

An Exterior View Of Modules And Sheaves

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Abstract. In this expository note I will explain an explicit version of the Bernstein-Gel'fand-Gel'fand correspondence between coherent sheaves on a projective space and graded modules over an exterior algebra. In particular I will describe the connection of free resolutions over the exterior algebra with cohomology of sheaves, and with the Chow form, topics taken from various joint work with Gunnar Fløystad and Frank-Olaf Schreyer.

In this note I will explain what one might call the “exterior” view of graded modules and sheaves, and some ways in which it has recently proved useful. Let $S = K[x_0, \dots, x_n] = \text{Sym}(W)$, where W is the vector space with basis x_0, \dots, x_n , and let $V = W^*$ be the dual of W . The basic idea is that graded modules over S can be regarded as complexes of a certain kind over the exterior algebra $E = \wedge V$ of V . This correspondence, actually an isomorphism of categories, is described in section 1 below. The correspondence is easy to define, and to compute explicitly. Nevertheless some properties that may seem deep for S -modules are transparent for the corresponding complexes over E .

One way to extend the idea is to ask “what about the other complexes over E ?” Pursuing this, one is soon lead to extend the correspondence from S -modules to complexes of S -modules, and it becomes an isomorphism of certain derived categories. This is the content of the theorems of [Bernstein et al. 1978] and [Beilinson 1978]. These papers, which appear next to each other, were both inspired by a lecture of Yuri Manin. They were the beginning of the idea of looking for equivalences of derived categories which is now important in many fields, from the representations of Artin algebras (see for example [Happel 1988]) to String Theory (see for example [Sharpe 1999]). For further information see [Eisenbud et al. 2001]. In this note we will not pursue the derived category point of view, but rather stick to two concrete applications: computing cohomology (from [Eisenbud and Schreyer 2001]) and finding resultants and Chow forms (from [Eisenbud et al. 2001])

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There are other recent applications of the exterior point of view ranging from combinatorics to the theory of free resolutions. For further information, the reader might consult [Aramova et al. 2000, Decker and Eisenbud 2001, Eisenbud and Weyman 2002, Eisenbud et al. 1999, Eisenbud et al. 2000].

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1. Complexes from Modules

The connection between sheaves on \mathbf{P}^n and modules over the exterior algebra arises from the following construction. Let $S = K[x_0, \dots, x_n] = \text{Sym } W$, where W is an $n + 1$ -dimensional vector space with basis x_0, \dots, x_n . Let $V = W^*$ be the dual of W , and let $E = \wedge V$ be the symmetric algebra on V . We grade S by giving the elements of W degree 1, and giving the elements of V degree -1 .

THEOREM 1. *The category of graded S -modules is naturally isomorphic to the category of linear free complexes over E .*

Here a *linear free complex* is a sequence of maps of graded free E modules

$$\mathbf{F} : \dots \rightarrow F^{i-1} \xrightarrow{\delta} F^i \xrightarrow{\delta} F^{i+1} \rightarrow \dots$$

with $\delta^2 = 0$, where F^i is generated in degree $f + i$ for some fixed f . Thus, after a choice of bases, we may regard a linear free complex as a sequence of matrices of linear forms over E whose pairwise products are zero.

PROOF. The isomorphism may be described as follows. Let $M = \oplus M_i$ be a graded S -module. The i -th term of the corresponding complex \mathbf{F} is the free module $F^i := E \otimes M_i$, generated in degree i . The differential $\delta : F^i \rightarrow F^{i+1}$ is determined by its action on the generators M_i of F^i . The map δ will take M_i into the graded component $V \otimes M_{i+1}$ of $F^{i+1} = E \otimes M_{i+1}$. To define δ we use the multiplication map $\mu : W \otimes M_i \rightarrow M_{i+1}$. Choose a basis e_0, \dots, e_{n+1} of V that is dual to the basis x_0, \dots, x_{n+1} of W , and define $\delta(m) = \sum e_{\otimes} \mu(x_i \otimes m)$. It is easy to check directly that the statement $\delta^2 = 0$ is equivalent to the associative and commutative laws for μ (see [Eisenbud et al. 2001], Section 1 for a basis-free argument), proving the theorem. \square

The reader should be warned that in [Eisenbud et al. 2001] we use a different grading convention that amounts to tensoring the complex \mathbf{F} defined above with the 1-dimensional vector space $\wedge^{n+1} W$ (regarded as a 1-dimensional space of elements of degree $n + 1$.)

As a first application we use Theorem 1 to turn the computation of the cohomology of a sheaf into the computation of a free resolution over E . Recall that a graded S -module M is said to be *m -regular in the sense of Castelnuovo-Mumford* if, for all $j \geq 0$, the j -th module in a minimal free resolution of M is generated in degrees $\leq m + j$.

THEOREM 2. *Let \mathcal{F} be a coherent sheaf on $\mathbf{P}^n = \mathbf{P}(W)$ represented by a finitely generated graded S -module M . If M is m -regular in the sense of Castelnuovo-Mumford, then the complex*

$$\mathbf{F}_{\geq m+1} : E \otimes M_{m+1} \rightarrow E \otimes M_{m+2} \rightarrow \dots$$

is acyclic. If we complete it to a doubly infinite complex of graded E modules

$$\mathbf{T}(M) : \cdots \rightarrow T^{m-1} \rightarrow T^m \rightarrow T^{m+1} = E \otimes M_{m+1} \rightarrow \cdots$$

by adjoining a minimal free resolution

$$\cdots \rightarrow T^{m-1} \rightarrow T^m$$

of the kernel of $E \otimes M_{m+1} \rightarrow E \otimes M_{m+2}$, then $\mathbf{T}(M)$ is independent (up to homotopy) on the choice of m , and there are natural identifications

$$T^s = \bigoplus_{j \geq 0} H^j(\mathcal{F}(s-j) \otimes E).$$

In fact the complex $\mathbf{T}(M)$ depends only on the sheaf \mathcal{F} represented by M , so we will write it as $\mathbf{T}(\mathcal{F})$. It is called the *Tate Resolution* of \mathcal{F} .

2. Introduction to Resultants and Chow Forms

The resultant of $r + 1$ homogeneous forms f_i of degree d in $r + 1$ variables is the (unique) monic polynomial of lowest possible degree in the coefficients of the f_i that vanishes if and only if the f_i have a common zero in \mathbf{P}^r . For example, the resultant of

$$\begin{aligned} f_0 &= ax^2 + bxy + cy^2 \\ f_1 &= a'x^2 + b'xy + c'y^2 \end{aligned}$$

is the *Sylvester determinant*

$$\mathcal{R}(f_0, f_1) = \det \begin{pmatrix} a & b & c & 0 \\ 0 & a & b & c \\ a' & b' & c' & 0 \\ 0 & a' & b' & c' \end{pmatrix}.$$

Of course this polynomial really depends only on the vector space spanned by f_0 and f_1 in the space of forms. This space is determined by its *Plücker* coordinates, the three 2×2 minors of the matrix

$$A = \begin{pmatrix} a & b & c \\ a' & b' & c' \end{pmatrix}.$$

To make this explicit, denote the minor of the matrix A that is obtained by deleting the i -th column by Δ_i . If we permute the rows of the Sylvester matrix to get

$$\begin{pmatrix} a & b & c & 0 \\ a' & b' & c' & 0 \\ 0 & a & b & c \\ 0 & a' & b' & c' \end{pmatrix}$$

and expand the determinant along the first two rows, we see that

$$\mathcal{R}(f_0, f_1) = \Delta_1 \Delta_3 - \Delta_2^2.$$

Resultants have been studied since [**Leibniz 1693**] because they allow one to eliminate variables in order to solve systems of equations: for example if $f_0 = 0$ and $f_1 = 0$ are two quadratic equations in two variables, then, homogenizing with respect to one of the variables, we can write them in the form above where a, b, c, a', b', c' are polynomials in 1 variable. The equation $\mathcal{R}(f_0, f_1) = 0$ is then an

equation in 1 variable which we may think of as the result of eliminating a variables from the equations $f_0 = f_1 = 0$. It can be solved numerically (and sometimes symbolically). Substituting the result into f_0 and f_1 , we again obtain polynomials in 1 variable whose solutions allow us to determine the answer to the original problem.

After centuries of work by the likes of Cayley, Macaulay, Noether, and many others, no polynomial formula as explicit as the Sylvester determinant is known in general. However, various formulas expressing this polynomial as a rational function are known. (See for example [Gelfand et al. 1994] for an exposition or [d'Andrea and Dickenstein 2001] for some recent work.) The method of exterior algebras gives polynomial formulas in some new cases. The first example occurs for three quadratic polynomials in two variables. Homogenizing for simplicity, we regard them as three homogeneous quadratic forms in three variables x, y, z . Using the method described at the end of this paper, one can show that the resultant is the Pfaffian of the skew-symmetric 8×8 matrix

$$\begin{pmatrix} 0 & [245] & [345] & [135] & [045] & [035] & [145] & [235] \\ -[245] & 0 & -[235] & [035] & [025] & [015] & [125] & -[125]+[045] \\ -[345] & [235] & 0 & [134] & [035] & [034] & [135] & [234] \\ -[135] & -[035] & -[134] & 0 & [023] & [013] & [123]-[034] & -[123] \\ -[045] & -[025] & -[035] & -[023] & 0 & [012] & -[015] & -[024]+[015] \\ -[035] & -[015] & -[034] & -[013] & -[012] & 0 & [023]-[014] & -[023] \\ -[145] & -[125] & -[135] & -[123]+[034] & [015] & -[023]+[014] & 0 & -[124]+[035] \\ -[235] & [125]-[045] & -[234] & [123] & [024]-[015] & [023] & [124]-[035] & 0 \end{pmatrix}$$

(this is a certain polynomial in the entries, which is equal to the square root of the determinant.) Here the monomials in the three variables are ordered $x^2, xy, xz, y^2, yz, z^2$ and the brackets $[ijk]$ denote the corresponding Plücker coordinates of the net of quadrics.

There is a more geometric version of the resultant, called the Chow form (or Cayley form) of a projective variety, which is the natural context for our work. Suppose that $X \subset \mathbf{P}^n$ is a variety of dimension k . A general plane of codimension k in \mathbf{P}^n meets X in finitely many points (the number is the degree of X), and thus a general plane of codimension $k+1$ will not meet X at all. It turns out that the set of codimension $k+1$ -planes that do meet X is a hypersurface (divisor) in the Grassmannian $\mathbf{G} := \mathbf{G}(k+1, n)$ of planes of codimension $k+1$ in \mathbf{P}^n , called the Chow divisor of X . Such a divisor is defined by the vanishing of a homogeneous form in the Plücker coordinates on \mathbf{G} , and this form is called the Chow form.

For a very simple example, consider the case $k = n - 1$, where X is a hypersurface defined, say by the equation $F(x_0, \dots, x_n) = 0$. In this case a codimension $k+1$ -plane in \mathbf{P}^n is just a point; and the condition that it meet X is that it be contained in X . The Plücker coordinates of the point are its ordinary coordinates; and the Chow form of X is the form F itself.

The Chow form generalizes the resultant in the following sense. If we take X to be \mathbf{P}^k in \mathbf{P}^n by the d -uple embedding, where $n = \binom{k+d}{k} - 1$, then a hyperplane in \mathbf{P}^n may be regarded as a form of degree d on \mathbf{P}^k , and the common zeros of several of these forms are their common intersection with X . Thus the Chow form of X is the resultant of $k+1$ forms of degree d in $k+1$ variables.

Applying this idea to the case of two quadric forms in two variables, we are led to consider the 2-uple embedding

$$\mathbf{P}^1 \rightarrow \mathbf{P}^2; \quad (s, t) \mapsto (s^2, st, t^2).$$

Since $(s^2)(t^2) - (st)^2 = 0$ the image X is the conic $x_0x_2 - x_1^2 = 0$. Since the conic is a hypersurface in the plane its Chow form is $x_0x_2 - x_1^2$ and we have already seen that this is the resultant of two quadrics, written in Plücker coordinates. For (much) more on this construction, see [Gelfand et al. 1994].

3. Chow Forms and Sheaves

Suppose again that X is a k -dimensional variety in \mathbf{P}^n , and that \mathbf{G} is the Grassmannian of codimension $d + 1$ -planes. A sheaf \mathbf{F} whose support is X (such as a vector bundle on X) gives rise to a representation of the Chow form of X as follows. We consider the *incidence correspondence* or *partial flag variety*

$$\Gamma = \{(x, L) \in \mathbf{P}^n \times \mathbf{G} \mid x \in L\}.$$

This variety comes with two natural projections

$$\mathbf{P}^n \xleftarrow{\pi_1} \Gamma \xrightarrow{\pi_2} \mathbf{G}.$$

It is immediate from the definition that the Chow divisor of X is $\pi_2(\pi_1^{-1}X)$. Thus the sheaf $\text{Chow}(\mathcal{F}) := (\pi_2)_*\pi_1^*\mathcal{F}$ is supported precisely on the Chow divisor, and its annihilator is generated by the Chow hypersurface.

This representation of the Chow form does not immediately suggest how one might write down a formula. However, if $\text{Chow}(\mathcal{F})$ happens to be vector bundle on the Chow divisor, then since it is locally of projective dimension 1 on \mathbf{G} it will have a presentation $\text{Chow}(\mathcal{F}) = \text{coker } \phi : \mathcal{E}' \rightarrow \mathcal{E}$, where \mathcal{E} and \mathcal{E}' are vector bundles of the same rank s on \mathbf{G} . It then follows that the Chow form of \mathcal{F} is the radical of the determinant of ϕ : more precisely, if \mathcal{F} has generic rank r on X , then $\text{Chow}(\mathcal{F})$ also has generic rank r on the Chow divisor, and the determinant of ϕ is the r -th power of the Chow form. This—with $r = 1$ —is the miracle that happens in all the (few) cases where nice formulas for the Chow form are known.

The construction above can, however, be generalized to yield formulas for the Chow form as a ratio of determinants in every case. The idea depends on the notion of the determinant of a complex, which also goes back to Cayley.

4. The Determinant of a Finite Free Complex

Let

$$\mathbf{L} : 0 \rightarrow L_m \xrightarrow{\rho_m} L_{m-1} \longrightarrow \cdots \longrightarrow L_1 \xrightarrow{\rho_1} L_0 \rightarrow 0$$

be a finite free complex over a commutative ring R , which we assume for simplicity is a domain. Suppose that \mathbf{L} becomes an exact sequence of vector spaces when tensored with the quotient field of R ; in this case we say that \mathbf{L} is *generically exact*.

If $m = 1$ then by generic exactness the ranks of the free modules F_1 and F_0 must be equal; say they are both t . The rank of the map ρ_1 must also be t . We define the determinant of \mathbf{L} to be the determinant of any matrix representing ρ_1 . Of course this is only unique up to a unit; the determinant of \mathbf{L} is thus best thought of as the divisor defined by any such determinant. A little more generally, if there is just one nonzero map in the complex \mathbf{L} , say ρ_i , then we define $\det \mathbf{L}$ to be $(\det \rho_i)^{(-1)^{i-1}}$.

This element of the quotient field of R is again well-defined up to a unit of R , and thus represents a well-defined element in the group of Cartier Divisors of R .

Now suppose that $m = 2$. It is tempting to define the determinant of \mathbf{L} as $(\det \rho_1)/(\det \rho_2)$ but this doesn't quite make sense, as the ρ_i are not represented by square matrices. Set $r_i = \text{rank } \rho_i$. By generic exactness we have $r_1 + r_2 = \text{rank } L_1$. Cayley's brilliant idea was to choose bases, and then look for a set of r_2 basis elements of L_1 such that the $r_2 \times r_2$ minor Δ_2 of ϕ_2 involving the corresponding r_2 rows is nonzero.

$$0 \longrightarrow R^{r_2} \xrightarrow{\begin{pmatrix} * \\ B' \end{pmatrix}} R^{r_1+r_2} \xrightarrow{\begin{pmatrix} A' & | & * \end{pmatrix}} R^{r_1} \longrightarrow 0$$

It turns out that then the $r_0 \times r_0$ minor Δ_1 of ϕ_1 involving the complementary set of r_0 columns is nonzero, and we define $\det \mathbf{L} = \Delta_1/\Delta_2$. For example, in the illustration above, we might take $\det \mathbf{L} = \det A'/\det B'$, as long as $\det B' \neq 0$. It is at first astounding that this process gives an element of the quotient field of R that is well-defined up to units of R ; that is, a Cartier divisor.

For example, suppose that $R = K[w, x, y, z]$ and

$$\mathbf{L} : 0 \rightarrow R \xrightarrow{\begin{pmatrix} xy \\ -wy \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} wz & xz \end{pmatrix}} R \rightarrow 0.$$

If we follow the prescription above, choosing the "first" minor of

$$\rho_2 = \begin{pmatrix} xy \\ -wy \end{pmatrix}$$

we get $\det \mathbf{L} = xz/xy = z/y$. If we take the second minor instead, we get $\det \mathbf{L} = wz/(-wy) = -z/y$, which is the same up to a unit of R as claimed.

Once having seen this case, it is not hard to invent the next ones. In general the determinant of \mathbf{L} is an alternating product of the form

$$\det \mathbf{L} = \frac{\det \rho'_1 \det \rho'_3 \det \rho'_5 \cdots}{\det \rho'_2 \det \rho'_4 \det \rho'_6 \cdots}.$$

Here, if we set $r_i = \text{rank } \rho_i$ then each ρ'_i is an $r_i \times r_i$ submatrix of ρ_i and the set of columns taken for ρ'_i is complementary to the set of rows taken for ρ'_{i+1} . The following figure illustrates how the submatrices $\rho'_1 = A'$, $\rho'_2 = B'$, and $\rho'_3 = C'$ might be distributed in the case $m = 3$:

$$0 \rightarrow R^{r_3} \xrightarrow{\begin{pmatrix} * \\ C' \end{pmatrix}} R^{r_3+r_2} \xrightarrow{\begin{pmatrix} * & | & * \\ B' & | & * \end{pmatrix}} R^{r_2+r_1} \xrightarrow{\begin{pmatrix} A' & | & * \end{pmatrix}} R^{r_1} \rightarrow 0$$

Grothendieck recognized that the determinant of a complex is an extremely robust notion, stable under all sorts of operations; for example, it is well-defined for generically exact complexes of vector bundles and is even an invariant of the class of the complex in the derived category. A full account can be read in [Knudsen and Mumford 1976]. The main property that we need is this one:

PROPOSITION 3. *Suppose the R is a commutative ring and \mathbf{L} is a generically exact finite complex of free modules such that $H^0 \mathbf{L}$ has codimension 1, and is generically free of rank r on its support. If $H^i \mathbf{L}$ has codimension ≥ 2 for $i \neq 0$, then $\det \mathbf{L}$ is the r -th power of the generator of the pure codimension 1 part of the annihilator of L .*

5. Chow Forms and Exterior Algebras

We are finally ready to explain how to derive formulas for the Chow form from sheaves. We suppose throughout that X is a k -dimensional variety in \mathbf{P}^n , and use the notation $\mathbf{P}^n \xleftarrow{\pi_1} \Gamma \xrightarrow{\pi_2} \mathbf{G}$, as in Section 3. First a nonconstructive version, due to Knudsen and Mumford [1976]. The reader who doesn't know about the higher derived image as a complex can simply skip this statement.

THEOREM 4. *If \mathcal{F} is a sheaf of generic rank r on X , then the Chow form of \mathcal{F} is the determinant of any free complex in the quasi-isomorphism class of $(\mathbf{R}\pi_2)_* \pi_1^* \mathcal{F}$.*

And now the constructive version: Again, let \mathcal{F} be a sheaf of generic rank r on X , and let \mathbf{T} be the Tate resolution of \mathcal{F} , as defined in Theorem 2. We define an additive functor \mathbf{U} from graded free modules over E to vector bundles over \mathbf{G} by sending $E(i)$ to $\wedge^i U$, where $U \subset \mathcal{O}_{\mathbf{G}} \otimes W$ is the tautological sub-bundle of rank $k + 1$ on \mathbf{G} . A homomorphism $a : E(i) \rightarrow E(j)$ is represented by an element $\alpha \in \wedge^{i-j} V$ (remember that the degree of elements of V is -1 .) The element α also induces, by contraction, a map $\wedge^i W \rightarrow \wedge^j W$. Tensoring with $\mathcal{O}_{\mathbf{G}}$, it is not hard to check that α induces a map that we will call $\mathbf{U}(a)$ on the exterior powers of the universal sub bundles:

$$\begin{array}{ccc} \wedge^i U & \subset & \mathcal{O}_{\mathbf{G}} \otimes \wedge^i W \\ \mathbf{U}(a) \downarrow & & \downarrow 1 \otimes \alpha \\ \wedge^j U & \subset & \mathcal{O}_{\mathbf{G}} \otimes \wedge^j W. \end{array}$$

By linearity this construction defines the desired functor from graded free modules over E to vector bundles (actually sums of exterior powers of the tautological bundle) on \mathbf{G} . Notice that $\mathbf{U}(E(i)) = 0$ unless $0 \leq i \leq \text{rank } U = k + 1$, so \mathbf{U} takes the Tate resolution $\mathbf{T}(\mathcal{F})$ of a coherent sheaf \mathcal{F} on \mathbf{P}^n to a finite complex $\mathbf{UT}(\mathcal{F})$ which can be computed efficiently and explicitly in a program such as Macaulay2.

THEOREM 5. *If \mathcal{F} is a coherent sheaf on \mathbf{P}^n , and*

$$\mathbf{P}^n \xleftarrow{\pi_1} \Gamma \xrightarrow{\pi_2} \mathbf{G}$$

is the incidence correspondence as above, then $(\mathbf{R}\pi_2)_ \pi_1^* \mathcal{F} \cong \mathbf{UT}(\mathcal{F})$. In particular, if \mathcal{F} has k -dimensional support X , then the Chow form of X is the determinant of the complex $\mathbf{U}(\mathbf{T}(\mathcal{F}))$.*

Theorem 5 has been implemented in Macaulay2; see [Eisenbud and Schreyer 2001]. It leads to the new formulas mentioned in Section 2 and to many others as well.

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