \\ 5 & 1 & -3 & -7 & -11 \end{vmatrix},$$

$$\begin{vmatrix} -11 & -6 & -1 & 4 & 9 \\ -7 & -4 & -1 & 2 & 5 \\ -3 & -2 & -1 & 0 & 1 \\ 1 & 0 & -1 & -2 & -3 \\ 5 & 5 & -1 & -4 & -4 \end{vmatrix}, \begin{vmatrix} -5 & -3 & -1 & 1 & 3 \\ -3 & -2 & -1 & 0 & 1 \\ -1 & -1 & -1 & -1 & -1 \\ 1 & 0 & -1 & -2 & -3 \\ 3 & 1 & -1 & -3 & -5 \end{vmatrix}.$$

We claim that the first three hyperbolas can be realized by vector bundles $\mathscr{F}_j$ with natural cohomology of rank 3, 2 and 2 respectively. The fourth cannot be a χ of a bundle, because that would be a line bundle, whose Hilbert polynomial factors. However there is a rank 2 bundle $\mathscr{F}_4$ with natural cohomology and $\chi\mathscr{F}_4(a,b) = 2ab-2$. To prove existence we consider the monad on $\mathbb{P}^1 \times \mathbb{P}^1$ for the desired bundles $\mathscr{F}_j$ in position $(-1,0)$. Due to the appropriate vanishing we expect a short exact sequence

$$0 \to \mathscr{F}_j(-1,0) \to \begin{array}{c} H^2\mathscr{F}_j(-2,-1)\otimes\mathscr{O}(-1,-1) \\ \oplus \\ H^1\mathscr{F}_j(-2,0)\otimes\mathscr{O}(-1,0) \end{array} \xrightarrow[\varphi_j]{} \begin{array}{c} H^1\mathscr{F}_j(-1,-1)\otimes\mathscr{O}(0,-1) \\ \oplus \\ H^1\mathscr{F}_j(-1,0)\otimes\mathscr{O}(0,0) \end{array} \to 0$$

in all four cases. A generic choice of matrices φ_j gives the desired bundles $\mathscr{F}_j$, as one can check computationally.

There is one relation among the cohomology tables of these bundles:

$$2h\mathscr{F}_1 + h\mathscr{F}_4 = 2h\mathscr{F}_2 + 2h\mathscr{F}_3,$$

so that these functions span only a 3-dimensional subspace of the $\mathbb{Q}$-vector space all maps $\mathbb{Z}^2 \times \{0,1,2\} \to \mathbb{Q}$. It is easy to see in this case that the intersection of this 3 dimensional subspace with the Boij–Söderberg cone is generated by our four rays. Indeed the facets of the convex cone spanned by these four rays are defined by the vanishing of the obviously non-negative functionals

$$\begin{aligned} \mathscr{G} &\mapsto h^1\mathscr{G}(0,1), \\ \mathscr{G} &\mapsto h^0\mathscr{G}(0,2), \\ \mathscr{G} &\mapsto h^0\mathscr{G}(2,0), \\ \mathscr{G} &\mapsto h^1\mathscr{G}(1,0). \end{aligned}$$

These vanish, in this subspace, on the facets spanned by rays $\{1,2\},\{2,4\},\{4,3\}$ and $\{4,1\}$ respectively.

Conjecture 5.4. *For any polynomial* $p(x,y) = (x-\alpha)(y-\beta) - \gamma \in \mathbb{Q}[x,y]$ *with* $\gamma > 0$ *there exists a vector bundle* $\mathscr{F}$ *with natural cohomology and Hilbert polynomial* $\chi\mathscr{F}(a,b) = rp(a,b)$ *for sufficiently large ranks* r.

More precisely we conjecture that these bundles can be obtained from a suitable matrix φ with entries of bidegree $(1,0),(0,1)$ and $(1,1)$ only, as the bundles above. This amounts to a maximal rank conjecture for such matrices.

Conjecture 5.5. *The cone of cohomology tables is spanned by cohomology tables of supernatural bundles.*

Unlike the case of cohomology tables of bundles on projective space, the extremal rays described in Conjecture 5.5 above can have accumulation points. For integral points $(a_1,b_1),(a_2,b_2)$, (a_3,b_3) the polynomial

$$p(x,y) = \det \begin{pmatrix} xy & x & y & 1 \\ a_1b_1 & a_1 & b_1 & 1 \\ a_2b_2 & a_2 & b_2 & 1 \\ a_3b_3 & a_3 & b_3 & 1 \end{pmatrix} / \det \begin{pmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{pmatrix}$$

converges for $b_1 \to \infty$ to the polynomial

$$\det \begin{pmatrix} xy & x & y & 1 \\ a_1 & 0 & 1 & 0 \\ a_2b_2 & a_2 & b_2 & 1 \\ a_3b_3 & a_3 & b_3 & 1 \end{pmatrix} / \det \begin{pmatrix} 0 & 1 & 0 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{pmatrix}$$

which has the integral asymptote $x = a_1$. In some sense an integral asymptote corresponds to an integral zero at infinity. Now these limit polynomials in turn converge for $a_2 \to \infty$ to a polynomial with both asymptotes integral and one more zero.

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