Relative Beilinson Monad and Direct Image for Families of Coherent Sheaves

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Abstract

The higher direct image complex of a coherent sheaf (or finite complex of coherent sheaves) under a projective morphism is a fundamental construction that can be defined via a Cech complex or an injective resolution, both inherently infinite constructions. Using exterior algebras and relative versions of theorems of Beilinson and Bernstein-Gel'fand-Gel'fand, we give an alternate description in finite terms.

Using this description we can characterize the generic complex, over the variety of finite free complexes of a given shape, as the direct image of an easily-described vector bundle. We can also give explicit descriptions of the loci in the base spaces of flat families of sheaves in which some cohomological conditions are satisfied—for example, the loci where vector bundles on projective space split in a certain way, or the loci where a projective morphism has higher dimensional fibers.

Our approach is so explicit that it yields an algorithm suited for computer algebra systems.

Introduction

Let A be a Noetherian ring, let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^n_A = \mathbb{P}^n \times \operatorname{Spec} A$, and let π be the projection map $\mathbb{P}^n_A \to \operatorname{Spec} A$:

$$\begin{array}{ccc} \mathcal{F} & & \mathbb{P}^n_A = \mathbb{P}^n \times \operatorname{Spec} A \\ & & \Big\downarrow \pi \\ R\pi_*\mathcal{F} & & \operatorname{Spec} A \end{array}$$

A locally free complex $R\pi_*\mathcal{F}$ of coherent sheaves on $\operatorname{Spec} A$ (or, equivalently, of A-modules) is defined up to quasi-isomorphism by pushing forward an injective resolution of $\mathcal{F}$ and choosing a bounded complex of finitely generated A-modules quasi-isomorphic to it. In this paper we will give an alternate construction, using only (finite parts of) free resolutions. This makes it possible to compute, for example, the loci in flat families of sheaves where certain cohomological conditions are satisfied, such as the loci where certain decompositions occur in a family of vector bundles, or to detect higher dimensional fibers.

For example, if $\mathcal{F}$ is flat over A, and A is local with maximal ideal $\mathfrak{m}$, then by a Theorem of Grothendieck [EGA,7.7], see also [Mumford,II.5] or [Hartshorne,III.12], the complex $R\pi_*\mathcal{F} \in D^b(A)$ is represented by a minimal complex

$$0 \to A^{h^0} \to A^{h^1} \to \ldots \to A^{h^n} \to 0$$

of free A-modules, which is unique up to isomorphism and commutes with base change. Hence

$$h^i = \dim_K \mathrm{H}^i(\mathcal{F} \otimes K),$$

where $K = A/\mathfrak{m}$ denotes the residue class field of A. If $\mathcal{F}$ is given explicitly, then our techniques algorithmically compute the matrices in this complex.

To formulate the main result we introduce some notation. Let $W = \pi_*\mathcal{O}_{\mathbb{P}^n_A}(1)$ be the free A-module of rank $n+1$ underlying $\mathbb{P}^n_A$, and let $x_0, \ldots, x_n$ be a basis of W. The scheme $\mathbb{P}^n_A$ is $\operatorname{Proj} S$ where $S = \operatorname{Sym} W \cong A[x_0, \ldots, x_n]$. Let $M = \sum_d M_d$ be a graded S-module whose sheafification is $\mathcal{F}$, and let $\operatorname{reg}(M)$ denote its Castelnuovo-Mumford regularity (as defined, in this relative case, below.)

Let $V = \operatorname{Hom}_A(W, A)$ be the dual of W, with basis $e_0, \ldots, e_n$ dual to $x_0, \ldots, x_n$. Let

$$E = \Lambda V = \oplus_{i=0}^{n+1} \Lambda^i V$$

be the exterior algebra on V. We give the variables e_i degree -1.

Corresponding to the graded S-module M there is a complex of E-modules

$$\cdots \to E \otimes M_d \to E \otimes M_{d+1} \to E \otimes M_{d+2} \to \cdots$$

where $\otimes$ denotes $\otimes_A$, the module $E \otimes M_d$ is in cohomological degree d and is generated by M_d regarded as an A-module concentrated in degree d. The differentials in the complex are given by

$$a \otimes m \mapsto \sum_i e_i a \otimes x_i m.$$

Theorem 0.1 (Main Theorem). *Suppose* $s \geq \max(0, \operatorname{reg}(M))$*, and set* $P^s = \ker(E \otimes M_{s+1} \to E \otimes M_{s+2})$*. If* $\mathbf{T}$ *is a projective resolution of* P^s*, regarded as an* E*-module concentrated in cohomological degree* s*, then*

$$R\pi_* \mathcal{F} \cong (\mathbf{T} \otimes_E A)_0.$$

In particular,

$$R^i \pi_* \mathcal{F} \cong \operatorname{Tor}^E_{s-i}(P^s, A)_0,$$

The proof is given in the first three sections of the paper. Its main ingredients are

1. the fact that the relative Beilinson monad for $\mathcal{F}$ is π_*-acyclic (that is, $R^i\pi_*$ vanishes on all terms of the monads for all $i > 0$).
2. an effective construction of the Beilinson monad.

In the remainder of the paper we carry out this construction in three examples. The first concerns the versal deformation of a rank r vector bundle of the form $\mathcal{O}^{r-1} \oplus \mathcal{O}(d)$ on $\mathbb{P}^1$ (this example can, of course be treated by other means, and we sketch one simple alternative at the end of the section.) The result gives determinantal equations for the loci, in the base space of the deformation, of bundles of a given splitting type; we conjecture that these determinantal equations actually generate the prime ideals corresponding to the loci in question. The case $r = 2$ has a considerable history, and can already be found, in equivalent form and without proof, in Room [1938].

Our second example treats the directs images of certain sheaves on the resolution of an elliptic singularity; this example seems amenable only to computation.

The last example was the most surprising to us. It leads to a new description of the variety of complexes, in Theorem 4.8, and the following result:

Theorem 0.2. *Let* A *be a Noetherian ring, and let*

$$\mathbb{F}: \quad 0 \to A^{\beta_0} \to A^{\beta_1} \to \ldots \to A^{\beta_n} \to 0$$

be a finite complex of finitely generated free A*-modules of length* n*. There is a vector bundle* $\mathcal{F}$ *on* $\mathbb{P}^n_A$ *such that* $\mathbb{F}$ *represents* $R\pi_*\mathcal{F}$*.*

In fact the bundle $\mathcal{F}$ can be given quite explicitly; see the proof of Theorem 4.7.

An implementation of the resulting algorithms in the system `Macaulay2` and all the examples treated as illustrations of the main result can be downloaded from `http//www.math.uni-sb.de/~ag-schreyer/computeralgebra`.

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Notation: Throughout this paper A will denote a Noetherian ring, and S will denote the polynomial ring $A[x_0, \dots, x_n] = \mathrm{Sym}(W)$, where W is the free A-module of rank $n+1$ generated by the x_i. We grade S with A in degree 0 and the x_i in degree 1. There is a canonical projection $\pi : \mathbb{P}^n_A \to \operatorname{Spec} A$ corresponding to the inclusion $A \subset S$. We denote by $\mathcal{F}$ a coherent sheaf on the projective space $\mathbb{P}^n_A = \operatorname{Proj} S$, and we write $R\pi_*\mathcal{F}$ for direct image, a complex of A modules determined up to quasi-isomorphism. The sheaf associated to a finitely generated graded A-module M will be denoted $\widetilde{M}$.

We write V for the A-dual of W, considered as a free module in degree -1, and $e_0, \dots, e_n$ will be a basis of V dual to $x_0, \dots x_n$. Let E be the exterior algebra $E = \bigwedge V = A\langle e_0, \dots, e_n\rangle$. If A is a local ring with maximal ideal $\mathfrak{m}$ then E is a local ring with maximal ideal $\mathfrak{m}_E = \mathfrak{m}E + (e_0, \dots, e_n)$.

1 Regularity and BGG

In this section we make precise the generalizations we need of some known results. Except for Proposition 1.2 the proofs are essentially the same as in the known cases, and we omit them.

Relative Castelnuovo-Mumord Regularity

The definition of Castelnuovo-Mumford regularity for the case where A is a field generalizes to our case as follows: If N is a graded S-module such that $N_d = 0$ for $d \gg 0$, then we define the *regularity* of N by

$$\mathrm{reg}(N) = \max\{d \mid N_d \neq 0\}.$$

For a finitely generated graded S-module, on the other hand, we define

$$\mathrm{reg}(M) = \max_i(\mathrm{reg}\,\mathrm{Tor}_i^S(A, M) - i).$$

As in the classical case, these definitions are equivalent in the case where they are both applicable. This is easy to see from the following re-interpretation of regularity in terms of local cohomology.

Proposition 1.1.

$$\operatorname{reg}(M) = \max_i(\operatorname{reg}\operatorname{Tor}_i^S(A, M) - i) = \max_j(\operatorname{reg}\mathrm{H}^j_{(x)}M + j)$$

and for each i

$$\operatorname{reg}\operatorname{Tor}_i^S(A, M) - i \geq \operatorname{reg}\mathrm{H}^{n+1-i}_{(x)}M + n + 1 - i$$

where (x) *denotes the ideal* $(x_0, \dots, x_n) \subset S$ *and* $\mathrm{H}^j_{(x)}$ *is the local cohomology functor.* □

Since the proof is similar to that in the classical case (see for example Eisenbud [2004, Corollary 4.5]), we omit it. We use it to deduce a consequence of regularity that has no interesting analogue in the case where A is a field.

Proposition 1.2. *If the sheafication* $\mathcal{F} = \widetilde{M}$ *is flat over* A*, then* M_d *is projective for* $d > \operatorname{reg}(M)$*.*

Proof. The exact sequence

$$0 \to \mathrm{H}^0_{(x)}M \to M \to \bigoplus_{d\in\mathbb{Z}} \pi_*\mathcal{F}(d) \to \mathrm{H}^1_{(x)}M \to 0$$

shows that

$$M_d \to \pi_*\mathcal{F}(d)$$

is an isomorphism for $d > \operatorname{reg}(M)$. Since M_d is a finitely generated module, it suffices to show that $\pi_{\mathcal{F}}(d) = M_d$ is flat, given that $\mathcal{G} := \mathcal{F}(d)$ is flat. This holds very generally: If $I \subset A$ is any ideal, then by the flatness of $\mathcal{G}$ the map the map $\pi^{-1}I \otimes_{\pi^{-1}A} \mathcal{G} \to \pi^{-1}A \otimes_{\pi^{-1}A} \mathcal{G} = \mathcal{G}$ is a monomorphism, where we have written A instead of $\mathcal{O}_{\operatorname{Spec} A}$. Writing $\mathcal{O}$ for $\mathcal{O}_{\mathbb{P}^n_A}$ we see that the composite map

$$\pi^*\mathcal{I} \otimes \mathcal{G} = \pi^{-1}\mathcal{I} \otimes_{\pi^{-1}A} \mathcal{O} \otimes_{\mathcal{O}} \mathcal{G} = \pi^{-1}\mathcal{I} \otimes_{\pi^{-1}A} \mathcal{G} \to \pi^{-1}A \otimes_{\pi^{-1}A} \mathcal{G} = \mathcal{G}$$

is a monomorphism. Since the map $I \otimes_A \pi_*\mathcal{G} \to \pi_*\mathcal{G}$ is the direct image of $\pi^*I \otimes \mathcal{G} \to \mathcal{G}$, it too is a monomorphism, so $\pi_*\mathcal{G}$ is flat. □

Bernstein-Gel'fand-Gel'fand Correspondence

Using this notion of regularity, we easily generalize the BGG correspondence. We need it in the form given for the case where A is a field in which we studied in [EFS, 2003] following [BGG, 1978]. The proofs are essentially the same as in the case treated there. For the reader's convenience we formulat the necessary statements, which involve a pair of adjoint functors

$$cplx(S) \underset{\mathbf{L}}{\overset{\mathbf{R}}{\longleftrightarrow}} cplx(E)$$

To an S-module $M = \oplus_d M_d$ we first associate

$$\mathbf{R}(M): \quad \cdots \to \mathrm{Hom}_A(E, M_d) \to \mathrm{Hom}_A(E, M_{d+1}) \to \cdots ,$$

which is (in all interesting cases) an infinite complex of E-modules with the differential given by the action of $t = \sum_{j=0}^{n} x_j \otimes e_j$. Since

$$\mathrm{Hom}_A(E, -) \cong E \otimes \wedge^{n+1} W \otimes -$$

canonically, this complex is isomorphic up to shift in grading to the complex

$$\cdots \to E \otimes M_d \to E \otimes M_{d+1} \to \cdots$$

of the introduction. The definition extends to the case where M is a bounded complex of S-modules by taking $\mathbf{R}(M)$ the total complex of the induced double complex.

Similarly $\mathbf{L}$ is defined by associating to an graded E-module $P = \oplus_j P_j$ the complex

$$\mathbf{L}(P): \quad \cdots \to S \otimes_A P_j \to S \otimes_A P_{j-1} \to \cdots .$$

Theorem 1.3. *Let M be a finitely generated S-module.*

a) The truncated complex $\mathbf{R}(M_{\geq s})$ is acyclic for $s \geq \operatorname{reg} M$.

b) For $s \geq \operatorname{reg} M$ and

$$P = \ker(\mathrm{Hom}_A(E, M_{s+1}) \to \mathrm{Hom}_A(E, M_{s+2}))$$

the complex $\mathbf{L}(P)$ is a resolution of $M_{>s}$ by (not necessarily) free S-modules in the sense that

$$\mathbf{L}(P): \ldots \to S \otimes_A P_{s+2} \to S \otimes_A P_{s+1} \to M_{>s} \to 0$$

is exact. The map to $M_{>s}$ is given by

$$S \otimes_A P_{s+1} = S \otimes \mathrm{Hom}_A(\Lambda^0 V, M_{s+1}) \to M_{>s}, \; f \otimes_A \varphi \mapsto f\varphi(1).$$

2 The Tate Resolution

We next construct the Tate resolution of a family of sheaves, a doubly infinite graded complex of projective E-modules. When A is a field, the Tate resolution is defined in [EFS 2003] by joining the complex $\mathbf{R}(M_{>s})$ to a free resolution of $P = \ker(\mathrm{Hom}_A(E, M_{s+1}) \to \mathrm{Hom}_A(E, M_{s+2}))$, but in the relative case the modules in $\mathbf{R}(M_{>s})$ are not free, and we have to do some additional work.

To simplify, we change notation and replace the modules $\mathrm{Hom}_A(E, M_d)$ used in the last section with $E \otimes M_d$. Since there are canonical isomorphisms $\mathrm{Hom}_A(E, M_d) \cong \mathrm{Hom}_A(E, A) \otimes M_d \cong E \otimes \wedge^{n+1} W \otimes M_d$, and $\wedge^{n+1} W \cong A(n+1)$, this only amounts to a shift of degrees by $n+1$.

Henceforward, set

$$P^s = \ker(E \otimes M_{s+1} \to E \otimes M_{s+2})$$

(if P is the module defined in Theorem 1.3, then $P^s = P \otimes \wedge^{n+1} V$). Let $\mathcal{F} = \widetilde{M}$ and pick $s \geq r := \mathrm{reg}(M)$. We call any projective resolution

$$\mathbf{T}_{\preceq s} : \quad \cdots \to T^{s-1} \to T^s \to P^s \to 0$$

of P^s as a (graded) E-module a *generator-truncated Tate resolution* of $\mathcal{F}$, since it is the subcomplex of a complete Tate resolution, defined below, consisting of all those free summands whose generators have degrees $\leq s$.

Theorem 0.1 says that if $s \geq \max(d, \mathrm{reg}\, M)$ then the degree d part of $(A \otimes_E \mathbf{T}_{\preceq s})$ is $R\pi_* \mathcal{F}(d)$. In particular, it is independent of the choice of s. Our first to give prove this independence directly and describe the graded structure of $\mathbf{T}_{\preceq s})$ in more detail. In the case where A was a field, $\mathbf{T}_{\preceq s}$ itself is an invariant, but this is only true in our setting when $\mathcal{F}$ is flat.

Any finitely generated projective graded E-module has the form $E \otimes_A N$ where N is a projective graded A-module. Regarding A as being concentrated in degree 0, we may write $N = \oplus_j N_j$ as a sum of projective A-modules with N_j in degree j.

We can collect information about the terms $T^i = \sum_j E \otimes N^i_j$ in the projective resolution $\mathbf{T}_{\preceq s}$ in a "Betti Table"

$$
\begin{array}{cccccccccccc}
\cdots & \cdots & \cdots & \cdots & \cdots & N_0^n & \cdots & N_{r-1-n}^{r-1} & & & & \\
\cdots & \cdots & \cdots & \cdots & \iddots & \cdots & \cdots & \vdots & & & & \\
\cdots & \cdots & \cdots & N_0^1 & \cdots & \cdots & \cdots & N_{r-2}^{r-1} & & & & \\
\cdots & \cdots & N_0^0 & \cdots & \cdots & \cdots & \cdots & N_{r-1}^{r-1} & N_r^r & \cdots & \cdots & N_s^s \ . \\
\cdots & N_0^{-1} & \cdots & \cdots & \cdots & \cdots & \cdots & N_r^{r-1} & \cdots & \cdots & N_s^{s-1} & \\
\iddots & \cdots & \cdots & \cdots & \cdots & \cdots & N_r^{r-2} & \cdots & \cdots & \iddots & & \\
\cdots & \cdots & \cdots & \cdots & \cdots & \iddots & \cdots & \cdots & \iddots & & &
\end{array}
$$

In case A is local, the N_j^i are free, and we sometimes put just the numbers $\operatorname{rank} N_j^i$ into the table.

The columns in the table correspond to terms in a single homological degree, that is, to a single T^i. Maps pointing directly to the right are linear in the variables e_i, while maps of higher degree in the e_i point to the right and downwards. The degree 0 maps in $\mathbf{T}$ go up and to the right, along the 45^0 diagonals; thus the complex $(A \otimes_E \mathbf{T}_{\preceq s})_d$ is

$$
\begin{array}{ccccccc}
 & & & & & & \nearrow \\
 & & & & & N_d^1 & \\
 & & & & \nearrow & & \\
 & & & N_d^0 & & & \\
 & & \nearrow & & & & \\
 & N_d^{-1} & & & & & \\
\nearrow & & & & & &
\end{array}
$$

The empty spaces in the table, and in particular the absence of dots above the top row shown, indicate that the terms that might have occured there are in fact 0. We will prove this in Proposition 2.2 below.

The following proposition allows us to construct a complete Tate resolution.

Proposition 2.1. *Suppose that $s \geq \operatorname{reg} M$ and let $\mathbf{T}(\mathcal{F})_{\preceq s}$ be the generator-truncated Tate resolution at s. Then $\mathbf{T}(\mathcal{F})_{\preceq s+1}$ can be chosen as the mapping cone of*

$$\mathbf{T}(\mathcal{F})_{\preceq s} \to E \otimes_A F(-s-1))$$

where $F : \ldots \to F^{s-1} \to F^s \to F^{s+1} \to M_{s+1}(s+1) \to 0$ is a projective resolution as of $M_{s+1}(s+1)$ as an A-module. If A is local and both $\mathbf{T}(\mathcal{F})_{\preceq s}$

and F are minimal free resolutions, then so is the mapping cone. In particular the homotopy type of the complex $(A \otimes_E \mathbf{T}_{\preceq s})_d$ depends only on the sheaf $\mathcal{F}$, and not on the choice of s.

Proof. By Theorem 1.3 we have a short exact sequence of E-modules,

$$0 \to P^s \to E \otimes M_s \to P^{s+1} \to 0$$

and projective resolutions $\mathbf{T}(\mathcal{F})_{\preceq s} \to P^s$ and $E \otimes F(-s-1) \to E \otimes M_s$ of E-modules. The mapping cone is then projective resolution of P^{s+1} and differs just by addition of free modules generated in degree exactly $s+1$. In the local case, where $\mathbf{T}(\mathcal{F})_{\preceq s}$ and F are chosen minimal, the free modules in $\mathbf{T}(\mathcal{F})_{\preceq s}$ are all generated in degrees $\leq s$, the generator degree of P_s, since the elements of E have degree ≤ 0. Thus the comparison maps from $\mathbf{T}(\mathcal{F})_{\preceq s}$ to $E \otimes F$ cannot contain degree 0 components, and the mapping cone is again minimal.

Any projective resolution of P_{s+1} differs from the mapping cone by a homotopy equivalence, which induces a homotopy equivalence on the d-th diagonal $(A \otimes \mathbf{T}(\mathcal{F})_{\preceq s})_d$. $\square$

Proposition 2.1 shows that it is possible to choose $\mathbf{T}(\mathcal{F})_{\preceq s+1}$ to be a complex containing $\mathbf{T}(\mathcal{F})_{\preceq s}$, and we define the *complete Tate resolution* $\mathbf{T}(\mathcal{F})$ of $\mathcal{F}$ as the inductive limit. When A is local we may choose the $\mathbf{T}(\mathcal{F})_{\preceq s}$ to be minimal resolutions, yielding a minimal Tate resolution determined by $\mathcal{F}$ up to isomorphism. The term $(\mathbf{T}(\mathcal{F})_{\preceq s})^k$ is the submodule of $\mathbf{T}(\mathcal{F})^k$ generated by the free generators of degrees $\leq s$.

The module $(\mathbf{T}(\mathcal{F})_{\preceq s})^k$ is finitely generated, whereas $\mathbf{T}(\mathcal{F})^k$ need not be. We will see that $\mathbf{T}(\mathcal{F})$ contains all the direct image complexes $R\pi_*(\mathcal{F}(m))$ as subquotients. To obtain a complex representing $R\pi_{\mathcal{F}}(d)$ we have to compute only finitely many terms.

Proposition 2.2. *1. Let $s \geq \operatorname{reg}(M)$, and let $\mathbf{T}(\mathcal{F})_{\preceq s} \to P^s \to 0$ be a generator-truncated Tate resolution. The homology $\mathrm{H}^i(\mathbf{T}(\mathcal{F})_{\preceq s} \otimes_E A)_j = \operatorname{Tor}^E_{s-i}(P^s, A)_j$ can be nonzero only for $i - n \leq j \leq i$; if $i \geq \operatorname{reg}(M)$, then can be nonzero only for $i = j$.*

2. If $\mathcal{F}$ is flat over A there is a Tate resolution for which $E \otimes N^i_j$ is nonzero only in this range.

3. *In general, there exists a Tate resolution for which N^i_j is nonzero only in the range $i-n \le j \le i+\operatorname{pd} A$; if $i \ge \operatorname{reg}(M)$ then it is nonzero only for $i \le j \le i + \operatorname{pd} A$.*

Proof. We use the projective resolution of $A = E/(e_0,\dots,e_n)$ as an E-module to compute the Tor. It has the form

$$\dots \to D_dV \otimes E \to \dots \to D_2V \otimes_A E \to V \otimes_A E \to E \to A \to 0$$

where $D_dV = \operatorname{Hom}(S_d(W), A)$, the d-th divided power module (see for example [EFS 2003]). Since $P^s \subset E \otimes M_{s+1}$ and $E \otimes M_s \to P^s$ is surjective by Theorem 1.3, P_s can be nonzero only in degrees $s-n,\dots,s$. Since $D_dV \otimes E$ is generated in degree $-d$, we get the first bound of statement 1.

If we may use the inductive process of Proposition 2.1 to construct a Tate resolution starting from a generator-truncated Tate resolution $\mathbf{T}(\mathcal{F})_{\preceq \operatorname{reg} M}$, then we get one with $N^i_j = 0$ for all $i \ge \operatorname{reg} M$ and $j \ne i$. This is precisely statement 2 in the case $i \ge \operatorname{reg} M$. It implies the remainder of statement 1, and the lower bound for j when $i \ge \operatorname{reg} M$ in 3. If we take the resolutions F used in Proposition 2.1 to have length at most $\operatorname{pd} A$, then we construct a Tate resolution satisfying the upper bound for j in part 3 as well.

Next we prove the lower bound for 2.) and 3.) for the case $i < reg(M)$. The bounded-above subquotient complexes

$$A \otimes N^i_j \to A \otimes N^{i+1}_j \to \dots \to 0$$

for $j \le i-n$ are split exact by 1.) and the projectivity of the N^i_j as A-modules. We apply descending induction on the i for which there is a term $N^i_j \ne 0$ with $j < i-n$. Take the maximal i such that $N^i_j \ne 0$ for some $j < i-n$. Choose j minimal. Then by 1.) $E \otimes N^{i-1}_j \to E \otimes N^i_j$ is surjective, which implies that the image $\operatorname{im}(E \otimes N^i_j \to \mathbf{T}(\mathcal{F})^{i+1}_{\preceq s})$ is contained in the image of $E \otimes (\oplus_{\ell > j} N^i_\ell)$. Thus the term $E \otimes N^i_j$ is superflous. Replacing N^i_j by zero and N^{i-1}_j by $\ker(N^{i-1}_j \to N^i_j)$ gives a Tate resolution with fewer terms. This argument applied for j repeatedly, proves the induction step for i.

In the flat case, we know by Proposition 1.2 that the M_d are flat A modules, and hence projective, for $d > \operatorname{reg}(M)$. The upper bound $j \le i$ follows in the range $i > \operatorname{reg}(M)$ by the inductive construction in Proposition 2.1. For $i \le \operatorname{reg}(M)$ we argue again by descending induction. Suppose the bound is correct for $i \ge d$. Consider $P^d = \ker(\mathbf{T}(\mathcal{F})^{d+1}_{\preceq s} \to \mathbf{T}(\mathcal{F})^{d+2}_{\preceq s})$.

Breaking the Tate resolution into graded pieces we find the bounded exact complex of A-modules

$$0 \to P_d^d \to \bigoplus_{\ell=0}^{1} \Lambda^\ell V \otimes N_{d+\ell}^{d+1} \to \bigoplus_{\ell=0}^{2} \Lambda^\ell V \otimes N_{d+\ell}^{d+2} \to \cdots \to 0$$

All but possibly the first term are projective A-modules, so P_d^d is projective as well and we may choose $N_d^d = P_d^d$ and $N_d^{d-1} = 0$. This proves the upper bound for $j \leq i$ in case 2.)

The proof of the upper bound for $j \leq i + \operatorname{pd} A$ in case of 3.) is very similar. As a start we have that $\operatorname{pd}_A M_s \leq b := \operatorname{pd} A$ giving the bound for $i \geq s - b$ and $s > \operatorname{reg}(M)$. Beyond that range we look at the exact bounded complex of graded A-modules

$$0 \to P_{d+b}^d \to \bigoplus_{\ell=0}^{1} \Lambda^\ell V \otimes N_{d+b+\ell}^{d+1} \to \bigoplus_{\ell=0}^{2} \Lambda^\ell V \otimes N_{d+b+\ell}^{d+2} \to \cdots \to 0$$

Hence P_{b+d}^d is projective as b th syzygy module and we may take $N_{d+b}^d = P_{d+b}^d$ and $N_{d+b}^{d-1} = 0$. $\square$

3 The Beilinson Monad

In this section we construct a relative Beilinson monad for $\mathcal{F}$: it is a complex $\mathbf{U}(\mathcal{F})$ of π_*-acyclic coherent sheaves on $\mathbb{P}_A^n$, unique up to homotopy, whose only homology is $\mathcal{F}$. The construction is parallel to that given in [ES 2003], but involves new subtleties, especially when $\mathcal{F}$ is not flat over A.

Let U denote the universal subbundle on $\mathbb{P}_A^n$.

$$0 \to U \to W \otimes \mathcal{O} \to \mathcal{O}(1) \to 0.$$

In other terms $U = \Omega^1_{\mathbb{P}_A^n/\operatorname{Spec} A}(1)$. Consider the category of projective E-modules. Their homomorphisms are given by the formula

$$\operatorname{Hom}_E(E \otimes N_j, E \otimes N_\ell) = \Lambda^{\ell-j} V \otimes_A \operatorname{Hom}_A(N_j(j), N_\ell(\ell)),$$

where as always it is understood that N_j sits in degree j and N_ℓ in degree ℓ. This allows us to define an additive functor

$$\mathbf{U} : \{\text{projective } E\text{-modules}\} \to coh(\mathbb{P}^n_A)$$

by defining

$$\mathbf{U}(E \otimes_A N_j) = \Lambda^{-j} U \otimes \pi^* N_j(j).$$

For a morphism $\phi \in \Lambda^{\ell-j} V \otimes \mathrm{Hom}_A(N_j(j), N_\ell(\ell))$ we define $\mathbf{U}(\phi)$ as the map

$$\begin{tikzcd} E \otimes N_j \arrow[d, "\phi"'] \\ E \otimes_A N_\ell \end{tikzcd} \qquad \begin{tikzcd} \Lambda^{-j} U \otimes \pi^* N_j(j) \arrow[r] \arrow[d, "\mathbf{U}(\phi)"'] & \Lambda^{-j} W \otimes \pi^* N_j(j) \arrow[d, "\phi"'] \\ \Lambda^{-\ell} U \otimes \pi^* N_\ell(\ell) \arrow[r] & \Lambda^{-\ell} W \otimes \pi^* N_\ell(\ell) \end{tikzcd}$$

induced by contraction.

If we apply this functor to a Tate resolution, then we obtain a bounded-above complex

$$\mathbf{U}(\mathcal{F}) := \mathbf{U}(\mathbf{T}(\mathcal{F}))$$

which depends only on $n+1$ diagonals of $\mathbf{T}(\mathcal{F})$, since $\Lambda^{-j} U = 0$ unless $0 \geq j \geq -n$. In particular $\mathbf{U}(\mathcal{F}) = \mathbf{U}(\mathbf{T}(\mathcal{F})_{\preceq s})$ for any $s \geq \max(0, \mathrm{reg}(M))$. We call $\mathbf{U}(\mathcal{F})$ a *relative Beilinson monad* for $\mathcal{F}$. Its crucial properties are the following:

Theorem 3.1. *Let $M = \sum_d M_d$ be a finitely generated graded S-module and let $\mathcal{F}$ be the associated coherent sheaf on $\mathbb{P}^n_A$. For $s \geq \max(0, \mathrm{reg}(M))$ let $\mathbf{T}(\mathcal{F})_{\preceq s}$ be a generator-truncated Tate resolution. The complex*

$$\mathbf{U}(\mathcal{F}) = \mathbf{U}(\mathbf{T}(\mathcal{F})_{\preceq s})$$

is a bounded-above π_*-acyclic monad of coherent sheaves for $\mathcal{F}$, which is determined by $\mathcal{F}$ up to homotopy. If A is local and $\mathbf{T}(\mathcal{F})_{\preceq s}$ chosen minimal then $\mathbf{U}(\mathcal{F})$ is determined by $\mathcal{F}$ up to isomorphism. Furthermore, if

1. *$\mathcal{F}$ is A-flat, or*

2. $\mathrm{pd}\, A < \infty$

holds, then we can choose $\mathbf{T}(\mathcal{F})_{\preceq s}$ such that the complex $\mathbf{U}(\mathcal{F})$ is bounded-below as well.

Proof. All terms in the complex $\mathbf{U}(\mathcal{F})_{\preceq s}$ are π_*-acyclic, because $R\pi_*\Lambda^i U = 0$ for $i > 0$. Further, $R\pi_*\Lambda^0 U = R^0\pi_*\mathcal{O}_{\mathbb{P}^n_A} = A$. These formulas follow as in the classical case by considering the truncated Koszul complexes

$$0 \to \Lambda^{n+1}W \otimes \mathcal{O}(i-n-1) \to \cdots \to \Lambda^{i+1}W \otimes \mathcal{O}(-1) \to \Lambda^i U \to 0.$$

To prove that $\mathrm{H}^*(\mathbf{U}(\mathcal{F})) = \mathrm{H}^0(\mathbf{U}(\mathcal{F})) \cong \mathcal{F}$ we use a double complex argument. Theorem 1.3 says that $\widetilde{\mathbf{L}}(P^s \otimes \Lambda^{n+1}W)$ is a complex with $\mathcal{F}$ as only homology. Here and below $\widetilde{G}$ stands for the sheafication of a (complex) of graded S-modules G. Consider the subcomplex $(\mathbf{T}(\mathcal{F})_{\preceq s})_{\geq -n}$ of elements in $\mathbf{T}(\mathcal{F})_{\preceq s}$ of internal degree $\geq -n$, and apply the functor $\widetilde{\mathbf{L}}$. The resulting double complex has as vertical pieces sums sheafifications of possibly truncated Koszul complexes $\widetilde{L}((E \otimes N^i_j)_{\geq -n})$:

$$\begin{array}{ccc}
0 & & 0 \\
\uparrow & & \uparrow \\
\sum_{\ell-j=n} \pi^*(\Lambda^\ell V \otimes N^i_j(j)) \otimes \mathcal{O}(n) & \to & \sum_{\ell-j=n} \pi^*(\Lambda^\ell V \otimes N^{i+1}_j(j)) \otimes \mathcal{O}(n) \\
\uparrow & & \uparrow \\
\sum_{\ell-j=n-1} \pi^*(\Lambda^\ell V \otimes N^i_j(j)) \otimes \mathcal{O}(n-1) & \to & \sum_{\ell-j=n-1} \pi^*(\Lambda^\ell V \otimes N^{i+1}_j(j)) \otimes \mathcal{O}(n-1) \\
\uparrow & & \uparrow
\end{array}$$

These complexes are either exact or resolutions of $\Lambda^{-j}U \otimes \pi^*(N^i_j(j)) \otimes \mathcal{O}(n+1)$ in case $0 \geq j \geq -n$. Thus the vertical homology of the double complex is the complex $\mathbf{U}(\mathcal{F}) \otimes \mathcal{O}(n+1)$. On the other hand the horizontal homology is the complex $\widetilde{L}(P^s)$ by the exactness of $\mathbf{T}(\mathcal{F})_{\preceq s+1}$ and its homology is $\mathcal{F}(n+1)$ by Theorem 1.3, since $P^s = P(-n-1)$. A diagram chase in the double complex proves $\mathrm{H}^*(\mathbf{U}(\mathcal{F}) \otimes \mathcal{O}(n+1)) = \mathrm{H}^0(\mathbf{U}(\mathcal{F}) \otimes \mathcal{O}(n+1)) \cong \mathcal{F}(n+1)$ as desired.

The last statement follows from Proposition 2.2. □

Corollary 3.2. *The complex of projective A-modules $\pi_*\mathbf{U}(\mathcal{F})$ represents $R\pi_*\mathcal{F}$ in the derived category of bounded-above, finitely generated complexes of A-modules. In case A is local and we choose $\mathbf{T}(\mathcal{F})_{\preceq s}$ minimal, $\pi_*\mathbf{U}(\mathcal{F})$ is the unique minimal free representative. In particular $\pi_*\mathbf{U}(\mathcal{F})$ has no homology in negative degrees and $R^i\pi_*\mathcal{F} = \mathrm{H}^i\pi_*\mathbf{U}(\mathcal{F})$ for $i \geq 0$.*

Proof. We have exact complexes

$$\ldots \to \mathbf{U}(\mathcal{F})^{-2} \to \mathbf{U}(\mathcal{F})^{-1} \to \mathcal{B} \to 0,$$

$$0 \to \mathcal{K} \to \mathbf{U}(\mathcal{F})^0 \to \mathbf{U}(\mathcal{F})^1 \to \dots$$

and

$$0 \to \mathcal{B} \to \mathcal{K} \to \mathcal{F} \to 0.$$

Since $\mathbf{U}(\mathcal{F})$ is π_*-acyclic and $R^{n+1}\pi_*\mathcal{G} = 0$ for any coherent sheaf we get an exact complex $\dots \to \pi_*\mathbf{U}(\mathcal{F})^{-2} \to \pi_*\mathbf{U}(\mathcal{F})^{-1} \to \pi_*\mathcal{B} \to 0$ and $R^i\pi_*\mathcal{B} = 0$ for $i > 0$. Thus the result follows from

$$0 \to \pi_*\mathcal{B} \to R\pi_*\mathcal{K} \to R\pi_*\mathcal{F} \to 0$$

and $R\pi_*\mathcal{K} = \pi_*\mathbf{U}(\mathcal{F})^{\geq 0}$. □

Completion of the proof of Theorem 0.1. Since

$$\pi_*\Lambda^\ell U = \begin{cases} 0, & \text{if } \ell \neq 0 \\ A, & \text{if } \ell = 0 \end{cases}$$

we obtain

$$R\pi_*\mathcal{F} = \pi_*\mathbf{U}(\mathcal{F}) = ((\mathbf{T}(\mathcal{F})_{\preceq s} \otimes_E A)_0,$$

as desired. □

Remark 3.3. Let $\mathcal{B}^0 = \operatorname{coker}(\mathbf{U}(\mathcal{F})^{-1} \to \mathbf{U}(\mathcal{F})^0)$. The proof shows that

$$0 \to \pi_*\mathcal{B}^0 \to \pi_*\mathbf{U}(\mathcal{F})^1 \to \dots \to \pi_*\mathbf{U}(\mathcal{F})^n \to 0$$

is a bounded complex representing $R\pi_*\mathcal{F}$.

Grothendieck's motivation for introducing $R\pi_*\mathcal{F}$ as a complex was to make a base change property true. It is amusing to note that the property follows directly from our construction.

Corollary 3.4 (Base change). *Suppose $\mathcal{F}$ is flat over A. Then $R\pi_*\mathcal{F}$ commutes with base change in the sense that $\varphi^* R\pi_*(\mathcal{F})$ represents $R\pi_*(\widetilde{\varphi}^*\mathcal{F})$ for any ring homomorphism $A \to A'$ and the induced diagram*

$$\begin{tikzcd} \operatorname{Spec} A' \times \mathbb{P}^n \arrow[r, "\widetilde{\varphi}"] \arrow[d] & \operatorname{Spec} A \times \mathbb{P}^n \arrow[d] \\ \operatorname{Spec} A' \arrow[r, "\varphi"] & \operatorname{Spec} A \end{tikzcd}$$

Proof. Because all the terms of $\mathbf{T}(\mathcal{F})$ are projective A-modules, they are flat, and we may take the Tate resolution of $\varphi^*\mathcal{F} = \mathcal{F} \otimes_A A'$ to be $\mathbf{T}(\mathcal{F}) \otimes_A A'$. □

Corollary 3.5. *Suppose A is local with maximal ideal $\mathfrak{m}$ and $\mathcal{F}$ flat over A. Then the k-th summands in the minimal Tate resolution and Beilinson monad are*

$$(\mathbf{T}(\mathcal{F}))^k = \sum_{i=0}^{n} E \otimes A(-k+i))^{h^i(k-i)},$$

and

$$(\mathbf{U}(\mathcal{F}))^k = \sum_{i=0}^{n} (\wedge^{i-k} U)^{h^i(k-i)},$$

where $h^i(k-i) = \dim_{A/\mathfrak{m}} \mathrm{H}^i(\mathcal{F}(k-i) \otimes_A A/\mathfrak{m})$, as in the case where A is a field.

4 Examples

Example 4.1 (Vector Bundles on $\mathbb{P}^1$). As a first example we take the versal deformation $\mathcal{F}$ of the bundle $\mathcal{E} = \mathcal{O} \oplus \mathcal{O}(-2)$ where $\mathcal{O} = \mathcal{O}_{\mathbb{P}^1}$ is the structure sheaf of the projective line over a field K, and compute the complex $R\pi_*\mathcal{F}$.

The base space of this deformation has as tangent space

$$\mathrm{Ext}^1(\mathcal{E}, \mathcal{E}) \cong \mathrm{H}^1(\mathcal{H}\mathrm{om}(\mathcal{E}, \mathcal{E})) = \mathrm{H}^1(\mathcal{H}\mathrm{om}(\mathcal{O}, \mathcal{O}(-2)) = K,$$

and since the deformations are unobstructed the base space of the versal deformation is the germ of $\mathbb{A}^1$. We thus work over $A = K[[a]]$.

By Corollary 3.5 the Betti table of the Tate resolution $\mathbf{T}(\mathcal{F})$ is

$j \setminus i$	-2	-1	0	1	2	3
1	6	4	2	1		
0			1	2	4	6

Hence the minimal representative of $R\pi_*\mathcal{F}$ is the complex

$$0 \longrightarrow A^1 \xrightarrow{\alpha} A^1 \longrightarrow 0$$

for some map α (which is not hard to guess, but which we will compute to illustrate our method in this easy example.)

From the Betti table we see directly that the regularity of $\mathcal{F}$ is 2, and we can compute $R\pi_*\mathcal{F}$ starting from the map of modules

$$\phi_2 : E \otimes \mathrm{H}^0\mathcal{F}(2) \to E \otimes \mathrm{H}^0\mathcal{F}(3)$$

over the exterior algebra E.

We write $x, y \in W = \pi_*\mathcal{O}(1)$ for fiber coordinates on $\mathbb{P}^1_A$, where now $\mathcal{O}$ denotes the structure sheaf of $\mathbb{P}^1_A$, and e, f for their dual coordinates in E. The sheaf $\mathcal{F}(2)$ is an extension

$$0 \to \mathcal{O} \to \mathcal{F}(2) \to \mathcal{O}(2) \to 0,$$

Lifting a basis for $\mathrm{H}^0\mathcal{O}(2)$, we may choose a basis of the free A-module $\mathrm{H}^0\mathcal{F}(2)$ denoted by $1, x^2, xy, y^2$. In terms of this basis, a presentation matrix may be written

$$\begin{array}{c} 1 \\ x^2 \\ xy \\ y^2 \end{array} \begin{pmatrix} -ax & 0 \\ y & 0 \\ -x & y \\ 0 & -x \end{pmatrix}$$

We choose as basis of $\mathrm{H}^0\mathcal{F}(3)$ the elements

$$x \cdot 1,\ y \cdot 1,\ x \cdot x^2,\ x \cdot xy,\ x \cdot y^2,\ y \cdot y^2.$$

From the given relations we see that $y \cdot xy = x \cdot y^2$. However, $y \cdot x^2 = x \cdot xy + a(x \cdot 1)$. Thus, in terms of these bases, the map ϕ_2 has matrix

$$\begin{array}{c} \\ x \cdot 1 \\ y \cdot 1 \\ x \cdot x^2 \\ x \cdot xy \\ x \cdot y^2 \\ y \cdot y^2 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & x^2 & xy & y^2 \end{array} \\ \begin{pmatrix} e & af & 0 & 0 \\ f & 0 & 0 & 0 \\ 0 & e & 0 & 0 \\ 0 & f & e & 0 \\ 0 & 0 & f & e \\ 0 & 0 & 0 & f \end{pmatrix} \end{array}$$

The further syzygy matrices are

$$\phi_1 = \begin{pmatrix} ef & af & 0 \\ 0 & e & 0 \\ 0 & f & e \\ 0 & 0 & f \end{pmatrix}, \phi_0 = \begin{pmatrix} a & e & f \\ e & 0 & 0 \\ f & 0 & 0 \end{pmatrix}, \phi_{-1} = \begin{pmatrix} fe & 0 & 0 & 0 \\ af & e & f & 0 \\ 0 & 0 & e & f \end{pmatrix},$$

Hence $\phi_{-1} =$ transpose ϕ_1 (in the sense appropriate to the exterior algebra), and $\phi_{-2} =$ transpose ϕ_2 as well.

Finally, by Theorem 0.1, $R\pi_*\mathcal{F} = (A \otimes_E \mathbf{T}(\mathcal{F}))_0$ is the complex

$$0 \longrightarrow A^1 \xrightarrow{(a)} A^1 \longrightarrow 0.$$

Similar computations can be done very quickly for much larger examples by `Macaulay2`.

For a further theoretical treatment we choose the family of globally generated vector bundles rank r and degree d on $\mathbb{P}^1$. The most special bundle in this family is $\mathcal{F}_0 = \mathcal{O}^{r-1} \oplus \mathcal{O}(d)$. Every other bundle in this family arises as an extension

$$0 \to \mathcal{O}^{r-1} \to \mathcal{F}_a \to \mathcal{O}(d) \to 0,$$

with $a \in \operatorname{Ext}^1(\mathcal{O}(d), \mathcal{O}^r) \cong \mathrm{H}^1(\mathcal{O}^{r-1}(-d)) \cong \mathrm{H}^0(\mathcal{O}(d-2)^{r-1})^*$. Thus as a base space of this family we may choose $\operatorname{Spec} A$ with

$$A = K[a_i^s,\, 0 \le i \le d-2, 1 \le s \le r-1],$$

and then take $\mathcal{F}$ to be the universal extension on $\mathbb{P}^1 \times A$.

We will specify $\mathcal{F}$ explicitely via its Beilinson monad. The Tate resolution has Betti table

$j \setminus i$	-d-1	-d	-d+1	...	-2	-1	0	1
1	d(r-1)+r	d(r-1)	(d-1)(r-1)	...	2(r-1)	r-1		
0		1	2	...	d-1	d	d+r	d+2r

at the special point $0 \in \operatorname{Ext}^1(\mathcal{O}(d), \mathcal{O}^{r-1})$. A few examples computed with `Macaulay2` make it possible to guess the pattern of the differentials, which we now going to verify. The differentials of

$$\mathbf{T}(\mathcal{O}): \qquad \cdots \xrightarrow{{}^tC^3} E^2(3) \xrightarrow{{}^tC^2} E(2) \xrightarrow{(ef)} E \xrightarrow{C^1} E^2(-1) \xrightarrow{C^2} E^3(-2) \xrightarrow{C^3} \cdots$$

are given by special $(\ell+1) \times \ell$ *Toeplitz* (or *Hankel*) matrices

$$C^\ell = \begin{pmatrix} e & 0 & & \dots & 0 \\ f & e & \ddots & & \vdots \\ 0 & f & \ddots & \ddots & \\ \vdots & \ddots & \ddots & e & 0 \\ & & \ddots & f & e \\ 0 & \dots & & 0 & f \end{pmatrix}$$

and there transposed. Hence $\mathbf{T}(\mathcal{F})$ is a deformation of the complex $\mathbf{T}(\mathcal{F}_0) = \oplus_1^{r-1}\mathbf{T}(\mathcal{O}) \oplus \mathbf{T}(\mathcal{O})[d](d)$ build from Toeplitz matrices. To describe the most relevant piece we consider for pairs (k, ℓ) with $k + \ell = d$ the $k \times \ell$ Hankel matrices

$$B^s_{k\ell} := \begin{pmatrix} a^s_0 & a^s_1 & a^s_2 & \dots & a^s_{\ell-1} \\ a^s_1 & a^s_2 & a^s_3 & \dots & a^s_\ell \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a^s_{k-1} & a^s_k & a^s_{k+1} & \dots & a^s_{k+\ell-2} \end{pmatrix}.$$

Proposition 4.2. *The $-(k+1)$ th differential*

$$E(k+2)^{(k+1)(r-1)} \oplus E(k+1)^\ell \xrightarrow{D} E(k+1)^{k(r-1)} \oplus E(k)^{\ell+1}\ ,$$

in the Tate resolution of $\mathcal{F}$ for $1 \le k \le d-1$ and $\ell = d-k$ is given by the block matrix

$$D = D^{-k-1} = \left(\begin{array}{cccc|c} {}^tC^k & 0 & \dots & 0 & B^1_{k\ell} \\ 0 & {}^tC^k & \ddots & \vdots & B^2_{k\ell} \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \dots & 0 & {}^tC^k & B^{r-1}_{k\ell} \\ \hline 0 & \dots & 0 & 0 & -C^\ell \end{array}\right)$$

(in suitable coordinates).

Proof. We first prove that the matrices $D^{-d}, D^{-d+1}, \dots, D^{-2}$ define a complex $\mathbf{T}$. Indeed $D^{-k} \cdot D^{-k-1} = 0$ holds because

$${}^tC^{k-1} \cdot B^s_{k\ell} - B^s_{k-1,\ell+1} \cdot C^\ell = 0.$$

With the relative Beilinson monads we can recover the corresponding coherent sheaf from any two consecutive matrices: The monad $\mathbf{U}(\mathbf{T}(-k)[-k])$ is the total complex of a double complex

$$\begin{array}{ccc} \mathcal{O}(-1)^{k(r-1)} & \longrightarrow & \mathcal{O}^{(k-1)(r-1)} \\ {\scriptstyle B_\ell}\uparrow & & \uparrow{\scriptstyle B_{\ell+1}} \\ \mathcal{O}(-1)^\ell & \longrightarrow & \mathcal{O}^{\ell+1} \end{array}$$

whose rows do not depend on the parameters. Here

$$B_\ell = \begin{pmatrix} B^1_{k\ell} \\ \vdots \\ B^{r-1}_{k\ell} \end{pmatrix}.$$

The homology of the top row, which is a subcomplex, is

$$\mathcal{O}(-k)^{r-1} = \ker(\mathcal{O}(-1)^{k(r-1)} \to \mathcal{O}^{(k-1)(r-1)}),$$

while the coorresponding quotient complex, the bottom row, has homology

$$\operatorname{coker}(\mathcal{O}(-1)^{\ell} \to \mathcal{O}^{\ell+1}) = \mathcal{O}(\ell) = \mathcal{O}(d-k).$$

Thus $\mathrm{H}^*\mathbf{U}(\mathbf{T}(-k)[-k]) = \mathrm{H}^0\mathbf{U}(\mathbf{T}(-k)[-k]) =: \mathcal{F}_k$ and the homology of $\mathbf{U}(\mathbf{T}(-k)[-k])$ fits into a short exact sequence

$$(*_k) \qquad 0 \to \mathcal{O}(-k)^{r-1} \longrightarrow \mathcal{F}_k \longrightarrow \mathcal{O}(d-k) \to 0\,.$$

Since the complex defined by the $D^{-d}, \ldots, D^{-2}$ has the right Betti number and the matrices have linearly independent rows, we conclude that they form part of the Tate resolution of a single sheaf $\mathcal{F}$, and, that $\mathcal{F}_k = \mathcal{F}(-k)$ with the extensions $(*_{k+1}) = (*_k) \otimes \mathcal{O}(-1)$. To prove that $\mathcal{F}$ is the universal extension, it suffices to prove this for anyone of the sheaves $\mathcal{F}_k = \mathcal{F}(-k)$. We choose $\mathcal{F}_d = \ker(\mathcal{O}(-1)^{d(r-1)} \oplus \mathcal{O} \overset{D^{-d}}{\to} \mathcal{O}^{(d-1)(r-1)})$. The boundary map in

$$\to \mathrm{H}^0(\mathbb{P}^1_A, \mathcal{F}_d) \to \mathrm{H}^0(\mathbb{P}^1_A, \mathcal{O}) \to \mathrm{H}^1(\mathbb{P}^1_A, \mathcal{O}(-d)^{r-1}) \to$$

is the composition

$$\mathrm{H}^0(\mathbb{P}^1_A, \mathcal{O}) \xrightarrow{B_1} \mathrm{H}^0(\mathbb{P}^1_A, \mathcal{O}^{(d-1)(r-1)}) \cong \mathrm{H}^1(\mathbb{P}^1_A, \mathcal{O}(-d)).$$

So the boundary map vanishes at a point a iff all coordinates a_i^s vanish at a. We conclude, that the a_i^s represent linearly independent extension classes, and, since we have the right number $(d-1)(r-1)$ of parameters, that $(*_d)$ is the universal extension. $\square$

Corollary 4.3. *Let $\mathcal{F}$ on $\mathbb{P}^1_A$ be the universal extension*

$$0 \to \mathcal{O}^{r-1} \to \mathcal{F} \to \mathcal{O}(d) \to 0$$

Then for each k in the range $1 \leq k \leq d-1$ the direct image complex of $\mathcal{F}(-k-1)$ is

$$R\pi_*\mathcal{F}(-k-1) : 0 \to A^{d-k} \xrightarrow{B_{d-k}} A^{k(r-1)} \to 0 \text{ with } B_{d-k} = \begin{pmatrix} B^1_{k,d-k} \\ \vdots \\ B^{r-1}_{k,d-k} \end{pmatrix}$$

Outside this range the direct image complexes are concentrated in one degree.

The corollary allows to describe the loci of extension classes of a given splitting type in $\mathrm{Ext}^1(\mathcal{O}(d), \mathcal{O}^{r-1})$ by rank conditions on the matrices B_k in the various direct image complexes.

We treat the example $(d, r) = (6, 3)$. The possible splitting type correspond to partition of d into at most r parts. In our special case this are the following strata with an arrow $p \to q$ indicating that the strata p lies in the closure of the strata q:

$$\begin{array}{cc}
\underline{\text{Partitions}} & \underline{\text{Ranks}} \\
\begin{tikzcd}
 & (6,0,0) \arrow[d] & \\
 & (5,1,0) \arrow[d] & \\
 & (4,2,0) \arrow[dl] \arrow[dr] & \\
(4,1,1) \arrow[dr] & & (3,3,0) \arrow[dl] \\
 & (3,2,1) \arrow[d] & \\
 & (2,2,2) &
\end{tikzcd}
&
\begin{tikzcd}
 & r_1 = 0 \arrow[d] & \\
 & r_2 < 2 \arrow[d] & \\
 & r_3 < 3, r_5 < 2 \arrow[dl] \arrow[dr] & \\
r_3 < 3 \arrow[dr] & & r_5 < 2 \arrow[dl] \\
 & r_4 < 4 \arrow[d] & \\
 & \text{open strata} &
\end{tikzcd}
\end{array}$$

In which strata an extension a lies is determined by the ranks $r_i = \operatorname{rank} B_i(a)$ of the $(d-i)(r-1) \times i$ matrices B_i evaluated at a.

$j \setminus i$	-7	-6	-5	-4	-3	-2	-1	0
1	15	12	10	8	6	4	2	
0		1	2	3	4	5	6	9

Indeed by the Base Change Theorem 3.4 and its Corollary 3.5 the r_i determine the dimensions $h^0(\mathbb{P}^1, \mathcal{F}_a(-i-1))$, which in turn determine the splitting type according to the following elementary Lemma.

Lemma 4.4. *Let $\mathcal{E}$ be a vector bundle on $\mathbb{P}^1$, and let*

$$h = h_{\mathcal{E}} : \mathbb{Z} \to \mathbb{Z},\ n \mapsto h^0\mathcal{E}(n)$$

be its Hilbert function. Then

$$\mathcal{E} \cong \oplus_{j \in \mathbb{Z}} \mathcal{O}(-j)^{h''(j)}\,,$$

where $h''(j) = h(j) - 2h(j-1) + h(j-2)$ denotes the second difference function. □

The claim on the strata indicated in the table above follows. Note that there is no single matrix, whose rank determine all splitting types, and for one strata there is not a single matrix on which a rank condition gives the defining equations.

To exhibit the beautiful pattern in this family of matrices more visibly, we drop the upper index notation a_i^s and use coordinates $a_0, \dots, a_4, b_0, \dots, b_4$ instead. With this notation we have

$$B_5 = \left(\begin{array}{ccccc} a_0 & a_1 & a_2 & a_3 & a_4 \\ \hline b_0 & b_1 & b_2 & b_3 & b_4 \end{array}\right), B_4 = \left(\begin{array}{cccc} a_0 & a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 & a_4 \\ \hline b_0 & b_1 & b_2 & b_3 \\ b_1 & b_2 & b_3 & b_4 \end{array}\right), B_3 = \left(\begin{array}{ccc} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ \hline b_0 & b_1 & b_2 \\ b_1 & b_2 & b_3 \\ b_2 & b_3 & b_4 \end{array}\right).$$

There are various relations between the ideals of minors of these matrices. The most interesting one is the primary decomposition of ideal 3×3 minors of the square matrix B_4:

$$minors(3, B_4) = minors(3, B_3) \cap minors(2, B_5).$$

We discovered this relation by computation using Macaulay2, which provides a proof in a few positive characteristics; the relation was recently proven non-computationally by Moty Katzman (private communication).

In terms of projective geometry the closed strata have the following descriptions as cones over projective varieties:

$$\begin{array}{ccccc}
 & \underline{\text{Geometry:}} & & \text{codim} & \text{degree}\\
 & \emptyset & & 10 & 1\\
 & \downarrow & & & \\
 & S(4,4) & & 7 & 8\\
 & \downarrow & & & \\
 & Sec(\mathbb{P}^1 \hookrightarrow \mathbb{P}^4)\times\mathbb{P}^1 & & 5 & 12\\
 \swarrow & & \searrow & & \\
Sec(S(4,4)) & & \mathbb{P}^4\times\mathbb{P}^1 & 4 & 15,5\\
 \searrow & & \swarrow & & \\
 & Sec^3(S(4,4)) & & 1 & 4\\
 & \downarrow & & & \\
 & \mathbb{P}^9 & & 0 & 1
\end{array}$$

Here $S(4,4) \subset \mathbb{P}^4 \times \mathbb{P}^1 \subset \mathbb{P}^9$ denotes the 2-dimensional rational normal scroll defined by the 2×2 minors of the matrix

$$^tB_2 = \left(\begin{array}{cccc|cccc} a_0 & a_1 & a_2 & a_3 & b_0 & b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 & a_4 & b_1 & b_2 & b_3 & b_4 \end{array} \right),$$

and $Sec(X)$ respectively $Sec^3(X)$ refers to the secant respectively 3-secant variety of X. Note that the fibers of $S(4,4) \to \mathbb{P}^1$ are rational normal curves of degree 4, and that

$$S(4,4) \cong \mathbb{P}^1 \times \mathbb{P}^1 \xhookrightarrow{|(4,1)|} \mathbb{P}^9.$$

Returning to the general case we conjecture:

Conjecture 4.1. *Let $B_1, \ldots, B_{d-1}$ be the non-trivial matrices of the direct image complexes of the versal deformation of $\mathcal{O}^{r-1} \oplus \mathcal{O}(d)$. For any collection of positive integers $r_{k_1}, \ldots, r_{k_s}$, the ideal*

$$\sum_{t=1}^{s} minors(r_{k_t}, B_{k_t})$$

is radical.

The case $r = 2$ was asserted by Room [1938], and proven in characteristic 0 by Peskine and Szpiro. See Conca [1998] for a general proof and other references.

The minimal primes of the ideal

$$\sum_{k=1}^{d-1} minors(r_k, B_k)$$

are easy to describe, and are (radicals of) ideals of the same form. First of all, the locus of extensions

$$0 \to \mathcal{O}^{r-1} \to \mathcal{E} \to \mathcal{O}(d) \to 0$$

on $\mathbb{P}^1_K$ such that $\mathcal{E}$ has a given splitting type, $\mathcal{E} \cong \oplus\mathcal{O}(a_i)$, is always irreducible. Its closure is locus of extensions such that each twist of $\mathcal{E}$ has at least as many global sections as the corresponding twist of $\oplus\mathcal{O}(a_i)$. The corresponding prime ideal is thus the radical of the corresponding sum of ideals $minors(r_k, B_k)$. Conversely, any sum of the ideals $minors(r_k, B_k)$ defines the locus of extensions such that various twists of $\mathcal{E}$ have at least a certain number of independent global sections. With some care one can give the irredundant decompositions in terms of splitting types.

In particular, the prime ideals of the form

$$Radical\Big(\sum_{k=1}^{d-1} minors(r_k, B_k)\Big)$$

are precisely the primes that define the closures of the strata of points $p \in \operatorname{Spec} A$ where $\mathcal{F}_p$ has a given splitting type. If the conjecture is true, of course, these sums of determinantal ideals are already radical.

One can see from this analysis that an ideal $minors(r_{k_t}, B_{k_t})$ can have components of different dimensions; in particular, it need not be Cohen-Macaulay.

It is worth remarking that one can also treat this family of examples without exterior methods. For example, from the exact sequence $0 \to \mathcal{O}^{r-1}(-k) \to \mathcal{F}(-k) \to \mathcal{O}(d-k) \to 0$ we get a triangle of direct images that expresses $R\pi_*\mathcal{F}(-k)$ as the mapping cone of a certain map

$$R\pi_*\mathcal{O}(d-k) \to R\pi_*\mathcal{O}^{r-1}(-k)[1].$$

At most one of the modules $R^i\pi_*\mathcal{O}(d-k) = \mathrm{H}^i(\mathcal{O}(d-k))$ is nonzero, and similarly for $\mathcal{O}^{r-1}(-k)$, so each of $R\pi_*\mathcal{O}(d-k)$ and $R\pi_*\mathcal{O}^{r-1}(-k)[1]$ reduces to a single free module. In the "interesting" range $-d \le k \le -2$ where both these modules are nonzero, the map between the modules is the connecting homomorphism $\mathrm{H}^0(\mathcal{O}(d-k)) \to \mathrm{H}^1(\mathcal{O}^{r-1}(-k))$ that we have called B_{d-k+1}. This connecting homomorphism is easy to compute concretely, especially since the computation reduces to the case $r = 1$.

Example 4.5 (Blow-up of an elliptic singularity). For an example that seems much harder to treat by simple methods, consider the singularity defined by

$$B = \{abc + a^4 + b^4 + c^4 = 0\} \subset \mathbb{A}^3.$$

The singularities of B are resolved by blowing up the origin $\sigma : \widetilde{\mathbb{A}}^3 \subset \mathbb{P}^2 \times \mathbb{A}^3 \to \mathbb{A}^3$ once. We consider the strict $\overline{B} = \overline{\sigma^{-1}(B \setminus \{0\})} \subset \mathbb{P}^2 \times B$ and the total transform $B' = \sigma^{-1}(B)$ of B. The Tate resolution of $\mathcal{O}_{\overline{B}}$ has Betti table

$j \setminus i$	-4	-3	-2	-1	0	1	2	3	4
1	15	12	9	6	3	1			
0	27	21	15	9	4	3	6	9	12
-1	21	15	9	4	4	9	15	21	*
-2	15	9	4	4	9	15	21	*	*
$\vdots$	$\iddots$	$\iddots$	$\iddots$	$\iddots$	$\iddots$	$\iddots$	$\iddots$	$\iddots$	

with eventually periodic diagonals by [Eisenbud, 1980].

The Tate resolution of $\mathcal{O}_{B'}$ looks quite different:

$j \setminus i$	-4	-3	-2	-1	0	1	2	3	4
2	10	6	3	1					
1	15	8	3						
0	6	3	1	1	1	3	6	10	15
-1					3	8	15	24	*
-2				1	3	6	10	*	*

with bounded $\mathbf{T}^k$, although $A = \mathcal{O}_{B,0}$ has not finite projective dimension. A closer inspection of the complexes gives

$$R\pi_*\mathcal{O}_{B'}(k) = R\sigma_*\mathcal{O}_{\widetilde{\mathbb{A}}^3}(k) \otimes_{\mathcal{O}_{\mathbb{A}^3}} \mathcal{O}_B,$$

a formula which holds, although the Base Change Theorem 3.4 does not apply. So for $k \geq 0$ we have

$$R^0\pi_*\mathcal{O}_{B'}(k) = \mathfrak{m}^k_{\mathbb{A}^3,0} \otimes_{\mathcal{O}_{\mathbb{A}^3,0}} \mathcal{O}_B,$$

while

$$R^0\pi_*\mathcal{O}_{\overline{B}}(k) = \mathfrak{m}^k_{B,0}.$$

Example 4.6 (Variety of complexes). One might hope that the direct image of a vector bundle on $\mathbb{P}^n_A$, say in the case of a local ring A, would have special properties compared to an arbitrary complex of free A-modules. But it turns out that such images are general; in fact one can get any complex as the push-forward of quite a simple bundle:

Theorem 4.7. *Every bounded minimal free complex*

$$0 \to A^{\beta_0} \to A^{\beta_1} \to \ldots \to A^{\beta_n} \to 0$$

over a local Noetherian ring arises as the direct image complex of a locally free sheaf on $\mathbb{P}^n_A$.

By flat base change, it suffices to prove the result for the *generic complex*

$$\mathbb{F}: \qquad 0 \to B^{\beta_0} \to \cdots \to B^{\beta_n} \to 0$$

defined over the ring

$$B = \mathbb{Z}[a^p_{ij}; 1 \le i \le \beta_{p+1}, 1 \le j \le \beta_p, p = 0, \ldots, n-1]/(\,(a^{p+1}_{ij})(a^p_{jk}) = 0\,),$$

with the map $B^{\beta_p} \to B^{\beta_{p+1}}$ given by the map with matrix (a^p_{jk}). The ring B is the affine coordinate ring of the "variety of complexes"(see for example [DS 1981]). The next Theorem is thus a strengthening of Threorem 4.7.

Theorem 4.8. *The generic complex* $\mathbb{F}$ *over the ring* B *above is the direct image of the versal deformation of the vector-bundle*

$$\bigoplus_0^n (\wedge^p \Omega_{\mathbb{P}^n_{\mathbb{Z}}/\mathbb{Z}})^{\beta_p}.$$

Note that Example 4.1 is the special case of Theorem 4.8 a complex of length 1 with free modules of rank 1,

$$0 \to B^1 \to B^1 \to 0.$$

Proof. To simplify the notation, we write Ω^i for $\wedge^i \Omega_{\mathbb{P}^n_{\mathbb{Z}}/\mathbb{Z}}$. We first show that the ring B (or more properly its completion at the origin) is the base of the versal deformation $\mathcal{F}$ of the vector bundle $\mathcal{F}_0 = \bigoplus_{p=0}^n (\Omega^p)^{\beta_p}$.

Proposition 4.9.

$$\mathrm{Ext}^1(\Omega^p,\Omega^q)=\begin{cases}\mathrm{H}^1(\Omega^1)=\mathbb{Z} & \text{if } 1\le q=p+1\le n\\ 0 & \text{otherwise}\end{cases}$$

$$\mathrm{Ext}^2(\Omega^p,\Omega^q)=\begin{cases}\mathrm{H}^2(\Omega^2)=\mathbb{Z} & \text{if } 2\le q=p+2\le n\\ 0 & \text{otherwise}\end{cases}$$

Thus the tangent space of the versal deformation of $\mathcal{F}_0$ is

$$\mathrm{Ext}^1(\mathcal{F}_0,\mathcal{F}_0)=\bigoplus_{p=0}^{n-1}\mathrm{Hom}(\mathbb{Z}^{\beta_p},\mathbb{Z}^{\beta_{p+1}}),$$

and the obstruction space is

$$\mathrm{Ext}^2(\mathcal{F}_0,\mathcal{F}_0)=\bigoplus_{p=0}^{n-2}\mathrm{Hom}(\mathbb{Z}^{\beta_p},\mathbb{Z}^{\beta_{p+2}}).$$

The obstruction map is the given by composition

$$(\phi_0,\ldots,\phi_{n-1})\mapsto(\phi_1\cdot\phi_0,\ldots,\phi_{n-1}\cdot\phi_{n-2}),$$

and the base space B of the deformation is the coordinate ring of the variety of complexes of free modules of ranks $\beta_0,\ldots,\beta_n$.

Proof. The generator of $\mathrm{Ext}^1(\Omega^{p-1},\Omega^p)$ corresponds to the extension

$$0\to\Omega^p\to\bigwedge^p(\mathcal{O}^{n+1}(-1))\to\Omega^{p-1}\to 0$$

that appears in the Koszul complex, and the generator of $\mathrm{Ext}^2(\Omega^{p-1},\Omega^{p+1})$ can also be realized in the Koszul complex as the extension

$$0\to\Omega^{p+1}\to\bigwedge^{p+1}(\mathcal{O}^{n+1}(-1))\to\bigwedge^p(\mathcal{O}^{n+1}(-1))\to\Omega^{p-1}\to 0,$$

which is thus the Yoneda product of the generators of $\mathrm{Ext}^1(\Omega^{p-1},\Omega^p)$ and $\mathrm{Ext}^1(\Omega^p,\Omega^{p+1})$.

The quadratic obstruction map

$$\mathrm{Ext}^1(\mathcal{F}_0,\mathcal{F}_0)\to\mathrm{Ext}^2(\mathcal{F}_0,\mathcal{F}_0)$$

is the map given by squaring, using the Yoneda product. The only nonzero contributions come from the maps

$$\begin{array}{c}\mathrm{Ext}^1((\Omega^{p-1})^{\beta_{p-1}},(\Omega^p)^{\beta_p})\times \mathrm{Ext}^1((\Omega^p)^{\beta_p},(\Omega^{p+1})^{\beta_{p+1}})\\ \downarrow \\ \mathrm{Ext}^2((\Omega^{p-1})^{\beta_{p-1}},(\Omega^{p+1})^{\beta_{p+1}})\end{array}$$

which, with natural choice of bases, is matrix multiplication by the computation above, so we see that the obstructions to second-order deformation are as claimed.

To see that there are no higher terms in the equations of the base space of the versal deformation, we observe that $\mathcal{F}_0$ can be graded by giving a summand Ω^p degree p. With this grading, each nonzero element of $\mathrm{Ext}^1(\mathcal{F}_0,\mathcal{F}_0)$ is of degree 1 and the elements of $\mathrm{Ext}^2(\mathcal{F}_0,\mathcal{F}_0)$ are similarly of degree 2. Since the versal deformation must be homogeneous for this grading, no higher-degree terms can occur in the equations. □

We also make use of the computation of the Tate resolution associated to the sheaf Ω^p: from [EFS 03], Proposition 5.5, we know it has the form

$j \setminus i$	$\dots$	$p-2$	$p-1$	p	$p+1$	$p+2$	$\dots$
n	$\dots$	u_{p-2}	u_{p-1}	0	0	0	$\dots$
$\vdots$							
$p+2$				0			
$p+1$				0	$\iddots$		
p				1			
$p-1$			$\iddots$	0			
$p-2$				0			
$\vdots$							
0	$\dots$	0	0	0	v_{p+1}	v_{p+2}	$\dots$

with $u_{p-1}=\binom{n+1}{n+1-p}$ and $v_{p+1}=\binom{n+1}{p+1}$.

It remains to show that the generic complex over B is $R\pi_*\mathcal{F}$, where $\mathcal{F}$ on $\mathbb{P}^n_A$ is the versal deformation of $\mathcal{F}_0=\oplus_{j=1}^{\beta_p}\Omega^p$ on $\mathbb{P}^n_{\mathbb{Z}}$.

From the Betti table above we see that the Tate resolution of $\mathcal{F}_0$ has a Betti table of the following shape, where the entries not shown in the center of the table are all zero:

$j \setminus i$	-1	0	1	2	$\dots$	n-1	n	n+1	$\dots$
n	δ_{-1}	δ_0	δ_1	δ_2	$\dots$	δ_{n-1}	β_n		
n-1						β_{n-1}			
$\vdots$					$\iddots$				
2				β_2					
1			β_1						
0		β_0	γ_1	γ_2	$\dots$	γ_{n-1}	γ_n	γ_{n+1}	$\dots$

Because the base ring B is naturally graded, we can use the result for local rings in Corollary 3.5 to conclude that the Tate resolution of $\mathcal{F}$ has the same entries. We want to prove that the component

$$E^{\beta_p} \to E^{\beta_{p+1}}$$

of the differential $\mathbf{T}(\mathcal{F})^p \to \mathbf{T}(\mathcal{F})^{p+1}$, which appears on the diagonal, is given by the matrix (a_{ij}^p).

Consider the subspace corresponding to the factor ring $B_p = B/I_p$ with ideal I_p generated by the linear forms $(a_{ij}^q; q \neq p)$. The restriction of the family $\mathcal{F}$ to this subspace is

$$\mathcal{F} \otimes \mathcal{O}_{B_p} \cong \left(\oplus_{q \neq p, p+1} (\Omega^q)^{\beta_q}\right) \oplus \mathcal{G}$$

where $\mathcal{G}$ is the versal deformation of

$$\mathcal{G}_0 = (\Omega^p)^{\beta_p} \oplus (\Omega^{p+1})^{\beta_{p+1}}.$$

The Tate resolution of $\mathcal{G}$ is a deformation of the Tate resolution of $\mathcal{G}_0$; the shape of its Tate resolution can be deduced from that of Ω^p and that of Ω^{p+1} as above.

We focus on the two differentials

$$\begin{array}{ccccc} E^{\beta_{p+1}(n-p)\binom{n+2}{n+1-p}}(n+1-p) & & E^{\beta_{p+1}\binom{n+1}{n-p}}(n-p) & & E^{\beta_{p+1}} \\ \oplus & \xrightarrow{c} & \oplus & \xrightarrow{d} & \oplus \\ E^{\beta_p\binom{n+1}{n+1-p}}(n+1-p) & & E^{\beta_p} & & E^{\beta_p\binom{n+1}{p+1}}(-p-1) \end{array},$$

with

$$c = \begin{pmatrix} c_1 & c_{12} \\ 0 & c_2 \end{pmatrix} \text{ and } d = \begin{pmatrix} d_1 & d_{12} \\ 0 & d_2 \end{pmatrix}.$$

By [EFS, 2003] Proposition 5.5 we know that c_2 is a direct sum of β_p copies of the $1\times\binom{n+1}{n+1-p}$ matrix consisting of all monomials of degree $n+1-p$ in the exterior variables $e_0,\dots e_n$, and that d_2 consist of a direct sum β_p copies of the $\binom{n+1}{p+1}\times 1$ matrix consisting of all monomials of degree $p+1$ in the exterior variables. (So $d_2\cdot c_2=0$ because the composition has degree $n+2$.) Similarly d_1 is a direct sum of β_{p+1} copies of the $\binom{n+1}{n-p}\times 1$ matrix of all monomials of degree $n+1-(p+1)$ in $e_0,\dots,e_n$ and c_1 consists of β_{p+1} copies of the linear syzygies matrix of these monomials.

The deformation sits in the components c_{12} and d_{12} of the matrices corresponding to the extensions. We want to prove that

$$d_{12}=(a^p_{ij}).$$

If we take this choice for d_{12} then we can build a complex by taking c_{12} as a suitable matrix of bihomogeneous forms in the variables a^p_{ij} and $e_0,\dots,e_n$, because $(e_0,\dots,e_n)^{n+1-p}\subset(e_0,\dots,e_n)^{n-p}$. The two differentials, deformed in this way, extend to a map of (doubly infinite) resolutions, and thus to a deformation of the whole Tate resolution. This defines a sheaf $\mathcal{G}'$ on $\mathbb{P}^n\times B_p$.

We argue now directly that $\mathcal{G}'$ over B_p defined in this way is the versal deformation of $\mathcal{G}_0$. Indeed for any other deformation $\mathcal{G}''$ over $\operatorname{Spec} T$ of $\mathcal{G}_0$ the direct image complex of $\mathcal{G}''$ induces a morphism $\varphi:\operatorname{Spec} T\to B_p$ by taking the substitution $\mathbb{Z}[a^p_{ij}]\to T$ obtained from the matrix d''_{12} in the complex $R\pi_*\mathcal{G}''$ on $\operatorname{Spec} T$, and $\mathcal{G}''\cong(\mathrm{id}_{\mathbb{P}^n}\times\varphi)^*\mathcal{G}$. This proves the (semi) universal property of $\mathcal{G}'$. It is universal because the grading of $\mathcal{G}'$ and its base ring $\mathbb{Z}[a^p_{ij}]$ given by degree in the a^p_{ij} prevents there being any automorphisms except for conjugation of the maps (a^p_{ij}) by invertible matrices in the obvious way. Thus $\mathcal{G}'=\mathcal{G}$.

The Base Change Theorem 3.4 and the grading of $\mathcal{F}$ and its base ring $\mathbb{Z}[a^p_{ij}; p=0,\dots,n-1]$ by degree in the a^p_{ij} shows that (a^p_{ij}) occurs as a differential in $R\pi_*\mathcal{F}$, since $\mathcal{G}$ is a summand of $\mathcal{F}\otimes\mathcal{O}_{B_p}$, as required. $\square$

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