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Order ideals and a generalized Krull height theorem

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Abstract. Let N be a finitely generated module over a Noetherian local ring $(R, \mathbf{m})$. We give criteria for the height of the order ideal $N^*(x)$ of an element $x \in N$ to be bounded by the rank of N. The Generalized Principal Ideal Theorem of Bruns, Eisenbud and Evans says that this inequality always holds if $x \in \mathbf{m}N$. We show that the inequality even holds if the hypothesis becomes true after first extending scalars to some local domain and then factoring out torsion. We give other conditions in terms of residual intersections and integral closures of modules. We derive information about order ideals that leads to bounds on the heights of trace ideals of modules—even in circumstances where we do not have the expected bounds for the heights of the order ideals!

Introduction

Let $(R, \mathbf{m})$ be a Noetherian local ring and let N be a finitely generated R-module. For $x \in N$ we define the *order ideal* of x, written $N^*(x)$, to be the set of images of x under homomorphisms $N \to R$.

The classical Krull height theorem (Krull 1928) says that r elements of R either generate an ideal of height at most r, or the unit ideal. This may be interpreted by saying that if N is free of rank r, then the order ideal of any element $x \in \mathbf{m}N$ has height at most r. Eisenbud and Evans (1976) conjectured that the same statement would be true for any module N; they proved the conjecture for rings containing a field, and Bruns (1981) subsequently gave a general argument. This work leaves the question addressed in the present paper:

Under which circumstances does the order ideal of a minimal generator of N have height at most the rank of N?

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We give criteria to settle this question in many cases, and use them in turn to prove related results bounding the heights of some residual intersections and trace ideals. An interesting feature is the need for studying Rees algebras and integral dependence of modules.

To get a feeling for the central question, consider the case where R is a standard graded Noetherian ring over an infinite field, N is a finitely generated graded R-module that represents a vector bundle $\mathcal{E}$ on $X = \mathrm{Proj}(R)$, and N is generated (perhaps only as a sheaf) by elements of degree 0. In this setting the order ideal of an element of degree 0 is just the vanishing locus of the corresponding section of the vector bundle; and there is a section which vanishes in codimension greater than r if and only if $\mathcal{E}$ admits a sub-bundle isomorphic to $\mathcal{O}_X$ if and only if the top Chern class $c_r(\mathcal{E})$ vanishes (for information on Chern classes see Fulton (1984, Chapter 3)).

For example, consider the cotangent bundle $\Omega^1_{\mathbf{P}^{n-1}}$ of projective $n-1$-space. It has no global sections, but its twist $\mathcal{E} = \Omega^1_{\mathbf{P}^{n-1}}(2)$ is a bundle of rank $n-1$ generated by global sections. The corresponding module N over the polynomial ring R in n variables is the kernel of the map $R^n(1) \to R(2)$ sending the i^{th} basis element to the i^{th} variable; it is generated in degree 0. The exact sequence

$$0 \to \mathcal{E} \to \mathcal{O}^n(1) \to \mathcal{O}(2) \to 0$$

and the fact that the "Chern polynomial"

$$c_t(\mathcal{E}) = 1 + c_1(\mathcal{E})t + c_2(\mathcal{E})t^2 + \cdots \in A^*(\mathbf{P}^{n-1}) = \mathbf{Z}[t]/(t^n)$$

is multiplicative show that $c_t(\mathcal{E}) \equiv (1+t)^n/(1+2t) \bmod (t^n)$. In particular $c_{n-1}(\mathcal{E}) = \sum_{i=0}^{n-1}(-1)^i\binom{n}{n-1-i}2^i$. A little arithmetic gives

$$c_{n-1}(\mathcal{E}) = \begin{cases} 0 & \text{if } n \text{ is even;} \\ 1 & \text{if } n \text{ is odd.} \end{cases}$$

Thus we expect N to have an order ideal of height $\geq n$ if and only if n is even. In Section 3 we present an algebraic analysis proving this result (and settling a number of related cases).

A first suggestion of the role of integrality in the theory is illustrated by a question of Huneke and Koh: They asked whether the order ideals of elements in the integral closure of $\mathbf{m}N$ always have height at most the rank of N. We prove a still more general statement; the following is a special case of Theorem 3.1:

Theorem. *Let R be an affine domain, let N be a finitely generated R-module of rank r, and let $x \in N$. Let $R \to S$ be a homomorphism from R to a Noetherian local domain $(S, \mathbf{n})$ and write SN for $S \otimes_R N$ modulo S-torsion. If the image of x in SN lies in $\mathbf{n}SN$, then the height of $N^*(x)$ is at most the rank of N.*

In Section 3 we provide systematic methods for constructing examples where $N^*(x)$ has height greater than $\mathrm{rank}(N)$. We illustrate our methods by constructing, among other things, a graded module N of rank 5 over a polynomial ring in 6 variables such that every homogeneous element of N has order ideal of height at most 5, but N contains inhomogeneous elements with order ideal of height 6!

To describe the more refined results of the paper we continue with the assumption that R is local. A basic construction of this paper (in a special case) is that of the *perpendicular module* of N: If F is a free module of minimal rank n with a surjection $\pi : F \to N$ we define $N^\perp = \mathrm{Coker}(\pi^*)$. Set $M = N^\perp$. We observe (Remark 2.2) that the order ideals of generators of N correspond to the colon ideals of the form $U :_R M = \{a \in R \mid aM \subset U\} = \mathrm{ann}(M/U)$ where $U \subset M$ is a $n-1$ generated submodule with M/U cyclic. Thus the existence of order ideals of elements of N having extraordinary height is the same as the existence of submodules $U \subset M$ as above with $U :_R M$ of extraordinary height. A classical argument of McAdam (1983), as generalized in Section 1, shows under mild assumptions that M is then integral over U (see Theorem 1.2); in particular the analytic spread $\ell(M)$ of M is strictly less than the minimal number of generators $\mu(N)$ (see Section 1 for the definitions of integral dependence and analytic spread). Under some circumstances we show that the condition $\ell(M) = \mu(N)$ is actually necessary and sufficient for all $x \in N$ to have order ideals of height $\leq \mathrm{rank}(N)$ (Propositions 3.6 and 3.9).

Using the related constructions, we are able to deduce information about colon ideals from information on order ideals and vice versa. An example is the following special case of Proposition 4.3, which gives conditions under which all "residual intersections" of a module have the expected height:

Proposition. *Let R be a regular local ring containing a field and let M be a finitely generated torsion free R-module of rank e. If s is an integer such that $\mathrm{Ext}^i_R(M, R) = 0$ for $2 \leq i \leq s-1$ then, for every submodule $U \subsetneq M$ with $\mu(U) - e + 1 \leq s$,*

$$\mathrm{ht}(U :_R M) \leq \max\{0, \mu(U) - e + 1\}.$$

In Section 4 we also investigate order ideals of modules of low rank. We show under mild hypotheses that if N has rank ≤ 2, or N is a k^{th} syzygy module of rank k, then all elements of N have order ideals of height $\leq \mathrm{rank}(N)$ (Proposition 4.1); under somewhat more stringent conditions we get a similar result for modules of rank 3 (Proposition 4.2).

In the last section we consider the relationship of trace ideals and order ideals. In the case of a module of rank 3 (or a k^{th} syzygy module of rank $k+1$) satisfying mild conditions there may well be order ideals that are too large; but we prove that the radical of such an order ideal must contain the whole *trace ideal* of N, defined as the sum of all order ideals $\mathrm{tr}(N) = N^*(N) = \{f(x) \mid f \in N^* \text{ and } x \in N\}$

(Proposition 5.4). Finally we turn to the question of the possible heights of trace ideals. The surprising result is that we can give a stronger bound for the height of the trace ideal of N if the height of some order ideal exceeds rank(N) than in the contrary case (Theorem 5.5).

1. Rees algebras

In this section we recall the general notion of Rees algebra of a module introduced in our (2003), and provide some results about integral dependence.

Let R be a Noetherian ring and let M be a finitely generated R-module. If Q is a prime ideal of R we write $\mu_Q(M)$ for the minimal number of generators of M_Q over R_Q. When $(R, \mathbf{m})$ is a local ring we set $\mu(M) = \mu_{\mathbf{m}}(M)$. By $-^*$ we denote the functor $\mathrm{Hom}_R(-, R)$. We say that an R-linear map $f : M \to F$ is a *versal map* from M to a free module if F is a free R-module and f^* is surjective. The latter condition means that every R-linear map from M to a free R-module factors through f. In our (2003, 0.1 and 1.3) we define the *Rees algebra* of M to be $\mathcal{R}(M) = \mathrm{Sym}(M)/(\cap_g L_g)$ where the intersection is taken over all maps g from M to free R-modules, and L_g denotes the kernel of $\mathrm{Sym}(g)$. Equivalently, $\mathcal{R}(M)$ is isomorphic to the image of the map $\mathrm{Sym}(f) : \mathrm{Sym}(M) \to \mathrm{Sym}(F)$, where $f : M \to F$ is any versal map to a free module. The Rees algebra of an arbitrary finitely generated module exists and is unique up to canonical isomorphisms of graded R-algebras; in fact the construction is functorial. On a less trivial note we prove in our (2003, 1.4 and 1.6) that the above definition gives the usual notion of Rees algebras of ideals, and that over a $\mathbb{Z}$-torsion free ring R any embedding $g : M \to G$ into a free module G can be used to define the Rees algebra $\mathcal{R}(M)$ as the image of $\mathrm{Sym}(g)$.

Let in addition $U \subset L$ be submodules of M, let U', L' be the images of U, L in $\mathcal{R}(M)$, and consider the R-subalgebras $R[U'] \subset R[L']$ of $\mathcal{R}(M)$. According to our (2003, 2.1) we say L is *integral* over U *in* M if the ring extension $R[U'] \subset R[L']$ is integral; the largest such module L (which exists and is unique) is called the *integral closure* of U *in* M; finally, we say M is *integral* over U or U is a *reduction* of M, if M is integral over U in M. In our (2003, 2.2) we record the following "valuative criterion of integrality":

Proposition 1.1. *Let R be a Noetherian ring, let M be a finitely generated R-module, let $U \subset L$ be submodules of M, and let $f : M \to F$ be a versal map from M to a free R-module. The following are equivalent:*

(1) *L is integral over U in M.*
(2) *For every minimal prime Q of R, the module L' is integral over U' in M', where $'$ denotes images in F/QF.*
(3) *For every map $M \to G$ to a free R-module and for every homomorphism $R \to S$ to a domain S, the module L' is integral over U' in M', where $'$ denotes tensoring with S and taking images in $S \otimes_R G$.*

(4) *For every homomorphism $R \to V$ to a rank one discrete valuation ring V whose kernel is a minimal prime of R, we have $U' = L'$, where $'$ denotes tensoring with V and taking images in $V \otimes_R F$.*
(5) *For every map $M \to G$ to a free R-module and every homomorphism $R \to V$ to a rank one discrete valuation ring V, we have $U' = L'$, where $'$ denotes tensoring with V and taking images in $V \otimes_R G$.*

Let R be a Noetherian local ring with residue field k and let M be a finitely generated R-module. In our (2003, 2.3) we define the *analytic spread* $\ell(M)$ of M to be the Krull dimension of $k \otimes_R \mathcal{R}(M)$. In case k is infinite one has $\ell(M) = \min\{\mu(U) \big| U$ a reduction of $M\}$; furthermore $\ell(M) \leq \mu(M)$ and equality holds if and only if M admits no proper reduction.

The next theorem is due to McAdam (1983, 4.1) in the case of ideals. Various cases with modules are treated in Rees (1987, 2.5), Kleiman and Thorup (1994, 10.7), Katz (1995, 2.4), and Simis, Ulrich and Vasconcelos (2001, 5.6). Our proof is a reduction to the case treated by Rees. The result plays an important role in this paper.

Theorem 1.2. *Let R be a locally equidimensional universally catenary Noetherian ring, let M be a finitely generated R-module, and let U be a submodule of M generated by t elements. If there exists a minimal prime Q of R such that M/QM is not integral over the image of U in M/QM, then*

$$\mathrm{ht}(U :_R M) \leq \max\{0, t + 1 - \mu_Q(M)\}.$$

Proof. We localize to assume that R is local and equidimensional. Write $R' = R/Q$, $M' = M/QM$, and let U' be the image of U in M'. We have $\mathrm{ht}(U :_R M) \leq \mathrm{ht}(U' :_{R'} M')$, because $R'(U :_R M) \subset U' :_{R'} M'$ and R is an equidimensional catenary local ring. Thus we may replace R, U, M by R', U', M' to assume that R is a universally catenary local domain. We may also suppose that M is torsion free and that $U :_R M \neq 0$, hence $\mathrm{rank}(U) = \mathrm{rank}(M)$. With these assumptions the assertion was proved by Rees (1987, 2.5). □

Corollary 1.3. *Let R be an equidimensional universally catenary Noetherian local ring, let M be a finitely generated R-module, and let U be a proper submodule of M generated by t elements. If $\ell(M) = \mu(M)$, then there exists a minimal prime Q of R such that M/QM is not integral over the image of U in M/QM. For any such Q,*

$$\mathrm{ht}(U :_R M) \leq \max\{0, t + 1 - \mu_Q(M)\}.$$

Proof. We may assume that the residue field of R is infinite. It follows from our hypothesis that M is not integral over U. By Proposition 1.1 and the functoriality of the Rees algebra, there exists a minimal prime ideal Q such that M/QM is not integral over the image of U. The assertion now follows from Theorem 1.2. □

2. Perpendicular modules

Central to this paper is the following:

Definition 2.1. Let R be a Noetherian ring and let N be a finitely generated R-module with a choice of generators $x_1, \dots, x_n$. Map a free R-module with basis $e_1, \dots, e_n$ onto N by sending e_i to x_i and denote this map by π. We define $[x_1, \dots, x_n]^\perp = \mathrm{Coker}(\pi^*)$, and write $x_i^\perp$ for the image of e_i^* in $\mathrm{Coker}(\pi^*)$. When R is local and the $x_1, \dots, x_n$ are minimal generators, we set $N^\perp = [x_1, \dots, x_n]^\perp$ and call it the *perpendicular module* to N (indeed, this module only depends on N).

We remark that every perpendicular module is torsionless (contained in a free module). Conversely, any finitely generated torsionless module is the perpendicular module of a finitely generated torsionless module with respect to some set of generators: If M is an R-module generated by $x_1, \dots, x_n$ then $[x_1^\perp, \dots, x_n^\perp]^\perp$ is the image of the natural map $M \to M^{**}$, which gives the identification $M = [x_1^\perp, \dots, x_n^\perp]^\perp$ in case M is torsionless.

Remark 2.2. Perpendicular modules and colons. With notation as in 2.1, let $M = [x_1, \dots, x_n]^\perp$ and let U be the submodule of M generated by $x_1^\perp, \dots, x_{i-1}^\perp, x_{i+1}^\perp, \dots, x_n^\perp$. We have

$$N^*(x_i) = U :_R M.$$

Indeed, the right hand side is clearly equal to the i^{th} row ideal (the ideal generated by the elements of the i^{th} row) of any matrix φ presenting M with respect to the generating set $x_1^\perp, \dots, x_n^\perp$. The i^{th} row ideal equals $N^*(x_i)$ because the image of an element f of N^* under π^* has i^{th} component equal to $f(x_i)$. More generally, for any $x = \sum_{i=1}^n a_i x_i \in N$ with $\underline{a} = [a_1, \dots, a_n] \in R^n$ one has $N^*(x) = R(\underline{a} \cdot \varphi)$.

An immediate consequence is that if R is local and N has no free summand, then $\mu(M) = n$. More generally, if Q is any prime of R we have $\mu_Q(M) = n - \mathrm{rf}_Q(N)$, where $\mathrm{rf}_Q(N)$ denotes the maximal rank of an R_Q-free direct summand of N_Q.

Combining these remarks with Theorem 1.2 we obtain the following consequence for order ideals:

Corollary 2.3. *Let R be a locally equidimensional universally catenary Noetherian ring, let N be a finitely generated R-module, and let Q be a minimal prime of R. Choose a generating set $x_1, \dots, x_n$ of N, write $M = [x_1, \dots, x_n]^\perp$, and let $U = Rx_1^\perp + \dots + Rx_{n-1}^\perp \subset M$. If M/QM is not integral over the image of U, then* $\mathrm{ht}(N^*(x_n)) \le \mathrm{rf}_Q(N)$.

Proof. Remark 2.2 shows that $N^*(x_n) = U :_R M$, and by Theorem 1.2, $\mathrm{ht}(U :_R M) \le n - \mu_Q(M)$. Finally, $n - \mu_Q(M) = \mathrm{rf}_Q(N)$. □

The following version of the semicontinuity theorem for heights of ideals in a family will be useful in bounding the heights of order ideals:

Proposition 2.4. *Let $(R, \mathbf{m})$ be an equidimensional universally catenary Noetherian local ring, let $\underline{Z} = Z_1, ..., Z_n$ be indeterminates over R, and write $R' = R(\underline{Z})$.*

(1) *If φ is a matrix over R with n rows, then* $\mathrm{ht}(R(\underline{a} \cdot \varphi)) \le \mathrm{ht}(R'(\underline{Z} \cdot \varphi))$ *for every vector $\underline{a}$ of n elements in R.*
(2) *If N is an R-module with generating set $x_1, \ldots, x_n$ and $y = \sum_{i=1}^{n} Z_i \otimes x_i \in R' \otimes_R N$, then* $\mathrm{ht}(N^*(x)) \le \mathrm{ht}((R' \otimes_R N)^*(y))$ *for every $x \in N$.*

Proof. To see (1) write $S = R[\underline{Z}]_{(\mathbf{m}, \{Z_i - a_i\})}$. The ring $R/(\underline{a} \cdot \varphi)$ is obtained from $S/(\underline{Z} \cdot \varphi)$ by factoring out the ideal generated by the S-regular sequence $Z_i - a_i, 1 \le i \le n$. Since S is equidimensional and catenary, it follows that $\mathrm{ht}(R(\underline{a} \cdot \varphi)) \le \mathrm{ht}(S(\underline{Z} \cdot \varphi))$. Localizing further we deduce (1).

To prove (2) we apply (1) to a matrix φ presenting $[x_1, \ldots, x_n]^\perp$ with respect to the generating set $x_1^\perp, \ldots, x_n^\perp$, and invoke Remark 2.2. □

Let N be a finitely generated module over a Noetherian ring R. We say that N satisfies G_s, where s is a positive integer, if N_Q is free of constant rank r for every minimal prime Q of R and $\mu_Q(N) \le \dim(R_Q) + r - 1$ for every prime Q of R with $1 \le \dim(R_Q) \le s - 1$. In case G_s holds for every s, the module N is said to satisfy G_∞.

We say that N has a *rank* and write $\mathrm{rank}(N) = r$ if N_Q is free of rank r for every associated prime Q of R. The module N is said to be *orientable* if N has rank r and $(\Lambda^r N)^{**} \cong R$. This differs slightly from the definition given in Bruns (1987, p. 882). Our definition implies that N_Q is the direct sum of a free R_Q-module and a torsion module for every prime Q such that $\mathrm{depth}(R_Q) \le 1$. Thus if two modules in a short exact sequence are orientable, then so is the third as long as the right-hand module is torsion free locally in depth one.

Proposition 2.5. *With notation as in 2.1, let $M = [x_1, \ldots, x_n]^\perp$. The module N has a rank or is orientable if and only if M has the same property.*

Proof. We use the exact sequence

$$0 \to N^* \xrightarrow{\pi^*} (R^n)^* \longrightarrow M \to 0 \tag{2.6}$$

from which it follows that M has a rank if and only if N^* has a rank. If Q is an associated prime of R, then N_Q is free if and only if N_Q^* is free. (To see this, notice that if $f \in N_Q^*$ is a free generator, then the image of $f : N_Q \to R_Q$ is faithful, and hence f is surjective.) Thus N^* has a rank if and only if N does.

Since M is torsion free, (2.6) shows that M is orientable if and only if N^* is. But N^* is orientable if and only if N is. □

3. Principal ideal theorems

We are now ready to prove our first main result. Recall that $\mathrm{rf}_Q(N)$ denotes the maximal rank of a free summand of N_Q.

Theorem 3.1. Generalized height theorem. *Let R be a locally equidimensional universally catenary Noetherian ring, let N be a finitely generated R-module, and let $x \in N$. Let $R \to S$ be a homomorphism of rings from R to some Noetherian local domain $(S, \mathbf{n})$ and write SN for $S \otimes_R N$ modulo S-torsion. If the image of x in SN lies in $\mathbf{n}SN$, then*

$$\mathrm{ht}(N^*(x)) \leq \min_Q\{\mathrm{rf}_Q(N)\}$$

where the minimum is taken over all minimal primes Q of R mapping to zero in S.

Proof. Replacing S by a rank one discrete valuation ring V containing S and centered on $\mathbf{n}$ we may assume that $S = V$. Notice that the order ideal of $1 \otimes x$ in the V-module $V \otimes_R N$ is a proper ideal.

Choose a generating set $x_1, \ldots, x_n = x$ of N and a presentation $R^m \xrightarrow{\psi} R^n$ of N with respect to this generating set. Consider the perpendicular module $M = \mathrm{Im}(\psi^*) = [x_1, \ldots, x_n]^\perp \subset R^{m*}$ and its submodule $U = Rx_1^\perp + \cdots + Rx_{n-1}^\perp$. Let Q be a minimal prime of R mapping to zero in $S = V$. According to Corollary 2.3 it suffices to prove that M/QM is not integral over the image of U.

In fact, writing $M' = \mathrm{Im}(\mathrm{Hom}_R(\psi, V)) = \mathrm{Im}(V \otimes_R \psi^*) \subset V \otimes_R R^{m^*}$, we have $M' = [1 \otimes x_1, \ldots, 1 \otimes x_n]^\perp$ and M/QM maps to M' with the image of $x_i^\perp$ being sent to $(1 \otimes x_i)^\perp$. Thus by the equivalence of (1) and (5) in Proposition 1.1, it suffices to show that $U' \neq M'$ for $U' = V(1 \otimes x_1)^\perp + \cdots + V(1 \otimes x_{n-1})^\perp$ (see also Rees (1987, 1.5(ii))). However according to Remark 2.2, $U' :_V M'$ is the order ideal of $1 \otimes x$ in $V \otimes_R N$. Since this ideal is proper we conclude that $U' \neq M'$. □

Theorem 3.1 says that whenever the height of the order ideal of x exceeds the expected value, then the injection of Rx into N must be 'valuatively split', meaning that after passing to an arbitrary valuation, the induced map does split. Put differently, either the order ideal of x has the expected height or else the height of the order ideal over any valuation ring becomes infinite. The set of elements having this property, but not splitting themselves, is a remarkable class, and exactly the class we wish to study.

Corollary 3.2. *Let R be a locally equidimensional universally catenary Noetherian ring and let N be a finitely generated R-module. Let $x \in N$ and suppose that* $\mathrm{ht}(N^*(x)) > \max_Q\{\mathrm{rf}_Q(N)\}$*, where the maximum is taken over all minimal primes Q of R.*

(1) *If $N' \to N$ is an arbitrary epimorphism of R-modules and $x' \in N'$ is an element mapping to x, then for every integer j, $\operatorname{Fitt}_{j+1}(N')$ is integral over $\operatorname{Fitt}_j(N'/Rx')$.*
(2) *If I is an arbitrary integrally closed ideal in R, then the natural map $(R/I) \otimes_R (Rx) \to (R/I) \otimes_R N$ is injective.*

Proof. To prove (1), we need to verify that for every homomorphism from R to a discrete valuation ring V, $V \operatorname{Fitt}_{j+1}(N') = V \operatorname{Fitt}_j(N'/Rx')$, or equivalently, $\operatorname{Fitt}_{j+1}(V \otimes_R N') = \operatorname{Fitt}_j(V \otimes_R N'/V(1 \otimes x'))$. However by Theorem 3.1, the image of x generates a free V-summand of rank one in VN. Thus $1 \otimes x'$ generates a free V-summand of rank one in $V \otimes_R N'$, and the asserted equality of Fitting ideals is obvious.

To prove (2) we have to show that $Rx \cap IN \subset Ix$. Let $x_1, \dots, x_n = x$ be a generating set of N. If $r \in R$ and $rx \in IN$, then $rx = \sum_{i=1}^n s_i x_i$ for some $s_i \in I$. Hence $s_1 x_1 + \dots + s_{n-1} x_{n-1} + (s_n - r)x = 0$. Take N' to be the R-module presented by the transpose of the vector $[s_1, \dots, s_{n-1}, s_n - r]$, and let x' be the last generator. Part (1) shows that $s_n - r$ is in the integral closure of the ideal $(s_1, \dots, s_{n-1})$. Therefore $r \in I$, since $s_i \in I$ and I is integrally closed. Hence $rx \in Ix$ as asserted. □

Remark 3.3. In terms of matrices, Corollary 3.2(1) can be stated as follows: Let $\underline{x} = x_1, \dots, x_n$ be a generating set of N with $x_n = x$, let ψ be a matrix with n rows satisfying $\underline{x} \cdot \psi = 0$, and let ψ' be the matrix obtained from ψ by deleting the last row. Then for every integer i, $I_i(\psi)$ is integral over $I_i(\psi')$.

The second corollary answers in the affirmative a question asked by the second author and Jee Koh in the late 1980's:

Corollary 3.4. *Let $(R, \mathbf{m})$ be an equidimensional universally catenary Noetherian local ring and let N be a finitely generated R-module. If x lies in $\overline{\mathbf{m}N}$, the integral closure of $\mathbf{m}N$ in N, then $\operatorname{ht}(N^*(x)) \le \min\{\mu_Q(N)\}$, where the minimum is taken over all minimal primes Q of R so that N_Q is free.*

Proof. Let Q be a minimal prime of R so that N_Q is free. Choose a local embedding $R/Q \hookrightarrow V$, where $(V, \mathbf{n})$ is a rank one discrete valuation ring, and a versal map $f : N \to F$ from N to a free module. As x lies in the integral closure of $\mathbf{m}N$ in N, Proposition 1.1 shows that $x' \in \mathbf{m}N'$, where $'$ denotes tensoring with V and taking images in $V \otimes_R F$. On the other hand since N_Q is free, f_Q is split injective and therefore $N' \cong VN$ as defined in Theorem 3.1. Thus the image of x lies in $\mathbf{m}VN \subset \mathbf{n}VN$, and Theorem 3.1 immediately gives the conclusion. □

Of course, Theorem 3.1 gives as an immediate corollary the theorem of Bruns, Eisenbud, and Evans in case the ring is equidimensional and catenary. The usual proofs reduce to this case, and from the paper of Bruns (1981, Theorem 1) one obtains that if $(R, \mathbf{m})$ is a Noetherian local ring, N is a finitely generated R-module, and $x \in \mathbf{m}N$, then

$$\operatorname{ht}(N^*(x)) \leq \max_{Q}\{\mu_Q(N)\}$$

where the maximum ranges over the minimal primes Q of R. However, we observe that the proof reduces at once to the complete domain case and yields an even stronger result:

Theorem 3.5. *Let $(R, \mathbf{m})$ be a Noetherian local ring and let N be a finitely generated R-module. If $x \in \mathbf{m}N$, then* $\dim(R/N^*(x)) \geq \max_Q\{\dim(R/Q) - \operatorname{rf}_Q(N)\}$ *where the maximum is taken over all minimal primes Q of R. Thus*

$$\operatorname{ht}(N^*(x)) \leq \min_Q\{\dim(R) - \dim(R/Q) + \operatorname{rf}_Q(N)\}.$$

Proof. If R is an equidimensional complete local ring and $x \in \mathbf{m}N$, then the second inequality follows from Theorem 3.1, and implies the first assertion at once.

To prove the theorem for an arbitrary Noetherian local ring, we only need to show the first inequality. We first reduce to the case where R is complete by passing to the completion $\hat{R}$ of R. Let Q be any minimal prime in R and choose a minimal prime $\hat{Q}$ of $\hat{R}Q$ such that $\dim(\hat{R}/\hat{Q}) = \dim(R/Q)$. Notice that $\hat{Q}$ is a minimal prime in $\hat{R}$ contracting to Q. Since the map $R_Q \to \hat{R}_{\hat{Q}}$ is flat and local, it follows that $\operatorname{rf}_Q(N) = \operatorname{rf}_{\hat{Q}}(\hat{N})$. If the result holds for $\hat{R}$ and $\hat{N}$, we obtain

$$\dim(R/N^*(x)) = \dim(\hat{R}/\hat{N}^*(x)) \geq \dim(\hat{R}/\hat{Q}) - \operatorname{rf}_{\hat{Q}}(\hat{N}) = \dim(R/Q) - \operatorname{rf}_Q(N).$$

Henceforth we assume that R is complete.

Again choose an arbitrary minimal prime ideal Q in R, let $J \subset R$ denote the Q-primary component of 0, and let $K \subset R$ be the preimage of $(N/JN)^*(x)$, the order ideal of the image of x in the R/J-module N/JN. Clearly $N^*(x) \subset K$, hence $\dim(R/N^*(x)) \geq \dim(R/K)$. As R/J is complete and equidimensional we have $\dim(R/K) \geq \dim(R/Q) - \operatorname{rf}_{Q/J}(N/JN)$. Finally $\operatorname{rf}_{Q/J}(N/JN) = \operatorname{rf}_Q(N)$ since $J_Q = 0$. □

Next we wish to find conditions on a module N over a local ring which guarantee that $N^*(x)$ has the expected height for *every* $x \in N$. Obviously, such a module should not have any nontrivial free summands, which means that $\mu(N^{\perp}) = \mu(N)$. We can turn this necessary condition into a sufficient one if we replace $\mu(N^{\perp})$ by $\ell(N^{\perp})$:

Proposition 3.6. *Let R be an equidimensional universally catenary Noetherian local ring and let N be a finitely generated R-module. If $\ell(N^{\perp}) = \mu(N)$, then for every $x \in N$,*

$$\operatorname{ht}(N^*(x)) \leq \max_{Q}\{\operatorname{rf}_Q(N)\}$$

where the maximum is taken over all minimal primes Q of R.

Proof. By Theorem 3.5 we may assume that x can be extended to a minimal generating set $x_1, \ldots, x_n = x$ of N. Write $U = Rx_1^{\perp} + \cdots + Rx_{n-1}^{\perp} \subset M = [x_1, \ldots, x_n]^{\perp} \cong N^{\perp}$. As $\ell(M) = n$, M cannot be integral over U. Hence by Proposition 1.1 and the functoriality of the Rees algebra, there exists a minimal prime Q of R such that M/QM is not integral over the image of U. The assertion now follows from Corollary 2.3. □

Here is a result bounding the analytic spread of a module from below. We will ultimately use it to give a partial converse of Proposition 3.6.

Proposition 3.7. *Let R be a Noetherian local ring with infinite residue field and let M be a finitely generated R-module such that M_Q is free of rank e for every minimal prime Q of R. Let U be the submodule of M generated by t general linear combinations of a set of generators of M. If $\ell(M) \leq t$ and M satisfies G_{t-e+2}, then* $\mathrm{ht}(U :_R M) \geq t - e + 2$.

Proposition 3.7 follows at once from the following more general version, which we phrase as a lower bound for the analytic spread of a module. This bound sharpens the obvious inequality $\ell(M) \geq e$ that holds for any finitely generated module M over a Noetherian local ring R if M_Q is free of rank e for some minimal prime Q of R.

Proposition 3.7bis. *Let R be a Noetherian local ring with infinite residue field and let M be a finitely generated R-module. Let $X \subset \mathrm{Spec}(R)$ be the nonfree locus of M, and assume that for integers $e \geq 0$ and $t \geq 0$ we have $\mu_P(M) \leq \dim(R_P) + e - 1$ whenever $P \in X$ and $\dim(R_P) \leq t - e + 1$. Let U be the submodule of M generated by t general linear combinations of a set of generators of M. If $\mathrm{ht}(U :_R M) \leq t - e + 1$, then $\ell(M) \geq t + 1$.*

Proof. We may assume that $e > 0$; indeed, if $e = 0$ and $\mathrm{ht}(U :_R M) = t + 1$ we factor out an element of $U :_R M$ not in any minimal prime of R, thus reducing to the case $e = 1$. Using basic element theory one can then show that $M_P = U_P$ for every $P \in X$ with $\dim(R_P) = t - e + 1$, see for instance Miyazaki and Yoshino (2001, 3.2 and its proof). Now suppose that $\ell(M) \leq t$. Then U is a reduction of M and hence $M_P = U_P$ for every prime $P \notin X$. Thus $\mathrm{ht}(U :_R M) > t - e + 1$, which yields a contradiction. □

The following consequence of Proposition 3.7bis shows that under good circumstances the analytic spread is monotonic for inclusions:

Corollary 3.8. *Let R be an equidimensional universally catenary Noetherian local ring and let M be a finitely generated R-module such that M_Q is free of rank e for every minimal prime Q of R. Write $\ell = \ell(M)$ and assume that M satisfies $G_{\ell-e+1}$. If $M' \subset M$ is any submodule with $\mathrm{ht}(M' :_R M) \geq \ell - e + 1$, then $\ell(M') \geq \ell(M)$.*

Proof. We may assume that the residue field of R is infinite. Let $t = \ell - 1$, and let U be the submodule generated by t general linear combinations of a set of generators of M'. Since $t < \ell(M)$, the module M is not integral over U. Thus by Proposition 1.1 and Theorem 1.2, there exists a minimal prime Q of R such that $\mathrm{ht}(U :_R M) \leq \max\{0, t+1-\mu_Q(M)\} \leq t-e+1$. Since $\mathrm{ht}(M' :_R M) \geq t-e+2$ we have $\mathrm{ht}(U :_R M') = \mathrm{ht}(U :_R M) \leq t - e + 1$. Now Proposition 3.7bis gives $\ell(M') \geq t + 1 = \ell(M)$. □

Here is the promised partial converse of Proposition 3.6:

Proposition 3.9. *Let R be a Noetherian local ring with infinite residue field and let N be a finitely generated R-module such that N_Q is free of rank r for every minimal prime Q of R. Assume that $N^{\perp}$ satisfies G_∞. If* $\mathrm{ht}(N^*(x)) \leq r$ *for every $x \in N$, then $\ell(N^{\perp}) = \mu(N)$.*

Proposition 3.9 is an immediate consequence of the following more general result:

Proposition 3.9.bis. *Let R be a Noetherian local ring with infinite residue field and let N be a finitely generated R-module. Let $X \subset$* $\mathrm{Spec}(R)$ *be the nonfree locus of N, and assume that for an integer r we have* $\mathrm{rf}_P(N) \geq r + 1 - \dim(R_P)$ *whenever $P \in X$. If* $\mathrm{ht}(N^*(x)) \leq r$ *for every $x \in N$, then $\ell(N^{\perp}) = \mu(N)$.*

Proof. We may assume $N \neq 0$. Set $M = N^{\perp}$. Write $n = \mu(N) \geq \mu(M)$ and let U be a submodule of M generated by $n-1$ general linear combinations of generators of M. By Remark 2.2, there exists an element $x \in N$ such that $U :_R M = N^*(x)$. The latter ideal has height at most r by assumption. Applying Proposition 3.7bis with $e = n - r$ and $t = n - 1$ we conclude that $\ell(M) \geq n$, hence $\ell(M) = n$. □

Combining Propositions 3.6 and 3.9bis one obtains the following. Assume that R is an equidimensional universally catenary Noetherian local ring of dimension $d > 0$ with infinite residue field, and let N be a finitely generated R-module that is free of constant rank r locally on the punctured spectrum. For every $x \in N$ one has $\mathrm{ht}(N^*(x)) \leq r < d$ if and only if $\ell(N^{\perp}) = \mu(N)$. To see this also notice that $\ell(N^{\perp}) \leq \dim(R) + (\mu(N) - r) - 1 = \mu(N) - r + d - 1$ (see Simis, Ulrich and Vasconcelos (2003, the proof of 2.3)).

Proposition 3.9 gives a systematic way of constructing modules of rank r with elements whose order ideals have height exceeding r.

Example 3.10. Let R be a Noetherian local ring with infinite residue field and let I be an R-ideal of positive height. Suppose that I satisfies G_∞ and $\ell(I) \neq \mu(I)$. Let $N = I^{\perp}$. Notice that $\mu(N) = \mu(I)$, $N^{\perp} = I$, and N_Q is free of rank $r = \mu(I)-1$ for every minimal prime Q of R. By Proposition 3.9, N contains an element x such that the height of $N^*(x)$ is strictly greater than r. Of course $N^*(x)$ is proper, since N has no nontrivial free summand.

Consider a monomial curve in $\mathbf{P}^3_k$ that is not arithmetically Cohen-Macaulay and let I be its defining ideal localized at the homogeneous maximal ideal. The analytic spread of I is 3 by Gimenez, Morales and Simis (1993, 2.8), and I satisfies G_4 by Herzog (1970, 3.9). If in addition $\mu(I) = 4$, then I is G_∞ and $\ell(I) \neq \mu(I)$. The module $N = I^\perp$ has rank 3, and – at least if the ground field k is infinite – will have an order ideal of height 4.

To be explicit, let $I \subset R = k[a, b, c, d]_{(a,b,c,d)}$ be the localized defining ideal of the monomial curve $t \mapsto (1, t^{\alpha-1}, t^{\alpha+1}, t^{2\alpha})$, for even numbers $\alpha > 0$. It is easy to check that this curve lies on the smooth quadric $ad - bc = 0$, and has divisor class $(\alpha - 1, \alpha + 1)$. Its ideal is thus minimally generated by 4 elements and all the conditions above are satisfied. (Actually the same is true for any curve in this divisor class, monomial or not.) The module N may be explicitly described as the image of the right-hand map in the exact sequence

$$0 \to R \xrightarrow{\begin{pmatrix} ad - bc \\ c^{\alpha+1} - b^{\alpha-1}d^2 \\ ac^\alpha - b^\alpha d \\ b^{\alpha+1} - a^2c^{\alpha-1} \end{pmatrix}} R^4 \xrightarrow{\begin{pmatrix} -b^\alpha & 0 & a & c \\ -ac^{\alpha-1} & 0 & b & d \\ -b^{\alpha-1}d & a & -c & 0 \\ -c^\alpha & b & -d & 0 \end{pmatrix}} R^4$$

obtained by dualizing the first two steps of the minimal free resolution of R/I. From this we see at once that the third generator of N has order ideal (a, b, c, d), of height 4.

We now give an example of a graded module N in which all homogeneous elements have order ideals of height at most the rank of N, although there are inhomogeneous elements whose order ideals have bigger height.

Example 3.11. Let k be an infinite field and let $R = k[z_1, \dots, z_6]$ be a polynomial ring, graded with all the variables in degree 1. Set

$$I = I_2 \begin{pmatrix} z_1 & z_2^2 & z_3^2 & 0 \\ 0 & z_4^2 & z_5^2 & z_6^2 \end{pmatrix}.$$

The perpendicular module $N = I^\perp$ has rank 5. We claim first that all the order ideals of homogeneous elements of N have height at most 5. By Remark 2.2, Corollary 2.3 and Theorem 3.5 it suffices to show that the ideal I has no reductions generated by 5 homogeneous elements.

Suppose to the contrary that J is a reduction of I generated by 5 homogeneous elements. The ideal generated by the lowest degree part of I has no proper reduction. This ideal must be contained in J since the ideal generated by the lowest degree elements of J is a reduction of the ideal generated by the lowest degree elements of I. On the other hand, (z_1) contains the lowest degree part of I. Let $^-$ denote images in $R/(z_1)$. The ideal $\overline{J}$ is a reduction of $\overline{I}$. Thus $\overline{I}$ would have a reduction generated by 2 elements. However, this is impossible: $\overline{I}$ is generically a complete intersection of height 2, and not a complete intersection, so by Cowsik and Nori (1976) any reduction of $\overline{I}$ requires at least 3 generators.

On the other hand, if we regrade the ring with degree$(z_1) = 2$, all the generators of I become homogeneous of the same degree. Since the generators of I satisfy the Plücker relation, I has a reduction J generated by 5 elements that are homogeneous in the new grading and form part of a homogeneous minimal generating set of I. As I is a complete intersection on the punctured spectrum, the ideal $J : I$ has height 6, and thus N has an element whose order ideal has height 6 by Remark 2.2. This element is homogeneous in the new grading.

We finish this section with two classes of examples arising from the Koszul complex and the Buchsbaum-Rim complex, respectively.

Let $R = k[z_1, \dots, z_d]$ be a polynomial ring in $d > 1$ variables over a field k, graded with the variables in degree 1, and write Ω^i for the i^{th} syzygy module of the maximal ideal $\mathbf{m} = (z_1, \dots, z_d)$. The module Ω^i has minimal free presentation $\wedge^{i+2} R^d \to \wedge^{i+1} R^d$; in particular $\Omega^i/\mathbf{m}\Omega^i = \wedge^{i+1} k^d$. Using the self-duality of the Koszul complex, we see at once that $(\Omega^i)^\perp = \Omega^{d-i-2}$ for $0 \le i \le d-2$. Note that Ω^{d-1} is a free module.

Proposition 3.12. *If* $2 \le i \le d-2$, *or if* $i = 1$ *and* d *is odd, then all elements of* Ω^i *have order ideals of height at most the rank* $\binom{d-1}{i}$ *of* Ω^i.

Proof. We may localize R at $\mathbf{m}$. If $2 \le i \le d-3$ there is nothing to prove, since Ω^i has no free summand and its rank is at least the dimension of R. If $i = d-2$ then $(\Omega^i)^\perp = \mathbf{m}$, which is generated by d analytically independent elements. Similarly, if $i = 1$ and d is odd, then $(\Omega^1)^\perp = \Omega^{d-3}$ is generated by $\binom{d}{2}$ analytically independent elements according to Simis, Ulrich and Vasconcelos (1993, 3.1(b)). Now Proposition 3.6 yields the desired inequality in either case. □

We can prove a more precise result for homogeneous generators: We say that an element of Ω^1 has rank b if its image in $\Omega^1/\mathbf{m}\Omega^1 = \wedge^2 k^d$ represents a linear transformation $(k^d)^* \to k^d$ of rank b. Since these linear transformations are alternating, the rank b is an even number. Any homogeneous minimal generator of rank $2c$ can be written as $e_1 \wedge e_2 + \cdots + e_{2c-1} \wedge e_{2c}$, where the e_i are homogeneous minimal generators of R^d.

Proposition 3.13. *The height of the order ideal of a homogeneous minimal generator* x *of* Ω^1 *is equal to the rank of* x. *In particular, if* d *is even, there are elements of* Ω^1 *with order ideals of height* $d > \operatorname{rank}(\Omega^1) = d-1$.

Proof. Let $e_1, \dots, e_d$ be homogeneous generators of R^d. If $d = 2c$ is even, then the image of $e_1 \wedge e_2 + \cdots + e_{2c-1} \wedge e_{2c}$ in Ω^1 has rank $2c$, so the second statement follows from the first.

The module Ω^1 is the image of the map $\wedge^2 R^d \to \wedge^1 R^d$ in the Koszul complex of $z_1, \dots, z_d$. To prove the first statement, it suffices to consider the order ideal of the element x that is the image of $e_1 \wedge e_2 + \cdots + e_{2c-1} \wedge e_{2c}$. The column corresponding to $e_{2i-1} \wedge e_{2i}$ has $\pm z_{2i}$ and $\pm z_{2i-1}$ in the $2i - 1$ and $2i$ places, respectively. Thus x is mapped to an element of R^d whose nonzero coordinates are $\pm z_1, \ldots, \pm z_{2c}$. Since the dual of the Koszul complex is acyclic, the components of the inclusion map $\Omega^1 \to R^d$ generate all the maps from Ω^1 to R, and we see that the order ideal of x is $(z_1, \ldots, z_{2c})$. □

In contrast to Proposition 3.13 the next example shows that the kernel N of a generic map $R^s \to R^t$ has only order ideals of height at most $\operatorname{rank}(N)$ as long as $t > 1$.

Proposition 3.14. *Let $(R, \mathbf{m})$ be a local Gorenstein ring and let $t \leq s$ be integers. Let χ be a t by s matrix with entries in $\mathbf{m}$ and $\operatorname{ht}(I_t(\chi)) = s - t + 1$. Set $N = \operatorname{Ker}(\chi)$. Except in the case where $t = 1$ and s is even, $\operatorname{ht}(N^*(x)) \leq \operatorname{rank}(N)$ for every $x \in N$.*

Proof. We may assume that $s - t = \operatorname{rank}(N) > 1$. Let $x_1, \ldots, x_n$ be a minimal generating set of N, let $S = R[Z_1, \ldots, Z_n]$ be a polynomial ring, and write $y = \sum_{i=1}^n Z_i \otimes x_i \in S \otimes_R N$. Further, let $I = (S \otimes_R N)^*(y)$ be the order ideal of y and J its unmixed part. By Proposition 2.4(2) it suffices to show that $\operatorname{ht}(I_{S\mathbf{m}}) \leq \operatorname{rank}(N)$.

To this end write $M = N^\perp$. As in the proof of Remark 2.2 one sees that $\operatorname{Sym}(M) \cong S/I$. Since M satisfies G_∞, one has $\dim(\operatorname{Sym}(M)) = \dim(R) + \operatorname{rank}(M)$ by Huneke and Rossi (1986, 2.6(ii)), which gives $\operatorname{ht}(I) = \operatorname{rank}(N) = s - t$. In this setting, Migliore, Nagel and Peterson show in (1999, 1.5 and its proof) that $I = J$ if $s - t$ is even, whereas $J/I \cong \operatorname{Sym}_{(s-t-1)/2}(S \otimes_R \operatorname{Coker}(\chi))$ if $s - t$ is odd. Thus for $s - t > 1$ odd and $t > 1$ one has $\mu_{S\mathbf{m}}(J/I) > 1$, which yields $J_{S\mathbf{m}} \neq S_{S\mathbf{m}}$. Hence in either case

$$\operatorname{ht}(I_{S\mathbf{m}}) = \operatorname{ht}(J_{S\mathbf{m}}) = \operatorname{ht}(J) = \operatorname{ht}(I) = \operatorname{rank}(N). \qquad \square$$

4. Order ideals of low rank modules

It turns out that order ideals of elements in modules of low rank are particularly well-behaved. For example if N is a module of rank 1, then (modulo torsion) N is isomorphic to an ideal I containing a non zerodivisor. If $x \in I$ is a non zerodivisor of R then $I^*(x) = Rx :_R I$ which is either the unit ideal or of grade 1. If on the other hand x is a zerodivisor contained in an associated prime Q of R, then $I^*(x)$ is also contained in Q and thus has grade 0. The following propositions extend this kind of result to modules of rank 2 and 3 as well as k^{th} syzygies of rank k having finite projective dimension. The case of rank 2 modules over regular local rings had already been treated in Evans and Griffith (1982, p. 377).

Proposition 4.1. *Let R be a Noetherian ring and let N be a finitely generated R-module. Assume either*

(1) *N is orientable of rank* 2*; or*
(2) *R contains a field and N is a* k^{th} *syzygy of rank k having finite projective dimension.*

If $x \in N$ *then either* $N^*(x) = R$ *or* $\operatorname{grade}(N^*(x)) \leq \operatorname{rank}(N)$.

Proof. We may assume that R is local. Consider the exact sequence

$$0 \to Rx \to N \to X \to 0.$$

We suppose that $\operatorname{grade}(N^*(x)) > \operatorname{rank}(N)$. It follows that the annihilator of x is 0, and we will show that X is free. Hence $N^*(x) = R$ as required.

In case (1) we may assume that $N = N^{**}$. If we localize the above sequence at an arbitrary prime Q with $\operatorname{depth}(R_Q) \leq 2$, then the sequence splits and in particular X_Q is reflexive. If $\operatorname{depth}(R_Q) \geq 3$ then $\operatorname{depth}(X_Q) \geq 2$. It follows that X is orientable and reflexive of rank one, hence free, completing the proof in this case.

Now suppose we are in case (2). After localizing at a prime Q with $\operatorname{depth}(R_Q) \leq k$, the above sequence splits. Since N_Q is free by the Auslander-Buchsbaum formula, it follows that X_Q is free as well for any such Q. If Q is a prime with $\operatorname{depth}(R_Q) > k$, then $\operatorname{depth}(N_Q) \geq k$ and therefore $\operatorname{depth}(X_Q) \geq k$. Furthermore X has finite projective dimension. Thus by Hochster and Huneke (1990, 10.9), the module X is a k^{th} syzygy. It has rank $k-1$, so the version of the Evans-Griffith Syzygy Theorem due to Hochster and Huneke (1990, 10.8) and Evans and Griffith (1989, 2.4) implies that X is free. □

The assumption of orientability in Proposition 4.1(1) is necessary: Let k be a field, $R = k[Z_0, \ldots, Z_3]/(Z_0Z_3 - Z_1Z_2)$, and let z_i denote the image of Z_i in R. Let M be the ideal (z_0^2, z_0z_1, z_1^2). If $N = [z_0^2, z_0z_1, z_1^2]^{\perp}$ then N is an R-module of rank 2. By Remark 2.2 the element $(z_0z_1)^{\perp}$ has order ideal $(z_0, \ldots, z_3)$, which has height 3.

Proposition 4.2. *Let R be a Gorenstein ring and let N be an orientable R-module of rank* 3 *that satisfies* S_3 *and is free in codimension* 2*. If* $x \in N$ *then either* $N^*(x) = R$ *or* $\operatorname{ht}(N^*(x)) \leq 3$.

Proof. We may assume that R is local, and we write $n = \mu(N)$, $e = n - 3$. By Proposition 4.1 we may assume that N has no nontrivial free summand. We will prove that $\operatorname{ht}(N^*(x)) \leq 3$.

Suppose the contrary and write $M = N^{\perp}$. By Theorem 3.5, x can be extended to a minimal generating set of N. Using Remark 2.2, a row ideal in some minimal presentation matrix of M then has height > 3. Let $u_1, \ldots, u_n$ be generic elements in $R' \otimes_R M$ defined over a local ring R' that is obtained from R by a purely transcendental residue field extension, and set $F = R'u_1 + \cdots + R'u_{e-1} \subset U = R'u_1 + \cdots + R'u_{n-1}$. The genericity of $u_1, \ldots, u_n$ implies that $\operatorname{ht}(U :_{R'} (R' \otimes_R M)) > 3$ as shown in Proposition 2.4(1). To simplify notation we will write $R = R'$.

Because N is S_3, the Acyclicity Lemma of Peskine and Szpiro (1973, 1.8) shows that $\mathrm{Ext}^1_R(N^*, R) = 0$. From the exact sequence (2.6) we see that $\mathrm{Ext}^2_R(M, R) = 0$. Because N has no free summands, $\mu(M) = n$ and hence $U \neq M$. The module M is free locally in codimension 2 and torsion free, and by Proposition 2.5 it is orientable of rank e. Thus by Simis, Ulrich and Vasconcelos (2003, 3.2), F is free and M/F is isomorphic to an ideal $I \subset R$ with $\mathrm{ht}(I) \geq 2$ that is a complete intersection locally in codimension two. Clearly $\mathrm{Ext}^3_R(R/I, R) \cong \mathrm{Ext}^2_R(I, R) \cong \mathrm{Ext}^2_R(M, R) = 0$. Let J be the image of U in I. Notice that $\mu(J) = (n-1)-(e-1) = 3$. Furthermore $J \neq I$ and $\mathrm{ht}(J : I) > 3$, because $I/J \cong M/U$.

If $\mathrm{ht}(I) \geq 3$ then J is a complete intersection and $\mathrm{ht}(J : I) \leq 3$, and we are done. Otherwise $\mathrm{ht}(I) = 2$. Then the condition $\mathrm{Ext}^3_R(R/I, R) = 0$ implies that the factor ring of R by any link of I satisfies S_2; see Chardin, Eisenbud and Ulrich (2001, 4.4). Since we are in a case where I has height 2 and is a complete intersection in codimension 2, and $\mu(J) \leq 3$, we can apply Chardin, Eisenbud and Ulrich (2001, 3.4(a)) to obtain $\mathrm{ht}(J : I) \leq 3$ as required. □

In Proposition 4.2 the assumption of freeness in codimension 2 can be weakened to requiring that $\mathrm{rf}_Q(N) \geq 2$ whenever $\dim(R_Q) = 2$. However, the S_3 condition is necessary, as can be seen from the monomial curves discussed in Example 3.10.

We can use Proposition 4.1 to prove that, under a vanishing hypothesis on some $\mathrm{Ext}^i_R(M, R)$, the colon ideal $U :_R M$ has at most the expected grade. In preparation, recall that if M^* has a rank then M does too, see the proof of Proposition 2.5.

Proposition 4.3. *Let R be a Noetherian local ring containing a field, let M be a finitely generated torsion free R-module such that M^* and $\mathrm{Ext}^1_R(M, R)$ have finite projective dimension, and set $e = \mathrm{rank}(M)$. If s is an integer such that $\mathrm{Ext}^i_R(M, R) = 0$ for $2 \leq i \leq s-1$ then, for every submodule $U \subsetneq M$ with $\mu(U) - e + 1 \leq s$,*

$$\mathrm{grade}(U :_R M) \leq \max\{0, \mu(U) - e + 1\}.$$

Proof. We may assume that $\mathrm{rank}(U) = e$ since otherwise $\mathrm{grade}(U :_R M) = 0$. Then $s \geq \mu(U) - e + 1 \geq 1$. Lowering s if necessary, it suffices to prove that $\mathrm{grade}(U :_R M) \leq s$. If $s = 1$, then U is free and thus $\mathrm{grade}(U :_R M) \leq 1 = s$. Therefore we may assume that $s \geq 2$. Suppose that $\mathrm{grade}(U :_R M) > s$. Choose a submodule $U \subsetneq V \subset M$ such that V/U is cyclic. Clearly $\mathrm{grade}(U :_R V) > s$ and $\mathrm{grade}(V :_R M) > s$. The latter inequality implies that $\mathrm{Ext}^i_R(V, R) \cong \mathrm{Ext}^i_R(M, R)$ for every $0 \leq i \leq s-1$, forcing V^* and $\mathrm{Ext}^1_R(V, R)$ to have finite projective dimension and $\mathrm{Ext}^i_R(V, R) = 0$ for $2 \leq i \leq s-1$. We are free to replace M by V, and from now on we assume that M/U is cyclic.

Since M/U is cyclic and $\mu(U) \leq s+e-1$, there exists a generating set $u_1, \ldots, u_n$ of M such that the first $n-1$ elements generate U and $n = s+e$. The module M is torsionless since it is torsion free and has a rank. By the remark at the beginning of Section 2, there exists a finitely generated module N with generators $x_1, \ldots, x_n$ so that $M = [x_1, \ldots, x_n]^{\perp}$ and $u_i = (x_i)^{\perp}$. By Remark 2.2, $U :_R M = N^*(x_n)$. Using the exact sequence (2.6) we see that N^{**} has finite projective dimension since M^* and $\operatorname{Ext}^1_R(M, R)$ have finite projective dimension, and that N^{**} is an s^{th} syzygy since $\operatorname{Ext}^i_R(M, R) = 0$ for $2 \leq i \leq s-1$. As $\operatorname{rank}(N^{**}) = n-e = s$, Proposition 4.1(2) now proves that $\operatorname{grade}(N^*(x_n)) \leq s$, giving the conclusion. □

Corollary 4.4. *Let R be a regular local ring containing a field and let I be an ideal satisfying $\operatorname{Ext}^i_R(R/I, R) = 0$ for $3 \leq i \leq s$. Then for every ideal $J \subsetneq I$ with $\mu(J) \leq s$, $\operatorname{ht}(J : I) \leq \mu(J)$.*

Note that Corollary 4.4 is interesting only if the height of I is one or two, and reduces to the height two case. One should compare the corollary to the results of Chardin, Eisenbud, and Ulrich (2001, 3.4(a) and 4.2), which yield the same conclusion for ideals of any height g under the (incomparable) assumptions that I satisfies G_s and $\operatorname{Ext}^i_R(R/I^{i-g}, R) = 0$ for $g+1 \leq i \leq s$. We did not expect a result that avoids reference to the powers of I!

Corollary 4.5. *Let R be a Noetherian local ring containing a field, let M be a finitely generated torsion free R-module such that M^* and $\operatorname{Ext}^1_R(M, R)$ have finite projective dimension, and set $\operatorname{rank}(M) = e$. If s is an integer such that R satisfies S_{s+1}, the module M satisfies G_{s+1}, and $\operatorname{Ext}^i_R(M, R) = 0$ for $2 \leq i \leq s-1$, then*

$$\ell(M) \geq \min\{\mu(M), e+s\}.$$

Proof. We may assume that R has an infinite residue field. Write $n = \mu(M)$ and $t = \min\{n, e+s\} - 1$. We may suppose that $t \geq e$. Let U be a submodule of M generated by t general linear combinations of generators of M. As $t \leq n-1$ one has $U \neq M$ and therefore $\operatorname{ht}(U :_R M) \leq t-e+1$ by Proposition 4.3. But then Proposition 3.7bis implies that $\ell(M) \geq t+1$. □

5. Heights of trace ideals

The surprising fact pursued in this section may be informally summarized by saying that if an order ideal of an element in a module N is "bigger than it should be", then the trace ideal $\operatorname{tr}(N) = N^*(N)$ of N is not much larger than this order ideal.

Proposition 5.1. *Let R be a Noetherian ring, let $N = Ry_1 + \cdots + Ry_n$ be an R-module, and let x, y be elements of N. Write $Y_i = N/Ry_i$ and $Y = N/Ry$.*

(1) *If* $\operatorname{ht}(N^*(x)/Y^*(x)) > 1$*, then* $N^*(y) \subset \sqrt{N^*(x)}$.
(2) *If* $\operatorname{ht}(N^*(x)/Y_i^*(x)) > 1$ *for* $1 \le i \le n$*, then* $\sqrt{\operatorname{tr}(N)} = \sqrt{N^*(x)}$. *In particular* $\operatorname{ht}(\operatorname{tr}(N)) \le \mu(N^*)$, *unless* $\operatorname{tr}(N) = R$.

Proof. To prove (1) suppose there exists a prime ideal Q of R with $N^*(x) \subset Q$, but $N^*(y) \not\subset Q$. Replacing R by R_Q we may then assume that $N^*(x) \neq R$ and $N = R \oplus Y$. Writing $x = r + z$ with $r \in R$, $z \in Y$, we obtain $R \neq N^*(x) = Rr + Y^*(x)$. Thus $\operatorname{ht}(N^*(x)/Y^*(x)) \le 1$, which yields a contradiction. This proves (1).

Part (2) is an immediate consequence of (1) since $N^*(x) \subset \operatorname{tr}(N) = N^*(y_1) + \cdots + N^*(y_n)$. □

Remark 5.2. The height assumptions in Proposition 5.1 are automatically satisfied if R is locally equidimensional and catenary and if $\operatorname{ht}(N^*(x)) > \operatorname{bight}(Y^*(x)) + 1$ (for (1)) or $\operatorname{ht}(N^*(x)) > \operatorname{bight}(Y_i^*(x)) + 1$ (for (2)). Here $\operatorname{bight}(I)$ stands for the *big height* of an ideal I, which is the maximum of the heights of all minimal primes of I.

There is a corresponding statement for colon ideals, which the reader may want to compare to Corollary 3.2(1):

Proposition 5.3. *Let R be a Noetherian ring and let M be a finitely generated R-module. Let $U = Ru_1 + \cdots + Ru_s \subset M$ be a submodule and write $U_i = Ru_1 + \cdots + Ru_i$. If* $\operatorname{ht}(U_i :_R M/U_{i-1} :_R M) > 1$ *for* $1 \le i \le s$*, then* $\sqrt{\operatorname{Fitt}_{j+s}(M)} = \sqrt{\operatorname{Fitt}_j(M/U)}$ *for any* $j \ge 0$*. In particular* $V :_R M \subset \sqrt{U :_R M}$ *for every* s*-generated submodule* $V \subset M$.

Proof. The asserted equality is equivalent to the statement that $u_1, \ldots, u_s$ form part of a minimal generating set of M locally at each prime in the support of M/U. Thus it suffices to prove that for $1 \le i \le s$, the image of u_i is a minimal generator of M/U_{i-1} locally on the support of M/U_i. This reduces us to the case $s = 1$.

Suppose $u = u_1$ is not a minimal generator of M locally at a prime Q in the support of $M/U = M/U_1$. Replacing R by R_Q we may assume $(R, \mathbf{m})$ is local, $u \in \mathbf{m}M$, and $M/U \neq 0$. If φ is a matrix with n rows presenting M, we obtain a presentation matrix ψ of M/U by adding one column with entries in $\mathbf{m}$. Notice that $I_n(\psi) \neq R$. Now by Bruns (1981, Corollary 1) (see also Eisenbud and Evans (1976, 2.1)), $\operatorname{ht}(I_n(\psi)/I_n(\varphi)) \le 1$, which gives $\operatorname{ht}(\operatorname{ann}(M/U)/\operatorname{ann}(M)) \le 1$, contrary to our assumption. □

In general one has the inclusion $\operatorname{Fitt}_j(M/U) \subset \operatorname{Fitt}_{j+s}(M)$ for any s-generated submodule U of a finitely generated module M. One may ask which power of $\operatorname{Fitt}_{j+s}(M)$ is contained in $\operatorname{Fitt}_j(M/U)$ under the assumptions of Proposition 5.3.

The height assumption in Proposition 5.3 implies that $\operatorname{ht}(U :_R M/0 :_R M) \geq 2s$. On the other hand a lower bound for $\operatorname{ht}(U :_R M/0 :_R M)$ alone does not suffice to deduce the equality $\sqrt{\operatorname{Fitt}_{j+s}(M)} = \sqrt{\operatorname{Fitt}_j(M/U)}$. For instance, let $R = k[z_0, \ldots, z_n]$ be a polynomial ring over a field k, φ the 3 by n matrix

$$\begin{pmatrix} 0 & 0 & \cdots & 0 \\ z_0 & 0 & \cdots & 0 \\ z_1 & z_2 & \cdots & z_n \end{pmatrix},$$

which gives a map from R^n to $R^3 = Re_1 \oplus Re_2 \oplus Re_3$, M the cokernel of φ, and $U \subset M$ the submodule generated by the images of e_1 and e_2. In this case, one has $\operatorname{ht}(U :_R M/0 :_R M) = n$, whereas $\operatorname{Fitt}_2(M) = (z_0, \ldots, z_n) \not\subset \sqrt{\operatorname{Fitt}_0(M/U)} = (z_1, \ldots, z_n)$.

Proposition 5.4. *Let R be a universally catenary Noetherian ring and let N be a finitely generated R-module. Assume that one of the following conditions holds:*

(1) *R satisfies S_3 and N is orientable of rank 3; or*
(2) *R is Gorenstein and N is orientable of rank 4, satisfies S_3 and is free in codimension 2; or*
(3) *R satisfies S_{k+1} and contains a field, and N is a k^{th} syzygy of rank $k+1$ having finite projective dimension.*

If $x \in N$ satisfies $\operatorname{ht}(N^*(x)) > \operatorname{rank}(N)$ *then* $\sqrt{\operatorname{tr}(N)} = \sqrt{N^*(x)}$.

Proof. We may assume that R is local and $N^*(x) \neq R$. Notice that $\operatorname{rank}(N) \geq 1$. If we are in case (3) with $k = 0$ then N is orientable of rank 1, so $N^* \cong R$ and $N^*(x)$ has height 1, contradicting the hypothesis. Thus in any case R is S_2, hence equidimensional.

After passing to a purely transcendental extension of the residue field of R, we consider generic generators $y_1, \ldots, y_n$ of N and set $Y_i = N/Ry_i$. Note that $\operatorname{ht}(N^*(y_i)) > \operatorname{rank}(N)$ according to Proposition 2.4(2). By Proposition 5.1(2) it suffices to show that $\operatorname{ht}(N^*(x)/Y_i^*(x)) > 1$ for $1 \leq i \leq n$. After localizing at a suitable minimal prime of $N^*(x)$ we only need to prove that $\operatorname{ht}(Y_i^*(x)) \leq \operatorname{rank}(N) - 1$. Write $y = y_i$, $Y = Y_i$, and notice that $Y^*(x) \subset N^*(x) \neq R$. As $\operatorname{ht}(N^*(y)) > \operatorname{rank}(N)$, we have $Ry \cong R$ and assumptions (1), (2), and (3) pass from N to Y, except that $\operatorname{rank}(Y) = \operatorname{rank}(N) - 1$ (see for instance Hochster and Huneke (1990, 10.9)). Thus we may apply Propositions 4.1 and 4.2 to conclude that $\operatorname{ht}(Y^*(x)) \leq \operatorname{rank}(Y) = \operatorname{rank}(N) - 1$. □

We now consider bounds for the height of trace ideals. Of course if R is a regular local ring and N is an n-generated R-module all of whose order ideals have height at most $r = \operatorname{rank}(N)$, then $\operatorname{ht}(\operatorname{tr}(N)) \leq rn$ — this holds because heights of ideals are subadditive in regular local rings (Serre (1965, V, Théorème 3)). Without the regularity assumption however, no such strong inequality is true: Let k be a Noetherian ring, X an $m \times n$ matrix of variables over k, $R = k[\{X_{ij}\}]/I_{r+1}(X)$

where $1 \le r \le \min\{m, n-1\}$, and φ the image of the matrix X over R. The R-module $N = \operatorname{Im}(\varphi)$ has rank r and minimal number of generators n; furthermore every order ideal $N^*(x)$ has height at most r, but $\operatorname{ht}(\operatorname{tr}(N)) = r(m+n-r)$ — this follows from Remark 2.2 and the fact that $N^\perp \cong \operatorname{Coker}(\varphi^*)$ by Bruns (1982, Theorems 4 and 5). Quite generally, Remark 2.2 and Bruns (1981, Corollary 1) show that if R is a Noetherian local ring and N is a finitely generated R-module of rank r with $n = \mu(N)$ and $m = \mu(N^*)$, then $\operatorname{ht}(\operatorname{tr}(N)) \le r(m+n-r)$.

Curiously, if we assume that some order ideal has height bigger than the rank (which would seem to make the trace ideal larger) then we can get a bound which is asymptotically much sharper (as $(m+n)/r \to \infty$ it has order $(r-2)m+(r-3)n$ instead of $r(m+n)$). It is convenient to give at the same time a bound on $N^*(U)$ for any i-generated submodule U:

Theorem 5.5. *Let R be an equidimensional catenary Noetherian local ring and let N be an orientable R-module of rank r. Let $U \subset N$ be a submodule not containing any nontrivial free summand of N, let i be an integer with $i \ge \mu(U)$ and write $m = \mu(N^*)$. Then either* $\operatorname{ht}(N^*(x)) \le r$ *for every $x \in U$ or else*

(1) $\operatorname{ht}(N^*(U)) \le (r-2)(m-r+3) + (r-3)i$, *in case R satisfies S_3;*
(2) $\operatorname{ht}(N^*(U)) \le (r-3)(m-r+3) + (r-3)i$, *in case R is Gorenstein, and N satisfies S_3 and is free in codimension 2;*
(3) $\operatorname{ht}(N^*(U)) \le (r-k)(m-r+3) + (r-3)i$, *in case R contains a field and satisfies S_{k+1}, and N is a k^{th} syzygy of finite projective dimension.*

Proof. Assume that $\operatorname{ht}(N^*(x)) > r$ for some $x \in U$. We will prove (1)–(3) by induction on i. Using Propositions 4.1 and 4.2 we see that $r \ge 3$ in (1), $r \ge 4$ in (2), and $r \ge k+1$ in (3).

By Theorem 3.5 we can write $U = Rx_1 + \cdots + Rx_i$ with $\operatorname{ht}(N^*(x_1)) > r$. If $i = 1$, then $\operatorname{ht}(N^*(U)) = \operatorname{ht}(N^*(x_1)) \le m$, which yields the desired estimates in this case (as $r \ge 3$, or $r \ge 4$, or $r \ge k+1$, respectively). Hence we may assume $i \ge 2$. Write $V = Rx_1 + \cdots + Rx_{i-1}$. If $N^*(x_i) \subset \sqrt{N^*(V)}$ we can replace U by V and apply the induction hypothesis (note that $x_1 \in V$ and $\operatorname{ht}(N^*(x_1)) > r$). Otherwise, we may choose a prime Q containing $N^*(V)$ such that $N^*(x_i) \not\subset Q$.

First notice that

$$\operatorname{ht}(N^*(U)) = \operatorname{ht}(N^*(x_i) + N^*(V)) \le m + \operatorname{ht}(N^*(V)) \le m + \operatorname{ht}(N^*_Q(V)). \tag{5.6}$$

Set $X = N/Rx_i$, let z_j be the image of x_j in X for $1 \le j \le i-1$, and let Z denote the image of V in X. Since $N^*(x_i) \not\subset Q$, $N_Q \cong R_Q \oplus X_Q$. Writing $x_j = r_j + z_j$ under this isomorphism, we obtain $N^*_Q(x_j) = (r_j) + X^*_Q(z_j)$. As $N^*_Q(x_1) \ne R_Q$ it follows that $\operatorname{ht}(X^*_Q(z_1)) \ge \operatorname{ht}(N^*_Q(x_1)) - 1 > r - 1$. Furthermore

$$N^*_Q(V) = (r_1, \ldots, r_{i-1}) + X^*_Q(Z). \tag{5.7}$$

Since $N_Q^*(V) \neq R_Q$, (5.6) and (5.7) yield

$$\mathrm{ht}(N^*(U)) \leq m + (i-1) + \mathrm{ht}(X_Q^*(Z)).$$

We apply our induction hypothesis to $Z_Q \subset X_Q$. Notice that $\mu(Z_Q) \leq i-1$, $X_Q^*(Z) \neq R_Q$, X_Q is orientable of rank $r-1$, $\mu(X_Q^*) \leq m-1$, and $\mathrm{ht}(X_Q^*(z_1)) > r-1$. Formally setting $k=2$ for (1), $k=3$ for (2), and $k=k$ for (3), we obtain

$$\mathrm{ht}(N^*(U)) \leq m+(i-1)+((r-1)-k)((m-1)-(r-1)+3)+((r-1)-3)(i-1).$$

Now the desired formulas follow. □

For example, let I to be the ideal of the curve $t \mapsto (1, t, t^3, t^4)$ in $\mathbf{P}_k^3$ treated in Example 3.10. If $N = I^\perp$, then the inequality of Theorem 5.5(1) is sharp for every i (here $r=3, m=n=4$). We do not have examples of rank ≥ 4 where the inequality is sharp.

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