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# Periodic resolutions over exterior algebras ☆

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Dedicated to Claudio Procesi on the occasion of his 60th birthday

**Abstract**

In this paper we study modules with periodic free resolutions (that is, *periodic modules*) over an exterior algebra. We show that any module with bounded Betti numbers (that is, a module whose syzygy modules have a bounded number of generators) must have periodic free resolution of period $\leqslant 2$, and that for graded modules the period is 1. We also show that any module with a linear Tate resolution is periodic. We give a criterion of exactness for periodic complexes and a parameterization of the set of periodic modules.
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Modules over exterior algebras arise in topology, in the study of sheaves on projective spaces, and in the study of free resolutions over polynomial rings (see, for example, [3,5–8]). In this paper we analyze the simplest modules over exterior algebras, and find a parameterization for the set of these objects.

Let $K$ be a field, let $V$ be a $K$ vector space of dimension $n+1$, and let $E = \bigwedge V$ be the exterior algebra. As $E$ is injective as a module over itself it has no modules of finite projective dimension except free modules. Since $E$ is local, any $E$-module $M$ has a unique minimal free resolution, and thus the syzygy modules of $M$ are well-defined up to isomorphism. We say that $M$ is *periodic* (*of period* $t$) if $M$ is isomorphic to its $t$th syzygy. The simplest non-free $E$-modules are the periodic modules.

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To get an idea of what might be true, we recall the case of an exterior algebra on two variables over an algebraically closed field, where modules can be classified completely using the Kronecker–Weierstrass classification of matrix pencils ([12]; see also [9, Chapter XII] or [13, Section 3.2]):

**Theorem 0.1.** *If $K$ is algebraically closed and* $\dim V = 2$ *then every indecomposable, module over $E = \bigwedge V$ is either*

(a) *free*; *or*
(b) *a syzygy of the residue class field or its dual*; *or*
(c) *periodic of period* 1. *Any indecomposable periodic module $M$ is self-dual and has minimal presentation matrix of the form*

$$\begin{pmatrix} a & b & 0 & \dots & 0 \\ 0 & a & b & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & a & b \\ 0 & \dots & 0 & 0 & a \end{pmatrix}$$

*where $a, b \in V$ are independent. The module $M$ is determined up to isomorphism, by its number of generators $d$* (*the size of the matrix*) *and the unique* 1*-dimensional subspace $\langle a\rangle$ such that $M$ is not free as module over the subalgebra $E' := K + \langle a\rangle$* (*the element $b$ may be changed at will to any element independent of $a$*).

We can interpret this result by saying that, in the two variable case, periodic modules correspond to sheaves of finite length on $\boldsymbol{P}(V^*) \cong \boldsymbol{P}^1$: any such sheaf is a direct sum of the modules of the form $\mathcal{O}_{\boldsymbol{P}^1,a}/\mathfrak{m}^d_{\boldsymbol{P}^1,a}$; and we let this indecomposable module correspond to the cokernel of the matrix of linear forms of size $d$ as pictured in the proposition.

In general we show that any $E$-module $M$ with syzygies of bounded length is periodic of period at most 2. If $M$ is graded, the period is 1. The Bernstein–Gel'fand–Gel'fand correspondence implies that graded periodic modules correspond to complexes of coherent sheaves of finite length on the projective space $\boldsymbol{P}(V^*)$ (Theorem 3.1). In particular, an indecomposable graded periodic module corresponds to a single point of projective space and a complex of finite length sheaves supported on an infinitesimal neighborhood of that point, and the category of periodic modules splits into a direct product of subcategories corresponding to the points of projective space. Moreover, every graded periodic module is filtered in such a way that the successive quotients have the form $E/(a_i)$ for various linear forms $a_i \in E$.

In two variables every $E$-module can be graded; but in the general case there are interesting differences between the graded and ungraded modules. For ungraded periodic modules, the minimal period may be 2. Graded or not, any periodic module is naturally associated to a periodic module with linear resolution (of period 1), which determines a collection of points in $\boldsymbol{P}(W)$ as above. Although we have no Bernstein–Gel'fand–Gel'fand correspondence in the ungraded case, we show by direct arguments that the category of periodic modules still splits as a direct product of categories of periodic modules corresponding, in the sense above, to single points. We give parameterizations for both the graded and ungraded categories (Theorem 3.3).

This similarity of the decomposition in the graded and ungraded cases leads to an interesting speculation. Is there some additional structure on a complex of sheaves of finite length on projective space that would correspond to the deformation from a graded periodic module to an ungraded one?

The results in Section 2 of this paper are analogous to results about modules over a commutative "complete intersection" $S/(f_1, \ldots, f_c)$, with $S$ a regular local ring, proved in [4]. (See [2] for generalization.) However, the approach there leans heavily on the regular local ring $S$ so the exterior case requires a different idea.

## 1. Notation and background

Throughout this paper we write $K$ for a fixed field, and $V, W$ for dual vector spaces of finite dimension $v$ over $K$. We give the elements of $W$ degree 1, and elements of $V$ degree $-1$. We write $E = \bigwedge V$ and $S = \mathrm{Sym}(W)$ for the exterior and symmetric algebras; these algebras are graded by their *internal degrees* whereby $\mathrm{Sym}_i(W)$ has degree $i$ and $\bigwedge^j V$ has degree $-j$. *We suppose all $E$-modules considered to be finitely generated.* We often use the fact that the exterior algebra is Gorenstein, which follows from the fact that $\mathrm{Hom}_K(E, K) \cong E$ as $E$-modules. As a consequence, the $E$-dual of any exact sequence is exact.

There are well-known algebra isomorphisms $E \cong \mathrm{Ext}_S^*(K, K)$ and $S \cong \mathrm{Ext}_E^*(K, K)$. This "Koszul duality" is a key ingredient in the Bernstein–Gel'fand–Gel'fand correspondence. We next review the version of this correspondence that is explained in [5] and used below.

A complex of free modules over $E$ is called *minimal* if all the maps can be represented by matrices with entries in the maximal ideal. It is *linear* if all the entries are linear forms. Given any complex $\boldsymbol{F}$ of free modules, we define its linear part $\mathrm{lin}(\boldsymbol{F})$ to be the result of taking a minimal complex $\boldsymbol{F}'$ homotopic to $\boldsymbol{F}$ and erasing all nonlinear terms from the entries of matrices representing the differentials of $\boldsymbol{F}$. It is easy to see that $\mathrm{lin}(\boldsymbol{F})$ is a complex, and the operation $\boldsymbol{F} \mapsto \mathrm{lin}(\boldsymbol{F})$ is functorial in a suitable sense [5, Theorem 3.4]. We have

**Theorem 1.1** [5, Theorem 3.1]. *If $\boldsymbol{F}$ is a free resolution, then* $\mathrm{lin}(\boldsymbol{F})$ *is eventually exact.*

Let $\{e_i\}$ and $\{x_i\}$ be dual bases of $V$ and $W$, respectively. We write $\boldsymbol{R}$ for the functor from the category of graded $S$-modules $N = \bigoplus N_i$ to the category of linear free complexes over $E$ whose value at $N$ is the complex

$$\boldsymbol{R}(N): \quad \cdots \longrightarrow \mathrm{Hom}_K(E, N_i) \xrightarrow{\phi} \mathrm{Hom}(E, N_{i+1}) \longrightarrow \cdots$$

where $\phi(\alpha): e \mapsto \sum (x_i \alpha(e_i e))$. We have

**Theorem 1.2** [5, Proposition 2.1].

(a) *The functor* $\boldsymbol{R}$ *is an equivalence between the category of graded* $S$*-modules and the category of linear free complexes over* $E$.
(b) *The linear part of the injective resolution of a finitely generated* $E$*-module may be written* $\boldsymbol{R}(N)$ *with* $N$ *a finitely generated* $S$*-module.*
(c) *If* $N$ *is any finitely generated module, then* $\boldsymbol{R}(N)$ *is exact starting from* $\mathrm{Hom}_K(E, N_{r+1})$ *where* $r$ *is the Castelnuovo–Mumford regularity of* $N$.
(d) *The functor* $\boldsymbol{R}$ *can be extended to complexes in such a way that* $\mathrm{lin}(\boldsymbol{R}(\boldsymbol{F})) = \boldsymbol{R}(\mathrm{H}^*(\boldsymbol{F}))$.

For further background, see [5].

## 2. Modules with bounded Betti numbers

Let $M$ be a finitely generated $E$-module. It is easy to show by induction on the length of $M$ that $\mathrm{Tor}^E_\bullet(M, K)$ is a finitely generated module over $\mathrm{Ext}^\bullet_E(K, K)$ (such arguments go back at least to the work of Quillen on group cohomology). In particular, the function $i \mapsto \dim_K(\mathrm{Tor}^E_i(M, K))$ is the Hilbert function of a finitely generated $S$-module. It follows that these Betti numbers of $M$ eventually grow polynomially. Thus the Betti numbers are either eventually constant or eventually strictly increasing, and the syzygies of $M$ are of bounded length if and only if and only if the Betti numbers are eventually constant. In this section we show that in the eventually constant case the resolution is periodic of period at most 2.

We will make use of a *twisting functor* $\tau$ taking a (possibly ungraded) $E$-module $M$ to the $E$-module $\tau M$ which is the same as $M$ as a vector space, but on which elements of odd degree in $E$ act by the negative of their action on $M$. More formally, $\tau$ is the functor given by pullback along the algebra homomorphism $E \to E$ induced by the map $-1: V \to V$. It follows from the definition that $\tau^2$ is the identity.

The effect of $\tau$ on the matrix representing a map of free modules is to change the signs of all the odd degree terms of the entries of the matrix. As $\tau$ is exact, it follows that a presentation matrix of a module $\tau M$ may be obtained from a presentation matrix for $M$ by changing the signs of all odd degree terms of entries in the matrix. The following was pointed out to me by Frank Schreyer:

**Proposition 2.1.** *If $M$ is a graded $E$-module then $\tau M \cong M$.*

**Proof.** If we change the signs of all the odd degree columns *and* all the odd degree rows in a presentation matrix for $\tau M$, we get the presentation matrix of $M$; but this change of signs does not change the isomorphism class of the cokernel. □

On the other hand, if $M$ is the ungraded module $E/(a+bc)$, where $a, b, c$ are independent linear forms, then $\tau M \cong E/(a-bc)$. This module is *not* isomorphic to $M$: it has a different annihilator.

**Theorem 2.2.** *Let $M$ be a finitely generated $E$-module with no free summands. If the syzygies of $M$ have bounded length then the first syzygy of $M$ is $\tau M$. In particular, the minimal free resolution of $M$ is periodic, of period at most* 2, *and if $M$ is graded, then the period is* 1.

**Proof.** By a result of Grothendieck [11, Proposition 2.5.8] two modules that become isomorphic after a ground field extension are isomorphic already, so we may assume that the ground field $K$ is infinite. Let

$$\boldsymbol{F}: \quad \cdots \xrightarrow{\phi_2} F^{-1} \xrightarrow{\phi_1} F^0 \longrightarrow M \longrightarrow 0$$

be the minimal free resolution of $M$.

For $w \in W$, let $w^\perp \subset V$ be the annihilator of $w$. We have an extension

$$\eta_w(K): \quad 0 \to K \to E\big/\big(w^\perp\big) \to K \to 0.$$

Tensoring over $K$ with $M$ and using the diagonal $E \to E \otimes_K E$ to define the module structure on $E/(w^\perp) \otimes_K M$ we get an extension

$$\eta_w(M): \quad 0 \to \tau M \to \big(E\big/\big(w^\perp\big)\big) \otimes_K M \to M \to 0,$$

from which we may derive a map (defined up to homotopy) from $\boldsymbol{F}$ to the shift $\tau\boldsymbol{F}[1]$ of the resolution of $\tau M$:

$$\begin{array}{ccccccc}
\boldsymbol{F}: & \cdots \longrightarrow & F^{-2} & \longrightarrow & F^{-1} & \longrightarrow & F^0 \\
\downarrow w & & \downarrow & & \downarrow & & \\
\tau\boldsymbol{F}[1]: & \cdots \longrightarrow & \tau F^{-1} & \longrightarrow & \tau F^0 & &
\end{array}.$$

Since the syzygy modules of $M$ have bounded length, it follows that the numbers of generators is also bounded. Consequently, the Betti numbers $\operatorname{rank}(F^{-j})$ are bounded. By the remarks at the beginning of this section, the ranks of the $F^{-j}$ are eventually constant—that is, they are constant for sufficiently large $j$.

We will show that, for a sufficiently general choice of $w$, the vertical map $F^{-j} \to \tau F^{-j+1}$ is an isomorphism for sufficiently large $j$. Indeed, it will follow from this that the resolution $\boldsymbol{F}$ is eventually periodic of period (at most) 2. Truncating at the $m$th step for large $m$, and using the facts that $E$ is injective as an $E$-module and $M$ has no free summand, we get from $\boldsymbol{F}$ the beginning of the minimal injective resolution of the $m$th syzygy of $M$. Since the beginning of this resolution is periodic, the whole resolution is periodic and we see that $\boldsymbol{F}$ is periodic from the beginning.

To show that the vertical maps are eventually isomorphisms, it suffices, given the boundedness of the ranks, to prove that they induce maps $\mathrm{Hom}_E(\tau F^{-j+1}, K) \to \mathrm{Hom}_E(F^{-j}, K)$ that are eventually monomorphisms. These maps induce the maps

$$\eta_w(M)^* : \mathrm{Ext}^{j-1}\big(\tau(M), K\big) \to \mathrm{Ext}^j(M, K)$$

which are the connecting homomorphisms in the long exact sequences coming from $\eta_w(M)$; thus they depend only on $w$, not the choice of the maps in the diagram.

We first reinterpret $\eta_w(M)^*$ by observing that the maps

$$\begin{aligned} &\eta_w(M)^* : \mathrm{Ext}_E^{j-1}(\tau M, K) \to \mathrm{Ext}_E^j(M, K) \quad \text{and} \\ &\eta_w(K)_* : \mathrm{Ext}_E^{j-1}(\tau M, K) \to \mathrm{Ext}_E^j(\tau M, \tau K) = \tau\, \mathrm{Ext}_E^j(M, K) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad = \mathrm{Ext}_E^j(M, K) \end{aligned}$$

are equal. To see this, first take a filtration of $M$ with successive quotients isomorphic to $K$ and use the naturality of all the maps in question to reduce to the case $M = K$. The result now follows since $\mathrm{Ext}_E^*(K, K)$ is a commutative algebra.

On the other hand, the elements $\eta_w(K)$ generate the algebra $\mathrm{Ext}_E^*(K, K) = \mathrm{Sym}(W)$; in fact, $\eta_w(K)$ corresponds naturally to $w$ under this identification. Using a filtration of $M$ as before, we see that $\mathrm{Ext}_E^*(M, K)$ is a finitely generated $\mathrm{Ext}_E^*(K, K)$-module. Since the latter is Noetherian, it follows by prime avoidance and the infiniteness of $K$ that $\eta_w$ induces a monomorphism on $\mathrm{Ext}_E^j(M, K)$ for large $j$ and general $w$, as claimed. □

It was noted in [1] that the vector space dimension of a module of complexity $i$ is divisible by $2^{v-i}$ (indeed, it is free over the exterior algebra on a general subspace $V' \subset V$ with $\dim V' = v - i$), and in particular that the vector space dimension of a module with periodic resolution is divisible by $2^{v-1}$. Here is a more precise result:

**Corollary 2.3.** *If the $E$-module $M$ has a periodic free resolution then* $\dim_K(M) = d \cdot 2^{v-1}$ *where $d$ is the minimal number of generators of $M$.*

**Proof.** From Theorem 2.2 we get an exact sequence

$$0 \to \tau(M) \to F \to M \to 0$$

with $F$ free. The functor $\tau$ is an equivalence, so $\dim_K(\tau(M)) = \dim_K(M)$. Thus $2 \cdot \dim_K(M) = \dim_K(F) = d \cdot 2^v$. □

The *Tate resolution* of a module $M$ is obtained by splicing a free resolution and an injective resolution of module $M$. As $E$ is self-injective, the Tate resolution may be characterized as a minimal free complex

$$\cdots \longrightarrow F^{-1} \xrightarrow{\phi} F^0 \longrightarrow F^1 \longrightarrow \cdots$$

with no homology at all, such that $M$ is the cokernel of $\phi$. A sufficient—but not necessary—condition for a resolution to be periodic is the linearity of the Tate resolution:

**Theorem 2.4.** *Suppose that the finitely generated $E$-module $M$ has no free summands. If the Tate resolution of $M$ has only linear maps then it is periodic of period* 1.

**Proof.** Since $M$ has linear free resolution, it can be graded. By Theorem 1.2 the injective resolution of $M$ has the form $\boldsymbol{R}(N)$ for some finitely generated $S$-module $N$. The Tate resolution of $M$ is, up to shifts, the Tate resolution associated to the sheaf $\mathcal{F}$ associated to $N$ (see [5, Section 4]). By Theorem 4.1 of that paper our assumption about the Tate resolution of $M$ implies that $\mathcal{F}$ has no cohomology other than $\mathrm{H}^0(\mathcal{F}(d)) = \operatorname{socle} F^d$. It follows that $\mathcal{F}$ has 0-dimensional support, so the $\mathrm{H}^0(\mathcal{F}(d))$ all have the same dimension. Thus the ranks of the $F^d$ are constant, and we may apply Theorem 2.2. Since $M$ has linear presentation, it is graded, so the period is 1 rather than 2. □

## 3. Parameterization

The following special case of the Bernstein–Gel'fand–Gel'fand Theorem gives a view of the classification problem in the graded case, and shows in particular that it is of wild type:

**Theorem 3.1.** *The category of graded $E$-modules $M$ with periodic free resolution is equivalent to the derived category of bounded complexes of sheaves with finite length homology, or equivalently to the derived category of bounded complexes of sheaves of finite length, on $\boldsymbol{P}(W)$. If $M$ is indecomposable, then the sheaves of finite length in the corresponding complex may all be taken to be supported at a single point of $\boldsymbol{P}(W)$.*

**Proof.** For simplicity we dualize and work with injective resolutions over $E$. If $\boldsymbol{G}$ is a periodic resolution over $E$ then by Theorem 1.1 the linear part of $\boldsymbol{G}$ is also eventually a resolution. Since $\boldsymbol{G}$ is periodic, the linear part is periodic, and is a resolution from the beginning. Using the category equivalence $\boldsymbol{R}$ we see that the ranks of the free modules in $\boldsymbol{G}$ are the values of the Hilbert function of the $S$-module $N$ corresponding to the linear part of $\boldsymbol{G}$. Because taking the linear part corresponds under $\boldsymbol{R}$ to taking homology, $N$ is the direct sum of the homology modules of the complex corresponding to $\boldsymbol{G}$. Since the ranks are constant, $N$ has finite length. Using Theorem 2.2 and Theorem 1.2(c), the argument may be reversed.

The category of complexes of sheaves of finite length splits into the categories of sheaves of finite length supported at different points in $\boldsymbol{P}(W)$, so if $M$ is indecomposable, all the sheaves must be supported at a single point. □

**Example 3.2** (A graded periodic module and its complex of sheaves). Let $E = \bigwedge\langle a, b, c\rangle$. The complex

$$\cdots \longrightarrow E(-2) \oplus E(-3) \xrightarrow{\begin{pmatrix} a & -bc \\ 0 & a \end{pmatrix}} E(-1) \oplus E(-2) \xrightarrow{\begin{pmatrix} a & bc \\ 0 & a \end{pmatrix}} E \oplus E(-1)$$

is exact; this follows from the fact that the linear part is already exact.

According to Theorem 3.1 the module resolved by the complex above corresponds to a complex of sheaves of finite length on $\boldsymbol{P}(W)$. To exhibit such a complex, let $x, y, z$ be a basis of $W$ dual to the basis $a, b, c$ of $V$. The complex has reduced support $y = z = 0$, and thus lives on the open set $x = 1$ whose coordinate ring is $S/(x-1)$. Let $R = S/((x-1) + (y,z)^2)$. The complex

$$\boldsymbol{F}\colon \quad 0 \longrightarrow R/(z) \xrightarrow{1 \mapsto z} R/(y) \longrightarrow 0$$

has homology $\mathrm{H}^0(\boldsymbol{F}) = K$, $\mathrm{H}^1(\boldsymbol{F}) = K(-1)$. Regarding $\boldsymbol{F}$ as a complex of sheaves on $\boldsymbol{P}(W)$ and taking twisted global sections, we get a complex of $S$-modules

$$\boldsymbol{G}\colon \quad 0 \longrightarrow R/(z)\big[x, x^{-1}\big] \xrightarrow{1 \mapsto z} R/(y)\big[x, x^{-1}\big] \longrightarrow 0.$$

Direct computation shows that the minimal complex homotopic to $\boldsymbol{R}(\boldsymbol{G})$ is the periodic resolution above.

The next result gives a parameterization of all periodic modules, graded or not, in the case of an algebraically closed ground field. The algebraic closure is used to ensure that the point of $\boldsymbol{P}(W)$ associated with an indecomposable periodic graded module in Theorem 3.1 is given by an ideal of linear forms, and thus corresponds to a one-dimensional subspace $\langle a\rangle$ of $V$.

**Theorem 3.3.** *Suppose that the ground field $K$ is algebraically closed. If $M$ is an indecomposable $E$-module with periodic free resolution, then $M$ has a presentation matrix of the form*

$$\phi = aI + B + C$$

*where*

- *$I$ is the identity matrix and $a \in V$ is a nonzero linear form*;

- *$B$ is a strictly upper triangular matrix of linear forms*; *and*
- *$C$ is a matrix of elements in the square of the maximal ideal of $E$ such that* $(B + C) \cdot (B - \tau(C)) = 0$.

*Moreover, if $M$ is graded then $C$ can be taken strictly lower triangular. Conversely, any such matrix $\phi$ is part of a periodic free resolution*: *the kernel of $\phi$ is spanned by the columns of $\tau(\phi)$.*

Before giving the proof of Theorem 3.3 we analyze the special case of a module with periodic linear resolution, where Theorems 3.1 and 3.3 take a simpler form:

**Proposition 3.4.** *There is an equivalence of categories between modules with periodic linear free resolution over $E = \bigwedge V$, and coherent sheaves of finite length on $\boldsymbol{P}(W)$. If the ground field $K$ is algebraically closed, then the presentation matrix of an indecomposable module with linear periodic resolution may be written in the form*

$$\phi_1 = a \cdot I + B$$

*where $0 \neq a \in E_1$ is a linear form, $I$ is the $n \times n$ identity matrix, and $B$ is a strictly upper triangular matrix of linear forms with $B^2 = 0$. Conversely, the kernel of such a matrix $\phi_1$ is the image of $\phi_1$, so the cokernel of $\phi_1$ has a linear free resolution which is periodic of period* 1.

**Proof.** By Theorem 1.2, any linear periodic resolution of an $E$-module is the dual of $\boldsymbol{R}(N)$, for some 0-regular graded module $N$ with bounded Hilbert function. In particular, $N$ represents a sheaf of finite length on $\boldsymbol{P}(W)$, and this correspondence gives a duality between the (self-dual) category of coherent sheaves of finite length on $\boldsymbol{P}(W)$ and the category of $E$-modules with linear free resolution.

If $M$ is indecomposable then so is $N$. It follows that $N$ is supported at a single point $p \in P(W)$. If $K$ is algebraically closed then the equations of this point correspond to a hyperplane in $W$, that is, an element $a \in V$ determined up to a scalar. Take $x \in W$ to be any nonzero linear form outside this ideal. A graded module $N_0 \oplus N_1 \oplus \cdots$ representing the sheaf $N$ is 0-regular if and only if multiplication by $x$ is an isomorphism from $N_i$ to $N_{i+1}$ for all $i \geqslant 0$.

Each $N_i$ is a module over the local ring $\mathcal{O}_{\boldsymbol{P}(W),p}$. Choose a basis of $N_0$ adapted to the filtration by powers of the maximal ideal of $\mathcal{O}_{\boldsymbol{P}(W),p}$. Multiplication by $x$ gives corresponding bases for all the $N_i$ with respect to which multiplication by each linear form in the ideal of $p$ becomes strictly upper-triangular. The chosen bases of the $N_i$ give rise to bases of each of the free modules in $\boldsymbol{R}(N)$. A direct computation shows that with respect to these bases the differentials of $\boldsymbol{R}(N)$ are upper triangular, with diagonal elements equal to $a$. They are all equal, and we may write them in the form $aI + B$ as required. In particular $(aI + B)^2 = 0$, and since $aI$ and $B$ anticommute we get $B^2 = 0$.

Conversely, if $B$ is a strictly upper triangular matrix of linear forms with $B^2 = 0$, then $(aI + B)^2 = 0$. In fact the image of $aI + B$ is equal to its kernel (the matrix is a deformation of $aI$, for which this statement is obviously true). □

**Proof of Theorem 3.3.** Suppose that $M$ is a periodic $E$-module, and let $\phi' : E^n \to E^n$ be a minimal free presentation of $M$. By Theorem 2.2 the image of $\phi'$ is isomorphic to $\tau(M)$, and thus there is an isomorphism $\alpha : E^n \to E^n$ such that $\phi'\alpha\tau(\phi') = 0$. We set $\phi = \phi'\alpha$. Since $\tau(\phi'\alpha) = \tau(\phi')\tau(\alpha)$, we have $\phi \cdot \tau(\phi) = 0$. Since $\operatorname{coker} \tau(\phi) = \tau(M)$, we see that the image of $\tau(\phi)$ is the kernel of $\phi$. Applying $\tau$, we wee that the image of $\phi$ is the kernel of $\tau(\phi)$ as well, so $M$ has periodic resolution of the form

$$\cdots \xrightarrow{\phi} E^n \xrightarrow{\tau(\phi)} E^n \xrightarrow{\phi} E^n \longrightarrow M \longrightarrow 0.$$

By Theorem 1.1 the linear part of $\boldsymbol{F}$ is eventually exact, and since it is periodic it is exact from the start. Thus we may apply Proposition 3.4 to the linear part of the complex above, and we see that, after an scalar change of bases, $\phi$ will have the form $\phi = A + B + C$ where $A + B$ is a direct sum of matrices of the form $a_i I_i + B_i$, each $I_i$ is an identity matrix of a certain size, each $a_i$ is a nonzero linear form, each $B_i$ is a strictly upper triangular square-zero matrix of linear forms, and $C$ consists of elements in the square of the maximal ideal. If all the $a_i$ are equal to $a_1$, then $a_1 I$ commutes with $B + C$ so the equation $\phi \cdot \tau(\phi)$ is equivalent to the equation $(B + C)(B - \tau(C)) = 0$, and we have reached the desired conclusion. Thus it is enough to show that if $a_1 \neq a_2$ then $M$ is decomposable.

To complete the argument, we transform the matrix $\phi$ step by step, using row and column operations, into a direct sum of matrices (block diagonal form). Suppose that the linear components $a(i)$ and $a(j)$ of the diagonal entries $\phi_{i,i}$ and $\phi_{j,j}$ are distinct for some indices $i, j$ with $\phi_{i,j} \neq 0$. Let $d$ be the minimal degree of a leading term of a $\phi_{i,j}$ with this property. Choose $i_0, j_0$ among such $i, j$ so that $i_0$ is as large as possible, and (with this choice) $j_0$ is as small as possible. We will make a row and column operation affecting only the $i_0$th row and the $j_0$th column so that after we are done the number $d$ will not have decreased, and the degrees of the leading terms of $\phi_{i_0,j}$ and $\phi_{i,j_0}$ satisfying the conditions above and having $j \leqslant j_0$ or $i \geqslant i_0$ are of degree $> d$. Repeating this step over and over, and using the fact that the degrees of nonzero terms in $E$ are bounded by $\dim V$, we see that we eventually reach a block triangular matrix where the diagonal entries of each block have constant linear term as required.

Let $x$ be the leading term of $\phi_{i_0,j_0}$. The dot product of the $i_0$th row of $\tau(\phi)$ with the $j_0$th column of $\phi$ is zero; and the hypotheses imply that the unique lowest degree terms of this dot product are $-a(i_0)x$ and $a(j_0)\tau(x)$, which must thus be equal up to sign. It follows that $x$ is annihilated by $a(i_0)a(j_0)$, and thus $x = ba(i_0) + ca(j_0)$ for suitable $b, c \in E$. Subtracting $b$ times the $j_0$th row from the $i_0$th row, and subtracting $c$ times the $i_0$th column from the $j_0$th column of $\phi$ we increase the degree of the leading term of $\phi_{i_0,j_0}$ in the desired way.

If the cokernel of $\phi$ is actually graded, we may take the matrix $\phi$ to be homogeneous. Since the diagonal elements of $\phi$ are all equal to $a$, which has degree $-1$, each column degree is one less than the corresponding row degree. We may re-order the basis of the source of $\phi$ so that the degrees of the rows and the degrees of the columns are weakly decreasing while keeping the given order among basis elements of equal degree. We then perform the corresponding permutation on the basis of the target of $\phi$. In the matrix representing $\phi$ in the reordered bases, all the linear forms remain on or above the diagonal, and any element of degree $\leqslant -2$ is strictly above the diagonal, as required.

Conversely, suppose that $\phi = aI + B + C$ has the given form. The last condition on $\phi$ is equivalent to $\phi \cdot \tau(\phi) = 0$. Filtering $E^n$ by the flag corresponding to powers of the maximal ideal of $E$ we see that $\phi$ and $\tau\phi$ are deformations of $aI + B$ and $-aI - B$. By Proposition 3.4 the complex made from these matrices is exact. The exactness of the complex made by $\phi$ and $\tau\phi$ follows. □

We remark that if $\phi$ is a map in a periodic resolution as in Theorem 3.3 then the columns of $\phi$ form a standard basis of the image of $\phi$ if we order the monomials of $E^n$ by taking lowest degree monomials first and refining by any given total order. For this it is enough to show that the relations on the leading terms of the columns of $\phi$ lift to relations on the columns of $\phi$ themselves. This follows because the matrix of leading terms $\mathrm{LT}(\phi)$ is then a diagonal matrix with nonzero linear forms on the diagonal and the relations among the columns of $\mathrm{LT}(\phi)$ are precisely the columns of $\mathrm{LT}(\tau(\phi))$, which lift to the relations on the columns of $\phi$ given by $\tau(\phi)$.

We can describe the structure of graded modules with periodic resolution in another way. It would be interesting to know an analogous statement in the ungraded case.

**Corollary 3.5.** *If $M$ is a finitely generated graded $E$-module, then $M$ has periodic resolution if and only if $M$ has a filtration $M = M_0 \supset M_1 \supset \cdots$ such that $M_i/M_{i+1} \cong E/(a_i)$ for some nonzero linear forms $a_i$ such that a minimal generator of each $M_i/M_{i+1}$ is a minimal generator of $M$. The linear forms $a_i$ are determined by $M$ up to scalars and permutation.*

**Proof.** If $M$ has a filtration of the given type then $\dim_K(M) = 2^{v-1}d$ where $d$ is the minimal number of generators of $M$. From the existence of the filtration, we see that the ranks of the free modules in a minimal free resolution of $M$ are bounded, and it follows from Theorem 2.2 that $M \cong M' \oplus F$, the direct sum of a module $M'$ with periodic resolution and a free module $F$. Corollary 2.3 shows that the vector space dimension of $M'$ is $2^{v-1}$ times the minimal number of generators of $M'$. As the dimension of $F$ is $2^v$ times its minimal number of generators, we see that $F = 0$.

Conversely, suppose that $M$ has a periodic free resolution. Both the existence and uniqueness assertions for the filtration could be deduced from Theorem 3.1. Instead we give a direct proof: we may harmlessly suppose that $M$ is indecomposable (to reduce the uniqueness statement to this case, use the Krull–Schmidt–Remak Theorem). We choose generators and relations for $M$ as in Theorem 3.3. It is evident from the form of the matrix that $M$ has a filtration of $M = M_0 \supset M_1 \supset \cdots \supset M_d = 0$ of length $d$ equal to the minimal number of generators of $M$, such that each $M_i/M_{i+1}$ is a quotient of $E/(a)$, and further that the minimal generator of $M_i/M_{i+1}$ is the class of a minimal generator of $M$. Since $\dim_K(M) = d \cdot 2^{v-1} = d \cdot \dim_K(E/(a))$, we see that $M_i/M_{i+1} \cong E/(a)$ for $0 \leqslant i < d$ as required.

To prove the uniqueness statement, it suffices (by induction on $d$) to show that if $b \notin (a)$ is another linear form of $E$, then a module $M$ with filtration as above cannot contain a submodule $N \cong E/(b)$ whose minimal generator is a minimal generator of $M$. Write $V = Ka \oplus V'$ for some subspace $V'$ containing $b$. The module $E/(a)$ is free over the exterior subalgebra $E' := K\langle V'\rangle$, and from the given filtration of $M$ it follows that as an $M$ is a free $E'$-module of rank $d$. Since $N$ contains a minimal generator of $M$ (as an $E$-module, and thus as an $E'$-module), it contains a free summand of the $E'$-module $M$, and we see that $N \cong E'$ as $E'$-modules. But $E/(b)$ is annihilated by $b \in E'_1$, a contradiction. □

Part of Theorem 3.3 can be reformulated as an exactness criterion:

**Corollary 3.6.** *Let $F \cong E^t$ be a finitely generated free $E$-module. A minimal complex*

$$\boldsymbol{F}\colon \quad F \xrightarrow{\tau(\phi)} F \xrightarrow{\phi} F,$$

*is exact if and only if the linear part $\operatorname{lin}(\boldsymbol{F})$ is exact. If $\boldsymbol{F}$ is linear, then $\boldsymbol{F}$ is exact if and only if there is a filtration $F = F_0 \supset F_1 \supset \cdots \supset F_t = 0$ preserved by $\phi$ such that for all $i$ we have $F_i/F_{i+1} \cong E$ and $\phi$ induces a nonzero map on each $F_i/F_{i+1}$.*

**Proof.** We may think of $\boldsymbol{F}$ as a doubly infinite complex

$$\boldsymbol{F}\colon \quad \cdots \xrightarrow{\phi} F \xrightarrow{\tau(\phi)} F \xrightarrow{\phi} F \xrightarrow{\tau(\phi)} \cdots,$$

without changing the question of exactness. If $\boldsymbol{F}$ is exact, then by Theorem 1.1 the linear part of $\boldsymbol{F}$ is eventually exact, and thus exact. If $\boldsymbol{F}$ is linear, we may apply Theorem 3.3 and write a matrix representing $\phi$ as an upper triangular matrix with nonzero forms on the diagonal. The corresponding filtration satisfies the theorem.

Conversely, if the linear part of $\boldsymbol{F}$ is exact, then an easy deformation argument (see, for example, [8, Lemma 5.8]) shows that $\boldsymbol{F}$ is exact. If $\mathcal{F}$ is linear and has a filtration of the desired type, then its associated graded with respect to the filtration is exact, and this is enough to imply exactness. □

In the non-graded case there may really be no way to make the matrix $C$ in Theorem 3.3 upper triangular, and thus no filtration of the type given for graded modules:

**Example 3.7** (A non-triangular periodic module). Let $E = \bigwedge\langle a, b, c, d, e\rangle$. By Theorem 3.3, the complex

$$E^2 \xrightarrow{\begin{pmatrix} -a & bc \\ cd & -a \end{pmatrix}} E^2 \xrightarrow{\phi := \begin{pmatrix} a & bc \\ cd & a \end{pmatrix}} E^2$$

is exact, and is thus part of a periodic minimal free resolution. But the matrix $\phi$ is not upper triangular with respect to any basis; equivalently, the map $\phi$ does not stabilize any rank one free summand. (Proof: If for example $\phi(\alpha, \beta) = \gamma(\alpha, \beta)$, with $\alpha$ a unit, and $\gamma \in E$ then solving for $\gamma$ and substituting the result into the expression given we see that $bc\beta^2 = cd\alpha^2$, whence $cd$ is in the ideal generated by $bc$, a contradiction.) Similarly the complex

$$E^2 \xrightarrow{\begin{pmatrix} a & bcd \\ cde & a \end{pmatrix}} E^2 \xrightarrow{\psi := \begin{pmatrix} a & bcd \\ cde & a \end{pmatrix}} E^2$$

is exact, and in particular $\psi^2 = 0$, but the matrix $\psi$ cannot be made triangular.

**Problem.** What are the reduced equations that define the locus of homogeneous matrices as in Theorem 3.3 that square to zero? In the $2 \times 2$ case, there is no condition; in the $3 \times 3$ case of

$$\begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix},$$

the necessary equation is $bd = 0$.

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## References

[1] A. Aramova, L.A. Avramov, J. Herzog, Resolutions of monomial ideals and cohomology over exterior algebras, Trans. Amer. Math. Soc. 352 (2000) 579–594.
[2] L.L. Avramov, V.N. Gasharov, I.V. Peeva, Complete intersection dimension, Publ. Math. Inst. Hautes Études Sci. (1997) 67–114.
[3] Bernstein, Gel'fand, Gel'fand, Algebraic bundles on $\boldsymbol{P}^n$ and problems of linear algebra, Funct. Anal. Appl. 12 (1978) 212–214, Translated from Funktional. Anal. i Prilozhen. 12 (1978) 66–67.
[4] D. Eisenbud, Homological algebra on a complete intersection, with an application to group representations, Trans. Amer. Math. Soc. 260 (1980) 35–64.
[5] D. Eisenbud, G. Fløystad, F.-O. Schreyer, Sheaf cohomology and free resolutions over exterior algebras, Trans. Amer. Math. Soc., to appear, math.AG/0104203.
[6] D. Eisenbud, S. Popescu, S. Yuzvinsky, Hyperplane arrangement cohomology and monomials in the exterior algebra, Trans. Amer. Math. Soc., to appear, math.AG/9912212.
[7] D. Eisenbud, S. Popescu, F.-O. Schreyer, C. Walter, Exterior algebra methods for the minimal resolution conjecture, Duke Math. J. 112 (2002) 379–395.
[8] D. Eisenbud, F.-O. Schreyer, Chow forms and resultants via exterior syzygies, preprint, 2001, math.AG/0111040.
[9] F.R. Gantmacher, in: Matrix Theory, Vol. 2, Chelsea, New York, 1960.
[10] D. Grayson, M. Stillman, Macaulay2, http://www.math.uiuc.edu/Macaulay2/.
[11] A. Grothendieck, J. Dieudonné, Éléments de géométrie algébrique IV, part 2, Publ. Math. Inst. Hautes Études Sci. 24 (1965).
[12] Kronecker, Algebraische Reduktion der Scharen bilinearer Formen, Sitzungsber. Akad. Berlin (1890) 763–776.
[13] C. Ringel, Tame algebras and integral quadratic forms, in: Lecture Notes in Math., Vol. 1099, Springer-Verlag, Berlin, 1984.