TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 355, Number 11, Pages 4451–4473
S 0002-9947(03)03198-2
Article electronically published on June 10, 2003

FITTING'S LEMMA FOR $\mathbb{Z}/2$-GRADED MODULES

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Abstract. Let $\phi: R^m \to R^d$ be a map of free modules over a commutative ring R. Fitting's Lemma shows that the "Fitting ideal," the ideal of $d \times d$ minors of ϕ, annihilates the cokernel of ϕ and is a good approximation to the whole annihilator in a certain sense. In characteristic 0 we define a Fitting ideal in the more general case of a map of graded free modules over a $\mathbb{Z}/2$-graded skew-commutative algebra and prove corresponding theorems about the annihilator; for example, the Fitting ideal and the annihilator of the cokernel are equal in the generic case. Our results generalize the classical Fitting Lemma in the commutative case and extend a key result of Green (1999) in the exterior algebra case. They depend on the Berele-Regev theory of representations of general linear Lie superalgebras. In the purely even and purely odd cases we also offer a standard basis approach to the module $\operatorname{coker}\phi$ when ϕ is a generic matrix.

Introduction

The classical Fitting Lemma (Fitting [Fit]) gives information about the annihilator of a module over a commutative ring in terms of a presentation of the module by generators and relations. More precisely, let

$$\phi : R^m \to R^d$$

be a map of finitely generated free modules over a commutative ring R, and for any integer $t \geq 0$ let $I_t(\phi)$ denote the ideal in R generated by the $t \times t$ minors of ϕ. Fitting's result says that the module $\operatorname{coker}\phi$ is annihilated by $I_d(\phi)$, and that if ϕ is the generic map—represented by a matrix whose entries are distinct indeterminates—then the annihilator is equal to $I_d(\phi)$. Thus $I_d(\phi)$ is the best approximation to the annihilator that is compatible with base change. Moreover, $I_d(\phi)$ is not too bad an approximation to $\operatorname{ann}\operatorname{coker}\phi$ in the sense that $I_d(\phi) \supset (\operatorname{ann}\operatorname{coker}\phi)^d$, or more precisely $(\operatorname{ann}\operatorname{coker}\phi)I_t(\phi) \subset I_{t+1}(\phi)$ for all $0 \leq t < d$. In this paper we will prove corresponding results in the case of $\mathbb{Z}/2$-graded modules over a skew-commutative $\mathbb{Z}/2$-graded algebra containing a field K of characteristic 0. Let R be a $\mathbb{Z}/2$-graded skew-commutative K-algebra: that is, $R = R_0 \oplus R_1$ as vector spaces, R_0 is a commutative central subalgebra, $R_iR_j \subset R_{i+j \pmod 2}$, and every element of R_1 squares to 0. Any homogeneous map ϕ of $\mathbb{Z}/2$-graded free

Received by the editors March 20, 2002 and, in revised form, May 29, 2002.
2000 *Mathematics Subject Classification.* Primary 13C99, 13C05, 13D02, 16D70, 17B70.
The second named author is grateful to the Mathematical Sciences Research Institute for support in the period this work was completed. Both authors are grateful for the partial support of the National Science Foundation.

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R-modules may be written in the form

$$\phi : \quad R^m \oplus R^n(1) \xrightarrow{\left(\begin{smallmatrix} X & A \\ B & Y \end{smallmatrix}\right)} R^d \oplus R^e(1),$$

where X, Y are matrices of even elements of R, and A, B are matrices of odd elements. We will define an ideal $I_{\Lambda(d,e)}$ and show that it is contained in the annihilator of the cokernel of ϕ, with equality in the *generic case* where the entries of the matrices X, Y, A, B are indeterminates (that is, R is a polynomial ring on the entries of X and Y tensored with an exterior algebra on the entries of A and B). In the purely even and purely odd cases our results are characteristic-free, but in general we give examples to show that the annihilator can look quite different—for example, it might be generated by forms of different degrees—in positive characteristic.

Now let K be a field, and let $U = U_0 \oplus U_1$ and $V = V_0 \oplus V_1$ be $\mathbb{Z}/2$-graded vector spaces of dimensions (d, e) and (m, n) respectively. We consider the generic ring

$$\mathcal{S} = \mathcal{S}(V \otimes U) := S(V_0 \otimes U_0) \otimes S(V_1 \otimes U_1) \otimes \wedge(V_0 \otimes U_1) \otimes \wedge(V_1 \otimes U_0),$$

where S denotes the symmetric algebra and $\wedge$ denotes the exterior algebra, and the *generic*, or *tautological*, map

$$\Phi : \mathcal{S} \otimes V \to \mathcal{S} \otimes U^*.$$

This map Φ is defined by the condition that $\Phi|_V = 1 \otimes \eta : V \to V \otimes_K U \otimes_K U^* \subset R \otimes_K U^*$, where $\eta : K \to U \otimes_K U^*$ is the dual of the contraction $U^* \otimes_K U \to K$. We will make use of this notation throughout the paper. We will compute the annihilator of the cokernel of Φ. Of course if we specialize Φ to any map of free modules ϕ over a $\mathbb{Z}/2$-graded ring, preserving the grading, then we can derive elements in the annihilator of the cokernel of ϕ by specializing the annihilator of the cokernel of Φ.

In the classical case, where V and U have only even parts ($e = n = 0$), the annihilator is an invariant ideal for the action of the product of general linear groups $\mathrm{GL}(V) \times \mathrm{GL}(U)$. Such invariant ideals have been studied by DeConcini, Eisenbud, and Procesi in [DEP] and have a very simple arithmetic. In the general case, Berele and Regev [BR] have developed a highly parallel theory, using the $\mathbb{Z}/2$-graded Lie algebra $\mathfrak{g} = \mathfrak{gl}(V) \times \mathfrak{gl}(U)$ in place of $\mathrm{GL}(V) \times \mathrm{GL}(U)$. They show that the generic ring $\mathcal{S}$ is a semisimple representation of $\mathfrak{g}$ (even though not all the representations of $\mathfrak{g}$ are semisimple) and that the irreducible summands of $\mathcal{S}$ of total degree t are parametrized by certain partitions of the integer t, just as in the commutative case. The Berele-Regev theory is described in detail below, in Section 1 of this paper.

If Λ is a partition, we write I_Λ for the ideal of $\mathcal{S}$ generated by the irreducible representation corresponding to Λ. If ϕ is a matrix representing any map of $\mathbb{Z}/2$-graded free modules over a $\mathbb{Z}/2$-graded skew-commutative K algebra R, then there is a unique ring homomorphism $\alpha : \mathcal{S} \to R$ such that $\phi = \alpha(\Phi)$, and we write $I_\Lambda(\phi) := \alpha(I_\Lambda(\Phi))R$ for the ideal generated by the image of $I_\Lambda = I_\Lambda(\Phi)$. If $e = n = 0$, so that the ring $\mathcal{S}$ is a polynomial ring, the classical Fitting Lemma (see for example Eisenbud [Eis, Prop. 20.7]) shows that the annihilator of $\operatorname{coker} \Phi$ is the ideal of $d \times d$ minors $I_d(\Phi)$. In representation-theoretic terms, this is the ideal generated by the representation

$$\wedge^d V_0 \otimes \wedge^d U_0 \subset \mathrm{Sym}_d(V_0 \otimes U_0),$$

the irreducible representation associated to the partition with one term (d). In our notation, $I_d(\Phi) = I_{(d)} = I_{(d)}(\Phi)$. On the other hand, if $d = n = 0$, so that the ring $\mathcal{S}$ is an exterior algebra, Green [Gre, Proposition 1.3] shows that the representation

$$S_e(V_0) \otimes \wedge^e U_1 \subset \wedge^e(V_0 \otimes U_1)$$

is at least contained in the annihilator of $\operatorname{coker}\Phi$. This is the representation associated to the partition $(1^e) = (1, 1, \ldots, 1)$ with e parts (see Appendix 1 for a characteristic-free treatment). Here is the common generalization of these results, which is the main result of this paper:

Theorem 1. *Suppose that K is a field of characteristic 0, and let*

$$\phi : \quad R^m \oplus R^n(1) \xrightarrow{\left(\begin{smallmatrix} X & A \\ B & Y \end{smallmatrix}\right)} R^d \oplus R^e(1)$$

be a $\mathbb{Z}/2$-graded map of free modules over a $\mathbb{Z}/2$-graded skew-commutative K-algebra R.

a) *When $R = \mathcal{S}$ and $\phi = \Phi$, the generic map defined above, the annihilator of the cokernel of Φ, is $I_{\Lambda(d,e)}(\Phi)$, where $\Lambda(d, e)$ is the partition $(d+1, d+1, \ldots, d+1, d)$ of $(d+1)(e+1)-1$ into $e+1$ parts. In general we have $I_{\Lambda(d,e)}(\phi) \subset \operatorname{ann}\ \operatorname{coker}(\phi)$.*

b) *If $x_1, \ldots, x_e \in \operatorname{ann}\ \operatorname{coker}(\phi)$, then $x_1 \ldots x_e \in I_{\Lambda(0,e)}(\phi)$. Moreover, if $0 \le s \le d-1$, and $x_1, \ldots, x_{e+1} \in \operatorname{ann}\ \operatorname{coker}(\phi)$, then $x_1 \ldots x_{e+1} I_{\Lambda(s,e)}(\phi) \subset I_{\Lambda(s+1,e)}(\phi)$.*

The proof is given in sections 2 and 3 below. An alternate approach through standard bases is given in Appendix 3 in the purely even and purely odd cases, and this approach is characteristic-free. In the classical case ($e = n = 0$) we can also describe the annihilator of $\operatorname{coker}\Phi$ by saying that it is nonzero only if $m \ge d$, and then it is generated, as a $\mathfrak{gl}(V) \times \mathfrak{gl}(U)$-ideal, by an $m \times m$ minor of Φ. To simplify the general statement, we note that a shift of degree by 1 does not change the annihilator of the cokernel of Φ, but has the effect of interchanging m with n and d with e.

Corollary 2. *With notation as above, the annihilator of the cokernel of Φ is nonzero only if*

a) $m > d$ (or symmetrically $n > e$) or

b) $m = d$ and $n = e$.

In each of these cases the annihilator is generated as a $\mathfrak{g}$-ideal by one element Z of degree $de + d + e$ defined as follows:

In case a) when $m > d$,

$$Z = Z_1 \cdot X(1, \ldots, d \mid 1, \ldots, d),$$

where $X(1, \ldots, d \mid 1, \ldots, d)$ is the $d \times d$ minor of X corresponding to the first d columns and $Z_1 = \prod_{j \le e, k \le d+1} b_{j,k}$ is the product of all the elements in the first $d+1$ columns of B (and symmetrically if $n > e$).

In case b),

$$Z = W_1 \cdots W_e \cdot \det(X),$$

where W_s is the $(d+1) \times (d+1)$ minor of Φ containing X and the entry $y_{s,s}$, that is,

$$W_s = \det(X) y_{s,s} + \sum_{1 \le i, \le d} \pm \det(X(\hat{i}, \hat{j})) a_{i,s} b_{s,j}.$$

Corollary 2 follows from intermediate results in the proof of Theorem 1(a). We next give some examples of Theorem 1(a) and Corollary 2.

Example 1. Suppose that $d = n = 0$, so that the presentation matrix B has only odd degree entries. A central observation of Green [Gre] is that the "exterior minors" of Φ are in the annihilator of $\operatorname{coker} \Phi$ (see Appendix 1 for a direct proof of Green's result that is different from the one given by Green, and can serve as an introduction to the proof we give for Theorem 1(a) in general). The element Z of Corollary 2 is the product of the elements in the first column of Φ. Quite generally, it is not hard to see that the product of all elements in a K-linear combination of the columns of B is an exterior minor in Green's sense. The representation corresponding to the partition $(1, \dots, 1)$ of e is generated by $\binom{m+e-1}{e}$ such products; so $\binom{m+e-1}{e}$ exterior minors generate the annihilator in the generic case. For example, taking $m = 2$ and $e = 2$, the annihilator of the cokernel of the generic matrix

$$\begin{pmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{pmatrix}$$

where the variables all have odd degree, is minimally generated by the three exterior minors

$$b_{1,1}b_{2,1}, \quad b_{1,2}b_{2,2}, \quad (b_{1,1} + b_{1,2})(b_{2,1} + b_{2,2}).$$

Example 2. Now suppose that our generic matrix has size 2×2 with the first row even and the second row odd ($m = 2,\ n = 0,\ d = e = 1$):

$$\begin{pmatrix} x_{1,1} & x_{1,2} \\ b_{1,1} & b_{1,2} \end{pmatrix}.$$

In this case our result shows that the cokernel has annihilator equal to the product

$$(x_{1,1}, \quad x_{1,2})(b_{1,1}b_{1,2}, \quad x_{1,1}b_{1,2} - x_{1,2}b_{1,1}),$$

which is minimally generated by 4 elements. The element Z is $x_{1,1}b_{1,1}b_{1,2}$.

Example 3. As a final 2×2 example, consider the case $m = n = d = e = 1$, which for simplicity we write as

$$\begin{pmatrix} x & a \\ b & y \end{pmatrix}.$$

Here the annihilator of the cokernel is again minimally generated by 4 elements, namely

$$axy, \quad bxy, \quad (xy - ab)x, \quad (xy + ab)y.$$

The element Z is $(xy + ab)x$. In Appendix 2 we will explain the action of $\mathfrak{g}$ on these elements.

Positive characteristics. As we have remarked, Theorem 1 is characteristic-free in the purely even or odd cases (and in these cases $\operatorname{coker} \Phi$ is free over $\mathbb{Z}$). But already with $m = n = d = e = 1$ as in Example 3, the annihilator is different in characteristic 2: in characteristic zero the annihilator is generated by forms of degree 3, but in characteristic 2 the algebra R is commutative, and so the determinant $xy - ab$ is in the annihilator as well. The annihilator can differ in other characteristics as well. Macaulay2 computations show that the case $d = 1,\ e = p - 1,\ m = 2,\ n = 0$ is exceptional in characteristic p for $p = 3, 5$ and 7. Perhaps the same holds for all primes p. The cokernel of the generic matrix over the integers can also have $\mathbb{Z}$-torsion. For example, Macaulay2 computation shows that if $d = 1,\ e = 2,\ m = 3,\ n = 1$, then the cokernel of $\Phi_{\mathbb{Z}}$ has 2-torsion.

Our interest in extending the Fitting Lemma was inspired by Mark Green's paper [Gre]. Green's striking use of his result on annihilators to prove one of the Eisenbud-Koh-Stillman conjectures on linear syzygies turns on the fact that if N is a module over a polynomial ring $S = K[X_1, \ldots, X_m]$, then $T := \mathrm{Tor}_*^S(K, M)$ is a module over the ring $R = \mathrm{Ext}_S^*(K, K)$, which is an exterior algebra. Green in effect translated the hypothesis of the linear syzygy conjecture into a statement about the degree 1 part of the R-free presentation matrix of the submodule of T representing the linear part of the resolution of N, and then showed that the exterior minors generated a certain power of the maximal ideal of the exterior algebra, which was sufficient to prove the conjecture. Green's result only gives information on the annihilator in the case where the elements of the presentation matrix are all odd. Elements of even degree in an exterior algebra can behave (if the number of variables is large) very much like variables in a polynomial ring, at least as far as expressions of bounded degree are concerned. Thus to extend Green's work it seemed natural to deal with the case of $\mathbb{Z}/2$-graded algebras.

This work is part of a program to study modules and resolutions over exterior algebras; see Eisenbud-Fløystad-Schreyer [ES], and Eisenbud-Popescu-Yuzvinsky [EPY] for further information.

We would never have undertaken the project reported in this paper if we had not had the program Macaulay2 (www.math.uiuc.edu/Macaulay2) of Grayson and Stillman as a tool; its ability to compute in skew commutative algebras was invaluable in figuring out the pattern that the results should have and in assuring us that we were on the right track.

1. Berele-Regev theory

For the proof of Theorem 1 we will use the beautiful results of Berele and Regev [BR] giving the structure of R as a module over $\mathfrak{g}$. For the convenience of the reader we give a brief sketch of what is needed. We make use of the notation introduced above: $U = U_0 \oplus U_1$ and $V = V_0 \oplus V_1$ are $\mathbb{Z}/2$-graded vector spaces over the field K of characteristic 0 with $\dim U = (d, e)$ and $\dim V = (m, n)$.

The $\mathbb{Z}/2$-graded Lie algebra $\mathfrak{gl}(V)$ is the vector space of $\mathbb{Z}/2$-graded endomorphisms of $V = V_0 \oplus V_1$. Thus

$$\mathfrak{gl}(V) = \mathfrak{gl}(V)_0 \oplus \mathfrak{gl}(V)_1,$$

where $\mathfrak{gl}(V)_0$ is the set of endomorphisms preserving the grading of V and $\mathfrak{gl}(V)_1$ is the set of endomorphisms of V shifting the grading by 1. Additively,

$$\mathfrak{gl}(V)_0 = \mathrm{End}_K(V_0) \oplus \mathrm{End}_K(V_1),$$
$$\mathfrak{gl}(V)_1 = \mathrm{Hom}_K(V_0, V_1) \oplus \mathrm{Hom}_K(V_1, V_0).$$

The commutator of the pair of homogeneous elements $x, y \in \mathfrak{gl}(V)$ is defined by the formula

$$[x, y] = xy - (-1)^{\deg(x)\deg(y)} yx.$$

By a $\mathfrak{gl}(V)$-module we mean a $\mathbb{Z}/2$-graded vector space $M = M_0 \oplus M_1$ with a bilinear map of $\mathbb{Z}/2$-graded vector spaces $\circ : \mathfrak{gl}(V) \times M \to M$ satisfying the identity

$$[x, y] \circ m = x \circ (y \circ m) - (-1)^{\deg(x)\deg(y)} y \circ (x \circ m))$$

for homogeneous elements $x, y \in \mathfrak{gl}(V), m \in M$.

In contrast to the classical theory, not every representation of the $\mathbb{Z}/2$-graded Lie algebra $\mathfrak{gl}(V)$ is semisimple. For example, its natural action on mixed tensors $V^{\otimes k} \otimes V^{*\otimes l}$ is in general not completely reducible. However, its action on $V^{\otimes t}$ decomposes just as in the ungraded case:

Proposition 1.1. *The action of $\mathfrak{gl}(V)$ on $V^{\otimes t}$ is completely reducible for each t. More precisely, the analogue of Schur's double centralizer theorem holds, and the irreducible $\mathfrak{gl}(V)$-modules occurring in the decomposition of $V^{\otimes t}$ are in* 1-1 *correspondance with irreducible representations of the symmetric group Σ_t on t letters. These irreducibles are the Schur functors*

$$\mathcal{S}_\lambda(V) = e(\lambda)V^{\otimes t}$$

where $e(\lambda)$ is a Young idempotent corresponding to a partition λ in the group ring of the symmetric group Σ_t.

This notation is consistent with the notation above in the sense that the d-th homogeneous component of the ring $\mathcal{S}(V)$ is $\mathcal{S}_d(V)$ where d represents the partition (d) with one part.

Here we use the symbol $\mathcal{S}_\lambda$ to denote the $\mathbb{Z}/2$-graded version of the Schur functor S_λ; the latter acts on ungraded vector spaces. Recall that the functor $\wedge^\lambda$ is by definition the same as the functor $S_{\lambda'}$, where λ' denotes the partition that is conjugate to λ. (For example, the conjugate partition to (2) is $(1,1)$.) We will extend this by writing $\bigwedge^\lambda := \mathcal{S}_{\lambda'}$ for the $\mathbb{Z}/2$-graded version. The partition (d) with only one part will be denoted simply d; so, for example, $\mathcal{S}_2(V) = \bigwedge^{(1,1)} V = S_2(V_0) \oplus (V_0 \otimes V_1) \oplus \wedge^2 V_1$ and similarly $\bigwedge^2 V = \mathcal{S}_{(1,1)}V = \wedge^2 V_0 \oplus V_0 \otimes V_1 \oplus S_2(V_1)$. In each case the decomposition is as representations of the subalgebra $\mathfrak{gl}(V_0) \times \mathfrak{gl}(V_1) \subset \mathfrak{gl}(V)$. Similar decompositions hold for all $\mathcal{S}_d$ and $\bigwedge^d V$. (If we were not working in characteristic zero, we would use divided powers in place of symmetric powers in the description of $\bigwedge V$.)

Proposition 1.1 implies that the parts of the representation theory of $\mathfrak{gl}(V)\times\mathfrak{gl}(U)$ that involve only tensor products of V and U and their summands are parallel to the representation theory in the case $V_1 = U_1 = 0$, which is the classical representation theory of a product of the two general linear Lie algebras $\mathfrak{gl}(V_0) \times \mathfrak{gl}(U_0)$.

The proposition also implies that the decompositions into irreducible representations of tensor products of the $\mathcal{S}_\lambda(V)$, as well as the decompositions of their symmetric and exterior powers, correspond to the decompositions in the even case: we just have to replace the ordinary Schur functors $S, \wedge$ by their $\mathbb{Z}/2$-graded analogues $\mathcal{S}, \bigwedge$.

The formulas giving equivariant embeddings or equivariant projections may also be derived from the corresponding formulas in the even case by applying the principle of signs: The formulas in the even case involve many terms where the basis elements are permuted in a prescribed way. The basis elements have degree 0. To write down a $\mathbb{Z}/2$-graded analogue of such a formula, we simply allow the basis elements to have even or odd degree and we adjust the signs of terms in such a way that changing the order of two homogeneous elements x and y of V in the $\mathbb{Z}/2$-graded analogue of the formula will cost the additional factor $(-1)^{\deg(x)\deg(y)}$.

There is a $\mathbb{Z}/2$-graded analogue of the Cauchy decomposition, which follows as just described from Proposition 1.1 together with the corresponding result in the even case (proven in Macdonald [Mac], Chapter 1, and in DeConcini, Eisenbud, and Procesi [DEP]). Recall that $\mathfrak{g} = \mathfrak{gl}(V) \times \mathfrak{gl}(U)$.

Corollary 1.2. *The t-th component $\mathcal{S}_t(V \otimes U)$ of $\mathcal{S}(V \otimes U)$ decomposes as a $\mathfrak{g}$-module as*

$$\mathcal{S}_t(V \otimes U) = \bigoplus_{\lambda, |\lambda| = t} \mathcal{S}_\lambda(V) \otimes \mathcal{S}_\lambda(U).$$

Another application of the same principle shows that to describe the annihilator of the cokernel of Φ, and what generates it, it suffices to describe which representations $\mathcal{S}_\lambda V \otimes \mathcal{S}_\lambda U$ it contains:

Corollary 1.3. *If $I \subset \mathcal{S}(V \otimes U)$ is a $\mathfrak{g}$-invariant ideal, then I is a sum of subrepresentations $\mathcal{S}_\lambda V \otimes \mathcal{S}_\lambda U$. Moreover, the ideal generated by $\mathcal{S}_\lambda V \otimes \mathcal{S}_\lambda U$ contains $\mathcal{S}_\mu V \otimes \mathcal{S}_\mu U$ if and only if $\mu \supset \lambda$.*

Although it is not so simple to describe the vectors in $\mathcal{S}_t(V \otimes U)$ that lie in a given irreducible summand, we can, as in the commutative case, define a filtration that has these irreducible representations as successive factors. We start by defining a map $\rho_t : \bigwedge^t V \otimes \bigwedge^t U \hookrightarrow \mathcal{S}_t(V \otimes U)$ as the composite

$$\bigwedge^t V \otimes \bigwedge^t U \to \otimes^t V \ \otimes\ \otimes^t U \to \mathcal{S}_t(V \otimes U)$$

where the first map is the tensor product of the two diagonal maps (here we use the sign conventions for $\mathbb{Z}/2$-graded vector spaces) and the second map simply pairs corresponding factors. Thus

$$\rho_t(v_1 \wedge \ldots \wedge v_t \otimes u_1 \wedge \ldots \wedge u_t) = \sum_{\sigma \in \Sigma_t} \pm (v_1 \otimes u_{\sigma(1)}) \cdot \ldots \cdot (v_t \otimes u_{\sigma(t)})$$

where the sign $\pm$ is the sign of the permutation σ adjusted by the rule that switching homogeneous elements x, y from either V or U means we multiply by $(-1)^{\deg(x)\deg(y)}$. For example, if V and U were both even, the image of this map would be the span of the $t \times t$ minors of the generic matrix; when V is even and U is odd, the image is the span of the space of "exterior minors" as in Green [Gre].

For any partition $\lambda = (\lambda_1, \ldots, \lambda_s)$ we define $\mathcal{F}_\lambda$ to be the image of the composite map

$$m \circ (\rho_{\lambda_1} \otimes \ldots \otimes \rho_{\lambda_s}) : \bigwedge^{\lambda_1} V \otimes \bigwedge^{\lambda_1} U \otimes \ldots \otimes \bigwedge^{\lambda_s} V \otimes \bigwedge^{\lambda_s} U \to \mathcal{S}_{|\lambda|}(V \otimes U)$$

where m denotes the multiplication map in $\mathcal{S}(V \otimes U)$.

As in the even case, we order partitions of t by saying $\lambda < \mu$ if and only if $\lambda'_i > \mu'_i$ for the smallest number i such that $\lambda'_i \neq \mu'_i$. Finally, we define the subspaces

$$\mathcal{F}_{<\lambda} = \sum_{|\mu| = |\lambda|,\ \mu < \lambda} \mathcal{F}_\mu \qquad \subset \qquad \mathcal{F}_{\leq\lambda} = \sum_{|\mu| = |\lambda|,\ \mu \leq \lambda} \mathcal{F}_\mu.$$

In the classical case, $\mathcal{F}_{\leq\lambda}$ is spanned by certain products of minors of the generic matrix. The *straightening law* of Dubillet, Rota, and Stein [DRS] shows that we get a basis if we choose only "standard" products of these types, and the successive quotients in the filtration are the irreducible representations of $\mathrm{GL}(V) \times \mathrm{GL}(U)$. The analogue in our $\mathbb{Z}/2$-graded case is

Proposition 1.4. *The subspaces $\mathcal{F}_{\leq\lambda}$ define a $\mathfrak{g}$-invariant filtration on $\mathcal{S}_{|\lambda|}(V \otimes U)$. The quotient $\mathcal{F}_{\leq\lambda}/\mathcal{F}_{<\lambda}$ is isomorphic to $\bigwedge^\lambda V \otimes \bigwedge^\lambda U = \mathcal{S}_{\lambda'} V \otimes \mathcal{S}_{\lambda'} U$.*

There is also one element of each irreducible representation that is easy to describe: the *highest weight vector.* To speak of highest weight vectors we must choose ordered bases $\{u_1, \dots, u_d\}$ and $\{u'_1, \dots, u'_e\}$ of U_0 and U_1, and ordered bases $\{v_1, \dots, v_m\}$ and $\{v'_1, \dots, v'_n\}$ of V_0 and V_1.

Proposition 1.5. *Let $\lambda = (\lambda_1, \dots, \lambda_s)$ be a partition, and let $w_i^1 \in \bigwedge^{\lambda_i}(V)$ and $w_i^2 \in \bigwedge^{\lambda_i}(U)$ be the elements*

$$w_i^1 = \begin{cases} v_1 \wedge \dots \wedge v_{\lambda_i}, & \text{if } \lambda_i \le m, \\ v_1 \wedge \dots \wedge v_m \wedge {v'_i}^{(\lambda_i - m)}, & \text{otherwise}, \end{cases}$$

$$w_i^2 = \begin{cases} u_1 \wedge \dots \wedge u_{\lambda_i}, & \text{if } \lambda_i \le d, \\ u_1 \wedge \dots \wedge u_d \wedge {u'_i}^{(\lambda_i - d)}, & \text{otherwise}. \end{cases}$$

The element

$$c_\lambda = \prod_{i=1}^{s} \rho_{\lambda_i}(w_i^1 \otimes w_i^2) \quad \in \quad \mathcal{S}(V \otimes U)$$

is the highest weight vector from the irreducible component $\bigwedge^\lambda V \otimes \bigwedge^\lambda U = \mathcal{S}_{\lambda'} V \otimes \mathcal{S}_{\lambda'} U$, where λ' is the conjugate partition to λ.

The Berele-Regev theory allows us to give explicit generators for the annihilator, generalizing the ordinary minors of Φ in the commutative case. More generally, we can give generators for arbitrary J_λ.

We start with a double tableau (S, T), that is, two sequences of tensors $v_{i,1} \wedge \ldots \wedge v_{i,\lambda_i} \in \bigwedge^{\lambda_i} V$ and $u_{i,1} \wedge \ldots \wedge u_{i,\lambda_i} \in \bigwedge^{\lambda_i} U$ $(1 \le i \le s)$. We imagine that the elements $v_{i,j} \in V$ correspond to the i-th row of the tableau S of shape λ, and the elements $u_{i,j} \in U$ correspond to the i-th row of another tableau T of shape λ. We define

$$\rho(S \otimes T) = \prod_{1 \le i \le s} \rho_{\lambda_i}(v_{i,1} \wedge \ldots \wedge v_{i,\lambda_i} \otimes u_{i,1} \wedge \ldots \wedge u_{i,\lambda_i}).$$

We think of λ as a Ferrers diagram. If S is a tableau of shape λ and σ is a permutation of the boxes in λ, then $\sigma(S)$ is another tableau of shape λ (here we write σ as a product of transpositions, and introduce a minus sign whenever we interchange two elements of odd degree). Let $P(\lambda)$ be the group of permutations of the boxes in λ that preserve the columns of λ.

Proposition 1.6. *The representation $\bigwedge^\lambda V \otimes \bigwedge^\lambda U \subset R$ is generated by elements*

$$\pi(S, T) = \sum_{\sigma \in P(\lambda)} \rho(\sigma S \otimes T),$$

or, equivalently, by

$$\pi'(S, T) = \sum_{\sigma \in P(\lambda)} \rho(S \otimes \sigma T)$$

where S and T range over all tableaux of shape λ.

Proof. We show that the $\pi(S, T)$ generate $\bigwedge^\lambda V \otimes \bigwedge^\lambda U$; the proof for π' is similar. Since $\rho(S, T)$ is antisymmetric in the elements appearing in each row of S, the element $\sum_{\sigma \in P(\lambda)} \rho(\sigma(S), T)$ is the $\mathfrak{gl}(V)$-linear projection of $\rho(S, T)$ to the $\bigwedge^\lambda V$-isotypic component of R. By Corollary 1.2 we have $R = \bigoplus_\lambda \bigwedge^\lambda V \otimes \bigwedge^\lambda U$; so this isotypic component is $\bigwedge^\lambda V \otimes \bigwedge^\lambda U \subset R$. □

To find a minimal set of generators for J_λ inside this generating set, choose bases $\{u_i\}$,$\{u'_i\}$, $\{v_i\}$,$\{v'_i\}$ as above, and order the bases of U and V by $u_1 < \cdots < u_d < u'_1 < \cdots < u'_e$ and $v_1 < \cdots < v_m < v'_1 < \cdots < v'_n$. A double tableau (S,T) whose entries $u_{i,j}$ and $v_{i,j}$ come from these bases is called *standard* if the following conditions are satisfied:

$$\begin{aligned} v_{i,j} &< v_{i,j+1} \text{ if } v_{i,j} \text{ is even, } v_{i,j} \le v_{i,j+1} \text{ if } v_{i,j} \text{ is odd,} \\ v_{i,j} &\le v_{i+1,j} \text{ if } v_{i,j} \text{ is even, } v_{i,j} < v_{i+1,j} \text{ if } v_{i,j} \text{ is odd,} \end{aligned}$$

and similarly for the $u_{i,j}$.

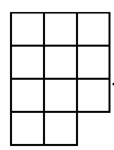
Proposition 1.7. *The ideal J_λ is minimally generated by the elements $\pi(S,T)$ where (S,T) ranges over the set of double standard tableaux of shape λ.*

Proof. Berele and Regev proved that the standard tableaux form a basis of $\bigwedge^\lambda U$. □

2. Proof of Theorem 1(a)

In this section U and V are $\mathbb{Z}/2$-graded vector spaces of dimensions (d,e) and (m,n) respectively, and Φ is the generic map, defined tautologically over $R = \mathcal{S}(V \otimes U)$.

We write $\Lambda(d,e)$ for the partition with $e{+}1$ parts $((d{+}1)^e, d) = (d{+}1, \ldots, d{+}1, d)$; that is, $\Lambda(d,e)$ corresponds to the Ferrers diagram that is a $(d+1) \times (e+1)$ rectangle minus the box in the lower right-hand corner. For example, $\Lambda(2,3)$ may be represented by the Ferrers diagram

$\Lambda(2,3) =$  .

For any partition λ we denote by I_λ the ideal in R generated by the representation $\bigwedge^\lambda V \otimes \bigwedge^\lambda U$. With this notation, Theorem 1(a) takes the form

Theorem 1(a). *The annihilator of the cokernel of Φ is equal to $I_{\Lambda(d,e)}$.*

Theorem 1(a) implies that the representations appearing in the annihilator of $\operatorname{coker}\Phi$ depend only on the dimension of U, not the dimension of V, as long as the dimension of V_0 is large (Corollary 2.2), and we begin by proving this. For the precise statement, we will use the following notation: Let V' be another $\mathbb{Z}/2$-graded vector space, and let Φ' be the generic map $R' \otimes V' \to R' \otimes U^*$ where $R' = \mathcal{S}(V' \otimes U)$. If V is a summand of V', so that $V' = V \oplus W$, then the ring $R = \mathcal{S}(V \otimes U)$ can be identified with a subring of R'. We want to compare the annihilators of the modules $\operatorname{coker}\Phi$ and $\operatorname{coker}\Phi'$.

Proposition 2.1. *If V is a $\mathbb{Z}/2$-graded summand of V', then* $\operatorname{ann}\operatorname{coker}\Phi = R \cap \operatorname{ann}\operatorname{coker}\Phi'$. *More precisely,*
a) $\operatorname{coker}\Phi$ *is an R-submodule of* $\operatorname{coker}\Phi'$, *and*
b) $\operatorname{coker}\Phi'$ *is a quotient of* $(\operatorname{coker}\Phi) \otimes R'$.

Proof. The first statement follows easily from a) and b).

For the proof of a) and b) we may write $V' = V \oplus W$, and we make use of the $\mathbb{N}$-grading of R' for which $V \otimes U$ has degree 0 and $W \otimes U$ has degree 1. (This grading has nothing to do with the $\mathbb{Z}/2$-grading used elsewhere in this paper!) The map Φ' is homogeneous of degree 0 if we twist the summands of its source appropriately,

$$\Phi' : R' \otimes W(-1) \oplus R' \otimes V \to^{(\Phi'_1, \Phi'_0)} R' \otimes U^*.$$

So we have an induced $\mathbb{N}$-grading on $\operatorname{coker} \Phi'$. Since $\Phi'_0 = \Phi$, we see that $(\operatorname{coker} \Phi')_0 = \operatorname{coker} \Phi$. Since the elements of R have degree 0, this is an R-submodule, as required for a).

For b) it suffices to note that $\operatorname{coker} \Phi'$ is obtained from $(\operatorname{coker} \Phi) \otimes R'$ by factoring out the relations corresponding to $W \otimes R'$. $\square$

Corollary 2.2. *With U, V, V', Φ, Φ' as above, suppose that V is such that $I_\lambda \neq 0$ in $\mathcal{S}(V \otimes U)$. If $I_\lambda \subset \operatorname{ann} \operatorname{coker} \Phi$, then $I_\lambda \subset \operatorname{ann} \operatorname{coker} \Phi'$.*

Proof. The inclusion $R = \mathcal{S}(V \otimes U) \subset R' = \mathcal{S}(V' \otimes U)$ carries $\bigwedge^\lambda V \otimes \bigwedge^\lambda U$ into $\bigwedge^\lambda V' \otimes \bigwedge^\lambda U$. The conclusion now follows from Proposition 2.1 a) and b). $\square$

Proof of Theorem 1(a). We first show that $I_{\Lambda(d,e)}$ is contained in the annihilator of $M := \operatorname{coker} \Phi$. By Corollary 2.2, it is enough, given U, to produce one nonzero element from $\bigwedge^{\Lambda(d,e)} V \otimes \bigwedge^{\Lambda(d,e)} U \subset \mathcal{S}(V \otimes U)$ that annihilates the cokernel of Φ for some space V. By Corollary 2.2 it suffices to prove this result in the case $m = d+1, n = 0$, that is, $\dim V = (d+1, 0)$.

Let $u_1, \dots, u_d$ be a basis of U_0, let $u'_1, \dots, u'_e$ be a basis of U_1, and let $v_1, \dots, v_{d+1}$ be a basis of $V = V_0$. We denote the variables from the $U_0 \otimes V_0$ block by $x_{i,k}$ $(1 \leq i \leq d,\ 1 \leq k \leq d+1)$, and the variables from the $U_1 \otimes V_0$ block by $b_{j,k}$ $(1 \leq j \leq e,\ 1 \leq k \leq d+1)$. Thus:

$$\Phi = \begin{pmatrix} X \\ B \end{pmatrix}, \quad X = (x_{i,k}), \quad B = (b_{j,k}).$$

Let

$$Z = Z_1 \cdot X(1, \dots, d \mid 1, \dots, d)$$

where $Z_1 = \prod_{j,k} b_{j,k}$ is the product of all the entries of B and $X(1, \dots, d \mid 1, \dots, d)$ is the $d \times d$ minor of the matrix X corresponding to the first d columns.

We now show that Z annihilates M. Indeed, by the classical Fitting Lemma we know that $X(1, \dots, d \mid 1, \dots, d)$ annihilates the even generic module $M/(V \otimes U_1)M$. Thus every basis element u_k multiplied by $X(1, \dots, d \mid 1, \dots, d)$ can be expressed modulo the image of Φ as a linear combination of $u'_1, \dots, u'_e$ with coefficients of positive degee in the variables $b_{j,k}$. Since these variables are odd, $Zu_k = 0$ in M.

To see that Zu'_l is also 0 in M, we use the classical Fitting Lemma again on the first d columns of the matrix of Φ to see that for any $1 \leq i \leq d$ the element $X(1, \dots, d \mid 1, \dots, d)u_i$ can be expressed, modulo the image of Φ, as a linear combination of the $b_{s,t}u'_j$. On the other hand, if we multiply the last column of the matrix of Φ by the product Z'_1 of all the $b_{j,k}$ except for the $b_{l,d+1}$, we get an expression for $Z_1 u'_l$, modulo the image of Φ, as a linear combination of $Z'_1 u_1 \dots, Z'_1 u_d$. Thus $Zu'_l = X(1, \dots, d \mid 1, \dots, d) Z_1 u'_l = 0$ in M, as required.

Next we prove that the element Z is a weight vector (not generally a highest weight vector) and lies in $\bigwedge^{\Lambda(d,e)} V \otimes \bigwedge^{\Lambda(d,e)} U$. Indeed, $X(1, \dots, d \mid 1, \dots, d)$ is a weight vector in $\bigwedge^d V \otimes \bigwedge^d U$. The element Z_1 is a weight vector in the representation $\bigwedge^{(d+1)^e} V \otimes \bigwedge^{(d+1)^e} U$. The product is thus contained in the ideal $\mathcal{F}_{\leq \Lambda(d,e)}$. The element Z has degree $(e+1)(d+1) - 1$, but it involves only $d+1$ elements from $V = V_0$. By Proposition 1.5 its weight can occur only in representations $\bigwedge^{\lambda} V \otimes \bigwedge^{\lambda} U \subset \mathcal{S}_{\lambda}(V \otimes U)$ with λ having all parts $\leq d+1$. Since $\Lambda(d,e)$ is the only partition λ with at most $e+1$ parts having $|\lambda| = (d+1)(e+1)-1$ and each $\lambda_i \leq d+1$, we are done. This argument shows that $I_{\Lambda(d,e)}$ is contained in the annihilator of the cokernel of Φ.

Now let μ be a partition not containing $\Lambda(d,e)$. To complete the proof of Theorem 1(a), we must show that the ideal I_μ does not annihilate $M = \operatorname{coker}\Phi$ or, equivalently, that the highest weight vector c_μ does not annihilate M.

Since μ does not contain $\Lambda(d,e)$, it does not contain one of the extremal boxes of $\Lambda(d,e)$. By shifting the gradings of V, U by 1 we do not alter the annihilator of the generic map, but we change the notation so that all partitions are changed to their conjugates. Thus we may assume that $\mu_e \leq d$. By Corollary 2.2 we may further assume that $n = 0$, so that $V = V_0$, and that $m >> 0$. To prove the theorem, we will carry out an induction on d.

If $d = 0$, we must show that the annihilator of the cokernel of Φ is contained in $I_{(1^e)}$; or, equivalently, that it contains no I_λ where λ has fewer than e parts. Set

$$Z_1 = \prod_{1 \leq j \leq e-1, 1 \leq k \leq m} b_{j,k}.$$

By Proposition 1.5, Z_1 is the highest weight vector in $\bigwedge^{m^{e-1}} V \otimes \bigwedge^{m^{e-1}} U$. The element $Z_1 u'_e$ is not in the image of Φ, because the coefficient of u'_e in any element from the image of Φ is in the ideal generated by $b_{e,1}, \dots, b_{e,m}$, while Z_1 is not in this ideal. Since Z_1 does not annihilate M, no I_λ such that λ has $< e$ parts can annihilate M.

In case $d > 0$ the matrix of Φ will contain an even variable $x_{1,1}$. To complete the induction we will invert this variable and use □

Lemma 2.3. a) *Over the ring $R_1 = R[x_{1,1}^{-1}]$ the map Φ can be reduced by row and column operations to the form*

$$\Phi' \oplus \ id: \ (V' \otimes_K R_1) \oplus R_1 \to U'^* \otimes_K \oplus R_1$$

where V is a $\mathbb{Z}/2$-graded vector space of dimension $(m-1, n)$ and U is a $\mathbb{Z}/2$-graded vector space of dimension $(d-1, e)$. Moreover, the ring R' generated over K by the entries of Φ' is isomorphic to $\mathcal{S}(V' \otimes U')$, and R_1 is a flat extension of R'.

b) *The localization of the ideal I_μ at $x_{1,1}$ is isomorphic to the extension of the ideal J'_ν from R' where ν is the partition obtained from μ by subtracting* 1 *from each nonzero part.*

Proof of Lemma 2.3. Column and row reduction give the following formulas for the entries of Φ':

$$x'_{i,k} = x_{i,k} - \frac{x_{1,k} x_{i,1}}{x_{1,1}}, \qquad a'_{i,l} = a_{i,l} - \frac{a_{1,l} x_{i,1}}{x_{1,1}},$$
$$b'_{j,k} = b_{j,k} - \frac{x_{1,k} b_{j,1}}{x_{1,1}}, \qquad y'_{j,l} = y_{j,l} - \frac{a_{1,l} b_{j,1}}{x_{1,1}}.$$

Consequently,

$$R_1 = R'[x_{1,1}, x_{1,1}^{-1}][x_{1,2}, \dots, x_{1,m}, \ a_{1,1}, \dots, a_{1,n}, \ x_{2,1}, \dots, x_{d,1}, \ b_{1,1}, \dots, b_{e,1}]$$

in the sense of $\mathbb{Z}/2$-graded algebras. This proves part a).

To prove part b), we first observe that the localization of the ideal $I_{(t)}(\Phi)$ gives the ideal $I_{(t-1)}(\Phi')$. Indeed, the ideal $I_{(t)}(\Phi)$ is generated by $\mathbb{Z}/2$-graded analogues of $t\times t$ minors of Φ. After localization it becomes the ideal $I_{(t)}(\Phi'\oplus id_{R_1})$ generated by the $\mathbb{Z}/2$-graded analogues of $t\times t$ minors of $\Phi'\oplus id_{R_1}$. Let us call the row and column of the matrix $\Phi'\oplus id_{R_1}$ corresponding to the summand R_1 the distinguished row and column. Every $\mathbb{Z}/2$-graded analogue of a $t\times t$ minor of $\Phi'\oplus id_{R_1}$ is either a $(t-1)\times(t-1)$ minor of Φ' (in case it contains the distinguished row and column), zero (if it contains the distinguished row but not the distinguished column, or vice versa), or a $t\times t$ minor of Φ' if it does not contain the distinguished row or column.

To show that the result generalizes to an arbitrary partition μ, we order the bases so that the distinguished row and column come first. We saw in Proposition 6 that the highest weight vectors in $\bigwedge^\mu V\otimes\bigwedge^\mu U$ are the products of minors of the matrix Φ on some initial subsets of rows and columns of Φ. So after localization each factor will contain both the distinguished row and the distinguished column of $\Phi'\oplus id_{R_1}$. ☐

Completion of the Proof of Theorem 1(a). Now suppose that $d>0$ and $n=0$. We may of course assume that $m\neq 0$, so that the matrix of Φ contains the even variable $x_{1,1}$. It is enough to prove that $I_\mu M\neq 0$ after inverting $x_{1,1}$. The ideal I_μ will localize to the ideal J'_ν where ν is equal to μ with all parts decreased by 1. The graded vector space U of dimension (d,e) will change to the $\mathbb{Z}/2$-graded vector space U' of dimension $(d-1,e)$. The desired conclusion follows by induction on d. ☐

3. Proof of Theorem 1(b)

If $\phi: R^m\to R^d$ is a matrix representing a map of free modules over a commutative ring, then, as we noted in the introduction, there are inclusions $\mathrm{ann}(M)\cdot I_i(\phi)\subset I_{i+1}(\phi)$ for $0\le i<d$, and thus, by induction, $\mathrm{ann}(M)^d\subset I_d(\phi)$; see for example Eisenbud [Eis]. To prove these inclusions, one first notes that the cokernel of ϕ is the same as the cokernel of

$$\psi: V_0\otimes R\oplus U_0^*\otimes R\to U_0^*\otimes R$$

where $\psi=(\phi,\, a\cdot Id)$. Thus $I_j(\phi)=I_j(\psi)$ and $I_{j+1}(\phi)=I_{j+1}(\psi)\supset a\cdot I_j(\phi)$. We will carry out the same approach in the $\mathbb{Z}/2$-graded case.

In this section we work with an arbitrary map

$$\phi: V\otimes R\to U^*\otimes R$$

of $\mathbb{Z}/2$-graded free modules over a $\mathbb{Z}/2$-graded commutative ring R. The first step is to show that, just as in the classical case, the ideal $I_\lambda(\phi)$ depends only on the cokernel of ϕ and on the number and degrees of the generators chosen.

Lemma 3.1. *If $\alpha: V'\otimes R\to V\otimes R$, then*

$$I_\lambda(\phi\alpha)\subset I_\lambda(\phi).$$

In particular, if $\psi: V'\otimes R\to U^*\otimes R$ has the same cokernel as ϕ, then $I_\lambda(\phi)=I_\lambda(\psi)$.

Proof. The second statement follows from the first, because each of the maps ϕ and ψ factors through the other.

To prove the first statement, we use the notation of Proposition 1.6. For any map $W\otimes R\to U^*\otimes R$, and any tableaux S and T of elements in W and U, both of shape λ, we let $\pi'_\psi(S,T)$ be the result of specializing the element $\pi'(S,T)$ defined for the generic map Φ when Φ is specialized to ψ. By Proposition 1.6, it is enough to show that when $W = V'$ the element $\pi'_{\phi\alpha}(S,T)$ is in $I_\lambda(\phi)$. We have

$$\rho_l(v'_1 \wedge \dots v'_l \otimes u_1 \wedge \dots u_l)$$
$$= \sum_{i_1<\cdots<i_l} \rho_l(v'_1 \wedge \dots v'_l \otimes v^*_{i_1} \wedge \dots v^*_{i_l})\rho_l(v_{i_1} \wedge \dots v_{i_l} \otimes u_1 \wedge \dots u_l)$$

where $v_1, \dots, v_{m+n}$ and $v^*_1, \dots, v^*_{m+n}$ are dual bases of V and V^*. Using this identity to rewrite the formula for $\pi'_{\phi\alpha}(S',T)$, where S' is a tableau of shape λ with entries in V' and T is a tableau of shape λ with entries in U, we see that $\pi'_{\phi\alpha}(S',T)$ is a linear combination of elements of the form $\pi'_\phi(S,T)$, where S is a tableau of shape λ with entries in V. □

Lemma 3.1 implies, in particular, that two presentations of the same module with the same numbers of even and odd generators have the same ideals $I_\lambda(\phi)$. Similar arguments show that we can allow for presentations with different numbers of generators as long as we change the partitions suitably: if we add d' even and e' odd generators, then we have to expand λ by adding d' columns of length equal to the length of the first column and e' rows of length equal to the length of the first row of the resulting partition (or vice versa). In this sense the ideals $I_\lambda(\phi)$ depend only on the cokernel of ϕ.

The main result of this section is the following.

Theorem 3.2. *Let R, U, V, ϕ be as in the beginning of the introduction, and let $M = \operatorname{coker}\phi$.*

a) *Let s be an integer, $0 \le s \le d-1$. If $x_1, \dots, x_{e+1} \in \operatorname{Ann}_R M$, then*

$$x_1 \dots x_{e+1} I_{\Lambda(s,e)}(\phi) \subset I_{\Lambda(s+1,e)}(\phi).$$

b) *If $x_1, \dots, x_e \in \operatorname{Ann}_R M$, then $x_1 \dots x_e \in I_{\Lambda(0,e)}(\phi)$.*

As in the classical case, we derive

Corollary 3.3. *Let M be a $\mathbb{Z}/2$-graded module over a $\mathbb{Z}/2$-graded ring R, with the presentation $\phi : V \otimes R \to U^* \otimes R$. Assume that $\dim U = (d,e), \dim V = (m,n)$. Let $x_1, \dots, x_{(d+1)(e+1)-1}$ be homogeneous elements from $\operatorname{Ann}_R M$. Then $x_1 \dots x_{(d+1)(e+1)-1} \in I_{\Lambda(d,e)}(\phi)$.*

Proof of Theorem 3.2. We begin with part b). We work with a presentation $(\phi,\psi) : V \otimes R \oplus W \otimes R \to U^* \otimes R$ where W is a $\mathbb{Z}/2$-graded vector space of dimension e with the i-th generator w_i going to x_i times the i-th generator u_i of U^*. The parity of the generators of W is adjusted so that ψ is of degree 0. Now, taking a double tableau (S,T) of the shape (1^e) with w_i and u_i in the i-th row, and applying the definition above, we see that the generator $\pi(S,T)$ is just $x_1 \dots x_e$.

To prove part a) we distinguish two cases. In the case $s < d-1$ we use the presentation $(\phi,\psi) : V \otimes R \oplus W \otimes R \to U^* \otimes R$ where W is a $\mathbb{Z}/2$-graded vector space of dimension $e+1$ with the i-th generator w_i going to x_i times the $(s+i+2)$-th generator u_i of U^*. The parity of the generators of W is adjusted so that ψ is of degree 0. We can assume without loss of generality that $\phi := \Phi$ is generic. Then it is enough to prove that $c_{\Lambda(s,e)} x_1 \dots x_{e+1} \in I_{\Lambda(s+1,e)}$, where $c_{\Lambda(s,e)}$ is the highest weight vector defined as in Proposition 1.5.

We pick a tableau (S,T) of shape $\Lambda(s+1,e)$ as follows. The entries $v_{i,j}$, $u_{i,j}$ in the i-th row are the same as in the canonical tableau, except the last ones. The last entry in the tableau v in the i-th row is w_i, and the last entry in the i-th row is u_{s+2}. The element $\pi'(S,T)$ is easily seen to be $c_{\Lambda(s,e)}x_1 \dots x_{e+1}$.

In the case $s = d-1$ we use the presentation $(\phi, \psi) : V \otimes R \oplus W \otimes R \to U^* \otimes R$ where W is a $\mathbb{Z}/2$-graded vector space of dimension $e+1$ with the the i-th generator w_i going to x_i times the $(d+i)$-th generator u_i of U^* for $1 \le i \le e$ and w_{e+1} going to x_{e+1} times u_d. The parity of the generators of W is adjusted so that ψ is of degree 0. We can assume without loss of generality that $\phi := \Phi$ is generic. Then it is enough to prove that $c_{\Lambda(d-1,e)}x_1 \dots x_{e+1} \in I_{\Lambda(d,e)}$, where $c_{\Lambda(d-1,e)}$ is the canonical tableau.

We pick a tableau (S,T) of shape $\Lambda(s+1,e)$ as follows. The entries $v_{i,j}$, $u_{i,j}$ in the i-th row are the same as in the canonical tableau, except the last ones. The last entry in the tableau v in the i-th row is w_i, and the last entry in the i-th row is u_{d+i} for $1 \le i \le e$, and u_d for the $(e+1)$-st row. The element $\pi'(S,T)$ is easily seen to be $c_{\Lambda(d-1,e)}x_1 \dots x_{e+1}$. □

4. The resolution of a generic $\mathbb{Z}/2$-graded module

In this section we work over the generic ring $R = \mathcal{S}$ as in the introduction, and we conjecture the form of a minimal free resolution over R of the cokernel C of the generic map Φ. This resolution is a natural generalization of the one constructed in [BE] in the commutative case. We work over a field K of characteristic 0. We define some $\mathbb{Z}/2$-graded free R-modules $\mathbf{F}_i$ as follows:

$$\mathbf{F}_0 = U^* \otimes R, \qquad \mathbf{F}_1 = V \otimes R,$$

$$\mathbf{F}_i = \bigoplus_{|\alpha|+|\beta|=i-2} \mathcal{S}_{\Theta(d,e,\alpha,\beta)}V \otimes \mathcal{S}_{\Lambda(d,e,\alpha,\beta)}U \otimes R$$

where $\Lambda(d,e,\alpha,\beta) = (d+1+\beta_1, d+1+\beta_2, \dots, d+1+\beta_e, e, \alpha'_1, \dots, \alpha'_s)$, $\Theta(d,e,\alpha,\beta) = (d+1+\alpha_1, d+1+\alpha_2, \dots, d+1+\alpha_e, e+1, \beta'_1, \dots, \beta'_s)$, and we sum over all pairs of partitions α, β with at most e parts.

Conjecture 4.1. *There exists an equivariant differential $d_i : \mathbf{F}_i \to \mathbf{F}_{i-1}$, linear for $i \ge 3$ or $i = 1$ and of degree $|\Lambda(d,e)|$ for $i = 2$, that makes $\mathbf{F}_\bullet$ into a minimal R-free $\mathbb{Z}/2$-graded resolution of C.*

In the even case ($U_1 = V_1 = 0$) the desired complex is the Buchsbaum-Rim resolution (see, for example, Buchsbaum and Eisenbud [BE]). We have checked the conjecture computationally, using Macaulay2, in a few more cases.

Appendix 1. Elementary proof of a special case

In this appendix we consider the case where the entries of Φ are all of odd degree, and give the most transparent proof we know of Theorem 1(a), the fact that a certain ideal I annihilates $\operatorname{coker}\Phi$. The result and its proof in this case are characteristic-free. The ideal I has a simple description as the "ideal of exterior minors" in the sense of Green [Gre], and Proposition A1.1 was originally proved by him in that paper in a different way. We give a simple way of generating the ideal of exterior minors, which works whenever the ground field K is infinite. This appendix will also serve as an introduction to the proof of Theorem 1(a). A different method of treating the purely odd and purely even cases, which we do not know how to generalize, is given in Appendix 3.

More precisely, we do the special case of Theorem 1(a) where $d = n = 0$ (the other case where Φ has odd matrices, namely where $e = m = 0$, differs just by shifting degrees). Here $U = U_1$ and $V = V_0$, so that the matrix Φ consists of a single block, with odd degree entries, and we wish to show that the ideal $I = I_{\Lambda(0,e)} = I_{(1^e)}$ annihilates the cokernel of Φ.

In this special $d = n = 0$ case the ring $\mathcal{S} = \mathcal{S}(V \otimes U)$ is the ordinary exterior algebra $\wedge(V \otimes U)$. For simplicity, we work throughout this appendix with the ordinary Schur functors $\wedge^t$ and S_t in place of their super analogues $\bigwedge^t$ and $\mathcal{S}_t$.

Over the integers the representation $S_e(V)\otimes\wedge^e U$ is not a summand of $\wedge(V\otimes U)$, but its saturation is easy to identify as $D_e(V) \otimes \wedge^e(U)$, where $D_e(V)$ is the e-th homogeneous component of the divided power algebra. It is perhaps simplest to think of $D_e(V)$ as the dual representation of $S_e(V^*)$, and with this as definition we can define the embedding $\iota:\ D_e(V) \otimes \wedge^e(U) \subset \wedge^e(V \otimes U)$ as the dual of the surjection $\pi:\ \wedge^e(V^* \otimes U^*) \to S_e(V^*) \otimes \wedge^e(U^*)$, which comes in turn by extending the identity map $\wedge(V^* \otimes U^*)_1 = (S(V^*) \otimes \wedge(U^*))_{1,1}$ to an algebra homomorphism $\wedge(V^*\otimes U^*) \to S(V^*)\otimes\wedge(U^*)$, using the fact that the elements of $(S(V^*)\otimes\wedge(U^*))_{1,1}$ square to 0 and anticommute. Explicitly, the map π can be specified by its action on pure vectors, which is

$$\pi:\ (\hat{v}_1 \otimes \hat{u}_1) \wedge \cdots \wedge (\hat{v}_e \otimes \hat{u}_e) \mapsto (\hat{v}_1 \cdots \hat{v}_e) \otimes (\hat{u}_1 \wedge \cdots \wedge \hat{u}_e).$$

We write $a_{i,j} = v_j \otimes u_i$ for the exterior variables. Theorem 1(a) may be stated for the case $d = n = 0$ as follows.

Proposition A1.1. *Let E be the exterior algebra on variables $a_{i,j}$, $i = 1,\dots,e$, $j = 1,\dots,m$, over a field K of arbitrary characteristic, and let*

$$\Phi = \begin{pmatrix} a_{1,1} & \cdots & a_{1,m} \\ \vdots & \ddots & \vdots \\ a_{e,1} & \cdots & a_{e,m} \end{pmatrix}$$

be the $e\times m$ generic matrix over E. The cokernel of Φ is annihilated by the element $b = b_1 \wedge b_2 \wedge \cdots \wedge b_e$ where

$$\beta = \begin{pmatrix} b_1 \\ \vdots \\ b_e \end{pmatrix}$$

is any K-linear combination of the columns of Φ. If K is infinite, then these elements generate the representation

$$D_e(V) \otimes \wedge^e(U) \subset \wedge^e(V \otimes U).$$

Proof. First consider the case $m = \dim V = 1$, where Φ has just one column, and let $\{v_1\}$ be a basis for V. Let $u_1,\dots,u_e$ be a basis of U and let $\overline{u}_i$ be the image of u_i in $\operatorname{coker}\Phi$. In this case $V \otimes U$ has dimension e, and so $\wedge^e(V\otimes U)$ is 1-dimensional, equal to $D_e(V) \otimes \wedge^e(U)$, and generated by the product $a_{1,1} \wedge \cdots \wedge a_{e,1}$.

We have $\sum a_{i,1}\overline{u}_i = 0$. Thus

$$\begin{aligned}(a_{1,1} \wedge \cdots \wedge a_{e,1})\overline{u}_e &= (a_{1,1} \wedge \cdots \wedge a_{e-1,1}) \wedge a_{e,1}\overline{u}_e \\ &= -\sum_{i\neq e}(a_{1,1} \wedge \cdots \wedge a_{e-1,1}) \wedge a_{i,1}\overline{u}_i = 0,\end{aligned}$$

since $a_{i,1} \wedge a_{i,1} = 0$.

In the general case $\dim V = m$, we note that the cokernel of the augmented matrix

$$\begin{pmatrix} a_{1,1} & \cdots & a_{1,m} & b_1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{e,1} & \cdots & a_{e,m} & b_e \end{pmatrix}$$

is the same as the cokernel of Φ, and thus is a quotient of the cokernel of the matrix β. By the case $m = 1$, the module $\operatorname{coker}\beta$ is annihilated by $b_1 \wedge \cdots \wedge b_e$; so its quotient $\operatorname{coker}\Phi$ is too.

Finally, we must show that, if K is infinite, the products b corresponding to columns β generate the representation $\iota(D_e(V) \otimes \wedge^e U)$. Let $v \in V$ be the element corresponding to the linear combination of the columns b, and let $V' \subset V$ be the 1-dimensional space spanned by v. The element b generates the image of the composite map

$$D_e(V') \otimes \wedge^e U \to D_e(V) \otimes \wedge^e U \to^{\iota} \wedge^e(V \otimes U).$$

We must show that the images of all such maps span $\iota(D_e(V) \otimes \wedge^e U)$. The space $\wedge^e U$ is 1-dimensional. Thus it suffices to show that the images of all $D_e(V')$ in $D_e(V)$ span $D_e(V)$. (Note that this would fail for $S_e(V)$ in place of $D_e(V)$ in characteristic p if $e = p$.)

Dually, it suffices to show that the intersection of the kernels of the maps $S_e(V^*) \to S_e(V^{'*})$ induced by all 1-dimensional projections $V^* \to V^{'*}$ is zero. Such a projection is a point in the projective space $\mathbf{P}(V)$, and the kernel is the set of polynomials of degree e vanishing at the point. The desired assertion follows because only the zero polynomial vanishes at all the points of $\mathbf{P}(V)$ when K is infinite. $\square$

Appendix 2. Comments on the action of $\mathfrak{g}$

It may at first be surprising that the generators given in Examples 1-3 of the Introduction are permuted by the action of $\mathfrak{g}$. So we will make the action explicit in one case, Example 3 (the other cases are similar and simpler).

When we think of $R = \mathcal{S}(V \otimes U)$ as a $\mathfrak{g}$-module, we think of $\mathfrak{g}$ acting on the left. But we may identify $U \otimes V$ with $\mathrm{Hom}(V, U^*)^* = \mathrm{Hom}(U^*, V)$, and thus identify R with the coordinate ring of the space $\mathrm{Hom}(V, U^*)$. In this identification it is natural to think of the Lie algebra $\mathfrak{g} = \mathfrak{gl}(V) \times \mathfrak{gl}(U)$ as $\mathfrak{gl}(V) \times \mathfrak{gl}(U^*)$, with the $\mathfrak{gl}(U^*)$ acting on the right. To make this identification, we use the *supertranspose*, which is the anti-isomorphism

$$\mathfrak{gl}(U) \to \mathfrak{gl}(U^*); \quad \begin{pmatrix} U_{0,0} & U_{0,1} \\ U_{1,0} & U_{1,1} \end{pmatrix} \mapsto \begin{pmatrix} U_{0,0}^t & U_{1,0}^t \\ -U_{0,1}^t & U_{1,1}^t \end{pmatrix}.$$

Now consider the case presented in Example 3 of the Introduction, whose notation we use. To see the action of $\mathfrak{g}$, let us act by two elements of the Lie algebra on the element axy. First we act with the element $v_{0,1}$ from $\mathfrak{gl}(V)$, changing an odd element to an even one. We get a sum of terms, each of which is axy with one changed factor; we can replace a by x or y by b. Thus we get terms xxy and axb. The first comes with positive sign ($v_{0,1}$ acts from the left and we replaced the first factor, and so there are no switches), the second term comes with negative sign (we replaced y by b, and so had to switch $v_{0,1}$ with a). Thus we get $x(xy - ab)$.

Let us also act on axy by the Lie algebra element $u_{1,0}$ from $\mathfrak{gl}(U^*)$ exchanging an even element with an odd one. We get a sum of terms where each term is axy with one factor changed; we can change x to b and a to y. We get terms yxy and aby. Both come without sign, since $u_{1,0}$ acts from the right and x, y have even degree. Thus we get $(xy + ab)y$.

Appendix 3. A standard basis approach to the generic module

In this appendix we describe a general method for analyzing the cokernel of the generic matrix by giving a *standard basis* in the case where all the variables of S are even or all are odd. It would be very interesting to give a generalization to the situation where there are both even and odd variables. In the cases treated here, the method, and the results, are characteristic-free.

To explain the method in the most familiar setting, we begin with the even case—the classical case of the cokernel of a matrix whose entries are distinct variables of a commutative polynomial ring. The standard basis approach to the ring S itself in this case is due to Doubillet, Rota, and Stein [DRS]; see also DeConcini, Eisenbud, and Procesi [DEP]. The standard basis for the generic module in the even case was obtained by Bruns and Vetter [BV] and by Bruns [Bru]. This standard basis was shown to be a Gröbner basis by Onn [Onn]. We give the proof based on representation theory because it exends to the odd case. Then we sketch the odd case, which is quite parallel. Throughout this appendix we think of tableaux as being filled with numbers, not with elements of the numbered basis as before.

The even case. Consider vector spaces $V = K^m, U = K^d$ and the generic map $\Phi : V \otimes_K S \to U^* \otimes_K S$ of free modules over the polynomial ring $S = \mathrm{Sym}_K(V \otimes U)$. We will give a good basis for the module $\operatorname{coker} \Phi$. Let $\{u_1, \dots, u_d\}$, $\{v_1, \dots, v_m\}$ be bases of U, V respectively. Let $\{u_1^*, \dots, u_d^*\}$ be the dual basis of U^*. Write $\Phi(v_j) = \sum_{i=1}^d x_{i,j} u_i^*$. The ring S can be identified with the commutative polynomial ring $K[x_{i,j}]_{1 \le i \le d, 1 \le j \le m}$. Recall that a *double tableau of shape* λ, where $\lambda = (\lambda_1 \ge \dots \ge \lambda_r \ge 0)$, is a pair of tableaux

$$P = (P_1, \dots, P_r), \quad Q = (Q_1, \dots, Q_r)$$

with $P_j = (p_{j,1}, \dots, p_{j,\lambda_j})$ and $Q_j = (q_{j,1}, \dots, q_{j,\lambda_j})$ satisfying $1 \le p_{i,j} \le d$ and $1 \le q_{i,j} \le m$ (and thus $\lambda_1 \le \min(m, d)$). Corresponding to the double tableau (P, Q) is the product $M_1 \cdots M_r \in S$, where M_j is the signed minor of the generic matrix $\Phi = (x_{i,j})_{1 \le i \le d, 1 \le j \le m}$ involving the rows indexed by P and the columns indexed by Q. We will henceforward think of the double tableau as elements of S. Thus, for example, $(p_1, \dots, p_r) \mid (q_1, \dots, q_r)$ denotes an $r \times r$ minor of Φ.

The tableau P is *standard* if $p_{i,j} < p_{i,j+1}$ for all i, j and $p_{i,j} \le p_{i+1,j}$ for all i, j. Similarly we define the standardness of Q, and a double tableau (P, Q) is standard if both P and Q are standard. The algebra S has a basis consisting of (the elements corresponding to) the *double standard tableaux.* A K-basis of the free module $U^* \otimes S$ is given by the products $(P, Q)u_i^*$ where (P, Q) is a standard double tableau and $1 \le i \le d$. Thus we can construct a basis for $\operatorname{coker} \Phi$ by taking an appropriate subset.

Definition. A product $(P, Q)u_i^*$ is *admissible* if the double tableau (P, Q) is standard and the first row of P does not contain the interval $[1, i]$. Notice that the additional condition is really a condition on P and u_i^*, and if it is satisfied, we will say that Pu_i^* is admissible.

Proposition A3.1. *The images of the admissible products in* $\operatorname{coker}\Phi$ *form a* K*-basis.*

Proof. We first show that the admissible products span $\operatorname{coker}\Phi$. Order the bases by setting $u_1 < \ldots < u_d$, $u_d^* < \ldots < u_1^*$, $v_1 < \ldots < v_m$. Order the products $(P, Q)u_i^*$ (where (P, Q) is a double tableau, not nessesarily standard) by reading lexicographically, first the element u_i^*, then P by rows and then Q by rows. It suffices to show that if a product is not admissible, it is a combination of earlier products.

The (i, j)-th entry of Φ may be written as $(i|j)$. Thus the relations $\sum_{i=1}^{d}(i|j)u_i^* = 0$ hold in $\operatorname{coker}\Phi$. We will use a generalization:

Lemma A3.2. *In* $\operatorname{coker}\Phi$*,*

$$\sum_{i=1}^{d}(a_1, \ldots, a_{n-1}, i, a_{n+1}, \ldots, a_r \mid b_1, \ldots, b_r)u_i^* = 0.$$

Proof. Use the Laplace expansion with respect to the n-th row of each minor.

Now assume that the product $(P, Q)u_i^*$ is not admissible. Use the relation from Lemma A3.2 for $n = i$, taking $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_r$ and $b_1, \ldots, b_r$ to be the first row of P minus i and the first row of Q, respectively. We may assume that the rows of P and Q are ordered, so that $a_1 = 1, \ldots, a_{i-1} = i - 1$. The relation shows that in $\operatorname{coker}\Phi$ the element $(P_1|Q_1)u_i^*$ is a linear combination of $u_{i+1}^*, \ldots, u_d^*$ with polynomial coefficients, because the other terms involve minors with repeated rows. These are earlier terms in our order. We multiply this relation by the other rows of (P, Q), and we see that $(P, Q)u_i^*$ is a linear combination of earlier terms in the same way.

It remains to show that the admissible products are linearly independent over K. It is enough to prove this over $\mathbf{Z}$, and thus it is enough to prove the linear independence over $\mathbf{Q}$. We will show that in a given degree the dimension of the module $\operatorname{coker}\Phi$ is at least equal to the number of admissible products. To do this we use the representation theory of $\mathrm{GL}(U)$.

We label the irreducible representation $S_\lambda U$ of $\mathrm{GL}(U)$ by its highest weight. This means they are labeled by the sequences $\lambda = (\lambda_1, \ldots, \lambda_d)$ of integers such that $\lambda_1 \geq \ldots \geq \lambda_d$, but we do not assume $\lambda_d \geq 0$. Irreducible representations of $\mathrm{SL}(U)$ correspond to weights λ with $\lambda_d = 0$ and are called Schur functors. We have

$$S_{(\lambda_1+1,\ldots,\lambda_d+1)}U = S_\lambda U \otimes \wedge^d U.$$

With this notation $\wedge^i U = S_{(1^i,0^{d-i})}U$ and $\wedge^i U^* = S_{(0^{d-i},(-1)^i)}U$; in particular, the naturally isomorphic representations $\wedge^i U = \wedge^d$ and $U \otimes \wedge^{d-i}U^*$ both have highest weight $(1^i, 0^{d-i})$. The representation with general λ_n can be expressed as a Schur functor tensored with an integer power of $\wedge^d U$. We can now express $\operatorname{coker}\Phi$ as a representation:

Proposition A3.3. *If* K *is a field of characteristic zero, then the decomposition of* $\operatorname{coker}\Phi$ *into irreducible representations of* $\mathrm{GL}(V) \times \mathrm{GL}(U)$ *is*

$$\operatorname{coker}\Phi = \bigoplus_{\lambda=(\lambda_1,\ldots,\lambda_{d-1})} S_\lambda V \otimes S_{(\lambda_1,\ldots,\lambda_{d-1},-1)}U.$$

Proof of Proposition A3.3. The free module $U^* \otimes S$ decomposes as

$$U^* \otimes S = \bigoplus_{\lambda=(\lambda_1,\ldots,\lambda_d),\ \lambda_d \geq 0} S_\lambda V \otimes S_\lambda U \otimes U^*.$$

Let us look at the isotypic component of $S_\lambda V$. Using Pieri's formula, we can decompose $S_\lambda U \otimes U^*$. Using the isomorphism $U^* = \wedge^{d-1} U \otimes \wedge^d U^*$, we get

$$S_\lambda U \otimes U^* = \bigoplus_{i \text{ such that } \lambda_i > \lambda_{i-1}} S_{\lambda_1,\ldots,\lambda_{i+1}\lambda_i-1,\lambda_{i-1}\ldots,\lambda_d} U.$$

The representation $V \otimes S$ contains only representations $S_\lambda U$ of $\mathrm{GL}(U)$ with all entries of λ nonnegative. On the other hand, if $\lambda_d = 0$, then $S_\lambda U \otimes U^*$ contains the representation $S_{\lambda_1,\ldots,\lambda_{d-1},-1} U$. Thus this representation must occur in $\operatorname{coker}\Phi$.

The following lemma concludes the proof of Proposition A3.3 and also gives the linear independence necessary to finish the proof of Proposition A3.1.

Lemma A3.4. *The dimension of $S_{(\lambda_1,\ldots,\lambda_{d-1},-1)} U$ is equal to the number of the products Pu_i^* where P is a standard tableau of shape λ and the first row of P does not contain the interval* $[1, i]$.

Proof. The representation $S_{(\lambda_1,\ldots,\lambda_{d-1},-1)} U$ has the same dimension as

$$S_{(\lambda_1,\ldots,\lambda_{d-1},-1)} U \otimes \bigwedge^d U = S_{(\lambda_1+1,\ldots,\lambda_{d-1}+1,0)} U.$$

This is equal to the number of standard tableaux of shape $(\lambda_1 + 1, \ldots, \lambda_{d-1} + 1, 0)$.

On the other hand, the admissibility condition on the product Pu_i^* is the same as the standardness condition on the tableau of shape λ starting with the row $1, 2, \ldots, i-1, i+1, \ldots, d$ and continuing with the rows of P. Thus the number of admissible products Pu_i^* is equal to the dimension of the representation $S_{\lambda_1,\ldots,\lambda_{d-1},-1} U$ in $S_\lambda U \otimes U^*$. □

This also completes the proofs of Lemma A3.2 and Proposition A3.1. □

Corollary A3.5. a) *The ideal I_d generated by $d \times d$ minors of Φ annihilates* $\operatorname{coker}\Phi$.

b) $\operatorname{coker}\Phi$ *is a torsion free module over the determinantal ring S/I_d.*

Proof. To prove a), consider the relation from Lemma A3.2 with $r = d$,

$$\{a_1, \ldots, a_{n-1}, a_{n+1}, \ldots, a_d\} = \{1, 2, \ldots, n-1, n+1, \ldots, d\}.$$

It says that

$$(1, 2, \ldots, d \mid b_1, \ldots, b_d) u_i^* = 0.$$

To prove b) we define an embedding of $\operatorname{coker}\Phi$ into a free S/I_d-module. We define the homomorphism

$$\Psi : \operatorname{coker}\Phi \to \wedge^{d-1} V^* \otimes \wedge^d U^* \otimes S/I_d$$

by setting

$$\Psi(u_i^*) = \sum_{i=1}^{d} (a_1, \ldots, a_{d-1} \mid b_1, \ldots, b_{d-1}) u_{a_1}^* \wedge \ldots \wedge u_{a_{d-1}}^* \wedge u_i^* \otimes v_{b_1}^* \wedge \ldots \wedge v_{b_{d-1}}^*.$$

Consider the complex

$$V \otimes S/I_d \xrightarrow{\Phi\otimes 1} U^* \otimes S/I_d \xrightarrow{\Psi} \wedge^{d-1} V^* \otimes \wedge^d U^* \otimes S/I_d.$$

One shows easily that $\Psi(\Phi \otimes 1) = 0$ (the coefficient of every image is a combination of $d \times d$ minors, so is zero in S/I_d). To prove the exactness of our complex, we notice that it is enough to prove the exactness over $\mathbf{Q}$. Indeed, our straightening law showed that the cokernel of $\Phi \otimes 1$ is a free module over $\mathbf{Z}$.

To prove the exactness over $\mathbf{Q}$ we use representation theory methods by decomposing to irreducibles. Let us look at our complex, and for a partition ν let us look at the $\mathrm{GL}(V)$-isotypic component of $S_\nu V$, with $\nu = (\nu_1, \dots, \nu_{d-1})$. It is enough to look at such components, since no other representations occur in the middle term of the complex. We get the following complex:

$$S_{\nu/(1)}U \to U^* \otimes S_\nu U \to \bigwedge^d U^* \otimes S_{(\nu_1+1,\dots,\nu_{d-1}+1)}.$$

The cokernel of the first map consists of the sole irreducible representation

$$S_{(\nu_1,\dots,\nu_{d-1},-1)}U$$

which occurs in the right-hand-side term. It is therefore enough to show that the map Ψ is nonzero on our isotypic component. The best way to do it is to see that the image of Ψ on the corresponding highest weight vector is nonzero. This highest weight vector is, however, easy to calculate, since it equals $(C_\nu, C_\nu)u_d^*$ where C_ν is the canonical tableau of shape ν having $(1, 2, \dots, \nu_i')$ in the i-th row. □

Remarks. Bruns and Vetter ([BV], chapter 13) show that $\operatorname{coker}\Phi$ is torsion free over S/I_d by a different argument. What is more, they show that $\operatorname{coker}\Phi$ is reflexive over S/I_d. This can also be proved by our method, by extending our sequence by one term:

$$\begin{aligned} V \otimes S/I_d &\xrightarrow{\Phi\otimes 1} U^* \otimes S/I_d \\ &\xrightarrow{\Psi} \wedge^{d-1} V^* \otimes \wedge^d U^* \otimes S/I_d \xrightarrow{\Theta} \wedge^d V^* \otimes \wedge^d U^* \otimes U^* \otimes S/I_d \end{aligned}$$

where Θ is defined on generators by the map

$$\begin{aligned} \wedge^{d-1}V^* \otimes \wedge^d U^* &\to \wedge^{d-1}V^* \otimes \wedge^d U^* \otimes V^* \otimes V \otimes U^* \otimes U \\ &\to \wedge^d V^* \otimes \wedge^d U^* \otimes U^* \otimes (V \otimes U), \end{aligned}$$

and then identifying $V \otimes U$ with S_1. We prove the exactness of the extended complex in the same way as above. This shows that $\operatorname{coker}\Phi$ is a second syzygy over S/I_d, and therefore reflexive. The dual module of $\operatorname{coker}\Phi$ turns out to be

$$\wedge^{m-d+1}(\operatorname{coker}\Phi^*).$$

The odd case. The results in this case are very similar; so we only sketch the proofs.

Again, let K be a field of arbitrary characteristic. Consider the vector spaces $V = K^n, U = K^d$. We consider the odd generic map $\Phi : V \otimes_K E \to U^* \otimes_K E$ of free modules over the exterior algebra $E = \wedge_K^\bullet(V \otimes U)$. We are interested in the generic module $\operatorname{coker}\Phi$.

Let $\{u_1, \dots, u_d\}$, $\{v_1, \dots, v_n\}$ be bases of U, V respectively. Let $\{u_1^*, \dots, u_d^*\}$ be the dual basis of U^*.

Let us write $\Phi(v_j) = \sum_{i=1}^d a_{i,j} u_i^*$. The ring E can be identified with the exterior algebra $\wedge^\bullet[a_{i,j}]_{1\le i\le d, 1\le j\le n}$.

It is shown in Akin, Buchsbaum, and Weyman [ABW] that E has a basis consisting of double standard tableaux, where now the notion of double standard tableaux is interpreted as follows: Suppose that (P, Q) is a double tableau of shape λ, where $\lambda = (\lambda_1, \ldots, \lambda_r)$, and $P = (P_1, \ldots, P_r), Q = (Q_1, \ldots, Q_r)$ with $P_j = (p_{j,1}, \ldots, p_{j,\lambda_j})$, $Q_j = (q_{j,1}, \ldots, q_{j,\lambda_j})$, for $1 \le p_{i,j} \le d$, $1 \le q_{i,j} \le m$. The corresponding element of E is the product $M_1 \ldots M_r$ where M_j is an exterior minor of the generic matrix $\tilde{X} = (a_{i,j})_{1\le i\le d, 1\le j\le n}$ corresponding to the rows $P_j = (p_{j,1}, \ldots, p_{j,\lambda_j})$ and (maybe repeated) columns $Q_j = (q_{j,1}, \ldots, q_{j,\lambda_j})$. Recall that P is standard if $p_{i,j} < p_{i,j+1}$ for all i, j and $p_{i,j} \le p_{i+1,j}$ for all i, j. The standardness of Q is defined by the conditions $q_{i,j} \le q_{i,j+1}$ for all i, j and $q_{i,j} < q_{i+1,j}$ for all i, j. (P, Q) is standard if both P and Q are standard.

From the representation-theoretic point of view, the map given by exterior minors of size s is the embedding

$$D_s V \otimes \wedge^s U \to \wedge^s (V \otimes U),$$

which is the specialization of our ρ_s to the odd case. Here $D_s V$ is the divided power of V.

The K-basis of the free module $U^* \otimes E$ is given by products $(P, Q)u_i^*$ where (P, Q) is a standard double tableau and $1 \le i \le d$.

Definition. A product $(P, Q)u_i^*$ is admissible if the double tableau (P, Q) is standard and the first row of P does not contain the interval $[1, i]$. Notice that the additional condition is really a condition on P and u_i^*. So we can also talk about admissible products Pu_i^*.

In the sequel we denote the exterior minor of X corresponding to the rows $a_1, \ldots, a_r$ and columns $b_1, \ldots, b_r$ by $(a_1, \ldots, a_r | b_1, \ldots, b_r)$.

Proposition A3.1′. *The admissible products form a K-basis of* $\operatorname{coker} \Phi$.

Proof. The proof proceeds as in the even case. The representation-theoretic content is identical; only the representation on the V side changes from $S_\lambda V$ to $S_{\lambda'} V$. The essential point is that in $\operatorname{coker} \Phi$ we have relations on the u_i^* that are analogous to those in Lemma A3.2: □

Lemma A3.2′. *In* $\operatorname{coker} \Phi$ *we have the relations*

$$\sum_{j=1}^{d} (a_1, \ldots, a_{n-1}, j, a_{n+1}, \ldots, a_r | b_1, \ldots, b_r) u_j^* = 0.$$

Proof. By linearity and the argument at the end of the proof of Proposition A1.1, it suffices to treat the case where $b_1 = b_2 = \cdots = b_r$. But, again as in A1.1, we have

$$(a_1, \ldots, a_{n-1}, j, a_{n+1}, \ldots, a_r | b_1, \ldots, b_1) = (a_1|b_1) \wedge (a_2|b_1) \wedge \cdots \wedge (a_r|b_1).$$

So the given relation is a multiple of the relation $\sum_{j=1}^{d} (j|b_1) u_j^* = 0$ given by column number b_1 of Φ. □

Corollary A3.5′. a) *The ideal $I_{(d)}$ generated by $d \times d$ exterior minors of Φ annihilates* $\operatorname{coker} \Phi$. *Moreover, u_d^* generates a free $E/I_{(d)}$-module in* $\operatorname{coker} \Phi$. *So the annihilator of* $\operatorname{coker} \Phi$ *is exactly $I_{(d)}$.*

b) $\operatorname{coker} \Phi$ *is a second syzygy over the ring $E/I_{(d)}$.*

Proof. To prove a), consider the relation from Lemma A3.2′ with $r = d$,

$$\{a_1, \ldots, a_{u-1}, a_{u+1}, \ldots, a_d\} = \{1, 2, \ldots, i-1, i+1, \ldots, d\}.$$

It says that

$$(1, 2, \ldots, d \; | b_1, \ldots, b_d) u_i^* = 0.$$

The last statement of a) follows because $(P|Q)u_d^*$ is admissible for every standard double tableau of shape λ with $\lambda_1 < d$, while $E/I_{(d)}$ is spanned by those with $\lambda_1 < d$.

To prove b) we define an embedding of $\operatorname{coker}\Phi$ into a free $E/I_{(d)}$-module. We define the homomorphism

$$\Psi : \operatorname{coker}\Phi \to S_{d-1}V^* \otimes \wedge^d U^* \otimes E/I_{(d)}$$

by setting

$$\Psi(u_i^*) = \sum (a_1, \ldots, a_{d-1} \; | b_1, \ldots, b_{d-1}) u_{a_1}^* \wedge \ldots \wedge u_{a_{d-1}}^* \wedge u_i^* \otimes v_{b_1}^* \ldots v_{b_{d-1}}^*.$$

Consider the complex

$$V \otimes E/I_{(d)} \xrightarrow{\Phi\otimes 1} U^* \otimes E/I_{(d)} \xrightarrow{\Psi} S_{d-1}V^* \otimes \wedge^d U^* \otimes E/I_{(d)}.$$

One shows easily that $\Psi(\Phi \otimes 1) = 0$ (the coefficient of every image is a combination of $d \times d$ minors, hence zero in $E/I_{(d)}$). To prove the exactness of our complex, we notice that this exactness is clear over $\mathbf{Q}$ by counting representations in an isotypic component of $S_\lambda V$. But our straightening law showed that the image of $\Phi \otimes 1$ is a free module over $\mathbf{Z}$ and that $(U^* \otimes E/I_{(d)})/\operatorname{Im}(\Phi \otimes 1)$ is also free over $\mathbf{Z}$. This proves the exactness of our complex, and so $\operatorname{coker}\Phi$ is torsion free.

We can extend our sequence to a longer sequence

$$\begin{aligned} V \otimes E/I_{(d)} \xrightarrow{\Phi\otimes 1} U^* \otimes E/I_{(d)} &\xrightarrow{\Psi} S_{d-1}V^* \otimes \wedge^d U^* \otimes E/I_{(d)} \\ &\xrightarrow{\Theta} S_d V^* \otimes \wedge^d U^* \otimes U^* \otimes E/I_{(d)} \end{aligned}$$

where Θ is defined on generators by the map

$$\begin{aligned} S_{d-1}V^* \otimes \wedge^d U^* &\to S_{d-1}V^* \otimes \wedge^d U^* \otimes V^* \otimes V \otimes U^* \otimes U \\ &\to S_d V^* \otimes \wedge^d U^* \otimes U^* \otimes (V \otimes U) \end{aligned}$$

and then identifying $V \otimes U$ with E_1. The exactness of the longer complex is proven in the same way as the analogous result in the even case. □

References

[ABW] K. Akin, D. A. Buchsbaum, and J. Weyman: Schur functors and Schur complexes, Adv. Math. 44 (1982), 207-278. MR **84c:**20021

[BR] A. Berele and A. Regev: Hook Young diagrams with applications to combinatorics and to representations of Lie superalgebras. Adv. in Math. 64 (1987) 118–175. MR **88i:**20006

[Bru] Bruns, Winfried: Generic maps and modules, Compositio Math. 47 (1982), 171-193. MR **84j:**13011

[BV] Bruns, Winfried and Udo Vetter: Determinantal rings. Spinger Lecture Notes in Math. 1327. Springer-Verlag, Berlin/Heidelberg/NY 1988. MR **89i:**13001

[BE] D. A. Buchsbaum and D. Eisenbud: Generic free resolutions and a family of generically perfect ideals. Advances in Math. 18 (1975), no. 3, 245-301. MR **53:**391

[DEP] C. DeConcini, D. Eisenbud and C. Procesi: Young diagrams and determinantal varieties, Invent. Math. 56 (1980), 129–165. MR **81m:**14034

[DRS] P. Doubilet, G.-C. Rota and Joel Stein: On the foundations of combinatorial theory. IX. Combinatorial methods in invariant theory. Studies in Appl. Math. 53 (1974), 185–216. MR **58:**16736

[Eis] D. Eisenbud: *Commutative Algebra with a View Toward Algebraic Geometry*. Springer-Verlag, New York, 1995. MR **97a:**13001

[EPY] D. Eisenbud, S. Popescu and S. Yuzvinsky: Hyperplane arrangement cohomology and monomials in the exterior algebra. Preprint, 2000.

[ES] D. Eisenbud, G. Fløystad and F.-O. Schreyer: Free resolutions and sheaf cohomology over exterior algebras. Preprint, 2000.

[Fit] H. Fitting: Die Determinantenideale eines Moduls. Jahresbericht der Deutschen Math.-Vereinigung 46 (1936) 195-229.

[GS] D. Grayson and M. Stillman: *Macaulay2*, a software system devoted to supporting research in algebraic geometry and commutative algebra. Contact the authors, or download from `http://www.math.uiuc.edu/Macaulay2`.

[Gre] M. Green: The Eisenbud-Koh-Stillman conjecture on linear syzygies. Invent. Math. 136 (1999) 411–418. MR **2000j:**13024

[Mac] I. G. MacDonald: *Symmetric functions and Hall polynomials.* Second edition. With contributions by A. Zelevinsky. Oxford Mathematical Monographs, Oxford University Press, New York, 1995. MR **96h:**05207

[Onn] S. Onn: Hilbert Series of Group Representations and Gröbner Bases for Generic Modules, Journal of Algebraic Combinatorics, vol. 3, pp. 187-206, 1994. MR **95b:**20010

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