HEIGHTS OF IDEALS OF MINORS

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Abstract. We prove new height inequalities for determinantal ideals in a regular local ring, or more generally in a local ring of given embedding codimension. Our theorems extend and sharpen results of Faltings and Bruns.

Introduction. Let φ be a map of vector bundles on a variety X. A well-known theorem of Eagon and Northcott [EN] gives an upper bound for the codimension of the locus where φ has rank $\leq s$ for any integer s.

Bruns [B] improved this result by taking into account the generic rank r of φ. We shall see below that unlike the Eagon-Northcott estimate, in most cases Bruns' theorem is sharp only when X is singular. The first goal of this paper is to give stronger results when X is nonsingular, and a little more generally.

Strengthening the Eagon-Northcott estimate in a different way from Bruns, Faltings [F] gave an improved bound for the case $s = r - 1$ under the additional assumption that X is nonsingular and the cokernel of φ is torsion free. We also improve Faltings' theorem to a result valid for all s.

Let R be a ring, and let $\varphi\colon R^m \to R^n$ be a matrix of rank r. We write $I_i = I_i(\varphi)$ for the ideal generated by the $i \times i$ minors of φ, and we assume that $i \leq r$ and $I_i \neq R$. Bruns' theorem says that

$$\text{height}(I_i) \leq (r - i + 1)(m + n - r - i + 1).$$

This formula is sharp for every m, n, r, i: take φ to be the image of the generic $n \times m$ matrix

$$\Phi = (x_{ij}) \qquad 1 \leq i \leq n, \quad 1 \leq j \leq m$$

over the ring $R = k[\{x_{ij}\}]/I_{r+1}(\Phi)$. Note that this ring is singular for $0 < T < \min\{m, n\}$.

For simplicity, for the remainder of this introduction we consider the case in which the ring is regular. Under this hypothesis we give a bound which in general improves Bruns' bound as follows:

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THEOREM A. *Let R be a regular local ring, and let φ: $R^m \to R^n$ be a matrix of rank r. We write $I_i = I_i(\varphi)$ for the ideal generated by the $i \times i$ minors of φ, and we assume that $i \leq r$ and $I_i \neq R$. Then*

$$\text{height}(I_i) \leq (r - i + 1)(\max\{m, n\} - i + 1) + i - 1.$$

Theorem A is a weak form of Corollary 3.6.1 below.

One should compare this result with the "trivial" case where the matrix φ contains only r nonzero rows (if $m \geq n$) or r nonzero columns (if $n \geq m$). In this case the codimension of the ideal of $i \times i$ minors is given by the "Eagon-Northcott" formula

$$\text{height}(I_i) \leq (r - i + 1)(\max\{m, n\} - i + 1),$$

which is an equality if the nonzero rows (columns) of φ are generic. This formula coincides with ours when $i = 1$.

A particularly interesting situation is that where the cokernel of φ is torsion free (or even a vector bundle on the punctured spectrum). In this torsion free case Faltings improved Bruns' bound (for $r \times r$ minors only) and showed

$$\text{height}(I_r) \leq n.$$

Generalizing this to arbitrary size minors, we obtain:

THEOREM B. *Let R be a regular local ring, and let φ: $R^m \to R^n$ be a matrix of rank r. We write $I_i = I_i(\varphi)$ for the ideal generated by the $i \times i$ minors of φ, and we suppose that $i \leq r$ and $I_i \neq R$. Assume that the cokernel of φ is torsion free (or more generally, is torsion free locally in codimension one, and is not the direct sum of a free module and a torsion module). Then*

$$\text{height}(I_i) \leq (r - i)(\max\{m, n\} - i + 1) + n.$$

Theorem B is a weak form of Corollary 3.6.2 below.

Theorem B is sharp in the case where $n = 3, m \geq 3, r = i = 2$ and φ is the generic alternating 3×3 matrix followed by a $3 \times (m - 3)$ matrix of zeros. If on the other hand φ has one generic column, $r - 1$ generic rows, and the rest of its entries 0, then

$$\text{height}(I_i) = (r - i)(m - i + 1) + \min\{m - i + 1, n - r + 1\}.$$

This actual value is close to the bounds provided by Theorems A and B. Some less degenerate examples are given in Section 4.

We can also ask for a bound on the height of one ideal of minors modulo the ideal of minors of the next larger size. We prove:

THEOREM C. *Let R be a regular local ring, and let φ: $R^m \to R^n$ be a matrix of rank r. We write $I_i = I_i(\varphi)$ for the ideal generated by the $i \times i$ minors of φ, and we suppose that $i \leq r$ and $I_i \neq R$. By symmetry we may assume that $m \geq n$. Then*

$$\mathrm{height}(I_i/I_{i+1}) \leq \max\{m - i + 1, n\} + r - i.$$

Theorem C is a weak form of Corollary 3.9.1 below.

This result is comparable to Theorems A and B (or their sharpenings) in the case $i = r$; but it does not follow from these results in general because R/I_{i+1} is not regular. However, if we have good information about the higher order minors of φ, as in the case where the cokernel of φ is an ideal, then Theorem C gives results on the height of $I_i(\varphi)$ that are better than those coming from Theorems A and B. In this way we reprove a theorem of Huneke [Hu] and extend it as follows:

COROLLARY D. *Let R be a regular local ring, and let J be an ideal of R of height g that is minimally generated by n elements.*

(a) (Huneke) *If J is not a complete intersection, that is $n > g$, then the locus of primes P such that J_P is not a complete intersection has codimension $\leq n + 2g - 1$.*

(b) *If R/J is a Cohen-Macaulay domain and $n > g + 1$, then the locus of primes P such that J_P cannot be generated by $g + 1$ elements has codimension $\leq 2n + 3g - 1$.*

Corollary D follows from a weak form of Example 3.11 below.

Huneke's result (which is sharp, for example, in case J is the ideal of 2×2 minors of a generic 2×3 matrix) improves a formula of Faltings [F] by 1. One should compare this to a famous conjecture of Hartshorne [Ha2] saying that if J is the homogeneous ideal of a smooth projective variety which is not a complete intersection, then the locus of primes P such that J_P is not a complete intersection has codimension $\leq 3g + 1$.

Both Theorems A and C are direct consequences of our other main result, which gives the bound on the codimension of the ideals of minors of a matrix $\bar{\varphi}$ over a ring $\bar{R} = R/I$ obtained by reducing φ modulo I.

THEOREM E. *Let R be a regular local ring, and let φ: $R^m \to R^n$ be a matrix of rank r. Let I be an ideal of R, and write $\bar{\varphi}$ for the matrix over the ring $\bar{R} = R/I$ obtained by reducing φ modulo I. Let $\bar{r}$ be the rank of $\bar{\varphi}$ and set $\delta = r - \bar{r}$. Suppose that $i \leq \bar{r}$ and $I_i(\bar{\varphi}) \neq \bar{R}$. By symmetry we assume that $m \geq n$. Then*

$$\mathrm{height}(I_i(\bar{\varphi})) \leq (\bar{r} - i)(m - i + \delta + 1) + \max\{m - i + 1, n\} + \delta.$$

Theorem E is a weak form of Theorem 3.1.1 below. As with Faltings' work, we do not need R to be regular, but can give bounds in terms of certain embedding codimensions.

We now describe the key ideas of our proofs. To establish height bounds for ideals of minors it is helpful to identify as "many" row ideals of φ as possible that have "small" height. As it turns out, the behavior of φ in this respect is determined by the analytic spread ℓ of $M = \mathrm{Coker}(\varphi)$ (see Section 1 for the definition of analytic spread). If ℓ has the maximal possible value n then every row ideal of any matrix φ minimally presenting M has height at most r, and (under weak conditions) the converse holds as well. Thus, whenever $\ell < n$ there have to exist row ideals whose height exceeds r. On the other hand we prove in this case that after a flat local base change, at least ℓ row ideals have height $\leq r - n + \ell < r$. To paraphrase, if the analytic spread of M is not maximal, then the behavior of the row ideals is more unbalanced, but not necessarily worse for our purposes. This is the content of Theorem 2.2, the main technical result of the paper. A complicated induction then completes the proofs of our formulas in Section 3.

We finish this introduction with a list of open problems specifically suggested by the results of this paper. Of course the biggest open problem is the conjecture of Hartshorne mentioned above.

Problem 1. Let φ be a symmetric $n \times n$ matrix of rank r, and suppose that 2 is invertible in R (but not necessarily that R is regular). We conjecture that for $i \leq r$,

$$\mathrm{height}(I_i) \leq \begin{pmatrix} n-i+2 \\ 2 \end{pmatrix} - \begin{pmatrix} n-r+1 \\ 2 \end{pmatrix}.$$

In Section 5 we prove this conjecture for the cases $i = 1$ and $i = n - 1$ if R is regular.

If the conjecture is true, it is sharp, for example for the generic symmetric matrix, taken modulo the ideal of $(r+1) \times (r+1)$ minors. This formula is the analogue of Bruns' bound for general matrices; it is computed as the difference between the heights of the ideals of $i \times i$ and of $(r+1) \times (r+1)$ minors of a generic symmetric matrix. Notice that the conjecture fails in characteristic 2, as can be seen by taking φ to be a generic alternating 3×3 matrix and $i = 2$.

Before stating the next problem, we recall that a generalized row (or column) of zeros in a matrix φ is a row (or column) of zeros after we change φ by invertible row (or column) operations. We assume again that R is regular.

Problem 2. Are there better bounds than the ones of Theorems A and B if we assume that φ has no (generalized) rows or columns of zeros?

Problem 3. Are there better bounds if φ is a matrix of linear forms?

Problem 4. Find sharp bounds assuming the ranks are small. For example, what about I_2 for a 4×4 matrix of rank 2? Is the height bounded by 3?

1. Basic results. In this section we fix our notation and review some basic facts, mainly about Rees algebras of modules, that will be used throughout.

Let R be a Noetherian ring and I an ideal of R. We write $\mathrm{ht}(I)$ for the height of I and $\mathrm{bight}(I)$ for its *big height*, which is the maximum of the heights of minimal primes of I. Let M a finitely generated R-module and φ an n by m matrix with entries in R. By the i^{th} *row ideal* of φ we mean the ideal generated by the entries of the i^{th} row of φ, and the *rank* of φ is the integer $r = \max\{i | I_i(\varphi) \neq 0\}$. We say that M has a *rank* and write $\mathrm{rank}(M) = e$ if $M \otimes_R K$ is a free K-module of rank e, with K denoting the total ring of quotients of R. Notice that if φ presents M and M has a rank, then $r + e = n$.

Let φ be a matrix presenting M and $\underline{T} = T_1, \ldots, T_n$ a row of variables. The row ideals of φ are related to the symmetric algebra $\mathrm{Sym}(M)$ of M via the homogeneous presentation $\mathrm{Sym}(M) \cong R[T_1, \ldots, T_n]/I_1(\underline{T} \cdot \varphi)$ (see also [EHU1], where this fact has been exploited systematically). Since the symmetric algebra fails to be equidimensional in general, we are lead to consider the *Rees algebra* $\mathcal{R}(M)$ of M instead. The general notion of Rees algebra has been introduced in [EHU2, 0.1]. In the present paper however we will restrict ourselves to considering modules that have a rank. In this case $\mathcal{R}(M)$ is equal to $\mathrm{Sym}(M)$ modulo R-torsion. We say that M is of *linear type* if the natural map from $\mathrm{Sym}(M)$ to $\mathcal{R}(M)$ is an isomorphism. If R has dimension d and E has a rank e, then $\dim \mathcal{R}(M) = d + e$ (see, e.g., [SUV, 2.2]). Suppose in addition that R is equidimensional, universally catenary and local. Under this assumption $\mathcal{R}(M)$ is equidimensional. Thus we may write $\mathcal{R}(M) \cong R[T_1, \ldots, T_n]/\mathfrak{b}$ with $\mathrm{bight}(\mathfrak{b}) = \mathrm{ht}(\mathfrak{b})$. In fact $\mathrm{ht}(\mathfrak{b}) = r$, the rank of any matrix with n rows that presents M.

If U is a submodule of M, we say that U is a *reduction* of M or, equivalently, M is *integral* over U if the ring $\mathcal{R}(M)$ is integral over its subalgebra $R[U]$. In case R is local with residue field k, the *analytic spread* $\ell(M)$ of M is defined to be the Krull dimension $\dim \mathcal{R}(M) \otimes_R k$. The two notions are related by the fact that $\ell(M) = \min\{\mu(U) | U$ a reduction of $M\}$ whenever k is infinite (here $\mu(-)$ denotes minimal number of generators). One always has $\mathrm{rank}(M) \leq \ell(M) \leq \mu(M)$ (see, e.g. [SUV, 2.3]), and the last inequality is an equality if and only if M has no proper reduction, at least in the case of an infinite residue field.

Before describing more refined estimates, we need to review the property G_s, where s is an integer: A module M of rank e is said to satisfy G_s if $\mu(M_P) \leq \dim R_P + e - 1$ for every prime ideal P with $1 \leq \dim R_P \leq s - 1$. What makes the concepts of integral dependence and analytic spread play a central role in this paper is their relation to the height of certain colon ideals:

THEOREM 1.1. [R, 2.5], [EHU3, 1.2 and 1.1] *Let R be an equidimensional universally catenary Noetherian local ring, let M be a finitely generated R-module having a rank e, and let U be a submodule of M with $\mu(U) \geq e-1$. If*

$$\mathrm{ht}(U :_R M) > \mu(U) - e + 1,$$

then U is a reduction of M.

This theorem yields the upper bound $\ell(M) \leq \mu(U)$ when the hypothesis is satisfied. Conversely, one has:

PROPOSITION 1.2. [EHU3, 3.7bis] *Let R be a Noetherian local ring with infinite residue field, let M be a finitely generated R-module having a rank e, and assume that M satisfies G_{s+1}. If $\mathrm{ht}(U :_R M) \leq \mu(U)-e+1$ for every submodule U generated by $e+s-1$ general linear combinations of generators of M, then $\ell(M) \geq e+s$.*

In a more general setting one still has the following weaker bound:

PROPOSITION 1.3. [SUV, 4.1.a] *Let R be a Noetherian local ring, and let M be a finitely generated R-module having a rank e. If M is not a direct sum of a free module and a torsion module, and M_P is free for every prime ideal P with* depth *$R_P \leq 1$, then $\ell(M) \geq e+1$.*

If $R \to S$ is a homomorphism of rings, J^c will denote the contraction to R of an S-ideal J, and $-_S$ will stand for the functor $- \otimes_R S$. We will denote $\mathrm{Hom}_R(-, R)$ by $-^*$. The *embedding codimension* $\mathrm{ecodim}(R)$ of a Noetherian local ring $(R, \mathfrak{m})$ is defined as the difference $\mu(\mathfrak{m}) - \dim R$; equivalently, writing $\hat{R} \cong S/J$ with $(S, \mathfrak{n})$ a regular local ring and J an S-ideal contained in $\mathfrak{n}^2$, one has $\mathrm{ecodim}(R) = \mathrm{ht}(J)$.

2. Choosing row ideals of small height. Let R be an equidimensional universally catenary Noetherian local ring with infinite residue field and M a finitely generated R-module having a rank e with $n = \mu(M)$, $\ell = \ell(M)$. Theorem 1.1 shows that if $\ell = n$, then *every* row ideal of any matrix minimally presenting M has height at most $r = n - e$. According to Proposition 1.2, the converse holds in case M satisfies G_{r+1}. Thus, whenever $\ell < n$ there tend to exist row ideals of height strictly greater than r. On the other hand, we will prove below that it is possible in this case to find "many" row ideals whose height is strictly less than r. More precisely, over a flat local extension ring S of R there exists a matrix ϕ minimally presenting M_S such that at least ℓ row ideals of ϕ have height at most $r - n + \ell = \ell - e$. These row ideals are constructed inside the defining ideals of Rees algebras of certain modules. The local homomorphism $R \to S$ has a complete intersection closed fiber, but regularity may fail to pass from R to S. This will require some extra care since the height of ideals in S may no longer be subadditive.

We begin by recording a weaker version of the above estimate, which has the advantage that S can be chosen to be a localization of a polynomial ring over R. This theorem was inspired by a result of Evans and Griffith saying that if R is a universally catenary domain with algebraically closed residue field and N is a finitely generated nonfree R-module of rank r, then there exists a minimal generator $x \in N$ with $\operatorname{ht}(N^*(x)) \leq r$ [EG, 2.12].

THEOREM 2.1. *Let R be an equidimensional universally catenary Noetherian local ring, and let M be a finitely generated R-module with rank e. Write $\ell = \ell(M)$ and $r = \mu(M) - e$. Then there exists a local homomorphism $R \to S$, with S a localization of a polynomial ring over R, and a minimal presentation matrix of M_S over S that has ℓ row ideals of height at most r.*

This result is a special case of the next theorem. Before stating the theorem we remark on some notation and terminology. Let $(R, \mathfrak{m})$ be a Noetherian local ring and M a finitely generated R-module with $n = \mu(M)$, $\ell = \ell(M)$. When we speak of a *generic generating set* for M defined over a local ring R' obtained from R by a purely transcendental residue field extension we mean the following: Let $X = (x_{ij})$ be a generic $n \times n$ matrix over R, and $R' = R(\{x_{ij}\}) = R[\{x_{ij}\}]_{\mathfrak{m}R[\{x_{ij}\}]}$. Fix a generating set $m_1, \ldots, m_n$ of M, and let $v_j = \sum_{i=1}^{n} x_{ij} m_i \in M_{R'}$. Then $M_{R'}$ is generated by $v_1, \ldots, v_n$, and these elements are said to be a generic generating set for M defined over R'. Furthermore, any ℓ of the v_j generate a minimal reduction of $M_{R'}$. (This follows from the fact that the correct number of generic elements always give Noether normalizations for finitely generated algebras over fields, which is explicitly shown in [FUV, 7.3].)

THEOREM 2.2. *Let $(R, \mathfrak{m})$ be an equidimensional universally catenary Noetherian local ring, let M be a finitely generated R-module with rank e, and set $n = \mu(M), \ell = \ell(M), r = n - e$. Let $v_1, \ldots, v_n \in M_{R'}$ be a generic generating set for M defined over a local ring R' that is obtained from R by a purely transcendental residue field extension, and let ψ be an $n \times m$ matrix presenting $M_{R'}$ with respect to $v_1, \ldots, v_n$. Further let T be an n by n matrix of the form*

$$T = \begin{bmatrix} \underline{T}_1 \\ \vdots \\ \underline{T}_n \end{bmatrix} = \left[\begin{array}{c|c} 1_{n-\ell} & 0 \\ \hline \multicolumn{2}{c}{T'} \end{array}\right]$$

with rows $\underline{T}_i$, where T' is a generic ℓ by n matrix over R'. Set $\phi = T\psi$.

There exists a local ring R'' obtained from R' by another purely transcendental residue field extension, a prime ideal Q of $A = R''[T]$ with $\det(T) \notin Q$ *and* $\mathfrak{m} \subset Q$, *and A_Q-regular sequences $\underline{a}_t$, for $n - \ell + 1 \leq t \leq n$, each of length $n - \ell$, so that the following holds: given an arbitrary (possibly empty) set $\Lambda = \{t_1, \ldots, t_d\}$ of integers $n - \ell + 1 \leq t_1 < \ldots < t_d \leq n$ and writing $B = R''[\{\underline{T}_i | \, i \notin \Lambda\}]_{Q^c}$, $\underline{a} = \underline{a}_{t_1}, \ldots, \underline{a}_{t_d}$, and $S = A_Q/(\underline{a})$, one has:*

(1) *The homomorphism $R \to B$ is local (and regular), and the homomorphism $B \to S$ is local and flat with complete intersection closed fiber.*

(2) *ϕ_S is a presentation matrix of M_S, and for $n - \ell + 1 \leq i \leq n$, the i^{th} row ideal J_i of ϕ_S has height at most $r - n + \ell = \ell - e$ if $i \in \Lambda$ and at most r otherwise.*

(3) *$\underline{a}$ form a regular sequence on A_Q/IA_Q for every proper ideal I of R.*

(4) $\operatorname{ecodim}(S_P) = \operatorname{ecodim}(R_{P \cap R})$ *for every prime ideal P of S with $P \notin V(J_{t_1} \cdot \ldots \cdot J_{t_d})$.*

Proof. Write $U = R' v_{n-\ell+1} + \cdots + R' v_n \subset M_{R'}$. Since U is generated by ℓ generic elements for M it follows that U is a minimal reduction of $M_{R'}$. To simplify notation we write R instead of R' from now on.

For $n - \ell + 1 \leq i \leq n$ let $\mathfrak{a}_i$ be the ideal of $A_i = R[\underline{T}_i]$ generated by the i^{th} row ideal of ϕ. We obtain isomorphisms

$$A_i/\mathfrak{a}_i \cong \operatorname{Sym}(M)$$

sending the (i,j) entry T_{ij} of T to v_j. Since $\operatorname{Sym}(M)$ maps onto $\mathcal{R}(M)$, there are A_i-ideals $\mathfrak{b}_i$ containing $\mathfrak{a}_i$ such that $A_i/\mathfrak{b}_i \cong \mathcal{R}(M)$. Observe that $\operatorname{bight}(\mathfrak{b}_i A) = \operatorname{bight}(\mathfrak{b}_i) = r$ (see the remarks at the beginning of Section 1).

Let $(R'', \mathfrak{m}'')$ be the local ring obtained from $R = R'$ by a purely transcendental residue field extension of transcendence degree $(n - \ell)(\sum \mu(\mathfrak{b}_i))$, let $k'' = R''/\mathfrak{m}''$, $\mathcal{E} = \otimes_R^\ell \mathcal{R}(U) \otimes_R k''$ and $\mathcal{F} = \otimes_R^\ell \mathcal{R}(M) \otimes_R k''$. The above isomorphisms induce an isomorphism

$$A/(\mathfrak{m}, \mathfrak{b}_{n-\ell+1}, \ldots, \mathfrak{b}_n) \cong \mathcal{F}.$$

Moreover, the natural map of k''-algebras $\mathcal{E} \to \mathcal{F}$ is module finite since U is a reduction of M. Its image is generated by the images in $\mathcal{F}$ of T_{ij} for $n - \ell + 1 \leq i \leq n$ and $n - \ell + 1 \leq j \leq n$. Hence these elements of $\mathcal{F}$ are algebraically independent over k'', because $\dim \mathcal{F} = \ell^2$. It follows that the image of $\Delta = \det(T)$ in $\mathcal{F}$ is not nilpotent. Thus there exists a prime ideal Q of A with $\Delta \notin Q$ and $(\mathfrak{m}, \mathfrak{b}_{n-\ell+1}, \ldots, \mathfrak{b}_n) \subset Q$.

For every t, $n - \ell + 1 \leq t \leq n$, let $\underline{a}_t \subset R''[\underline{T}_t]$ be a sequence of $n - \ell$ generic elements for $\mathfrak{b}_t \subset R[\underline{T}_t]$, defined using indeterminates over $R = R'$ as coefficients. Such sequences exist by the definition of R''. As $(\mathfrak{m}, \mathfrak{b}_t)A_t/\mathfrak{m}A_t$ is an ideal in a polynomial ring over a field of height $\dim(A_t/\mathfrak{m}A_t) - \ell(M) = n - \ell$, it follows that $\underline{a}_t$ form a regular sequence on $A_t/\mathfrak{m}A_t \otimes_k k''$.

We are now ready to verify statements (1)–(4) in the theorem. Write $\mathfrak{n}$ for the maximal ideal of B. As $\mathfrak{m} \subset Q$ we have that $\mathfrak{m} \subset \mathfrak{n}$ and thus the map $R \to B$ is a (regular) local homomorphism. Furthermore, $\underline{a} \subset Q$ and $A_Q/\mathfrak{n}A_Q$ is flat over $(A_{t_1}/\mathfrak{m}A_{t_1}) \otimes_k \cdots \otimes_k (A_{t_d}/\mathfrak{m}A_{t_d}) \otimes_k k''$. Thus $\underline{a}$ form a regular sequence on $A_Q/\mathfrak{n}A_Q$, the closed fiber of the flat local map $B \to A_Q$. Consequently, the (local) homomorphism $B \to S = A_Q/(\underline{a})$ is flat with complete intersection closed fiber, and $\underline{a}$ form a regular sequence on A_Q/IA_Q for any R-ideal $I \subset \mathfrak{m}$ [M, p. 177]. This proves (1) and (3).

To show (2), observe that the image of Δ is a unit in S since $\Delta \notin Q$. Thus ϕ_S is a presentation matrix of M_S. Obviously $J_i = \mathfrak{a}_i S \subset \mathfrak{b}_i S$. If $i \notin \Lambda$ then S is flat over A_i and hence $\mathrm{ht}(J_i) \leq \mathrm{ht}(\mathfrak{b}_i S) = \mathrm{ht}(\mathfrak{b}_i (A_i)_{Q \cap A_i}) \leq \mathrm{bight}(\mathfrak{b}_i) = r$. If on the other hand $i \in \Lambda$ then $\underline{a_i} \subset \mathfrak{b}_i$, which together with the A_Q-regularity of $\underline{a}$ gives $\mathrm{ht}(J_i) \leq \mathrm{ht}(\mathfrak{b}_i S) \leq \dim(S) - \dim(S/\mathfrak{b}_i S) = \dim(A_Q) - d(n-\ell) - (\dim(A_Q) - (d-1)(n-\ell) - \mathrm{ht}(\mathfrak{b}_i A_Q)) = \mathrm{ht}(\mathfrak{b}_i A_Q) - (n-\ell) \leq \mathrm{bight}(\mathfrak{b}_i A) - n + \ell = r - n + \ell = \ell - e$. This proves (2).

Finally, to show (4) notice that if $P \in \mathrm{Spec}(S) \setminus V(J_{t_1} \cdot \ldots \cdot J_{t_d})$, then $P \notin V(\mathfrak{b}_t S)$ for every $t \in \Lambda$. Thus by the generic choice of $\underline{a_t}$ in $\mathfrak{b}_t$, the ring S_P is a localization of a polynomial ring over $R_{P \cap R}$. $\square$

Notice that Theorem 2.1 follows from Theorem 2.2.2 by taking $\Lambda = \emptyset$. We will apply Theorem 2.2 in conjunction with the following generalization of a theorem of Serre:

Lemma 2.3.

(1) *Let $A \to S$ be a local homomorphism of Noetherian local rings with A regular and S equidimensional and universally catenary, let I be an ideal of A, and let J be an ideal of S. Then*

$$\mathrm{ht}(IS + J) \leq \mathrm{ht}(I) + \mathrm{ht}(J).$$

(2) *Let $B \to S$ be a local homomorphism of equidimensional and universally catenary Noetherian local rings, let K be an ideal of B, and let J be an ideal of S. Then*

$$\mathrm{ht}(KS + J) \leq \mathrm{ht}(K) + \mathrm{ht}(J) + \mathrm{ecodim}(B).$$

Proof. We may assume that the local rings A, B and S are complete by passing to their completions; our assumptions do not change (see [M, 31.7]), nor do the conclusions.

We prove (1). Suppose first that the map $A \to S$ is onto, and write $S = A/L$. Lift J to an ideal H in A, so that $J = H/L$. Since S is equidimensional and A is regular, $\mathrm{ht}(IS + J) = \mathrm{ht}((I+H)/L) = \mathrm{ht}(I+H) - \mathrm{ht}(L) \leq \mathrm{ht}(I) + \mathrm{ht}(H) - \mathrm{ht}(L) = \mathrm{ht}(I) + \mathrm{ht}(J)$, where the middle inequality follows from the subadditivity of height in regular local rings [S, Chap. V, Thm. 3].

To treat the general case we use a Cohen factorization of the map $A \longrightarrow S$. Indeed by [AFH, 1.1] there is a factorization $A \xrightarrow{g} R \xrightarrow{h} S$, where g is flat and local with regular closed fiber and h is surjective. Notice that R is regular by [M, 23.7] and that $\operatorname{ht}(I) = \operatorname{ht}(IR)$. Since R maps onto S, the assertion now follows.

We prove (2). Write $B = A/L$, where A is a regular local ring and $\operatorname{ht}(L) = \operatorname{ecodim}(B)$, and lift K to an ideal I in A so that $K = I/L$. Note that $IS = KS$ and $\operatorname{ht}(I) = \operatorname{ht}(K) + \operatorname{ecodim}(B)$. Now (2) follows from (1). $\square$

Next we give a short proof of a modified version of Theorem 2.2. It requires the following definition:

Definition 2.4. Let R be a Noetherian local ring with residue field k (or a positively graded k-algebra), let M be a finitely generated (graded) R-module having a rank, and write $\mathcal{R} = \mathcal{R}(M)$. We set

$$s(M) = \dim_k [(\mathcal{R} \otimes_R k)/\sqrt{0}]_1.$$

Remark 2.5. Observe that in general $\ell(M) \leq s(M) \leq \mu(M)$. If M is graded and generated by homogeneous elements of the same degree, then $\mathcal{R} \otimes_R k$ embeds into $\mathcal{R}$ and therefore $s(M) = \mu(M)$ as long as R is reduced and M is torsion free.

THEOREM 2.6. *Let R be an equidimensional universally catenary Noetherian local ring with algebraically closed residue field, let M be a finitely generated R-module with rank e, and write $r = \mu(M) - e$, $s = s(M)$. There exists a minimal presentation matrix of M that has s row ideals of height at most r.*

Proof. Write $\mathcal{R}$ for the Rees algebra of M, k for the residue field of R, and set $V = [(\mathcal{R} \otimes_R k)/\sqrt{0}]_1$, which we identify with affine space of dimension s. Consider the closed subset X of V whose coordinate ring is the homogeneous k-algebra $(\mathcal{R} \otimes_R k)/\sqrt{0}$. Since k is algebraically closed there exists a basis $v_1, \ldots, v_s$ of V contained in X, and then the lines $kv_1, \ldots, kv_s$ all lie on X.

Let $z_1, \ldots, z_n$ be a minimal generating set of M chosen so that z_i maps to v_i for $1 \leq i \leq s$, and let ϕ be a presentation matrix with respect to $z_1, \ldots, z_n$. Set J_i equal to the i^{th} row ideal of ϕ. We claim $\operatorname{ht}(J_i) \leq r$ for $1 \leq i \leq s$.

Let $A = R[T_1, \ldots, T_n]$ be a polynomial ring, let $\mathfrak{m}$ denote the maximal ideal of R, and for $1 \leq i \leq s$ consider the prime ideals $Q_i = (\mathfrak{m}, T_1, \ldots, \hat{T}_i, \ldots, T_n)$ of A. Mapping T_j to z_j for $1 \leq j \leq n$, we obtain presentations $\operatorname{Sym}(M) \cong A/\mathfrak{a}$ and $\mathcal{R} \cong A/\mathfrak{b}$, where $\mathfrak{a} \subset \mathfrak{b}$ are A-ideals. As X contains the line kv_i, we have $\mathfrak{b} \subset Q_i$ for $1 \leq i \leq s$. Thus $\operatorname{ht}(\mathfrak{a}_{Q_i}) \leq \operatorname{ht}(\mathfrak{b}_{Q_i}) \leq \operatorname{bight}(\mathfrak{b}) = r$. Let $\pi_i \colon A_{Q_i} \longrightarrow R(T_i)$ be the $R[T_i]$-epimorphism whose kernel is generated by the A_{Q_i}-regular sequence $T_1, \ldots, \hat{T}_i, \ldots, T_n$. Since $\operatorname{ht}(\pi_i(\mathfrak{a}_{Q_i})) + n - 1 = \operatorname{ht}(\pi_i(\mathfrak{a}_{Q_i}), T_1, \ldots, \hat{T}_i, \ldots, T_n) = \operatorname{ht}(\mathfrak{a}_{Q_i}, T_1, \ldots, \hat{T}_i, \ldots, T_n) \leq \operatorname{ht}(\mathfrak{a}_{Q_i}) + n - 1$, it follows that $\operatorname{ht}(\pi_i(\mathfrak{a}_{Q_i})) \leq \operatorname{ht}(\mathfrak{a}_{Q_i})$. But $\pi_i(\mathfrak{a}_{Q_i}) = J_i R(T_i)$, which gives $\operatorname{ht}(J_i) \leq r$. $\square$

We finish the section with two immediate consequences of Theorem 2.6. Both are first height estimates for ideals of minors of matrices, stated more conveniently in terms of Fitting ideals of modules.

COROLLARY 2.7. *Let R be a regular local ring with perfect residue field k, let M be a finitely generated R-module of rank e, and write $r = \mu(M) - e$, $s = s(M)$. For every $1 \leq i \leq s$,*

$$\operatorname{ht}(\operatorname{Fitt}_{i-1}(M)) \leq ir.$$

Proof. There exists a flat local homomorphism $R \to S$ where S is a regular local ring with algebraically closed residue field K [G, (10.3)]. Since S is flat over R and k is perfect, one has that $(\mathcal{R}(M_S) \otimes_S K)/\sqrt{0} \cong ((\mathcal{R}(M) \otimes_R k)/\sqrt{0}) \otimes_k K$ and therefore $s(M) = s(M_S)$. We replace R and M by S and M_S, and assume that k is algebraically closed.

By Theorem 2.6 there exists a minimal presentation matrix of M that has i row ideals $J_1, \ldots, J_i$ of height at most r. As $\operatorname{Fitt}_{i-1}(M) \subset J_1 + \cdots + J_i$ and R is a regular local ring, we conclude that $\operatorname{ht}(\operatorname{Fitt}_{i-1}(M)) \leq ir$ (see [S, Chap. V, Thm. 3]). □

COROLLARY 2.8. *Let R be a polynomial ring over a field, let M be a torsion free graded R-module of rank e minimally generated by n homogeneous elements of the same degree, and write $r = n - e$. For every $1 \leq i \leq n$, $\operatorname{ht}(\operatorname{Fitt}_{i-1}(M)) \leq ir$. In particular, for every submodule U of M generated by $t < n$ elements, $\operatorname{ht}(U :_R M) \leq (t+1)r$.*

Proof. We may assume that the ground field is perfect. Writing $\mathfrak{m}$ for the irrelevant maximal ideal of R we observe that $s(M_{\mathfrak{m}}) = s(M) = n$ by Remark 2.5. Furthermore $U :_R M \subset \sqrt{\operatorname{Fitt}_t(M)}$. The assertions now follow from Corollary 2.7. □

3. Heights of determinantal ideals. The classical theorem of Bruns [B, Cor. 1] states that in a Noetherian ring R, the height of the (proper) ideal of i by i minors of an n by m matrix of rank r (with $i \leq r$) cannot exceed the "generic" value $N(i, r, m, n)$ defined as follows: let X be a generic n by m matrix and set $N(i, r, m, n) := \operatorname{ht}(I_i(X)) - \operatorname{ht}(I_{r+1}(X)) = (r - i + 1)(m + n - r - i + 1)$. This is exactly the height of the ideal of i by i minors of the image of X in the ring $R[X]/I_{r+1}(X)$ (note the image of X has rank r in this ring). However, if we also insist that the base ring R be regular and the rank r of the matrix not be maximal, then it is by no means clear that this maximum is ever attained. The main results known for the regular case are due to Bruns [B, Thm. 3] and Faltings [F, Kor. 2], and their results apply only to the case $i = r$. In Corollary 3.6.1 below we establish a bound for the height of the (proper) ideal of i by i minors of an n by m matrix of rank r over a regular ring that is roughly $(r - i)(\max\{m, n\} - i + 1) + \max\{m - i + 1, n\}$.

A second, related problem is to estimate the height of the (proper) ideal of i by i minors modulo the ideal of $i+1$ by $i+1$ minors. Again, the best general bound is $N(i,i,m,n) = m+n-2i+1$, but one may expect better results if R is regular and the rank r of the matrix is not maximal. We address this issue in Corollary 3.9.1, where the bound $\max\{m-i+1, n\} + r - i$ is established.

Both problems are special cases of the following, more general question: How can one estimate the height of the ideal of i by i minors of a matrix of rank $\bar{r}$ that can be "lifted" to a matrix of rank r over a ring R? Theorem 3.1, the main result of this section, gives such a bound involving the difference $r - \bar{r}$ of the ranks and the embedding codimension of R. The proof of this result relies on the work of Section 2 about row ideals of small height. The theorem gives particularly strong estimates if the matrix can be lifted in such a way that the increase in the rank is outweighed by a decrease in the embedding codimension of the ambient ring.

THEOREM 3.1. *Let R be an equidimensional universally catenary Noetherian local ring, let φ be an n by m matrix of rank r with entries in R, and let I be an R-ideal. Assume that $M = \mathrm{Coker}(\varphi)$ has a rank, and write $\ell = \ell(M)$, $\bar{R} = R/I$, $\bar{\varphi} = \varphi_{\bar{R}}$, $\bar{r} = \mathrm{rank}(\bar{\varphi})$. Let $i \leq \bar{r}$ be an integer so that $I_i(\bar{\varphi}) \neq \bar{R}$. Set $\delta = r - \bar{r}$ and $\epsilon = \max_P\{\mathrm{ecodim}(R_P)\}$, where the maximum is taken over all prime ideals P of R not containing $I_i(\varphi)$.*

(1)

$$\begin{aligned}\mathrm{ht}(I_i(\bar{\varphi})) \;\leq\; & \max\{(\min\{n-\ell, \bar{r}\} - i + 1)(m - i + 1 + \max\{0, n - \ell - \bar{r}\}),\\ & \qquad (\bar{r} - i)(\max\{m, n+\epsilon\} - i + \delta + 1) + \ell + \delta + \mathrm{ecodim}(R)\}\\ \leq\; & (\bar{r} - i)(\max\{m, n+\epsilon\} - i + \delta + 1)\\ & + \max\{m - i + 1, \ell + \mathrm{ecodim}(R)\} + \delta.\end{aligned}$$

(2) *If the $\bar{R}$-module $\bar{M} = M_{\bar{R}}$ is not a direct sum of a free module and a torsion module, $\bar{M}_{\bar{P}}$ is free for every prime $\bar{P}$ of $\bar{R}$ with $\mathrm{depth}(\bar{R}_{\bar{P}}) \leq 1$, and M_P is of linear type for every associated prime P of I, then*

$$\mathrm{ht}(I_i(\bar{\varphi})) \leq (\bar{r} - i)(\max\{m, n+\epsilon\} - i + \delta + 1) + \ell + \delta + \mathrm{ecodim}(R).$$

Before proving the theorem we wish to make several comments. First notice that $\epsilon = 0$ in case R is regular locally on the punctured spectrum. If the $\bar{R}$-module $\bar{M}$ is a direct sum of a torsion module and a free module then trivially $\mathrm{ht}(I_i(\bar{\varphi})) \leq (\bar{r} - i + 1)(m - i + 1)$. It is also obvious that one can replace the bound of part (1) by the better formula of (2) whenever $i \geq n - \ell + 1$. Finally, the estimates of Theorem 3.1 are sharp for φ a generic matrix with entries in the localization of a polynomial ring over a regular ring and $I = I_{\bar{r}+1}(\varphi)$, if $n \leq m$ or $i = 1$.

Proof of Theorem 3.1. We first prove that the second inequality of (1) is true, namely that

$$\begin{aligned}&\max\{(\min\{n-\ell,\bar r\}-i+1)(m-i+1+\max\{0,n-\ell-\bar r\}),\\ &\quad(\bar r-i)(\max\{m,n+\epsilon\}-i+\delta+1)+\ell+\delta+\operatorname{ecodim}(R)\}\\ &\qquad\le(\bar r-i)(\max\{m,n+\epsilon\}-i+\delta+1)+\max\{m-i+1,\ell+\operatorname{ecodim}(R)\}+\delta.\end{aligned}$$

We prove that each term in the maximum on the left-hand side of the inequality is at most the right-hand side. This is clear for the second term. It remains to see why

$$\begin{aligned}&(\min\{n-\ell,\bar r\}-i+1)(m-i+1+\max\{0,n-\ell-\bar r\})\\ &\quad\le(\bar r-i)(\max\{m,n+\epsilon\}-i+\delta+1)+\max\{m-i+1,\ell+\operatorname{ecodim}(R)\}+\delta.\end{aligned}$$

By possibly lessening the right-hand side and increasing the left-hand side, it is enough to prove that

$$\begin{aligned}(\bar r-i+1)(m-i+1+\max\{0,n-\ell-\bar r\})&\le(\bar r-i)(m-i+\delta+1)+m-i+1+\delta\\ &=(\bar r-i+1)(m-i+\delta+1),\end{aligned}$$

and for this it suffices to prove that $\max\{0,n-\ell-\bar r\}\le\delta=r-\bar r$. Clearly $0\le\delta$. The inequality $n-\ell-\bar r\le r-\bar r$ is equivalent to the inequality $n-r\le\ell$, which is always true, since $n-r=e=\operatorname{rank}(M)\le\ell$ (see the remarks at the beginning of Section 1).

We use induction on n to prove the first inequality of Theorem 3.1. Suppose that $n=1$. As $1\le i\le\bar r\le r\le n$, we conclude that $i=\bar r=r=n=1$. In particular, $\operatorname{rank}(M)=n-r=0$, hence M is a torsion module and therefore $\ell=0$, since we always factor out torsion to compute the analytic spread. Now the inequality reads:

$$\operatorname{ht}(I_1(\bar\varphi))\le\max\{m,\operatorname{ecodim}(R)\}.$$

By the Krull height theorem, the height of $I_1(\bar\varphi)$ is at most its number of generators, which is bounded by m, proving the case $n=1$.

We may suppose that the entries of φ lie in the maximal ideal of R. We claim that we may further assume that $I=P$ is a prime ideal. Let P be a minimal prime of I having maximal dimension. We write r_P for the rank of $\varphi_{R/P}$. If $r_P<i$ then $I_i(\varphi)\subset P$, hence $\operatorname{ht}(I_i(\bar\varphi))=0$. In this case the first inequality of (1) holds since the right-hand side is nonnegative. Thus we may assume that $i\le r_P$. The ring R being catenary we have that $\operatorname{ht}(I_i(\bar\varphi))\le\operatorname{ht}(I_i(\varphi_{R/P}))$. Hence the left-hand side of the first inequality of (1) cannot decrease as we replace $\bar\varphi$ by $\varphi_{R/P}$. We prove

that as a function of $\bar{r}$, the right-hand side of (1) is nonincreasing as we decrease $\bar{r}$ to i. Since $i \le r_P \le \bar{r}$ this will prove our claim.

The right-hand side of (1) is a maximum of two terms. Decreasing $\bar{r}$ by one changes the second term, $(\bar{r}-i)(\max\{m, n+\epsilon\} - i + \delta + 1) + \ell + \delta + \operatorname{ecodim}(R)$, to $(\bar{r}-i-1)(\max\{m, n+\epsilon\} - i + \delta + 2) + \ell + \delta + 1 + \operatorname{ecodim}(R)$. Subtracting the second from the first gives the value $\max\{m, n+\epsilon\} + r - 2\bar{r} + 1$, which is always nonnegative. The first term, $(\min\{n-\ell, \bar{r}\} - i + 1)(m - i + 1 + \max\{0, n-\ell-\bar{r}\})$, can only increase if $n-\ell-\bar{r} \ge 0$. Then as $\bar{r}$ decreases by 1, $\max\{0, n-\ell-\bar{r}\}$ will increase by 1. However, in this case $\min\{n-\ell, \bar{r}\}$ will be $\bar{r}$ and will decrease by 1. Then the product has the form $(\bar{r}-i+1)(m-i+1+(n-\ell-\bar{r}))$, and when we replace $\bar{r}$ by $\bar{r}-1$ we obtain $(\bar{r}-i)(m-i+1+(n-\ell-\bar{r})+1)$. But $(\bar{r}-i+1)(m-i+1+(n-\ell-\bar{r})) \ge (\bar{r}-i)(m-i+1+(n-\ell-\bar{r})+1)$, since $m+n-\ell+1 \ge 2\bar{r}$ by our assumption that $n-\ell-\bar{r} \ge 0$.

Thus we may suppose that $\bar{R}$ is a domain, hence equidimensional. We use the notation of Theorem 2.2 and in addition set $\underline{a}_j = 0$ whenever $j \le n-\ell$. For $0 \le j \le n$ let ϕ_j be the j by m matrix consisting of the first j rows of ϕ, and define

$$t = \min\{j \mid I_i(\phi) \subset \sqrt{(I_i(\phi_j), I, \underline{a}_j)_Q}\}.$$

We apply Theorem 2.2 with $\Lambda = \emptyset$ if $t \le n-\ell$ and $\Lambda = \{t\}$ if $t \ge n-\ell+1$. Let J_t be the t^{th} row ideal of the matrix ϕ_S, and write $\bar{B} = B/IB$, $\bar{S} = S/IS$, $\bar{J}_t = J_t\bar{S}$. By Theorem 2.2, $R \subset B \subset S$ and $\bar{R} \subset \bar{B} \subset \bar{S}$ are flat local extensions, S and $\bar{S}$ are equidimensional and catenary, and $\operatorname{ecodim}(S_P) \le \epsilon$ for every prime P of S not containing $I_i(\varphi) \cdot J_t$. Notice that $I_i(\phi_{\bar{S}}) \subset \sqrt{I_i((\phi_t)_{\bar{S}})}$ according to the definition of t. In particular we may assume $i \le t$, as otherwise $I_i(\bar{\varphi}) \subset I_i(\varphi_{\bar{S}}) = I_i(\phi_{\overline{S}})$ is nilpotent and then $\operatorname{ht}(I_i(\bar{\varphi})) = 0$. Moreover $I_i(\phi_{\bar{S}}) \not\subset \sqrt{I_i((\phi_{t-1})_{\bar{S}})}$; for otherwise $I_i(\bar{\varphi}) \subset \sqrt{I_i((\phi_{t-1})_{\bar{B}})}$ since $\bar{B} \subset \bar{S}$ is a flat local extension, and then $I_i(\phi) \subset \sqrt{(I_i(\phi_{t-1}), I, \underline{a}_{t-1})_Q}$, contradicting the choice of t. Again by Theorem 2.2, $\operatorname{ht}(J_t) \le r-n+\ell$ if $t \ge n-\ell+1$. Furthermore as $I_i((\phi_{t-1})_S) + IS$ is extended from B and S is flat over B, Lemma 2.3.2 implies that

$$\operatorname{ht}(J_t + I_i((\phi_{t-1})_S) + IS) \le \operatorname{ht}(J_t) + \operatorname{ht}(I_i((\phi_{t-1})_S) + IS) + \operatorname{ecodim}(R).$$

Thus by our equidimensionality conditions,

$$\operatorname{ht}(\bar{J}_t + I_i((\phi_{t-1})_{\bar{S}})) \le \operatorname{ht}(J_t) + \operatorname{ht}(I_i((\phi_{t-1})_{\bar{S}})) + \operatorname{ecodim}(R).$$

Since $\bar{S}$ is a flat local extension of $\bar{R}$ and

$$I_i(\varphi_{\bar{S}}) = I_i(\phi_{\bar{S}}) \subset \sqrt{I_i((\phi_t)_{\bar{S}})} \subset \sqrt{\bar{J}_t + I_i((\phi_{t-1})_{\bar{S}})},$$

we conclude that

$$\operatorname{ht}(I_i(\bar{\varphi})) = \operatorname{ht}(I_i(\varphi_{\bar{S}})) \leq \operatorname{ht}(J_t) + \operatorname{ht}(I_i((\phi_{t-1})_{\bar{S}})) + \operatorname{ecodim}(R).$$

To simplify notation we will henceforth write $\phi, \phi_j, \bar{\phi}, \bar{\phi}_j$ instead of $\phi_S, (\phi_j)_S, \phi_{\bar{S}}, (\phi_j)_{\bar{S}}$. With this we have

$$\sqrt{I_i(\bar{\phi}_{t-1})} \subsetneq \sqrt{I_i(\bar{\phi}_t)} \tag{3.2}$$

$$\operatorname{ht}(I_i(\bar{\varphi})) = \operatorname{ht}(I_i(\bar{\phi}_t)) \leq \operatorname{ht}(J_t) + \operatorname{ht}(I_i(\bar{\phi}_{t-1})) + \operatorname{ecodim}(R). \tag{3.3}$$

Case 1. $t \leq n - \ell$. In this case the first equality of (3.3) gives $\operatorname{ht}(I_i(\bar{\varphi})) \leq \operatorname{ht}(I_i(\bar{\phi}_{n-\ell}))$. Therefore $\operatorname{ht}(I_i(\bar{\varphi})) \leq (n-\ell-i+1)(m-i+1)$, and according to [B, Cor. 1],

$$\operatorname{ht}(I_i(\bar{\varphi})) \leq (\bar{r} - i + 1)(m + n - \ell - \bar{r} - i + 1).$$

Now the first inequality of (1) follows.

Case 2. $t \geq n - \ell + 1$. In this case $\operatorname{ht}(J_t) \leq r - n + \ell$, and therefore (3.3) yields

$$\operatorname{ht}(I_i(\bar{\varphi})) \leq r - n + \ell + \operatorname{ht}(I_i(\bar{\phi}_{t-1})) + \operatorname{ecodim}(R). \tag{3.4}$$

By (3.2) there exists a prime ideal P of S with $I_i(\phi_{t-1}) + IS \subset P$ and $I_i(\phi_t) \not\subset P$. Since $I_i(\phi_t)$ is contained in $I_{i-1}(\phi_{t-1})$, in $I_i(\varphi)S$, and in $J_t + I_i(\phi_{t-1})$, one automatically has $I_{i-1}(\phi_{t-1}) \not\subset P$ as well as $I_i(\varphi) \cdot J_t \not\subset P$. By the latter, $\operatorname{ecodim}(S_P) \leq \epsilon$. Set $s = \max\{j | I_j(\phi) \not\subset P\}$. Clearly $1 \leq i \leq s \leq \bar{r}$. Recall that $I_{i-1}(\phi_{t-1})_P = S_P$ and $I_i(\bar{\phi}_{t-1})_P \neq \bar{S}_P$. Thus without changing the ideal $I_i(\bar{\phi}_{t-1})_P$, we may perform elementary row and column operations over S_P to assume that

$$\phi_{S_P} = \left[\begin{array}{c|cc} 1_{i-1} & \multicolumn{2}{c}{0} \\ \hline 0 & \phi' & \phi'' \\ \cline{2-3} & 0 & 1_{s-i+1} \end{array}\right],$$

where ϕ', ϕ'' have entries in the maximal ideal of S_P. Notice that the $n-s$ by $m-s$ matrix ϕ' has rank $r-s$ and $\bar{\phi}'$ has rank $\bar{r}-s$, with $\bar{\phi}', \bar{\phi}''$ standing for $\phi'_{\bar{S}_P}, \phi''_{\bar{S}_P}$.

Since $I_i(\bar{\phi}_{t-1})_P \subset I_1(\bar{\phi}')+I_1(\bar{\phi}'') \neq \bar{S}_P$ and $\bar{S}_P$ is equidimensional and catenary, we obtain

$$\begin{aligned}\operatorname{ht}(I_i(\bar{\phi}_{t-1})) &\leq \operatorname{ht}(I_i(\bar{\phi}_{t-1})_P) \leq \mu(I_1(\bar{\phi}'')) + \operatorname{ht}(I_1(\bar{\phi}'))\\ &\leq (s-i+1)(n-s) + \operatorname{ht}(I_1(\bar{\phi}')).\end{aligned}$$

Thus by (3.4),

$$\text{(3.5)} \qquad \operatorname{ht}(I_i(\bar{\varphi})) \leq r - n + \ell + (s-i+1)(n-s) + \operatorname{ht}(I_1(\bar{\phi}')) + \operatorname{ecodim}(R).$$

If $\bar{r} - s \geq 1$ we may apply the induction hypothesis to the matrix ϕ'. Using the weaker second inequality of (1) yields

$$\begin{aligned}\operatorname{ht}(I_1(\bar{\phi}')) &\leq ((\bar{r}-s)-1)(\max\{m-s,(n-s)+\epsilon\} - 1 + \delta + 1)\\ &\quad + \max\{(m-s)-1+1,(n-s)+\epsilon\} + \delta\\ &= (\bar{r}-s)(\max\{m, n+\epsilon\} - s + \delta).\end{aligned}$$

This inequality also holds if $\bar{r} - s = 0$, since then $I_i(\bar{\Phi}') = 0$. Now by (3.5),

$$\begin{aligned}\operatorname{ht}(I_i(\bar{\varphi})) &\leq r - n + \ell + (s-i+1)(n-s) + (\bar{r}-s)(\max\{m,n+\epsilon\} - s + \delta)\\ &\quad + \operatorname{ecodim}(R)\\ &\leq r - n + \ell + n - i + (\bar{r}-i)(\max\{m,n+\epsilon\} - i + \delta) + \operatorname{ecodim}(R),\end{aligned}$$

because $i \leq s \leq \bar{r}$ and $n - s \leq \max\{m, n+\epsilon\} - i + \delta$. It follows that

$$\operatorname{ht}(I_i(\bar{\varphi})) \leq (\bar{r}-i)(\max\{m,n+\epsilon\} - i + \delta + 1) + \ell + \delta + \operatorname{ecodim}(R),$$

proving the first inequality of (1) in Case 2 as well.

To show part (2) first notice that the $\bar{R}$-module $\bar{M}$ has a rank, as can be seen from the Abhyankar-Hartshorne connectedness lemma (see [Ha1, 2.2]). The natural map $\operatorname{Sym}(M) \to \operatorname{Sym}(\bar{M})$ induces an epimorphism $\mathcal{R}(M) \to \mathcal{R}(\bar{M})$ since M is of linear type locally at every associated prime of I. Therefore $\ell(M) \geq \ell(\bar{M})$. On the other hand $\ell(\bar{M}) \geq \operatorname{rank}(\bar{M})+1$ by Proposition 1.3. Therefore $\ell \geq n-\bar{r}+1$, and (2) follows from (1). □

COROLLARY 3.6. *Let R be an equidimensional universally catenary Noetherian local ring, and let φ be an n by m matrix of rank r with entries in R. Assume that $M = \operatorname{Coker}(\varphi)$ has a rank and write $\ell = \ell(M)$. Let $i \leq r$ be an integer such that $I_i(\varphi) \neq R$. Set $\epsilon = \max_P\{\operatorname{ecodim}(R_P)\}$, where the maximum is taken over all prime ideals P of R not containing $I_i(\varphi)$.*

(1)

$$\begin{aligned}\mathrm{ht}(I_i(\varphi)) &\leq \max\{(n-\ell-i+1)(m-i+1), (r-i)(\max\{m, n+\epsilon\}-i+1)\\ &\qquad +\ell+\mathrm{ecodim}(R)\}\\ &\leq (r-i)(\max\{m, n+\epsilon\}-i+1)+\max\{m-i+1, \ell+\mathrm{ecodim}(R)\}.\end{aligned}$$

(2) *If M is not a direct sum of a free module and a torsion module, and M_P is free for every prime P of R with* $\mathrm{depth}(R_P) \leq 1$, *then*

$$\mathrm{ht}(I_i(\varphi)) \leq (r-i)(\max\{m, n+\epsilon\}-i+1)+\ell+\mathrm{ecodim}(R).$$

Proof. Apply Theorem 3.1 with $I = 0$ and use the fact that $\ell \geq \mathrm{rank}(M)$. □

In the setting of Corollary 3.6, part (1) could also be deduced from (2) whenever M is free locally in depth one: for if M is a direct sum of a free module and a torsion module, then obviously $\mathrm{ht}(I_i(\varphi)) \leq (r-i+1)(m-i+1)$.

COROLLARY 3.7. *Let R be an equidimensional universally catenary Noetherian local ring, and let φ be an n by m matrix of rank r with entries in R. Assume that* $M = \mathrm{Coker}(\varphi)$ *has a rank and write $\ell = \ell(M)$.*

(1) [B, Thm. 3] *If M is not free, then* $\mathrm{ht}(I_r(\varphi)) \leq \max\{m-r+1, \ell+\mathrm{ecodim}(R)\}$.

(2) [F, Kor. 2] *If R is S_2 and M is not a direct sum of a free module and a torsion module, then* $\mathrm{ht}(I_r(\varphi)) \leq \ell+\mathrm{ecodim}(R)$.

Proof. Set $i = r$ in Corollary 3.6. This gives (1) immediately. To prove (2) notice that M_P is free for every prime P of R with $\mathrm{depth}(R_P) \leq 1$ unless the height of $I_r(\varphi)$ is at most one. On the other hand $1 \leq \mathrm{rank}(M) \leq \ell$. □

COROLLARY 3.8. *Let R be a universally catenary Noetherian local ring of dimension d satisfying S_2, and let M be a finitely generated R-module having a rank. Let Λ be the set of all prime ideals Q of R such that the R_Q-module M_Q is not a direct sum of a free module and a torsion module. If Λ is nonempty then*

$$d \leq \max_{Q\in\Lambda}\{\mu(M_Q)+\mathrm{ecodim}(R_Q)+\dim(R/Q)\}.$$

Proof. First note that R is equidimensional. We may factor out the torsion of M to assume that M is torsion free. Notice this does not change the set Λ. Choose Q minimal in Λ. Then M_P is free for all primes $P \subsetneq Q$. If φ is a matrix minimally presenting M_Q we let r be the rank of φ. Our choice of Q shows that $\sqrt{I_r(\varphi)R_Q} = QR_Q$. Furthermore Corollary 3.7.2 gives $\mathrm{ht}(I_r(\varphi)R_Q) \leq \mu(M_Q) + \mathrm{ecodim}(R_Q)$. Hence $d - \dim(R/Q) = \dim(R_Q) = \mathrm{ht}(I_r(\varphi)R_Q) \leq \mu(M_Q) + \mathrm{ecodim}(R_Q)$, from which the corollary follows. □

COROLLARY 3.9. *Let R be an equidimensional universally catenary Noetherian local ring, and let φ be an n by m matrix of rank r with entries in R. Assume that $M = \operatorname{Coker}(\varphi)$ has a rank and write $\ell = \ell(M)$. Let $i \leq r$ be an integer such that $I_i(\varphi) \neq R$.*

(1) $\operatorname{ht}(I_i(\varphi)/I_{i+1}(\varphi)) \leq \max\{m-i+1, \ell+\operatorname{ecodim}(R)\}+r-i$.

(2) *If $i \geq n-\ell+1$, then*

$$\operatorname{ht}(I_i(\varphi)/I_{i+1}(\varphi)) \leq \ell+r-i+\operatorname{ecodim}(R)$$

and in particular

$$\operatorname{ht}(I_i(\varphi)) \leq (r-i+1)(\ell+\operatorname{ecodim}(R))+\binom{r-i+1}{2}.$$

Proof. Apply Theorem 3.1.1 with I any minimal prime of $I_{i+1}(\varphi)$ that does not contain $I_i(\varphi)$. Notice that $\bar{r}=i$. Iterate to get the last statement. □

The reader may want to compare Corollary 3.9.2 to Corollary 2.7. The significance of both formulas is that they do not involve m. The above result leads to improved height bounds for $I_i(\varphi)$ if one knows a priori that for some $j \geq i$, the height of $I_j(\varphi)$ is "smaller than expected". Applying this observation to ideals one obtains:

COROLLARY 3.10. *Let R be an equidimensional universally catenary Noetherian local ring with residue field k, and let J be an R-ideal with $\operatorname{grade}(J) > 0$. Write $g = \operatorname{ht}(J)$, $\ell = \ell(J)$, $n = \mu(J)$, and $m = \dim_k \operatorname{Tor}_1^R(k,J)$. Let i be an integer with $g-1 \leq i \leq n-1$.*

(1) *If $i \leq \ell-1$, then*

$$\operatorname{ht}(\operatorname{Fitt}_i(J)) \leq (i-g+1)(\ell+g-1+\operatorname{ecodim}(R))+\binom{i-g+1}{2}+g.$$

(2) *If $i \geq \ell$, then*

$$\begin{aligned}\operatorname{ht}(\operatorname{Fitt}_i(J)) \;\leq\; & (\ell-g)(\ell+g-1+\operatorname{ecodim}(R))+\binom{\ell-g}{2}+g\\ & +(i-\ell+1)\max\left\{m-n+\frac{\ell+3i}{2}, \frac{3\ell+i}{2}-1+\operatorname{ecodim}(R)\right\}.\end{aligned}$$

Proof. Notice that $\operatorname{ht}(\operatorname{Fitt}_{g-1}(J)) \leq g$ and argue as in the proof of Corollary 3.9. □

EXAMPLE 3.11. *Let R be a regular local ring, and let J be a proper R-ideal with $g = \mathrm{ht}(J)$ and $\ell = \ell(J)$.*

(1) (*Non-complete-intersection locus,* [Hu, 1.1]) *If J is not a complete intersection then* $\mathrm{ht}(\mathrm{Fitt}_g(J)) \leq \ell + 2g - 1$.

(2) (*Non-almost-complete-intersection locus*) *If* $\mathrm{Ext}^g_R(J,R) = 0$, *$J_Q$ is a complete intersection for every prime ideal Q of R containing J with* $\dim(R_Q) = g$, *and J is not an almost complete intersection, then* $\mathrm{ht}(\mathrm{Fitt}_{g+1}(J)) \leq 2\ell + 3g - 1$.

Proof. We may assume that the residue field of R is infinite. In (1) we may suppose that $\mathrm{ht}(\mathrm{Fitt}_g(J)) \geq g + 1$. But then J satisfies G_{g+1}, and hence $\ell \geq g + 1$ by [CN]. The assertion follows from Corollary 3.10.1. Likewise in (2) one can assume that $\mathrm{ht}(\mathrm{Fitt}_{g+1}(J)) \geq g + 2$. Thus J satisfies G_{g+2}, and therefore $\ell \geq g + 2$ according to [CEU, 4.4 and 3.4(a)] and Proposition 1.2. Again we may apply Corollary 3.10.1. $\square$

4. A family of examples. We present a class of n by m matrices of rank r which show that the inequalities of Corollary 3.6 are fairly sharp for all values of i, r, m, n. Unlike the examples given in the introduction, these matrices have no generalized zeros.

EXAMPLE 4.1. *Let i, r, m, n be integers with $1 \leq i \leq r \leq n \leq m$ and let φ be the product of a generic n by r matrix with a generic r by m matrix. One has*

$$\mathrm{ht}(I_i(\varphi)) = \begin{cases} (r-i+1)(n-i+1) & \text{if } m \geq n + r - i + 1 \\ (r-i+1)(n-i+1) - \frac{(r+n-m-i+1)^2}{4} & \text{if } r + n - m - i + 1 > 0 \\ & \text{and even} \\ (r-i+1)(n-i+1) - \frac{(r+n-m-i+1)^2-1}{4} & \text{if } r + n - m - i + 1 > 0 \\ & \text{and odd.} \end{cases}$$

Proof. We may assume that the ambient ring R is obtained by adjoining the entries of the two generic matrices to a ring k. The height of $I_i(\varphi)$ cannot decrease when k is replaced by the residue field of any minimal prime of k, and it cannot increase if we pass to the residue field of $P \cap k$ for some minimal prime P of $I_i(\varphi)$ having minimal height. Thus it suffices to consider the case where k is a field, and we may even assume that k is algebraically closed.

Let X be the closed subset of $\mathbb{P}_k^{r(m+n)-1} = \mathbb{P}(\mathrm{Hom}_k(k^m, k^r) \times \mathrm{Hom}_k(k^r, k^n))$ defined by the homogeneous ideal $I_i(\varphi)$. Notice that $X = \{[(\alpha, \beta)] \mid \mathrm{rank}(\beta\alpha) \leq i - 1\}$, where $\alpha \in \mathrm{Hom}_k(k^m, k^r)$ and $\beta \in \mathrm{Hom}_k(k^r, k^n)$. For $0 \leq s \leq r - i + 1$ set $X_s = \{[(\alpha, \beta)] \mid \mathrm{rank}(\alpha) \leq s + i - 1,\ \mathrm{rank}(\beta) \leq r - s,\ \mathrm{rank}(\beta\alpha) \leq i - 1\}$. As X is the union of the closed subsets X_s, our formula will follow once we have shown that

$$\dim X_s = (r - s)(n + s) + (s + i - 1)(m - i + 1) + (i - 1)r - 1.$$

In doing so we even show that X_s is irreducible and we construct an explicit desingularization (see also [HU, the proof of 3.16] and [ACGH, Chapter II, Section 2]). Let Y be the flag variety $\mathrm{Fl}(s, s+i-1; k^r) = \{(U, V) \mid U \subset V \subset k^r\}$, where U and V are subspaces of dimension s and $s+i-1$, respectively. In $Y \times \mathbb{P}_k^{r(m+n)-1}$ consider the closed subset $Z = \{((U, V), [(\alpha, \beta)]) \mid \mathrm{Image}(\alpha) \subset V, \mathrm{Ker}(\beta) \supset U\}$. The projections onto the first and second factor of $Y \times \mathbb{P}_k^{r(m+n)-1}$ yield surjective morphisms

$$\begin{tikzcd} & Z \arrow[dl, "f"'] \arrow[dr, "g"] & \\ Y & & X_s. \end{tikzcd}$$

It is known that Y is irreducible of dimension $s(r-s)+(i-1)(r-s-i+1)$. The fibers of f over all closed points (U, V) of Y are isomorphic to $\mathbb{P}(\mathrm{Hom}_k(k^m, V) \times \mathrm{Hom}_k(k^r/U, k^n)) \cong \mathbb{P}_k^{m(s+i-1)+n(r-s)-1}$, hence are irreducible of constant dimension. Since, furthermore, $Z \subset Y \times \mathbb{P}_k^{r(m+n)-1}$, it follows that Z is irreducible (see [E, Exercise 14.3]). One necessarily has

$$\dim Z = \dim Y + \dim \mathbb{P}_k^{m(s+i-1)+n(r-s)-1} = (r-s)(n+s)+(s+i-1)(m-i+1)+(i-1)r-1,$$

as can be seen, for instance, from the lemma of generic flatness (see [E, 14.4]). On the other hand, since Z is irreducible and g is surjective, X_s is irreducible as well. As $\{[(\alpha, \beta)] \mid \mathrm{rank}(\alpha) \leq s+i-2 \text{ or } \mathrm{rank}(\beta) \leq r-s-1\} \cap X_s$ is a closed proper subset of X_s, it follows that for every closed point $[(\alpha, \beta)]$ in some dense open subset of X_s, the fiber of g over $[(\alpha, \beta)]$ consists of the single point $((\mathrm{Ker}(\beta), \mathrm{Image}(\alpha)), [(\alpha, \beta)])$. Thus again by generic flatness, $\dim X_s = \dim Z$, which proves our assertion. □

5. Some results on symmetric matrices. We prove the conjecture of Problem 1 in the extremal cases $i = 1$ and $i = n-1$ if the ring is regular.

PROPOSITION 5.1. *Let $(R, \mathfrak{m})$ be a regular local ring with residue field k, and let φ be a symmetric n by n matrix of rank r with entries in $\mathfrak{m}$.*

(1) $\mathrm{ht}(I_1(\varphi)) \leq rn - \binom{r}{2}$.

(2) *If* $\mathrm{char}\, k \neq 2$ *and* $r = n-1 \geq 1$*, then*

$$\mathrm{ht}(I_{n-1}(\varphi)) \leq 2.$$

Proof. To prove (1) we apply Theorem 2.1 to the module $M = \mathrm{Coker}(\varphi)$. One has $\ell(M) \geq \mathrm{rank}(M) = n-r$. By the theorem there exists a local homomorphism $R \to S$ with S a localization of a polynomial ring over R, and an invertible n by n matrix T over S so that $n - r$ row ideals $J_1, \ldots, J_{n-r}$ of $\Psi = T\varphi T^*$ have height at most r. By the symmetry of φ, $\mu(I_1(\Psi)/(J_1 + \cdots + J_{n-r})) \leq \binom{r+1}{2}$. Therefore $\mathrm{ht}(I_1(\varphi)) = \mathrm{ht}(I_1(\varphi)S) = \mathrm{ht}(I_1(\Psi)) \leq \mathrm{ht}(J_1 + \cdots + J_{n-r}) + \binom{r+1}{2} \leq (n-r)r + \binom{r+1}{2} = rn - \binom{r}{2}$, where the last inequality uses the subadditivity of height in regular local rings ([S, Chap. V, Thm. 3]).

To prove (2) we suppose that $\mathrm{ht}(I_{n-1}(\varphi)) \geq 3$. Since 2 is a unit in R we may assume that φ_{11}, the $(1,1)$ entry of φ, does not lie in $\mathfrak{m}I_1(\varphi)$. Having rank $n-1$, the matrix φ fits into an exact sequence

$$0 \longrightarrow R \xrightarrow{\psi} R^n \xrightarrow{\varphi} R^{n*}.$$

As $\mathrm{ht}(I_1(\psi)) \geq \mathrm{ht}(I_{n-1}(\varphi)) \geq 3$, the complex

$$F.\!: \ 0 \longrightarrow R \xrightarrow{\psi} R^n \xrightarrow{\varphi} R^{n*} \xrightarrow{\psi^*} R^*$$

is exact by the Buchsbaum-Eisenbud acyclicity criterion, see [BE1, Theorem]. Thus $I_1(\psi) = I_1(\psi^*) \subset I_1(\varphi)$. Furthermore $I_1(\psi^*)$ is a Gorenstein ideal of height 3, and hence according to [BE2, 2.1], there is an exact sequence

$$G.\!: \ 0 \longrightarrow R \xrightarrow{\psi} R^n \xrightarrow{\chi} R^{n*} \xrightarrow{\psi^*} R^*$$

with χ alternating.

The identity map on $\mathrm{Coker}(\psi^*)$ lifts to a morphism of complexes $\alpha.\!: F. \longrightarrow G.$ where $\alpha_0 = id$ and $\alpha_1 = id$. Notice that α_3 is multiplication by some $u \in R$. As $F.^*$ and $G.^*$ are acyclic complexes of free modules, there exists a morphism of complexes $\beta.\!: G.^* \longrightarrow F.^*$ with $\beta_{-3} = \alpha_3^* = u \cdot id$ and $\beta_{-2} = u \cdot id$. One has that $\alpha.^*$ and $\beta.$ are homotopic, hence $\alpha.$ and $\beta.^*$ are homotopic. Thus $\alpha_2 \equiv u \cdot id \bmod (I_1(\varphi) + I_1(\psi))$. Since $\varphi = \chi \circ \alpha_2$, it follows that

$$\varphi \equiv u \cdot \chi \bmod (I_1(\chi)I_1(\varphi) + I_1(\chi)I_1(\psi)),$$

hence

$$\varphi \equiv u \cdot \chi \bmod I_1(\varphi)^2.$$

But this is impossible because $\chi_{11} = 0$, whereas $\varphi_{11} \notin \mathfrak{m}I_1(\varphi)$. $\square$

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