THE REGULARITY OF TOR AND GRADED BETTI NUMBERS

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Abstract. Let $S = K[x_1, \dots, x_n]$, let A, B be finitely generated graded S-modules, and let $\mathbf{m} = (x_1, \dots, x_n) \subset S$. We give bounds for the regularity of the local cohomology of $\operatorname{Tor}_k(A, B)$ in terms of the graded Betti numbers of A and B, under the assumption that $\dim \operatorname{Tor}_1(A, B) \leq 1$. We apply the results to syzygies, Gröbner bases, products and powers of ideals, and to the relationship of the Rees and symmetric algebras. For example we show that any homogeneous linearly presented $\mathbf{m}$-primary ideal has some power equal to a power of $\mathbf{m}$; and if the first $\lceil (n-1)/2 \rceil$ steps of the resolution of I are linear, then I^2 is a power of $\mathbf{m}$.

1. Introduction. Let $S = K[x_1, \dots, x_n]$ and let A, B be finitely generated graded S-modules. If T is a finitely generated graded S-module we write $\operatorname{reg} T$ for the Castelnuovo-Mumford regularity of T, and we extend this to Artinian graded modules T by setting $\operatorname{reg} T = \max\{i \mid T_i \neq 0\}$. The main technical results of this paper, proved in Section 2, give upper bounds on the regularity of the local cohomology modules $\mathrm{H}^i_{\mathbf{m}}(\operatorname{Tor}_k(A, B))$ under the hypothesis that $\operatorname{Tor}_1(A, B)$ has Krull dimension ≤ 1. A special case says that if $A \otimes B$ has finite length then, for any k,

$$\operatorname{reg} \operatorname{Tor}_k(A, B) + n \leq \operatorname{reg} \operatorname{Tor}_p(A, K) + \operatorname{reg} \operatorname{Tor}_q(B, K)$$

for any p, q with

$$p + q = n + k, \ p \leq n, \ q \leq n.$$

In this formula $\operatorname{reg} \operatorname{Tor}_p(A, K)$ is just the maximal degree of a homogeneous generator of the minimal p-th syzygies of A. Such terms occur so often in this paper that we will adopt a special notation, and write

$$t_p(A) := \operatorname{reg} \operatorname{Tor}_p(A, K).$$

The rest of the paper is devoted to applications of the bounds proven in Section 2. By way of introduction, we will now sample the less technical con-

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sequences. Almost every result stated below occurs with more generality in the body of the paper.

We begin, in Section 3, with the regularity of the Tor modules. We show that if A and B are finitely generated graded S-modules such that $\dim \operatorname{Tor}_1(A,B) \leq 1$, then

$$\operatorname{reg} \operatorname{Tor}_k(A,B) \leq \operatorname{reg} A + \operatorname{reg} B + k,$$

which generalizes results of Geramita-Gimigliano-Pitteloud [1995], Chandler [1997], Sidman [2002], Conca-Herzog [2003], and Caviglia [2003]. Several of these results rely on an argument due to Lazarsfeld. For a geometric consequence, let $X, Y \subset \mathbf{P}^{n-1}$ be projective schemes. It is elementary that, if I and J are their homogeneous ideals, then the ideal of forms vanishing on $X \cap Y$ is equal to $I+J$ in degree $d \gg 0$. It follows from our results that if $\dim(X \cap Y) = 0$ then it suffices to take

$$d > t_p(S/I) + t_q(S/J) - n$$

for any p, q such that $p \leq \operatorname{codim} X$, $q \leq \operatorname{codim} Y$, and $p + q = n$.

In Section 4 we deduce relations between graded Betti numbers. For example, we show that if $A = B = S/I$ is a cyclic module of dimension ≤ 1, then the function $p \mapsto t_p(S/I)$ satisfies the weak convexity condition

$$t_n(S/I) \leq t_p(S/I) + t_{n-p}(S/I)$$

for $0 \leq p \leq n$.

We also compare the graded Betti numbers of a module and an ideal that annihilates it. We prove that if S/I is Cohen-Macaulay of codimension c, and I contains a regular sequence of elements of degrees $d_1 \leq \cdots \leq d_q$, then

$$t_c(S/I) \leq t_{c-q}(S/I) + d_1 + \cdots + d_q.$$

If I is generated in degrees $\leq d$, then we can take all the $d_i = d$, and we see that

$$t_c(S/I) - t_{c-q}(S/I) \leq qd.$$

In Section 5 we study the relationship between the graded Betti numbers of an ideal I and its initial ideal in reverse lexicographic order. For example, suppose that $I \subset S$ is a homogeneous $\mathbf{m}$-primary ideal. Setting $m = t_p(S/I)$, we show that the initial ideal of I in reverse lexicographic order contains $(x_1, \ldots, x_p)^{m-p+1}$. If the minimal free resolution of I is linear for q steps, I is generated in degree d, and L is any ideal generated by $n-q-1$ independent linear forms, then we show that

$$\mathbf{m}^d \subset I + L.$$

In other words, $\operatorname{reg}(I+L) \leq d$.

In Section 6 we explore the meaning of this last condition by characterizing the ideals I generated by quadrics such that $\mathbf{m}^2 \subset I+L$ for every ideal L generated by $n-q-1$ independent linear forms.

In Section 7 we study powers of linearly presented ideals. The following conjecture sparked this entire paper:

CONJECTURE 1.1 (Eisenbud and Ulrich). *If $I \subset S$ is a linearly presented $\mathbf{m}$-primary ideal generated in degree d, then $I^{n-1} = \mathbf{m}^{d(n-1)}$.*

We prove this conjecture when $n = 3$, and, in Section 8, for the case of monomial ideals. In general we can prove an asymptotic statement:

THEOREM 1.2. *If I is a linearly presented $\mathbf{m}$-primary ideal generated in degree d, then $I^t = \mathbf{m}^{dt}$ for all $t \gg 0$.*

This theorem relies on our specialization results in Section 5.

The following theorem proves Conjecture 1.1 in the case $n = 3$, and gives more precise information than Theorem 1.2. It is perhaps the most surprising result of this paper.

THEOREM 1.3. *Suppose I and J are homogeneous ideals in S of dimension ≤ 1, generated in degree d. If the resolutions of I and J are linear for $\lceil (n-1)/2 \rceil$ steps (for instance if I and J have linear presentation and $n \leq 3$), then IJ has linear resolution. In particular, if I and J are $\mathbf{m}$-primary then $IJ = \mathbf{m}^{2d}$.*

Here the last statement follows from the previous one because the powers of the maximal ideal are the only $\mathbf{m}$-primary ideals with linear resolutions. Based on this evidence, we extend Conjecture 1.1 to:

CONJECTURE 1.4. *If I is an $\mathbf{m}$-primary ideal, and the resolution of I is linear for q steps, then I^t is equal to a power of $\mathbf{m}$ for all $t \geq (n-1)/q$.*

A natural generalization of Conjecture 1.1 and Theorem 1.3 would be to say that if I is a linearly presented ideal of small dimension whose free resolution begins with q linear steps, then the t-th power of I has a resolution that begins with tq linear steps. This is false, even for $q = 1$. In fact in Section 7 we give an example, Example 7.10, of an $\mathbf{m}$-primary ideal I in 8 variables with linear presentation whose square is not even linearly presented. Sturmfels [2000] (see also Conca [2003]) previously gave examples of this phenomenon, but not for $\mathbf{m}$-primary ideals.

The torsion in $I \otimes I^t$ is $\operatorname{Tor}_2(S/I, S/I^t)$. In Section 9 we use this relationship to study the torsion in the symmetric algebra $\operatorname{Sym}(I)$. We were motivated by the following conjecture of Eisenbud and Ulrich for (not necessarily $\mathbf{m}$-primary) ideals $I \subset S$ with linear resolution:

CONJECTURE 1.5. *Assume that I has linear free resolution and is generated in degree d. If I is of linear type on the punctured spectrum* (*that is, the torsion of* $\operatorname{Sym}(I)$ *is supported only at* $\mathbf{m}$) *then for every t the torsion of* $\operatorname{Sym}_t(I)$ *is concentrated in degree dt; equivalently, the symmetric algebra of I is a subalgebra of the symmetric algebra of* $\mathbf{m}^d$.

We are able to show, for example, that if I is an $\mathbf{m}$-primary ideal generated in degree d, and has a free resolution that is linear for $\lceil n/2 \rceil$ steps, then, for every t, the torsion in $\operatorname{Sym}_t(I)$ is concentrated in degree dt. (Related ideas show that $\wedge^t I$ is a vector space concentrated in degree dt.) We show in Example 9.3 that, at least for $n = 3$, the bound $\lceil n/2 \rceil$ is sharp.

In Section 10 we explore a consequence for elimination theory, a method of finding the defining ideal of the image of a map $\alpha_V : \mathbf{P}^{n-1} \to \mathbf{P}^{N-1}$ given by an N-dimensional vector space $V \subset S_d$ of forms of degree d. We assume that the morphism α_V is everywhere defined, which means that V generates an ideal $I = SV$ that is $\mathbf{m}$-primary. Let $M = \dim_K \operatorname{Tor}_1(I, K)$ be the number of relations required for I, and let ϕ be a $N \times M$ matrix of linear forms that presents I. The matrix ϕ can be represented as an $n \times N \times M$ tensor over K, and thus also represents an $n \times M$ matrix of linear forms ψ over the polynomial ring in N variables representing $\mathbf{P}^{N-1}$. In this setting, we show that if the free resolution of the ideal I generated by V begins with at least $\lceil n/2 \rceil$ linear steps, then the annihilator of $\operatorname{coker} \psi$ is the ideal of forms in $\mathbf{P}^{N-1}$ that vanish on $\alpha_V(\mathbf{P}^{n-1})$.

If I is an ideal generated in degree d, and $I^t = \mathbf{m}^{dt}$, then the number of generators μ of I must satisfy

$$\binom{\mu + t - 1}{t} \geq \binom{n + td - 1}{n - 1}.$$

By Corollary 7.6, this relation is satisfied with $t = 2$ if the resolution of I is linear for $\lceil (n-1)/2 \rceil$ steps. In Section 11 we give a stronger lower bound for the number of generators of an ideal whose resolution is linear for $n - 2$ steps (the "almost linear" case). Lower bounds on the number of generators of ideals whose resolutions are linear for q steps would follow from Conjecture 1.4.

The truncation principle. Since the focus of this paper is on linearly presented ideals, we have stated many results only for this case. However, it is possible to make any ideal I into an ideal with linear resolution for q steps by truncating, and thus generalize many of the results. Rather than doing this throughout the paper, we illustrate it here.The following result is elementary:

PROPOSITION 1.6. *If $J = I \cap \mathbf{m}^u$ then $t_i(J) = \max\{u + i,\ t_i(I)\}$. Thus J has linear resolution for q steps if $u \geq t_i(I) - i$ for $0 \leq i \leq q$.*

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2. Degrees of syzygies. Throughout this paper, K is a field and $S = K[x_1, \dots, x_n]$ is a polynomial ring in n variables, graded with $\deg x_i = 1$ (but see Remark 2.4 below for the case of general grading). We write $\mathbf{m} = (x_1, \dots, x_n)$ for the homogeneous maximal ideal of S. All Tor and Ext modules are taken over the ring S. The Krull dimension of a module A is denoted $\dim A$ (we use $\dim_K$ for vector space dimension).

We write $\operatorname{reg} A$ for the (Castelnuovo-Mumford) regularity of a graded S-module A (see for example Eisenbud [2004]). If A is a finitely generated graded vector space, or more generally an Artinian graded S-module, then $\operatorname{reg} A = \sup\{i \mid A_i \neq 0\}$. If A is a finitely generated graded S-module then $\operatorname{reg} A$ is defined in terms of local cohomology by the formula

$$\operatorname{reg} A = \max_j \{\operatorname{reg} \mathrm{H}^j_{\mathbf{m}}(A) + j\}.$$

For example, if $A = 0$ then $\operatorname{reg} A = -\infty$. We may also compute $\operatorname{reg} A$ in terms of Tor (or in terms of a minimal free resolution) by the formula

$$\operatorname{reg} A = \max_p \{t_p(A) - p\}.$$

From local duality one see that the two ways of expressing the regularity are also connected "termwise" by the inequality $t_p(A) - p \geq \operatorname{reg} \mathrm{H}^{n-p}_{\mathbf{m}}(A) + n - p$.

The numbers $\operatorname{reg} \mathrm{H}^j_{\mathbf{m}}(A) + j$ and $t_p(A) - p$ will appear often in our formulas. The next two theorems express the basic technical result of this paper.

THEOREM 2.1. *Suppose that A and B are finitely generated graded S-modules such that $\dim \operatorname{Tor}_1(A, B) \leq 1$, and let j, k be integers. If $p \leq \operatorname{codim} A$, $q \leq \operatorname{codim} B$ and $p + q = n - j + k$ then*

$$\operatorname{reg} \mathrm{H}^j_{\mathbf{m}}(\operatorname{Tor}_k(A, B)) \leq t_p(A) + t_q(B) - n.$$

THEOREM 2.2. *Suppose that A and B are finitely generated graded S-modules such that $\dim \operatorname{Tor}_1(A, B) \leq 1$, and let j, k be integers. If $n - j + k \geq \operatorname{codim} A + \operatorname{codim} B$, then*

$$\operatorname{reg} \mathrm{H}^j_{\mathbf{m}}(\operatorname{Tor}_k(A, B)) \leq \max_{\substack{p+q=n-j+k \\ p \geq \operatorname{codim} A \\ q \geq \operatorname{codim} B}} \left\{ t_p(A) + t_q(B) \right\} - n.$$

In fact, both these theorems follow from a more general statement:

THEOREM 2.3. *Suppose that A and B are finitely generated graded S-modules such that* $\dim \operatorname{Tor}_1(A,B) \leq 1$*, and let j,k be integers. For any integers p,q with $p+q = n-j+k$*

$$\operatorname{reg} \mathrm{H}^j_{\mathbf{m}}(\operatorname{Tor}_k(A,B)) \leq \max\{X,Y,Z\}$$

where

$$\begin{aligned} X &= t_p(A) + t_q(B) - n, \\ Y &= \max_{\substack{p'+q'=n-j+k \\ p'>p}} \left\{ t_{p'}(A) + \operatorname{reg} \mathrm{H}^{n-q'}_{\mathbf{m}}(B) \right\}, \\ Z &= \max_{\substack{p'+q'=n-j+k \\ p'<p}} \left\{ \operatorname{reg} \mathrm{H}^{n-p'}_{\mathbf{m}}(A) + t_{q'}(B) \right\}. \end{aligned}$$

Proof of Theorem 2.1. Since $q' < q \leq \operatorname{codim} B$ in the expression for Y and $p' < p \leq \operatorname{codim} A$ in the expression for Z, the local cohomology modules in the expressions for Y and Z in Theorem 2.3 are zero. Because the regularity of the module 0 is $-\infty$ we have $Y = Z = -\infty$, and Theorem 2.3 reduces to Theorem 2.1. □

Proof of Theorem 2.2. Since $n-j+k \geq \operatorname{codim} A + \operatorname{codim} B$, we can pick p,q with $p \geq \operatorname{codim} A$, $q \geq \operatorname{codim} B$ and $p+q = n-j+k$. Replacing the terms $\operatorname{reg} \mathrm{H}^{n-q'}_{\mathbf{m}}(B)$ in Y with the possibly larger terms $t_{q'}(B) - n$ (and similarly for Z) in Theorem 2.3, we obtain Theorem 2.2. □

We postpone the proof of Theorem 2.3 until later in this section.

Remark 2.4. These formulas adapt easily to the case where the degrees of the x_i are not assumed to be 1: Setting $\sigma = \sum \deg x_i$ we must add $n-\sigma$ to the term X in Theorem 2.3, and we correspondingly add $n-\sigma$ to the right hand side of the formulas in Theorem 2.1 and Theorem 2.2. The proofs use the comparison $t_p(A) - p \geq \operatorname{reg} \mathrm{H}^{n-p}_{\mathbf{m}}(A) + \sigma - p$.

Finally, if the module B is Cohen-Macaulay, a special case of the inequality takes on a simple form no matter what the relation of $n-j+k$ and $\operatorname{codim} A + \operatorname{codim} B$:

COROLLARY 2.5. *Suppose that A and B are finitely generated graded S-modules such that* $\dim \operatorname{Tor}_1(A,B) \leq 1$*. If B is Cohen-Macaulay of dimension b then*

$$\operatorname{reg} \mathrm{H}^j_{\mathbf{m}}(\operatorname{Tor}_k(A,B)) \leq t_{b-j+k}(A) - b + \operatorname{reg} B.$$

For example, when B has finite length, this statement reduces to the easy formula $\operatorname{reg}(\operatorname{Tor}_k(A,B)) \leq t_k(A) + \operatorname{reg} B$.

Proof of Corollary 2.5. Take $q = n - b = \operatorname{codim} B$ in Theorem 2.3. Because B is Cohen-Macaulay, the only nonvanishing local cohomology of B is $\mathrm{H}^b_{\mathbf{m}}(B)$. Hence the terms $\operatorname{reg} \mathrm{H}^{n-q'}_{\mathbf{m}}(B)$ that appear in the expression for Y in Theorem 2.3 are all $-\infty$. The terms $t_{q'}(B)$ that appear in the expression for Z are all $-\infty$ as well, because when $p' < p$ the number q' is bigger than $n - b$, the projective dimension of B. □

The assumption $\dim \operatorname{Tor}_1(A,B) \leq 1$ is used in the proof of Theorem 2.3 to ensure the degeneration of a certain spectral sequence. The theorem can fail without this assumption, even in the case where $A = B = R/I$ is 2-dimensional and $n = 4$: for instance Example 4.5 does not satisfy Corollary 2.5 for $k = 0$.

We note that the hypothesis $\dim \operatorname{Tor}_1(A,B)$ is always satisfied if A, B are "dimensionally transverse" in the sense that $\operatorname{codim} A \otimes B \geq \operatorname{codim} A + \operatorname{codim} B$ (in which case equality holds), and A, B are both locally Cohen-Macaulay off a set of dimension ≤ 1.

For any graded S-module we write $\operatorname{mindeg} T = \inf\{i \mid T_i \neq 0\}$. If $T = 0$ we set $\operatorname{mindeg} T = \infty$.

Proof of Theorem 2.3. Let $\mathbf{F}$: $\cdots \to F_1 \to F_0$ be a minimal homogeneous free resolution of A and let $\mathbf{G}$: $\cdots \to G_1 \to G_0$ be a minimal homogeneous free resolution of B. The proof consists of an analysis of the double complex $\mathbf{F}^* \otimes \mathbf{G}^* = (\mathbf{F} \otimes \mathbf{G})^*$ where $*$ denotes $\operatorname{Hom}(-, S)$.

For any finite complex $\mathbf{K}$: $\cdots \to K_n \to K_{n-1} \to \cdots$ of free S-modules there is a spectral sequence with E_2 term $\operatorname{Ext}^s_S(\mathrm{H}_t(\mathbf{K}), S)$ converging to $\mathrm{H}^{s+t}(\mathbf{K}^*)$, obtained from the double complex $\operatorname{Hom}(\mathbf{K}, \mathbf{I})$, where $\mathbf{I}$ is an injective resolution of S. We apply this to $\mathbf{K} = \operatorname{Tot}(\mathbf{F} \otimes \mathbf{G})$. Since $\operatorname{Tor}_1(A,B)$ has Krull dimension at most 1, Auslander's Theorem [1961] on the rigidity of Tor shows that $\mathrm{H}_t(\mathbf{F} \otimes \mathbf{G}) = \operatorname{Tor}_t(A,B)$ has dimension ≤ 1 for every $t \geq 1$. It follows that $\operatorname{Ext}^s_S(\mathrm{H}_t(\mathbf{K}), S)$ is nonzero only when $t = 0$ or when $s = n - 1$ or $s = n$. The E_2 differential $\operatorname{Ext}^s_S(\mathrm{H}_t(\mathbf{K}), S) \to \operatorname{Ext}^{s+2}_S(\mathrm{H}_{t-1}(\mathbf{K}), S)$ thus vanishes and the spectral sequence degenerates at E_2. The degeneracy in turn shows that $\operatorname{Ext}^s_S(\mathrm{H}_t(\mathbf{K}), S)$ is a subquotient of $\mathrm{H}^{s+t}(\mathbf{K}^*)$.

By local duality

$$\begin{aligned} \mathrm{H}^j_{\mathbf{m}}(\operatorname{Tor}_k(A,B)) &= \mathrm{H}^j_{\mathbf{m}}(\mathrm{H}_k(\mathbf{K})) \\ &= \operatorname{Hom}_K(\operatorname{Ext}^{n-j}(\mathrm{H}_k(\mathbf{K}), S), K)(n), \end{aligned}$$

where Hom_K denotes the graded Hom functor over K. Since $\operatorname{Ext}^{n-j}(\operatorname{H}_k(\mathbf{K}), S)$ is a subquotient of $\operatorname{H}^{n-j+k}(\mathbf{K}^*)$, it follows that

$$\operatorname{reg} \operatorname{H}^j_{\mathbf{m}}(\operatorname{Tor}_k(A,B)) \leq -\operatorname{mindeg} \operatorname{H}^{n-j+k}(\mathbf{K}^*) - n.$$

To prove Theorem 2.3 we need to show that any homogeneous element $\zeta \in \operatorname{H}^{n-j+k}(\mathbf{K}^*)$ of degree

$$\deg \zeta < -\max\{X, Y, Z\} - n = \min\{-X - n, -Y - n, -Z - n\}$$

is zero. We have

$$-X - n = -t_p(A) - t_q(B)$$

and by local duality

$$-Y - n = \min_{\substack{p'+q'=n-j+k \\ p'>p}} \left\{ -t_{p'}(A) + \operatorname{mindeg} \operatorname{Ext}^{q'}(B, S) \right\}, \tag{*}$$

$$-Z - n = \min_{\substack{p'+q'=n-j+k \\ p'<p}} \left\{ \operatorname{mindeg} \operatorname{Ext}^{p'}(A, S) - t_{q'}(B) \right\}. \tag{**}$$

Let $z = \{z^{p',q'} \mid p' + q' = p + q\}$ be a homogeneous cycle of $\mathbf{K}^*$ representing ζ. Since

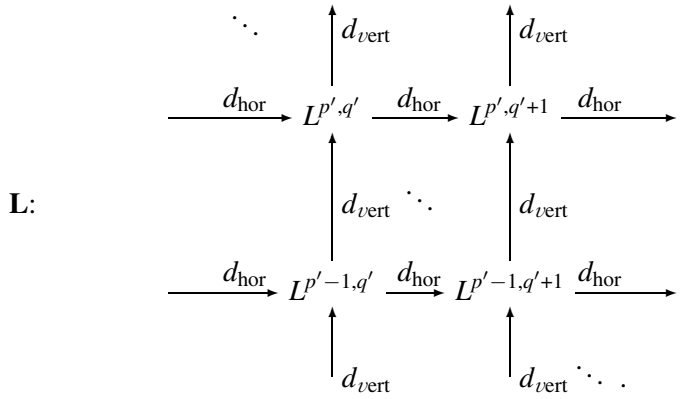
$$\begin{aligned} \operatorname{mindeg}(F_p^* \otimes G_q^*) &= \operatorname{mindeg}(F_p^* \otimes K) + \operatorname{mindeg}(G_q^* \otimes K) \\ &= -t_p(A) - t_q(B) \\ &> \deg \zeta, \end{aligned}$$

it follows that $z^{p,q} = 0$. To finish the proof we will show that the other components $z^{p',q'}$ are also zero.

By equation (**) the vertical homology of $\mathbf{K}^*$ is zero at $(\mathbf{K}^*)^{p',q'}$ in degree $\deg \zeta$ when $p' + q' = p + q$ and $p' < p$, while by equation (*) the horizontal homology of $\mathbf{K}^*$ is zero at $(\mathbf{K}^*)^{p',q'}$ in degree $\deg \zeta$ when $p' + q' = p + q$ and $p' > p$.

We may thus complete the proof by applying the following more general lemma to the complex $\mathbf{L}$ formed by taking the degree $\deg \zeta$ part of $\mathbf{K}^*$. The

result gives information about the total cycles in the double complex



LEMMA 2.6. *Let* $\mathbf{L}$ *be any bounded below double complex, with noation as above, suppose that* p, q *are chosen so that the vertical homology of* $\mathbf{L}$ *is zero at* $L^{p',q'}$ *when* $p' + q' = p + q$ *and* $p' < p$*, and the horizontal homology of* $\mathbf{L}$ *is zero at* $L^{p',q'}$ *when* $p' + q' = p + q$ *and* $p' > p$*. If* $\zeta \in \mathrm{H}^{p+q}(\mathrm{Tot}(\mathbf{L}))$ *represented by a cycle*

$$z = (z^{p',q'}) \in \oplus_{p'+q'=p+q} L^{p',q'}$$

satisfies $z^{p,q} = 0$*, then* $\zeta = 0$*.*

Proof. We have $d_{v\mathrm{ert}}(z^{p-1,q+1}) = -d_{\mathrm{hor}}(z^{p,q}) = 0$. By our assumption the vertical homology vanishes at $L^{p-1,q+1}$ so $z^{p-1,q+1} = d_{v\mathrm{ert}}(w)$ for some $w \in L^{p-2,q+1}$. Subtracting $d_{\mathrm{Tot}}w$ from z we get a homologous cycle y whose components $y^{p',q'}$ agree with $z^{p',q'}$ for $p' \geq p$, but $y^{p-1,q+1} = 0$. Repeating this process we see that z is homologous to a cycle x with $x^{p',q'} = z^{p',q'}$ for $p' \geq p$ while $x^{p',q'} = 0$ for $p' < p$.

Similarly, using the fact that the horizontal homology is zero at $L^{p',q'}$ for $p' > p$ and $p' + q' = p + q$, we can change x by a boundary to arrive at a cycle that is 0 in every component, so $\zeta = 0$. □

In the special case where B is a Gorenstein factor ring of S we can describe when Theorem 2.3 (in the form of Corollary 2.5) is sharp. Suppose ϕ: $F' \to F$ is a map of graded free modules such that $\operatorname{reg} F = d$. By a *generalized row of* ϕ *of minimal degree* we mean the composition of ϕ with a projection $F \to S(-d)$. By the *ideal of entries* in this row we mean the ideal that is the image of the corresponding map $F'(d) \to S$.

PROPOSITION 2.7. *Suppose that A is a finitely generated graded S-module with free resolution*

$$\cdots \longrightarrow F_t \xrightarrow{\phi_t} F_{t-1} \longrightarrow \cdots \xrightarrow{\phi_1} F_0$$

and J is an ideal such that S/J is Gorenstein of dimension b and A/JA has finite length. If $k \leq \operatorname{codim} A - b$ then

$$\operatorname{reg} \operatorname{Tor}_k(A, S/J) \leq t_{b+k}(A) - b + \operatorname{reg} S/J$$

with equality if and only if J contains the ideal of entries in some generalized row of minimal degree of ϕ_{b+k+1}.

Proof. The inequality is Corollary 2.5. Since $B = S/J$ is Cohen-Macaulay we have $\operatorname{reg} B = t_{n-b}(B) - n + b$. Since $A \otimes B = A/JA$ has finite length,

$$\operatorname{reg} \operatorname{Tor}_k(A, B) = -\operatorname{mindeg} \operatorname{Hom}_K(\operatorname{Tor}_k(A, B), K).$$

By local duality, we can rewrite this as $-(\operatorname{mindeg} \operatorname{Ext}^n(\operatorname{Tor}_k(A, B), S)) - n$.

We now use the notation and spectral sequence from the proof of Theorem 2.3. Because $A \otimes B$ has finite length, the E_2 page of the spectral sequence for the homology of $\mathbf{K}^*$ has nonzero terms only in one row and one column, and if follows that $\operatorname{Ext}^n(\operatorname{Tor}_k(A, B), S) = \mathrm{H}^{n+k}(\mathbf{K}^*)$.

From this we see that equality holds in Proposition 2.7 if and only if $\operatorname{mindeg} \mathrm{H}^{n+k}(\operatorname{Tot}(\mathbf{F}^* \otimes \mathbf{G}^*)) = \operatorname{mindeg}(F^*_{b+k} \otimes G^*_{n-b})$. Because B is Gorenstein we may write $G^*_{n-b} = S(e)$ for some e. Moreover $\mathbf{G}^*$ is a resolution of $\operatorname{Ext}^{n-b}(B, S) = B(e)$. It follows that $\mathrm{H}^{n+k}(\operatorname{Tot}(\mathbf{F}^* \otimes \mathbf{G}^*)) \cong \mathrm{H}^{b+k}((\mathbf{F}^* \otimes B)(e))$. Hence equality holds if and only if $\operatorname{mindeg}(F^*_{b+k} \otimes S(e)) = \operatorname{mindeg} \mathrm{H}^{b+k}((\mathbf{F}^* \otimes B)(e))$. Since $\mathbf{F}^*$ is a minimal complex, this is equivalent to saying that a generator of minimal degree of F^*_{b+k} is a cycle mod J; that is, J contains the ideal of entries in some generalized row of minimal degree of ϕ_{b+k+1}. □

3. Castelnuovo-Mumford regularity. The following is an extension of results of Sidman [2002] and Caviglia [2003], who treat the case $k = 0$ by different methods.

COROLLARY 3.1. *If A and B are finitely generated graded S-modules such that $\dim \operatorname{Tor}_1(A, B) \leq 1$, then*

$$\operatorname{reg} \operatorname{Tor}_k(A, B) \leq \operatorname{reg} A + \operatorname{reg} B + k.$$

Proof. We use the formula

$$\operatorname{reg} M = \max_j \{\operatorname{reg} \mathrm{H}^j_{\mathbf{m}}(M) + j\}$$

to compute $\operatorname{reg} \operatorname{Tor}_k(A, B)$, and

$$\operatorname{reg} A + \operatorname{reg} B = \max_{p,q} \{t_p(A) - p + t_q(A) - q\}.$$

The proof is then a straightforward application of the inequalities in Theorems 2.1 and 2.2. □

COROLLARY 3.2. *Suppose that A and B are finitely generated graded S-modules such that* $\dim \operatorname{Tor}_1(A,B) \le 1$. *If* $k + \dim B \le p \le \operatorname{codim} A$ *then*

$$\operatorname{reg} \operatorname{Tor}_k(A,B) \le t_p(A) + t_{n+k-p}(B) - n.$$

Proof. Since $p \le \operatorname{codim} A$ and $n + k - p \le \operatorname{codim} B$, Theorem 2.1 gives

$$\operatorname{reg} \operatorname{Tor}_k(A,B) \le \max_{j=0,1}\{t_p(A) + t_{n-j+k-p}(B) + j - n\}.$$

But $t_{n-j+k-p}(B) + j \le t_{n+k-p}(B)$, again because $n + k - p \le \operatorname{codim} B$. □

COROLLARY 3.3. *Suppose that A and B are graded S-modules such that* $\delta := \dim \operatorname{Tor}_1(A,B) \le 1$. *If B is a Cohen-Macaulay module of dimension b, then for* $k > 0$,

$$\operatorname{reg} \operatorname{Tor}_k(A,B) \le \max\{t_p(A) - p \mid b + k - \delta \le p \le b + k\} + \operatorname{reg} B + k.$$

Proof. Notice that $\dim \operatorname{Tor}_k(A,B) \le \delta$ by the rigidity of Tor (see Auslander [1961]). Thus the assertion follows from Corollary 2.5. □

As an application of Corollaries 3.1 and 3.3 with $k = 1$, we have:

COROLLARY 3.4. *If I and J are homogeneous ideals of S such that* $(IJ)_d = (I \cap J)_d$ *for* $d >> 0$, *then the equality holds for all* $d \ge \operatorname{reg} I + \operatorname{reg} J$. *If in addition* S/J *is Cohen-Macaulay of dimension b, then it suffices that*

$$d \ge t_b(I) - b + \operatorname{reg} J.$$

Proof. We use the formula $\operatorname{Tor}_1(S/I, S/J) = (I \cap J)/IJ$, and apply Corollaries 3.1 and 3.3. □

Suppose that $X, Y \subset \mathbf{P}^{n-1}$ are schemes. The ideal $I_{X\cap Y}$ of $X \cap Y$ is the saturation of the sum of the ideals of X and Y; that is, $I_{X\cap Y}$ and $I_X + I_Y$ agree in high degrees. Using Theorems 2.1 and 2.2 we can make this quantitative in the case where X and Y meet at most in dimension 0. Note that in this case $\operatorname{codim} X + \operatorname{codim} Y \ge n - 1$.

COROLLARY 3.5. *Let* $X, Y \subset \mathbf{P}^{n-1}$ *be schemes with ideals* $I, J \subset S$. *Suppose that* $\dim X \cap Y = 0$.

(a) *If* $\operatorname{codim} X + \operatorname{codim} Y \ge n$, *then any form of degree d vanishing on* $X \cap Y$ *is a sum of a form vanishing on X and a form vanishing on Y as long as*

$$d > t_p(S/I) + t_q(S/J) - n$$

for some integers p, q *satisfying* $p \le \operatorname{codim} X$, $q \le \operatorname{codim} Y$, *and* $p + q = n$.

(b) *If* $\operatorname{codim} X + \operatorname{codim} Y = n-1$, *then any form of degree d vanishing on* $X \cap Y$ *is a sum of a form vanishing on X and a form vanishing on Y as long as*

$$d > \max\{t_{1+\operatorname{codim} X}(S/I) + t_{\operatorname{codim} Y}(S/J), \\ t_{\operatorname{codim} X}(S/I) + t_{1+\operatorname{codim} Y}(S/J)\} - n.$$

Proof. Notice that $S/(I+J) = (S/I) \otimes (S/J)$. It follows that $S/(I+J)$ is saturated in degree d if $\mathrm{H}^0_{\mathbf{m}}(\operatorname{Tor}_0(S/I, S/J))_d = 0$. Cases (a) and (b) follow from Theorems 2.1 and 2.2, with $j = k = 0$. □

A similar result follows for any schemes X and Y whose intersection is "homologically transverse" except along a zero-dimensional set in $\mathbf{P}^{n-1}$ (but the sum of the codimensions of X and Y may then be $< n-1$, in which case more terms appear in case (b)).

4. Convexity of degrees of syzygies. Theorem 2.1 yields a kind of "triangle inequality" or convexity for degrees of syzygies that seems to be new even in the case where $A = B$ is a module of finite length.

COROLLARY 4.1. *Suppose that A and B are finitely generated graded S-modules such that* $\dim \operatorname{Tor}_1(A, B) \leq 1$, *then*

$$t_n(A \otimes B) \leq t_p(A) + t_{n-p}(B)$$

whenever $\dim B \leq p \leq \operatorname{codim} A$. *In particular, if* $A = B = S/I$ *is a cyclic module of dimension* ≤ 1, *then the function* $p \mapsto t_p(S/I)$ *satisfies the weak convexity condition*

$$t_n(S/I) \leq t_p(S/I) + t_{n-p}(S/I)$$

for $0 \leq p \leq n$.

When $\dim B > \operatorname{codim} A$ a similar result follows from Theorem 2.2.

Proof. For any finitely generated graded module M,

$$\operatorname{Tor}_n(M, K) = \ker\left(M(-n) \xrightarrow{\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}} M^n(-n+1)\right) = \operatorname{socle} M(-n),$$

as can be calculated from the Koszul resolution of K. Thus $\operatorname{reg} \operatorname{Tor}_n(A \otimes B) = \operatorname{reg} \mathrm{H}^0_{\mathbf{m}}(A \otimes B) + n$, and the assertion follows from Theorem 2.1. □

If a module A is annihilated by an $\mathbf{m}$-primary ideal J, then it is immediate that the degree of the socle of A is bounded above by the highest degree of a

generator of A plus the highest degree of the socle of S/J. This relation can be written as $t_n(A) \leq t_0(A) + t_n(S/J)$. The following result gives such a bound without the assumption that J is $\mathbf{m}$-primary.

COROLLARY 4.2. *Suppose that A is a finitely generated graded S-module of codimension c and that $\delta := \dim A - \operatorname{depth} A \leq 1$. Let J be a homogeneous ideal contained in the annihilator of A. If* $\operatorname{depth} S/J \geq \operatorname{depth} A$ *then for* $0 \leq q \leq \operatorname{codim} J$,

$$t_{c+\delta}(A) \leq t_{c+\delta-q}(A) + t_q(S/J).$$

In particular:

(a) *If the annihilator of A contains a regular sequence of forms of degree $d_1 \leq \cdots \leq d_q$ then*

$$t_{c+\delta}(A) \leq t_{c+\delta-q}(A) + d_1 + \cdots + d_q.$$

(b) *If J is generated in degree d with linear resolution, then*

$$t_{c+\delta}(A) \leq t_{c+\delta-q}(A) + d + q - 1.$$

Proof. We may harmlessly assume that K is infinite. If $\dim A > 1$, a general sequence of $\operatorname{depth} A$ linear forms is a regular sequence on both A and S/J, so we factor out these linear forms (and work over the corresponding factor ring of S) without changing the statement. Thus we may suppose $\dim A \leq 1$ and $\operatorname{depth} A = 0$, so $n = c + \delta$. Since the case $q = 0$ is trivial, we may assume that $q \geq 1$.

We now apply Theorem 2.1 with $k = j = 0$, $B = S/J$ and $p = n - q$. As $p \leq \operatorname{codim} A$ we obtain $\operatorname{reg} \mathrm{H}^0_{\mathbf{m}}(\operatorname{Tor}_0(A, B)) \leq t_p(A) + t_q(B) - n$. Since $\operatorname{reg} \mathrm{H}^0_{\mathbf{m}}(\operatorname{Tor}_0(A, B)) = \operatorname{reg} \mathrm{H}^0_{\mathbf{m}}(A) = t_n(A) - n$, this gives the first statement. Parts (a) and (b) follow immediately by computing $t_q(B)$ in the given cases. □

Example 4.3. Let X be an arithmetically Cohen-Macaulay scheme of codimension c in $\mathbf{P}^{n-1}$ with ideal I. If X is contained in a nondegenerate variety of codimension q and (minimal) degree $q + 1$, then by part (b) of Corollary 4.2,

$$t_c(S/I) \leq t_{c-q}(S/I) + q + 1.$$

Example 4.4 (G. Caviglia [2004]). The principle of part (a) of Corollary 4.2 does not hold for individual steps in the resolution. For example, if

$$I = (x_1^3, \ldots, x_4^3, (x_1 + \cdots + x_4)^3) \subset S = K[x_1, \ldots, x_4],$$

then $t_1(S/I) = 3$ while $t_2(S/I) = 7 > 3 + 3$. Notice that I is $\mathbf{m}$-primary.

Here is a class of two-dimensional ideals that exhibit even more extreme behavior:

Example 4.5 (G. Caviglia [2004]). If

$$I = (x_1^r, x_2^r, x_1x_3^{r-1} - x_2x_4^{r-1}) \subset S = K[x_1, \dots, x_4],$$

then $t_1(S/I) = r$ while $t_2(S/I) = r^2 > 2r$ for $r \geq 3$, and in fact $\operatorname{reg}(S/I) = r^2 - 2$.

This example also shows that the dimension bound on $\operatorname{Tor}_1(A, B)$ is necessary in Corollary 2.5 and Corollary 3.1. Set $A = S/(x_1^r, x_2^r)$ and $B = S/(x_1x_3^{r-1} - x_2x_4^{r-1})$. For $r \geq 3$,

$$\operatorname{reg} \operatorname{Tor}_0(A, B) = \operatorname{reg}(S/I) = r^2 - 2 > \operatorname{reg}(A) + \operatorname{reg}(B) = 3r - 3.$$

In this case $\dim \operatorname{Tor}_1(A, B) = 2$.

5. Specialization and degrees of syzygies. As an application of Corollary 3.3 we give a bound for the saturation and regularity of a plane section, generalizing Theorem 1.2 of Eisenbud-Green-Hulek-Popescu [2004]:

COROLLARY 5.1. *Let $X \subset \mathbf{P}^{n-1}$ be a scheme, and let $\Lambda \subset \mathbf{P}^{n-1}$ be a linear subspace such that the sheaf $\operatorname{Tor}_1(\mathcal{O}_X, \mathcal{O}_\Lambda)$ is supported on a finite set. Let $I \subset S$ be any homogeneous ideal defining X, and let $L \subset S$ be the ideal of Λ.*

(a) *The restriction map*

$$I_d \to \mathrm{H}^0(\mathcal{I}_{X\cap\Lambda,\Lambda}(d))$$

is surjective for all $d \geq t_{\dim\Lambda}(I) - \dim\Lambda$.

(b) *Let c be the codimension of $X \cap \Lambda$ in Λ. We have*

$$\begin{aligned} \operatorname{reg}(\mathcal{I}_{X\cap\Lambda}) &= \operatorname{reg}\left(\frac{\mathcal{I}_X + I_\Lambda}{\mathcal{I}_\Lambda}\right) \\ &\leq \max\{t_p(I) - p \mid c - 1 \leq p \leq \dim\Lambda - 1\}. \end{aligned}$$

The hypothesis that the sheaf $\operatorname{Tor}_1(\mathcal{O}_X, \mathcal{O}_\Lambda)$ is supported on a finite set is satisfied for general Λ of any dimension, or for any Λ such that $X \cap \Lambda$ is finite.

Proof. By Corollary 2.5 we have

$$\begin{aligned} \operatorname{reg} \mathrm{H}^j_{\mathbf{m}}(S/(I + L)) &= \operatorname{reg} \mathrm{H}^j_{\mathbf{m}}(\operatorname{Tor}_0(S/I, S/L)) \\ &\leq t_{\dim(S/L)-j}(S/I) - \dim S/L \\ &= t_{\dim\Lambda-j}(I) - \dim\Lambda - 1 \\ &< t_{\dim\Lambda-j}(I) - \dim\Lambda. \end{aligned}$$

Taking $j = 0$ in the inequalities, we see that $I + L$ is saturated in degree d when $d \geq t_{\dim \Lambda}(I) - \dim \Lambda$, proving part (a). Adding j to both sides and taking the maximum over j for $1 \leq j \leq \dim S/(I + L) = \dim (X \cap \Lambda) + 1$ we see that

$$\begin{aligned} \operatorname{reg} \mathcal{I}_{X\cap\Lambda} &= \max_{1\leq j}\{\operatorname{reg} \oplus_m \mathrm{H}^j(\mathcal{I}_{X\cap\Lambda}(m)) + j + 1\} \\ &\leq \max_{1\leq j\leq \dim S/(I+L)}\{\operatorname{reg} \mathrm{H}^j_{\mathbf{m}}(S/(I+L)) + j + 1\} \\ &\leq \max_{1\leq j\leq \dim S/(I+L)}\{t_{\dim\Lambda - j}(I) - \dim\Lambda + j\}, \end{aligned}$$

which is the desired inequality. □

We say that the resolution of a finitely generated graded S-module A generated in a single degree d is *linear for q steps* if it has the form

$$\cdots \xrightarrow{S}{}^{n_q}(-d-q) \longrightarrow \cdots \longrightarrow S^{n_0}(-d) \longrightarrow A \longrightarrow 0.$$

COROLLARY 5.2. *Let $I \subset S$ be a homogeneous ideal, let p be an integer with $0 \leq p \leq \operatorname{projdim}(S/I)$, and set $m = t_p(S/I)$. Let $L \subset S$ be any ideal generated by $n - p$ linearly independent linear forms. If $I + L$ contains a power of $\mathbf{m}$ (which will always be true if K is infinite, L is general and $p \leq \operatorname{codim} I$) then $I + L$ contains $\mathbf{m}^{m-p+1}$, and more generally*

$$\mathbf{m}^{m-p+s} \subset I + L^s.$$

For example, if I is generated in degree d and the minimal free resolution of I is linear for $p - 1$ steps, then

$$\mathbf{m}^d \subset I + L.$$

Proof. The resolution of L^s is linear, as one can see by computing the degree of the socle of S/L^s (in fact, the resolution can be obtained as an Eagon-Northcott complex, see Eisenbud [1995, pg.600]). Hence $t_{n-p}(S/L^s) = n - p + s - 1$. As $p \leq \operatorname{codim} I$, Theorem 2.1 gives $\operatorname{reg} \mathrm{H}^0_{\mathbf{m}}(S/I \otimes S/L^s) \leq m - p + s - 1$, which is the asserted result. □

Notice that the containment $\mathbf{m}^d \subset I + L$ in Corollary 5.2 actually gives that I and $\mathbf{m}^d$ coincide modulo L.

COROLLARY 5.3. *Let $I \subset S$ be a homogeneous ideal, let $L \subset S$ be any ideal generated by $n - p$ linearly independent linear forms, and let "$\bar{}$" denote images in $\overline{S} = S/L$. If $\overline{I}$ is $\overline{\mathbf{m}}$-primary then $t_p^{\overline{S}}(\overline{S}/\overline{I}) \leq t_p^S(S/I)$.*

COROLLARY 5.4. *Suppose that $I \subset S$ is a homogeneous $\mathbf{m}$-primary ideal, and let* in I *denote the initial ideal of I with respect to the reverse lexicographic order*

on the monomials of S, with the variables ordered $x_1 > x_2 > \ldots > x_n$. *If* $0 \le p \le n$ *and* $m = t_p(S/I)$, *then*

$$(x_1, \ldots, x_p)^{m-p+1} \subset \operatorname{in} I.$$

In particular, if I is generated in degree d and the resolution of I is linear for $p-1$ *steps, then the initial ideal of I in reverse lexicographic order contains* $(x_1, \ldots, x_p)^d$.

Proof. Corollary 5.2 shows that $\mathbf{m}^{m-p+1} \subset I + L$, where $L = (x_{p+1}, \ldots, x_n)$. Because the monomial order is reverse lexicographic, $\operatorname{in}(I + L) = (\operatorname{in} I) + L$ (see Eisenbud [1995, Proposition 15.12]). Thus $\mathbf{m}^{m-p+1} \subset (\operatorname{in} I) + L$, whence $(x_1, \ldots, x_p)^{m-p+1} \subset \operatorname{in} I$. □

In the case where I is $\mathbf{m}$-primary and linearly presented, Corollary 5.4 says that $(x_1, x_2)^d \subset \operatorname{in} I$. In generic coordinates we hope for a stronger inclusion:

CONJECTURE 5.5. *Suppose that the ideal* $I \subset S$ *is* $\mathbf{m}$*-primary, linearly presented, and generated in degree d. If K is infinite, then*

$$\mathbf{m}^d \subset I + (z_3, \ldots, z_n)^2$$

for sufficiently general linear forms $z_3, \ldots, z_n$, *or even*

$$(z_1, z_2)^{d-1}\mathbf{m} \subset \operatorname{Gin} I,$$

where $\operatorname{Gin} I$ *denotes the reverse lexicographic initial ideal with respect to generic coordinates* $z_1, \ldots, z_n$. *If the resolution of I is linear for* $p-1$ *steps, then we similarly conjecture that*

$$\mathbf{m}^d \subset I + (z_{p+1}, \ldots, z_n)^2$$

for sufficiently general linear forms z_i.

We were led to this conjecture studying Conjecture 1.1. In case $n = 3$ and S/I is Gorenstein, Conjecture 5.5 would follow from the Strong Lefschetz property. We have observed it experimentally in a large number of cases with $n = 3$ and $n = 4$.

COROLLARY 5.6. *Suppose that K has characteristic zero and* $I \subset S$ *is a homogeneous* $\mathbf{m}$*-primary ideal. If I is generated in degree d and the resolution of I is linear for* $n-2$ *steps, then* $\mu(\operatorname{Gin} I) = \mu(\mathbf{m}^d)$.

Proof. Corollary 5.2 shows that $I + (z) = \mathbf{m}^d + (z)$ for every linear form z in S. But then $\mu(\operatorname{Gin} I) = \mu(\mathbf{m}^d)$ by Conca-Herzog-Hibi [2004, Corollary 3.4 (b)]. □

6. Ideals generated by quadrics. If an ideal I generated in degree d has a resolution that is linear for q steps, then by Corollary 5.2 we have $\mathbf{m}^d \subset I + (z_{q+2}, \ldots, z_n)$ for every set of linearly independent linear forms $z_{q+2}, \ldots, z_n$. For ideals generated by quadrics, this latter condition is easy to interpret. For simplicity we assume throughout this section that the base field K is algebraically closed of characteristic not 2. We will identify a quadric and its associated symmetric bilinear form.

Recall that a m-dimensional vector space of quadrics in n variables (with a basis) can be described by a symmetric $n \times n$ matrix of linear forms in m variables; to get the symmetric matrix corresponding to the i-th quadric, just set all but the i-th variable equal to 0, and set the i-th variable equal to 1. We call a symmetric matrix of linear forms in m variables *symmetrically q-generic* if every generalized principal $(q+1) \times (q+1)$ submatrix has linearly independent entries on and above the diagonal (here a principal submatrix is one involving the same rows as columns, and a generalized submatrix of A is a submatrix of PAP^* for some invertible matrix P). These definitions are adapted from the notion of k-generic matrices in Eisenbud [1988]. In particular, symmetrically 1-generic matrices are the same as 1-generic matrices that happen to be symmetric. We say that a family of quadrics is *q-generic* if the corresponding matrix of linear forms is symmetrically q-generic.

It is convenient for our purpose to specify a space of quadrics via its orthogonal complement. A symmetric matrix A representing a quadric may be thought of as a linear transformation $A\colon\ W \to W^*$. The dual of the vector space $\mathrm{Hom}(W, W^*)$ is $\mathrm{Hom}(W^*, W)$ via the pairing $(A, B) = \mathrm{TraceAB}$. What this means in practice for symmetric matrices $A = (a_{ij})$, $B = (b_{ij})$ is that $(A, B) = \sum_{i,j} a_{ij} b_{ij}$. Thus from a space of (quadratic or) bilinear forms U we can construct a space $U^\perp$ of (quadratic or) bilinear forms. This is the degree 2 part of the the "annihilator ideal" that appears for example in Eisenbud [1995, Section 21.2].

The orthogonal complement construction allows us to give examples of q-generic families of quadrics for all q:

PROPOSITION 6.1. *A nonzero quadratic form Q has rank $\geq q+2$ if and only if the family $(Q)^\perp$ of quadratic forms orthogonal to Q is q-generic.*

Proof. If Q has rank $\leq q+1$ then, after a change of variables, Q will be represented by a diagonal matrix with at most $q+1$ nonzero entries. It follows that the matrices in $(Q)^\perp$ satisfy a nontrivial linear equation among the entries of some $(q+1) \times (q+1)$ principal submatrix, so the family is not $(q+1)$-generic.

Conversely, suppose the family $V = (Q)^\perp$ is not q-generic. In this case the symmetric matrix of linear forms corresponding to V has a $(q+1) \times (q+1)$ generalized principal submatrix whose entries on or above the diagonal are linearly dependent. The coefficients of this dependency relation define a nonzero quadratic form Q' of rank at most $q+1$ so that $V \subset (Q')^\perp$. Since both sides are

codimension 1 in S_2, they are equal, and it follows that Q' and Q generate the same 1-dimensional subspace. In particular they have the same rank. □

PROPOSITION 6.2. *Let $V \subset S_2$ be a vector space of quadrics in n variables. The ideal I generated by V has the property that $\mathbf{m}^2 \subset I + (z_{q+2}, \dots, z_n)$ for every set of linearly independent linear forms $z_{q+2}, \dots, z_n$ if and only if V is q-generic.*

Proof. Let A be the symmetric matrix of linear forms associated to V. The space of quadratic forms $\overline{V} \subset (S/(z_{q+2}, \dots, z_n))_2$ corresponds to the $(q+1) \times (q+1)$ generalized submatrix of A obtained by leaving out rows and columns corresponding to the linear forms z_i. Its $\binom{q+2}{2}$ entries on and above the diagonal are linearly independent if and only if it corresponds to a space of quadrics of dimension $\binom{q+2}{2}$, which is the dimension of $(S/(z_{q+2}, \dots, z_n))_2$. □

COROLLARY 6.3. *If the ideal I generated by m quadratic forms in n variables is $\mathbf{m}$-primary and satisfies $\mathbf{m}^2 \subset I + (z_3, \dots, z_n)$ for every set of independent linear forms $z_3, \dots, z_n$, then $m \geq 2n - 1$.*

Proof. The entries of a 1-generic $n \times n$ matrix must span a space of at least dimension $2n - 1$; see Eisenbud [1988, Proposition 1.3]. □

Example. The "catalecticant" (or Hankel) matrix

$$\begin{pmatrix} y_1 & y_2 & y_3 & \cdots \\ y_2 & y_3 & \cdots & \\ y_3 & \cdots & & \\ \vdots & & & \end{pmatrix}$$

is a symmetrically 1-generic matrix representing a $2n - 1$ dimensional space of quadrics.

COROLLARY 6.4. *Let $V \subset S_2$ be a vector space of quadrics. If V is not q-generic then the ideal I generated by V has a free resolution with at most $q - 1$ linear steps.*

In case V has codimension 1 in the space of all quadrics, Corollary 6.4 is sharp:

PROPOSITION 6.5. *Let $V \subset S_2$ be a codimension 1 subspace of the quadratic forms of S. The ideal generated by V has q linear steps in its resolution if and only if V is q-generic.*

Proof. Let Q be a quadratic form generating the orthogonal complement of V. Suppose that the rank of Q is $q + 2$. By Proposition 6.1 and Corollary 6.4, it suffices to show that the resolution of $I = (V)$ has q linear steps.

Let J be the annihilator of Q in the sense of Eisenbud [1995, Section 21.2]. Thus S/J is Gorenstein with "dual socle generator Q", and J contains exactly $n - q - 2$ independent linear forms.

If $q+2=n$, the resolution of S/J has the form

$$0 \to S(-n-2) \longrightarrow \oplus S(-n) \longrightarrow \cdots \longrightarrow \oplus S(-2) \longrightarrow S,$$

showing that $J=I$ and proving the proposition in this case.

For arbitrary q we see that the resolution of S/J is the tensor product of a Koszul complex on $n-q-2$ linear forms with a resolution of S/J', where S/J' is Gorenstein of codimension $q+2$ and has resolution similar to the one above. In particular, J is generated in degrees 1 and 2, so I may be written as $I=J\cap \mathbf{m}^2$. Hence the truncation principle Proposition 1.6 shows that S/I has q linear steps in its resolution as required. □

Using the theory of matrix pencils, it should be possible to analyze all the complements of codimension two sets of quadrics.

7. Regularity of products and powers. There has been considerable recent progress on the general subject of regularity bounds for powers of an ideal; for example see Trung-Wang [2002] and the references cited there.

In this section we give our results on Conjecture 1.4. We prove that *some* power of a linearly presented ideal I coincides with a power of $\mathbf{m}$, and that in case the resolution of I is linear for at least $\lceil (n-1)/2\rceil$ steps, then I^2 is a power of $\mathbf{m}$. We can also give some weak numerical evidence related to the number of generators of I. This section is devoted to these and related more general results.

THEOREM 7.1. *If $I\subset S=K[x_1,\dots,x_n]$ is a linearly presented $\mathbf{m}$-primary ideal generated in degree d (or, when the ground field is algebraically closed, if $\mathbf{m}^d\subset I+(z_3,\dots,z_n)$ for all sequences of $n-2$ linearly independent linear forms $z_3,\dots,z_n$), then $I^t=\mathbf{m}^{dt}$ for all $t\gg 0$.*

We will use the following criterion:

PROPOSITION 7.2. *Let $I\subset S$ be an ideal generated by a vector space $V\subset S_d$ for some d. If $I^s=\mathbf{m}^{ds}$ for some $s\geq 1$, then $I^t=\mathbf{m}^{dt}$ for all $t\geq s$. This condition is satisfied if and only if the linear series $|V|$ maps $\mathbf{P}^{n-1}$ isomorphically to its image in $\mathbf{P}(V)$.*

Proof of Proposition 7.2. To prove the first assertion it suffices, by induction, to treat the case $t=s+1$. Suppose that $I^s=\mathbf{m}^{ds}$. Since $I\subset \mathbf{m}^d$ we get $I\mathbf{m}^{d(s-1)}=\mathbf{m}^{ds}$. Thus $I^{s+1}=II^s=I\mathbf{m}^{ds}=I\mathbf{m}^{d(s-1)}\mathbf{m}^d=\mathbf{m}^{ds}\mathbf{m}^d=\mathbf{m}^{d(s+1)}$, as required.

To prove the last assertion, note that the image of $\mathbf{P}^{n-1}$ under the map ϕ defined by the linear series $|V|$ is by definition the variety with homogeneous coordinate ring

$$\bigoplus_t (V)^t \subset \bigoplus_t S_{dt}.$$

To say that ϕ is an isomorphism onto its image means that these two rings are equal in high degree; that is, $(V)^t = S_d$, or equivalently $I^t = \mathbf{m}^{dt}$ for large t. □

Proof of Theorem 7.1. We can harmlessly extend the ground field and assume that it is algebraically closed.

By Proposition 7.2 it suffices to show that the map ϕ defined by the linear series $|V|$ is an isomorphism. For this it is even enough to show that the restriction of ϕ to any line is an isomorphism: There is a line through any two points of $\mathbf{P}^{n-1}$ and a line containing any tangent vector to a point of $\mathbf{P}^{n-1}$, so if ϕ restricts to an isomorphism on each line then ϕ is one-to-one and unramified, whence an isomorphism.

A line $\ell \subset \mathbf{P}^{n-1}$ is defined by an ideal generated by the vanishing of $n-2$ linear forms, say $z_3, \dots, z_n$. The restriction $\phi|_\ell$ of ϕ to ℓ is defined by the degree d component of the ideal $(I+(z_3, \dots, z_n))/(z_3, \dots, z_n)$. By Corollary 5.2, this ideal equals $(\mathbf{m}^d + (z_3, \dots, z_n))/(z_3, \dots, z_n)$, so $\phi|_\ell$ is defined by the complete linear series of degree d, which is an isomorphism as required. □

To give the results about Conjecture 1.4 in their natural generality, we turn to the regularity of the product of two ideals.

The following fact was proved (in a superficially more special case) by Jessica Sidman [2002]:

THEOREM 7.3. *Suppose that I and J are homogeneous ideals of S and set $\delta = \dim \operatorname{Tor}_1(S/I, S/J)$. If $j \geq \delta$, then*

$$\operatorname{reg} H^j_{\mathbf{m}}(IJ) + j \leq \operatorname{reg} I + \operatorname{reg} J.$$

Thus if $\delta \leq 1$ then $\operatorname{reg} IJ \leq \operatorname{reg} I + \operatorname{reg} J$, and if $\delta \leq 2$ then $\operatorname{reg}(IJ)^{\mathrm{sat}} \leq \operatorname{reg} I + \operatorname{reg} J$.

Since $\operatorname{Tor}_1(A, B) = (I \cap J)/IJ$, the condition $\dim \operatorname{Tor}_1(A, B) \leq 1$ of Corollary 3.1 may be interpreted as saying that the codimension of $(IJ)_d$ in $(I \cap J)_d$ is bounded independently of d. Thinking of I, J as determining projective schemes $X, Y \subset \mathbf{P}^{n-1}$, we may also state the condition as saying that X and Y are "homologically transverse" except at a finite set of points of $\mathbf{P}^{n-1}$.

Proof. Extending the ground field if necessary, we may assume it is infinite. A general linear form is then annihilated only by an ideal of finite length modulo I, J, IJ or $I+J$. If $\delta \geq 2$ then factoring out such a general form, the left hand side of the displayed inequality can only increase and the right hand side can only decrease. Thus it suffices to treat the case $\delta \leq 1$.

Consider the exact sequence

$$0 \to IJ \to I \to I/IJ \to 0.$$

Note that $I/IJ = \operatorname{Tor}_0(I, S/J)$ and that $\operatorname{Tor}_1(I, S/J) = \operatorname{Tor}_2(S/I, S/J)$ has dimension at most $\delta \leq 1$ according to Auslander [1961]. By Corollary 3.1,

$\operatorname{reg} \operatorname{Tor}_0(I, S/J) \leq \operatorname{reg} I + \operatorname{reg} S/J$, and therefore

$$\begin{aligned} \operatorname{reg} IJ &\leq \max\{\operatorname{reg} I, \operatorname{reg} I/IJ + 1\} \\ &= \max\{\operatorname{reg} I, \operatorname{reg} I + \operatorname{reg} S/J + 1\} \\ &= \operatorname{reg} I + \operatorname{reg} J. \end{aligned}$$

□

THEOREM 7.4. *Suppose that I and J are homogeneous ideals of S with* $\dim \operatorname{Tor}_1(S/I, S/J) \leq 1$. *If p, q are integers such that $p \leq \operatorname{codim} I$, $q \leq \operatorname{codim} J$ and $p + q = n + 1$, then*

$$\operatorname{reg} IJ \leq \max\{\operatorname{reg} I,\ \operatorname{reg} J,\ t_p(S/I) + t_q(S/J) - n + 1\}.$$

Proof. From the short exact sequences

$$0 \to (I \cap J)/IJ \to S/IJ \to S/(I \cap J) \to 0$$
$$0 \to S/(I \cap J) \to S/I \oplus S/J \to S/(I + J) \to 0$$

we see that

$$\begin{aligned} \operatorname{reg} S/IJ &\leq \max\{\operatorname{reg} S/(I \cap J),\ \operatorname{reg} (I \cap J)/IJ\} \\ &\leq \max\{\operatorname{reg} S/I,\ \operatorname{reg} S/J,\ 1 + \operatorname{reg} S/(I + J),\ \operatorname{reg} (I \cap J)/IJ\}. \end{aligned}$$

Notice that $S/(I + J) = \operatorname{Tor}_0(S/I, S/J)$ and $(I \cap J)/IJ = \operatorname{Tor}_1(S/I, S/J)$. To bound the regularity of these modules we apply Corollary 3.2 with $0 \leq k \leq 1$.

From the hypothesis we see that $1 + \dim S/J \leq p \leq \operatorname{codim} I$. Hence by Corollary 3.2,

$$\operatorname{reg} \operatorname{Tor}_0(S/I, S/J) \leq t_p(S/I) + t_{q-1}(S/J) - n$$

and

$$\operatorname{reg} \operatorname{Tor}_1(S/I, S/J) \leq t_p(S/I) + t_q(S/J) - n.$$

Using the inequalities above, we obtain

$$\begin{aligned} \operatorname{reg} S/IJ \leq \max\{&\operatorname{reg} S/I,\ \operatorname{reg} S/J,\ 1 + t_p(S/I) + t_{q-1}(S/J) - n, \\ &t_p(S/I) + t_q(S/J) - n\}. \end{aligned}$$

Because $q \leq \operatorname{codim} S/J$ we have

$$t_{q-1}(S/J) \leq t_q(S/J) - 1.$$

Thus

$$\operatorname{reg} S/IJ \leq \max\{\operatorname{reg} S/I,\ \operatorname{reg} S/J,\ t_p(S/I) + t_q(S/J) - n\},$$

as required. □

COROLLARY 7.5. *Suppose that I and J are homogeneous ideals of S. If either* $\dim S/J = 0$ *and I is generated in degrees at most d, or* $\dim S/J = 1$ *and I is related in degrees at most $d+1$, then*

$$\operatorname{reg} IJ \leq \max\{\operatorname{reg} I, d + \operatorname{reg} J\}.$$

Proof. We may assume $I \neq 0$, and dividing I by its greatest common divisor we may then suppose that $\operatorname{codim} I \geq 2$. Now apply Theorem 7.4 with $p = 1$ in the first case, and $p = 2$ in the second case. □

COROLLARY 7.6. *Suppose that I and J are homogeneous ideals in S of dimension ≤ 1, generated in degree d. If the resolutions of I and J are linear for $\lceil (n-1)/2 \rceil$ steps (for instance if I and J have linear presentation and $n \leq 3$), then IJ has linear resolution. In particular, if I and J are $\mathbf{m}$-primary then $IJ = \mathbf{m}^{2d}$.*

Proof. Applying Corollary 3.2 with $k = 0$ we get

$$\operatorname{reg} S/I = \operatorname{reg} \operatorname{Tor}_0(S/I, S/I) \leq 2d - 2,$$

and similarly for $\operatorname{reg} S/J$. From Theorem 7.4 with $p = \lceil (n+1)/2 \rceil$ we see that $\operatorname{reg} IJ \leq 2d$. Since IJ is generated in degree $2d$, it follows that IJ has linear resolution. □

Taking $I = J$ we get the special case $s = \lceil (n-1)/2 \rceil$ of Conjecture 1.4:

COROLLARY 7.7. *Suppose that $I \subset S$ is a homogeneous ideal of dimension ≤ 1, generated in degree d. If the resolution of I is linear for $\lceil (n-1)/2 \rceil$ steps (for instance if I has linear presentation and $n \leq 3$), then I^t has linear resolution for all $t \geq 2$. In particular, if I is $\mathbf{m}$-primary then $I^2 = \mathbf{m}^{2d}$.*

If $I \subset S$ is an $\mathbf{m}$-primary ideal generated in degrees $\leq d$ then $\operatorname{reg} I^t \leq \operatorname{reg} I + (t-1)d$. (Reason: Write $e = \operatorname{reg} I$. Since $\mathbf{m}^e \subset I$, we have $\mathbf{m}^e \subset \mathbf{m}^{e-d} I$ and thus $\mathbf{m}^{e+(t-1)d} \subset \mathbf{m}^{e+(t-2)d} I$. Induction on t completes the argument.) But we can prove a little more. The following result is also a generalization of Corollary 7.7.

COROLLARY 7.8. *Let $I \subset S$ be a homogeneous ideal and let $t \geq 2$ be an integer. If* $\dim S/I = 0$ *and I is generated in degrees at most d, or* $\dim S/I = 1$ *and I is related in degrees at most $d+1$, then* $\operatorname{reg} I^t \leq \operatorname{reg} I + (t-1)d$. *More generally, for* $1 + \dim S/I \leq p \leq \operatorname{codim} I$,

$$\operatorname{reg} I^t \leq t_{p-1}(I) + t_{n-p}(I) - n + (t-2)d + 1.$$

Proof. We use induction on $t \geq 2$. We may assume that $\operatorname{codim} I \geq 2$. Corollary 3.2 shows that

$$\begin{aligned} \operatorname{reg} S/I &= \operatorname{reg} \operatorname{Tor}_0(S/I, S/I) \\ &\leq t_p(S/I) + t_{n-p}(S/I) - n \\ &< t_p(S/I) + t_{n+1-p}(S/I) - n, \end{aligned}$$

where the last inequality holds because $n + 1 - p \leq \operatorname{codim} I$. Similarly,

$$\begin{aligned} \operatorname{reg} I^{t-1}/I^t &= \operatorname{reg} \operatorname{Tor}_1(S/I, S/I^{t-1}) \\ &\leq t_p(S/I) + t_{n+1-p}(S/I^{t-1}) - n. \end{aligned}$$

Hence the exact sequence

$$0 \to I^{t-1}/I^t \to S/I^t \to S/I^{t-1} \to 0$$

and an induction on $t \geq 2$ show that

$$\begin{aligned} \operatorname{reg} S/I^t &\leq t_p(S/I) + t_{n+1-p}(S/I^{t-1}) - n \\ &\leq t_p(S/I) + \operatorname{reg} S/I^{t-1} + 1 - p. \end{aligned}$$

The base case $t = 2$ of the present corollary now follows from the first inequality. The induction step uses the second equality with $p = 1$ or $p = 2$, depending on whether $\dim S/I = 0$ and I is generated in degrees $\leq d$, or $\dim S/I = 1$ and I is related in degrees $\leq d + 1$. □

COROLLARY 7.9. *Suppose that $I \subset S$ is a homogeneous ideal of dimension ≤ 1. If I is generated in degree d and has linear presentation, and if some positive power of I has linear resolution, then all higher powers of I have linear resolution.*

Proof. We may assume that $\operatorname{codim} I \geq 2$. If I^{t-1} has linear resolution for some $t \geq 2$, the proof of Corollary 7.8 shows that

$$\operatorname{reg} S/I^t \leq d + \operatorname{reg} S/I^{t-1} = td - 1. \qquad \square$$

No such result holds for 2-dimensional ideals in $S = K[x_1, \ldots, x_4]$: Aldo Conca [2003] has shown for the ideal $I = (x_1 x_2^r, x_1 x_3^r, x_2^{r-1} x_3 x_4) + x_2 x_3 (x_2, x_3)^{r-1}$ with $r > 1$ that I^t has linear resolution for $t < r$, whereas I^r is not even linearly presented. See also Sturmfels [2000].

For a long time the authors believed that the powers of linearly presented **m**-primary ideals would also be linearly presented. Sadly this is not the case as the following example shows. We discovered this example through an analysis

of an example of Mike Stillman's. He found a linearly presented $\mathbf{m}$-primary ideal whose square does not have a resolution with two linear steps. Our example shows even more extreme behaviour.

Example 7.10. Let $S = K[x_1, \dots, x_8]$ and

$$J = (x_1^3, \dots, x_8^3, x_1x_2(x_3 - x_4), x_3x_4(x_5 - x_6), x_5x_6(x_7 - x_8), x_7x_8(x_1 - x_2)).$$

One has $t_2(S/J) = 6$ and $t_2(S/J^2) = 12$. The truncation principle Proposition 1.6 then shows that $I = \mathbf{m}^2J$ has a linear presentation, but I^2 is not even linearly presented. Note that I is $\mathbf{m}$-primary.

By comparing the number of generators of $\mathbf{m}^{d(n-1)}$ with the number of generators of the $(n-1)$ symmetric power of I, we see that Conjecture 1.1 implies that the minimal number of generators $\mu(I)$ is at least $d(n-1)+1$. This is exactly the number of generators of $(x_1, x_2)^{d-1}\mathbf{m}$ (Conjecture 5.5 would give a more precise version.)

The following proposition, when combined with Corollary 5.2, provides further numerical evidence.

PROPOSITION 7.11. *Let $I \subset S$ be an $\mathbf{m}$-primary ideal generated by m forms of degree d. If $\mathbf{m}^d \subset I + L$ for every ideal L generated by $n - p$ independent linear forms, then*

$$m \geq p(n-p) + \binom{p-1+d}{d}.$$

For example, if $p = 2, n = 3$ then $m \geq d + 3$, while if $p = 2, d = 2$ then $m \geq 2n - 1$ (see also Corollary 6.3).

Proof. Let $W = S_1$ be the vector space of linear forms in S, and let $V = I_d \subset S_d$. Consider the natural composite map of vector bundles on the Grassmannian G of $n - p$ dimensional subspaces Λ

$$V \to \operatorname{Sym}_d(W) \to \operatorname{Sym}_d(W/\Lambda).$$

The hypothesis implies that this map is locally everywhere surjective. Because $\operatorname{Sym}_d(W/\Lambda)$ is ample (see Hartshorne [1970, Chapter 3]) the theorem of Fulton and Lazarsfeld [1981] requires that $\dim G < \dim V - \operatorname{rank} \operatorname{Sym}_d(W/\Lambda) + 1$, which is the desired inequality. □

We finish this section with a remark about Rees algebras and reduction numbers. Recall that if $J \subset I$ are ideals of S, then the *reduction number* $\mathrm{r}_J(I)$ of I with respect to J is the smallest integer $0 \leq r \leq \infty$ with $I^{r+1} = JI^r$.

COROLLARY 7.12. *Let $I \subset S$ be a homogeneous $\mathbf{m}$-primary ideal generated in degree d and assume that $I \neq \mathbf{m}^d$.*

(a) *If I has linear presentation, then* $\operatorname{depth} \mathcal{R}(I) = 1$.

(b) *If the resolution of I is linear for $\lceil (n-1)/2 \rceil$ steps, then* $\mathrm{r}_J(I) = \max\{2, n - 1 - \lfloor (n-1)/d \rfloor\}$ *for every $\mathbf{m}$-primary ideal $J \subset I$ generated by n forms of degree d.*

Proof. (a) Consider the exact sequence of finitely generated $\mathcal{R}(I)$-modules

$$0 \longrightarrow \mathcal{R}(I) \longrightarrow \mathcal{R}(\mathbf{m}^d) \longrightarrow C \longrightarrow 0.$$

The module $C \neq 0$ has finite length by Theorem 7.1, showing that $\operatorname{depth} \mathcal{R}(I) = 1$.

(b) Since $\mathcal{R}(I)$ is not Cohen-Macaulay and $n \geq 2$, one has $\mathrm{r}_J(I) \geq 2$ according to Valabrega-Valla [1978, Proposition 3.1] and Goto-Shimoda [1982, Remark 3.10]. On the other hand, $I^t = \mathbf{m}^{dt}$ for every $t \geq 2$ by Corollary 7.7. Therefore

$$\mathrm{r}_J(I) = \max\{2, \mathrm{r}_J(\mathbf{m}^d)\}.$$

It remains to see that $\mathrm{r}_J(\mathbf{m}^d) = e := n - 1 - \lfloor (n-1)/d \rfloor$. As $\operatorname{reg} S/J = n(d-1)$ it follows that $\mathbf{m}^{de} \not\subset J$, whereas $\mathbf{m}^{d(e+1)} \subset J$ and hence $\mathbf{m}^{d(e+1)} = J\mathbf{m}^{de}$. Thus indeed $\mathrm{r}_J(\mathbf{m}^d) = e$. □

8. Monomial ideals. In this section we will prove Conjecture 1.4 for monomial ideals, and give a necessary and sufficient condition for a monomial ideal to satisfy the asymptotic version.

THEOREM 8.1. *Let $I \subset S = K[x_1, \ldots, x_n]$ be a linearly presented $\mathbf{m}$-primary monomial ideal, generated in degree d. If the minimal resolution of I is linear for q steps then $I^t = \mathbf{m}^{dt}$ for all $t \geq (n-1)/q$.*

Theorem 8.1 follows at once from the next two results:

PROPOSITION 8.2. *If I is an $\mathbf{m}$-primary monomial ideal that is generated in degree d and has linear resolution for q steps, then I contains the ideal*

$$J(d, q) = \sum_{i_1 < \cdots < i_{q+1}} (x_{i_1}, \ldots, x_{i_{q+1}})^d.$$

Proof. Since I is its own initial ideal, in any monomial order, the statement follows from Corollary 5.4. □

PROPOSITION 8.3. *For all $t \geq 1$, $J(td, tq) \subset J(d, q)^t$. In particular, if $t \geq \frac{n-1}{q}$ then $J(d, q)^t = \mathbf{m}^{dt}$.*

Proof. The second statement follows from the first because $J(d, q) = \mathbf{m}^d$ for $q \geq n - 1$.

By induction on i, it suffices to show that

$$J(i\,d, i\,q) \subset J(d,q) \cdot J((i-1)d, (i-1)q).$$

To this end, let $m = \prod x_j^{a_j} \in J(id, iq)$ be a monomial of degree id. By the definition of $J(id, iq)$, at most $iq+1$ of the a_j are nonzero. To simplify the notation we assume that $a_j = 0$ for $j > iq + 1$.

Not every sum of q of the $a_1, \ldots, a_{iq+1}$ can be strictly bigger than d; otherwise $id = \sum_j a_j \geq (d+1)i$, a contradiction. Choose q of the a_j whose sum σ is maximal with respect to being at most d. By relabeling we may assume these are $a_1, \ldots, a_q$.

Suppose first that there is no index $k > q$ such that $\sigma + a_k \geq d$. It follows from the maximality of σ, that $a_k \leq a_j$ whenever $j \leq q < k$. From this we see that the sum of any $q + 1$ of the a_j is at most $d - 1$. But then

$$id = \sum_{j=1}^{iq+1} a_j \leq (d-1) \left\lceil \frac{iq+1}{q+1} \right\rceil \leq (d-1)i,$$

a contradiction.

Thus there exists an index $k > q$ such that $\sigma + a_k \geq d$. It follows that $u := x_1^{a_1} \cdots x_q^{a_q} x_k^{d-\sigma} \in J(d,q)$, while $v := m/u \in J((i-1)d, (i-1)q)$, as required. □

Here is a criterion for the asymptotic version to hold.

PROPOSITION 8.4. *An* $\mathbf{m}$*-primary monomial ideal* $I \subset S$ *generated in degree* d *has a power equal to a power of* $\mathbf{m}$ *if and only if* $J := \mathbf{m}(x_1^{d-1}, \ldots, x_n^{d-1}) \subset I$. *Further,* $J^t = \mathbf{m}^{dt}$ *if and only if* $t \geq (d-2)(n-1)$.

The second statement is shown in the course of the proof of Herzog and Hibi [2003, Theorem 1.1] (the original formulation is for any $\mathbf{m}$-primary ideal $\mathbf{m}J'$ with J' generated in degree $d-1$). We include a proof for the reader's convenience.

Proof. First consider $J = \mathbf{m}J'$, where $J' = (x_1^{d-1}, \ldots, x_n^{d-1})$. The tth power of J' has resolution obtained from that of the t-th power of $\mathbf{m}$ by substituting x_i^{d-1} for x_i. Thus the regularity of J'^t is precisely $(d-1)(t+n-1) - n$, so J'^t contains $\mathbf{m}^{(d-1)(t+n-1)-n+1}$ but no lower power. Since the generators of J'^t have degree $(d-1)t$, we see that $J^t = \mathbf{m}^t J'^t = \mathbf{m}^{dt}$ if and only if $t \geq (d-1)(t+n-1) - n + 1 - (d-1)t$, that is, $t \geq (d-2)(n-1)$.

It remains to show if I has a power equal to a power of $\mathbf{m}$, then $J \subset I$. For $V = I_d$ the map ϕ defined by $|V|$ defines an isomorphism according to Proposition 7.2. Thus, writing $P = (x_1, \ldots, x_{n-1})$ and $Q = P \cap K[I_d]$, we have an equality of homogeneous localizations $S_{(P)} = K[I_d]_{(Q)}$. Therefore I must contain $x_i x_n^{d-1}$ for each i, and likewise $x_i x_j^{d-1}$ for every i, j. □

9. Torsion in symmetric and exterior powers. In general it is a difficult problem to understand the relations defining the *Rees algebra* $\mathcal{R}(I) := S \oplus I \oplus I^2 \oplus \cdots$ of an ideal $I \subset S$. As a start, we may write $\mathcal{R}(I)$ as a homomorphic image $\operatorname{Sym}(I)/\mathcal{A}$ of the symmetric algebra $\operatorname{Sym}(I)$. The relations defining $\operatorname{Sym}(I)$ are easily derived from the first syzygies of I: if $G_1 \to G_0 \to I \to 0$ is a free presentation, then $\operatorname{Sym}(I) = \operatorname{Sym}(G_0)/G_1 \operatorname{Sym}(G_0)$. That is, the defining ideal of $\operatorname{Sym}(I)$ in the polynomial ring $\operatorname{Sym}(G_0)$ is generated by the image of G_1, regarded as a space of forms that are linear in the variables corresponding to generators of G_0.

Thus the problem is to understand $\mathcal{A}$. Let $\mathcal{A}_t$ be the component of $\mathcal{A}$ in $\operatorname{Sym}_t(I)$, so that $\mathcal{A} = \oplus_{t\geq 2}\mathcal{A}_t$. It is easy to see that $\mathcal{A}_t$ is the S-torsion submodule of $\operatorname{Sym}_t(I)$. In this section we will study the regularity of $\mathcal{A}_t$ in the case where I is a homogeneous $\mathbf{m}$-primary ideal.

An ideal I is said to be of *linear type* if $\mathcal{A} = 0$. Following Herzog, Hibi and Vladoiu [2003] we say more generally that I is of *fiber type* if a generating set of relations of the *fiber ring* $\mathcal{R}(I)/\mathbf{m}\mathcal{R}(I)$ lifts to a generating set for $\mathcal{A}$. If I is generated by forms of degree d, then all the generators of $\mathcal{A}_t$ have degrees $\geq dt$. The simplest situation occurs when the regularity of $\mathcal{A}_t$ is dt.

THEOREM 9.1. *Let $I \subset S$ be a homogeneous $\mathbf{m}$-primary ideal.*

(a) *If I is generated in degrees at most d and related in degrees at most $e+1$, then* $\operatorname{reg}\mathcal{A}_t \leq \operatorname{reg} I + (t-2)d + e$ *for every t.*

(b) *Suppose that I is generated in degree d and has linear presentation. Let $s \geq 1$ be an integer such that $I^s = \mathbf{m}^{ds}$. We have* $\operatorname{reg}\mathcal{A}_{s+u} \leq \max\{\operatorname{reg}\mathcal{A}_s, sd\} + ud$ *for every $u \geq 0$.*

(c) *If the resolution of I is linear for $\lceil n/2 \rceil$ steps, then $\mathcal{A}_t$ is concentrated in degree dt for every t; in particular, I is of fiber type and $\mathcal{A}$ is annihilated by $\mathbf{m}$.*

In the course of their study of implicitization of surfaces, Busé and Jouanolou [2003, Proposition 5.5] proved a different bound for the torsion in the symmetric algebra $\operatorname{Sym}(I)$ for ideals I of dimension ≤ 1. This was later sharpened by Busé and Chardin [2003]. (Although the result was originally stated only for ideals with $n+1$ generators, this restriction is irrelevant. A forthcoming paper of Chardin will contain further generalizations.)

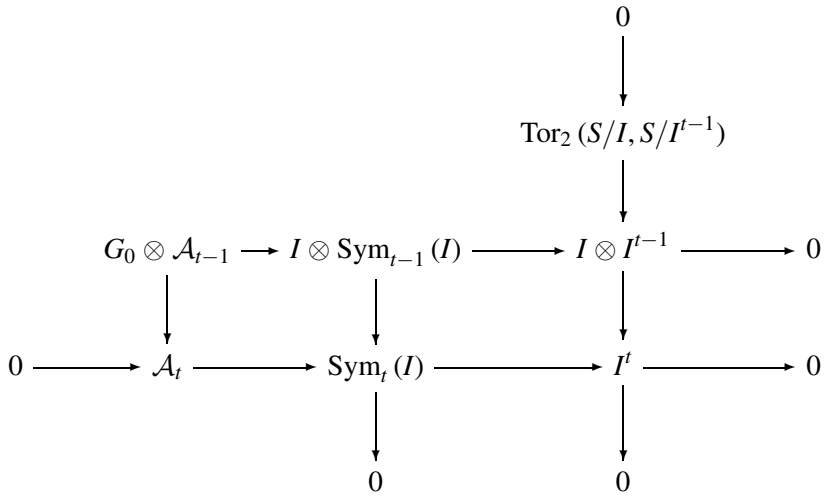
Our proof of Theorem 9.1 is based on a more general lemma:

LEMMA 9.2. *If $I \subset S$ is a homogeneous $\mathbf{m}$-primary ideal generated in degrees at most d then*

$$\operatorname{reg}\mathcal{A}_t \leq \max\{d + \operatorname{reg}\mathcal{A}_{t-1},\ \operatorname{reg}\operatorname{Tor}_2(S/I, S/I^{t-1})\}.$$

Proof of Lemma 9.2. Let $G_1 \to G_0 \to I$ be a minimal homogeneous free presentation, so that G_0 is generated in degrees $\leq d$. There is a commutative

diagram with exact rows and columns of the form



where the left most vertical map is given by the $\mathrm{Sym}(G_0)$-module structure on $\mathrm{Sym}(I)$. The Snake Lemma shows that $\mathcal{A}_t$ is an extension of a quotient of $G_0 \otimes \mathcal{A}_{t-1}$ by a quotient of $\mathrm{Tor}_2(S/I, S/I^{t-1})$. Since both these modules have finite length, the regularity of such an extension is bounded by the maximum of the two regularities as required. □

Proof of Theorem 9.1. We may assume that $n \geq 2$.

(a) We do induction on t. If $t \leq 1$ then $\mathcal{A}_t = 0$, so the assertion is trivial. For $t \geq 2$ we apply Lemma 9.2, and it suffices to prove $\operatorname{reg} \mathrm{Tor}_2(S/I, S/I^{t-1}) \leq \operatorname{reg} I + (t-2)d + e$. From Theorem 2.1 with $p = 2$ we obtain

$$\operatorname{reg} \mathrm{Tor}_2(S/I, S/I^{t-1}) \leq t_2(S/I) + t_n(S/I^{t-1}) - n \leq e + 1 + \operatorname{reg} S/I^{t-1}.$$

Hence $\operatorname{reg} \mathrm{Tor}_2(S/I, S/I^{t-1}) \leq \operatorname{reg} I^{t-1} + e \leq \operatorname{reg} I + (t-2)d + e$ by Corollary 7.8, as required.

(b) The same argument works, but this time we start the induction from $t = s$, and use the fact that $\operatorname{reg} I^{s+u-1} = (s+u-1)d$ for $u \geq 1$ according to Proposition 7.2.

(c) By Corollary 7.7 we know that $I^2 = \mathbf{m}^{2d}$, and from Lemma 9.2 we have $\operatorname{reg} \mathcal{A}_2 \leq \operatorname{reg} \mathrm{Tor}_2(S/I, S/I)$. By Theorem 2.1 with $p = \lceil n/2 \rceil + 1$ we obtain $\operatorname{reg} \mathrm{Tor}_2(S/I, S/I) \leq 2d$. Thus we can apply part (b) with $s = 2$ to obtain the desired result. □

Example 9.3. The conclusion of Theorem 9.1(c) does not hold for linearly presented monomial ideals in 3 variables. For example, let I be the ideal in $S = K[x_1, x_2, x_3]$ generated by all the monomials of degree 5 except $x_1^3x_2x_3, x_1x_2^3x_3, x_1x_2x_3^3$. The ideal I is linearly presented, but (since it is not $\mathbf{m}^3$) it does not have $\lceil n/2 \rceil = 2$ linear steps in its resolution. In this case the S-module $\mathcal{A}_2$ is generated

in degree $2d = 10$, but has regularity 11 instead of 10. (However $\mathcal{A}_t$ does have regularity $5t$ for all $t \geq 3$.)

Example 9.4. There exist linearly presented $\mathbf{m}$-primary ideals that are not even of fiber type, as shown by the ideal $I \subset S$ of Example 7.10. In this case the S-module A_2 is not generated in degree $2d = 10$, since otherwise $t_1(I^2) \leq t_1(I) + t_0(I) = 11$, and I^2 would be linearly presented.

Notice that the conclusion of Theorem 9.1(c) holds for linearly presented $\mathbf{m}$-primary Gorenstein ideals in $n = 4$ variables, due to the symmetry of the resolution in this case. Surprisingly, it works for $n = 3$ as well:

COROLLARY 9.5. *Let $I \subset K[x_1, x_2, x_3]$ be a homogeneous $\mathbf{m}$-primary Gorenstein ideal. If I is generated in degree d and has linear presentation, then $\mathcal{A}_t$ is concentrated in degree dt for every t; in particular, I is of fiber type and $\mathcal{A}$ is annihilated by $\mathbf{m}$.*

Proof. We know that $I^2 = m^{2d}$ by Corollary 7.7 and $\mathcal{A}_2 = 0$ by Huneke [1984, Corollary 4.9 and the discussion after Corollary 4.11]. Hence the assertion follows from part (b) of Theorem 9.1. □

The application of Theorem 2.1 to Tor_2 also yields a result on the regularity of exterior powers:

COROLLARY 9.6. *Suppose that* $\operatorname{char} K \neq 2$ *and $I \subset S$ is a homogeneous ideal. If* $\dim S/I \leq 1$ *then*

$$\operatorname{reg} \mathrm{H}^0_{\mathbf{m}}(\wedge^2 I) \leq \operatorname{reg} \mathrm{H}^0_{\mathbf{m}}(I \otimes I) \leq t_p(S/I) + t_q(S/I) - n$$

for any $p, q \leq \operatorname{codim} I$ such that $p + q = n + 2$. In particular, if I is an $\mathbf{m}$-primary ideal generated in degree d with linear free resolution for $\lceil n/2 \rceil$ steps, then $\wedge^t I$ is a vector space concentrated in degree dt for every $t \geq 2$.

Proof. For the first statement we simply observe that $\wedge^2 I$ embeds into $I \otimes I$ and that the torsion submodule of $I \otimes I$ is $\mathrm{Tor}_2(S/I, S/I)$. Now use Theorem 2.1.

To obtain the second statement for $t = 2$ we apply the first inequality with $p = \lceil (n+2)/2 \rceil$ and $q = \lfloor (n+2)/2 \rfloor$. For general $t \geq 2$ we use the surjection $\wedge^2 I \otimes \wedge^{t-2} I \to \wedge^t I$, and the fact that the S-module $\wedge^2 I \otimes \wedge^{t-2} I$ is annihilated by $\mathbf{m}$ and generated in degree dt. □

10. Application. Instant elimination. Let I be an ideal of S, generated by a vector space V of forms of degree d. We may think of V as a linear series on $\mathbf{P}^{n-1}$ and ask for the equations of the image scheme; we may also restrict V to a subscheme $X \subset \mathbf{P}^{n-1}$ to try to compute the image of X. These computations involve the elimination of variables: If $V = \langle f_1, \ldots, f_m \rangle$ then we are looking for

the relations on the elements $f_i t$ in $S_X[It] \subset S_X[t]$. Geometrically, the ideal I defines the base locus of a blowup, and we are looking for the defining relations on the *fiber* $\mathcal{R}_{S_X}(I)/\mathbf{m}\mathcal{R}_{S_X}(I)$.

In some interesting classical cases, there is a much easier way to do elimination. For example, if V is the linear series of d-ics through a set B of $\binom{d+1}{2}$ general points in the projective plane then the ideal I generated by V is linearly presented: indeed, by the Hilbert-Burch theorem, the free resolution of S/I has the form

$$0 \longrightarrow S(-d-1)^d \xrightarrow{\phi} S(-d)^{d+1} \longrightarrow S.$$

The $(d+1) \times d$ matrix ϕ of linear forms in 3 variables may be thought of as a $3 \times (d+1) \times d$ tensor over K. This tensor may also be identified with a matrix ψ of size $3 \times d$ in $d+1$ variables, called the *adjoint* (or *Jacobian dual*) matrix. The image of $\mathbf{P}^2$ under the rational map defined by V is isomorphic to $\mathbf{P}^2$ blown up at B. The defining ideal of this variety is generated by the 3×3 minors of ψ by Room [1938]; see also Geramita and Gimigliano [1991], and Geramita, Gimigliano and Pittleloud [1995], who do the case of determinantal sets of points in $\mathbf{P}^r$.

The idea of doing elimination in this way was generalized and put to practical use by Schreyer and his coworkers (Decker-Ein-Schreyer [1993], Popescu-Ranestad [1996], Popescu [1998]) in their study of surfaces of low degree in $\mathbf{P}^4$, in cases where the usual elimination methods were too demanding computationally. It is easy to see that the method works whenever I is of linear type (as an ideal of S_X, in the sense that the powers of I are equal to the symmetric powers). But the examples above are not of linear type.

Here is a general criterion for when the instant elimination process works. We regard $\operatorname{Sym}(I)$ and $\mathcal{R}(I)$ as bigraded algebras with degrees with an element of degree a in $\operatorname{Sym}_b(I)$ being given degree (a, b).

PROPOSITION 10.1. *Let $|V|$ be a linear series of forms of degree d on $\mathbf{P}^{n-1}$. Suppose that the ideal I generated by V has linear presentation, with matrix ϕ, and that ψ is the adjoint matrix. If the torsion in the symmetric algebra of I occurs only in degrees (a, b) such that $a = db$, then the annihilator of* $\operatorname{coker}\psi$ *is the ideal of forms in $\mathbf{P}(V)$ that vanish on the image of $\mathbf{P}^{n-1}$ under the rational map associated to $|V|$.*

Proof. Write $V = \langle f_1, \ldots, f_m \rangle$. We consider the epimorphism of bigraded algebras

$$K[X_1, \ldots, X_n, T_1, \ldots, T_m] \to \operatorname{Sym}(I); \quad X_i \mapsto x_i, \quad T_i \mapsto f_i \in \operatorname{Sym}_1(I)$$

where X_i is an indeterminate of degree $(1, 0)$ and T_i is an indeterminate of degree

$(d, 1)$. There are $K[T_1, \ldots, T_m]$-module isomorphisms

$$\operatorname{coker} \psi \cong \oplus_b (\operatorname{Sym}(I))_{(db+1,b)} \cong \oplus_b (\mathcal{R}(I))_{(db+1,b)},$$

where the last isomorphism follows from our assumption about the torsion of $\operatorname{Sym}(I)$. On the other hand, since $\mathcal{R}(I)$ is a domain, $\sum_b (\mathcal{R}(I))_{(db+1,b)}$ and $\sum_b (\mathcal{R}(I))_{(db,b)} = K[f_1 t, \ldots, f_m t]$ have the same annihilator. □

COROLLARY 10.2. *Let $|V|$ be a base point free linear series of forms of degree d on $\mathbf{P}^{n-1}$. Suppose that the free resolution of the ideal I generated by V is linear for $\lceil n/2 \rceil$ steps. Let ϕ be a linear presentation matrix of I. If ψ is the adjoint matrix of ϕ then the annihilator of* $\operatorname{coker} \psi$ *is the ideal of forms in $\mathbf{P}(V)$ that vanish on the image of $\mathbf{P}^{n-1}$ under the rational map associated to $|V|$.*

Proof. Apply Theorem 9.1 (c) and Proposition 10.1. □

11. Ideals with almost linear resolution. We can get a bound for the number of generators of an ideal with "almost linear" resolution as follows. Let $n = r + 1$ so that $S = K[x_0, \cdots, x_r]$, with $r \geq 2$ to avoid the trivial case, and suppose that the free resolution of S/I has the form

$$0 \longrightarrow \sum_{i=1}^{m_{r+1}} S(-d-r-b_i) \longrightarrow S^{m_r}(-d-r+1) \longrightarrow \cdots \longrightarrow S^{m_1}(-d) \longrightarrow S;$$

that is, I is generated in degree d, S/I has "almost linear resolution," and the socle elements of S/I lie in degrees $d + b_i - 1$, with $b_i \geq 0$. Assume further that S/I has finite length. Our goal is to find a lower bound for the number of generators of I (see also Corollary 6.3 and Proposition 7.11).

Computing the Hilbert polynomial $0 \equiv P_{S/I}(\nu)$ we get

$$0 = \binom{\nu + r}{r} + \sum_{i=1}^{r} (-1)^i m_i \binom{\nu - d - (i-1) + r}{r} + (-1)^{r+1} \sum_{i=1}^{m_{r+1}} \binom{\nu - d - b_i}{r}.$$

Taking $\nu = d - 1$, all but the first and last terms vanish, so

$$\binom{d + r - 1}{r} = (-1)^r \sum \binom{-b_i - 1}{r} = \sum \binom{b_i + r}{r}. \tag{1}$$

Taking $\nu = d$, all but the first two and the last terms vanish, so

$$m_1 = \binom{d + r}{r} - \sum \binom{b_i + r - 1}{r}, \tag{2}$$

or equivalently the Hilbert function satisfies $\mathrm{H}_{S/I}(d) = \sum \binom{b_i + r - 1}{r}$.

Continuing in this way we could inductively compute all the m_i in terms of the b_i. But already equations (1) and (2) suffice to give a lower bound for the number of generators:

PROPOSITION 11.1. *With notation as above,*

$$m_1 \geq \binom{d+r-1}{r-1} + \binom{d+r-2}{r-1}$$

with equality if and only if S/I is Gorenstein.

Proof. By equation (1) we have $d-1 \geq b_i$ for every i, and equality holds for some i if and only if $m_{r+1} = 1$, that is, S/I is Gorenstein (and there is only one b_i). Thus by equation (2)

$$\begin{aligned} m_1 &= \binom{d+r}{r} - \sum \binom{b_i+r-1}{r} \\ &= \binom{d+r}{r} - \sum \frac{b_i}{b_i+r}\binom{b_i+r}{r} \\ &\geq \binom{d+r}{r} - \frac{d-1}{d+r-1}\sum \binom{b_i+r}{r} \end{aligned}$$

with equality if and only if S/I is Gorenstein. By equation (1) we may rewrite the last line as

$$\binom{d+r}{r} - \frac{d-1}{d+r-1}\binom{d+r-1}{r} = \binom{d+r-1}{r-1} + \binom{d+r-2}{r-1}. \qquad \square$$

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