EXTERIOR ALGEBRA METHODS FOR THE MINIMAL RESOLUTION CONJECTURE

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Abstract
If $r \geq 6, r \neq 9$, we show that the minimal resolution conjecture (MRC) fails for a general set of γ points in $\mathbb{P}^r$ for almost $(1/2)\sqrt{r}$ values of γ. This strengthens the result of D. Eisenbud and S. Popescu [EP1], who found a unique such γ for each r in the given range. Our proof begins like a variation of that of Eisenbud and Popescu, but uses exterior algebra methods as explained by Eisenbud, G. Fløystad, and F.-O. Schreyer [EFS] to avoid the degeneration arguments that were the most difficult part of the Eisenbud-Popescu proof. Analogous techniques show that the MRC fails for linearly normal curves of degree d and genus g when $d \geq 3g-2,\ g \geq 4$, re-proving results of Schreyer, M. Green, and R. Lazarsfeld.

1. Introduction

From the Hilbert function of a homogeneous ideal I in a polynomial ring S over a field k, one can compute a lower bound for the *graded Betti numbers*

$$\beta_{i,j} = \dim_k \operatorname{Tor}_i^S(S/I, k)_j$$

because these numbers are the graded ranks of the free modules in the minimal free resolution of I. The graded Betti numbers are upper semicontinuous in families of ideals with constant Hilbert function, and in many cases this lower bound is achieved by the general member of such a family. The statement that it is achieved for a particular family T is called the minimal resolution conjecture (MRC) for T, following A. Lorenzini, who made the conjecture for the family of ideals of sufficiently general sets of γ points in $\mathbb{P}^r$ (all r, γ).

The MRC for a general set of γ points in $\mathbb{P}^r$ has received considerable attention (see Eisenbud and Popescu [EP1] for full references and discussion). In particular, it is known that the MRC is satisfied if $r \leq 4$ (see F. Gaeta [G1], [G2], A. Geramita and Lorenzini [GL], E. Ballico and Geramita [BG], C. Walter [W], F. Lauze [L])

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or $\gamma \gg r$ (see A. Hirschowitz and C. Simpson [HS]), but computer evidence produced by Schreyer, extended also by M. Boij in his 1994 thesis and by S. Beck and M. Kreuzer in [BK], suggested that it might fail for certain examples starting with 11 points in $\mathbb{P}^6$. Indeed, Eisenbud and Popescu [EP1] proved that if $r \geq 6,\ r \neq 9$, then the MRC fails for a general set of

$$\gamma = r + \lfloor (3 + \sqrt{8r+1})/2 \rfloor$$

points in $\mathbb{P}^r$.

This family of counterexamples to the minimal resolution conjecture begins with Schreyer's suggested 11 points in $\mathbb{P}^6$, and it seems to have contained all the examples in print in 1999. However, the thesis of Boij, completed in 1994 but published only in [B], contains computations suggesting one further counterexample: 21 points in $\mathbb{P}^{15}$.

In this paper we simplify and extend the idea of Eisenbud and Popescu to prove the existence of infinitely many new counterexamples, including the one suggested by Boij. We prove the following.

THEOREM 1.1
The MRC fails for the general set of γ points in $\mathbb{P}^r$ whenever $r \geq 6,\ r \neq 9$, and

$$r + 2 + \sqrt{r+2} \leq \gamma \leq r + (3 + \sqrt{8r+1})/2,$$

and also when $(r, \gamma) = (8, 13)$ or $(15, 21)$.

Thus we get about $(\sqrt{2} - 1)\sqrt{r}$ counterexamples in $\mathbb{P}^r$ for each $r \geq 6,\ r \neq 9$.

Our proof not only gives more, but it also avoids the subtle degeneration argument used by Eisenbud and Popescu. In the presentation below we skip some of the steps presented in Eisenbud and Popescu [EP1], and we fully treat only the new ideas, so it may be helpful to the reader to explain their strategy and the point at which it differs from ours. Their proof can be divided into three steps, starting from a set Γ of γ general points in $\mathbb{P}^r$.

Step I. Their crucial first step is to consider the Gale transform of $\Gamma \subset \mathbb{P}^r$, which is a set Γ' of γ points in $\mathbb{P}^s$ with $\gamma = r + s + 2$ determined "naively" as follows. If the columns of the $((r+1) \times \gamma)$-matrix M represent the points of Γ, then a $((s+1) \times \gamma)$-matrix N representing Γ' is given by the transpose of the kernel of M (see [EP1] for a precise definition). The free resolution of the canonical modules of the homogeneous coordinate rings of Γ and Γ' are related, and using this relation and the fact that (under our numerical hypotheses) the ideal of Γ' is not 2-regular, they construct a map ϕ_Γ from a certain linear free complex $F_\bullet(\mu)$ to the dual of the resolution of the ideal I_Γ of Γ. For the r and γ considered in Theorem 1.1, straightforward arithmetic shows that the injectivity of the top degree component of ϕ_Γ would force the graded Betti numbers of I_Γ to be too large for the MRC to hold. (We recall the definition of $F_\bullet(\mu)$ in Sec. 3.)

Step II. They show that ϕ_Γ is an injection of complexes if $F_\bullet(\mu)$ has a property somewhat weaker than exactness called *linear exactness* or *irredundancy* (see Def. 2.1).

Step III. For a generic set of points Γ, they show that $F_\bullet(\mu)$ is irredundant by a subtle degeneration argument: they degenerate the general set Γ to lie on a special curve C, and the argument finishes with a study of a refined stability property of the tangent bundle of the projective space restricted to C in a certain embedding of C connected with the Gale transform of the points.

In our new proof, Step I remains unchanged, but Step II is replaced by a more precise statement, which requires us to verify a weaker condition in Step III; this weak condition can be verified without any degeneration argument, making the replacement for Step III much simpler. More precisely, the injectivity of ϕ_Γ is needed only for a sufficiently large irredundant quotient of the complex $F_\bullet(\mu)$, and we give a criterion for this weaker condition using exterior algebra methods. The new criterion is checked in Step III with an argument inspired by Green's proof in [Gr2] of the linear syzygy conjecture. The generality of Γ enters via work of Kreuzer [K] showing that certain multiplication maps of the canonical module of the cone over Γ are 1-generic in the sense of Eisenbud [E].

In Section 2 we introduce and study the irredundancy of linear complexes using the Bernšteĭn-Gel'fand-Gel'fand (BGG) correspondence. In Section 3 we construct the complex whose irredundancy is the key to the failure of the MRC. Finally, in Section 4 we briefly explain how Gale duality allows this theory to be applied to sets of points, and we give the arithmetic part of the proof. As another application of our techniques, we recover the result on curves of Schreyer (unpublished) and of Green and Lazarsfeld [GrL] (see Th. 4.1).

Notation. Throughout this paper k denotes an arbitrary field. Let V be a finite-dimensional vector space over k. We work with graded modules over $S := \mathrm{Sym}(V)$ and $E := \wedge V^*$. We think of elements of V as having degree 1 and elements of V^* as having degree -1. We write $\mathfrak{m}$ for the maximal ideal generated by V in S.

2. Linear complexes and exterior modules

In this section we illustrate the exterior algebra approach to linear free resolutions with some basic ideas used in the rest of the paper, and with a new proof of the linear rigidity theorem of Eisenbud and Popescu [EP1].

Definition 2.1
A complex of graded free S-modules

$$F_\bullet : \quad \cdots \xrightarrow{\phi_3} F_2 \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0$$

is a called a *standard linear free complex* if F_i is generated in degree i for all i. (Note that all the differentials ϕ_i are then represented by matrices of linear forms.)

A *linear free complex* is any twisted (or shifted) standard linear free complex (i.e., a right-bounded complex of free modules F_i, so that for some integer s the generators of F_i are all in degree $s+i$ for all i).

We say that a linear free complex $F_\bullet$ is *irredundant* (see [EFS]) or *linearly exact* (see [EP1]) if the induced maps

$$F_{i+1}/\mathfrak{m}F_{i+1} \to \mathfrak{m}F_i/\mathfrak{m}^2F_i$$

are monomorphisms for $i > 0$.

One sees immediately that a standard linear free complex $F_\bullet$ as above is irredundant if and only if $\mathrm{H}_i(F_\bullet)_i = 0$ for all $i > 0$.

As explained in [EFS], irredundancy can be conveniently rephrased via the BGG correspondence as follows. If P is a graded E-module, we define a linear free S-complex $F_\bullet = \mathbf{L}(P)$ with free modules $F_i = S \otimes_k P_i$ generated in degree i and differentials

$$\phi_i : \quad F_i \to F_{i-1}, \qquad 1 \otimes p \mapsto \sum x_i \otimes e_i p \in S \otimes P_{i-1},$$

where $\{x_i\}$ and $\{e_i\}$ are (fixed) dual bases of V and V^*. Up to twisting and shifting, every linear free complex of S-modules arises from a unique E-module in this way. The linear complex $\mathbf{L}(P)$ is standard if $P_i = 0$ for all $i < 0$.

Given a graded E-module P, we write P^* for the dual graded E-module $P^* = \mathrm{Hom}_k(P, k)$. We take k in degree zero, so that $(P^*)_i$ is the dual vector space to P_{-i}.

PROPOSITION 2.2
If

$$F_\bullet = \mathbf{L}(P) : \quad \cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0$$

is a standard linear free complex of S-modules, then $F_\bullet$ is irredundant if and only if P^* is generated as an E-module in degree zero.

Thus any linear free complex $F_\bullet = \mathbf{L}(P)$ as in Proposition 2.2 has a unique maximal irredundant quotient $F'_\bullet$, which is functorial in $F_\bullet$, constructed as follows. If Q is the E-submodule of P^* generated by P_0^*, then $F'_\bullet := \mathbf{L}(Q^*)$. Alternatively, $F'_\bullet = \mathbf{L}(P/N)$, where N is the submodule with graded pieces $N_i = \{n \in P_i \mid \wedge^i V^* \cdot n = 0\}$. Observe that we have $F'_0 = F_0$.

The next lemma is the basic tool that allows us to deduce that a minimal free resolution is large. A weaker version of the result was implicit in [EP1].

LEMMA 2.3
Suppose that $\alpha_\bullet : F_\bullet \to G_\bullet$ is a map from an irredundant standard linear free complex to a minimal free complex. If α_0 is a split monomorphism, then α_i is a split monomorphism for all $i \geq 0$.

Proof
By induction, it suffices to prove that α_1 is a split monomorphism. Since both $F_\bullet$ and $G_\bullet$ are minimal (i.e., the differentials are represented by matrices of elements of $\mathfrak{m}$), there is a commutative diagram

$$\begin{array}{ccc} F_1/\mathfrak{m}F_1 & \longrightarrow & \mathfrak{m}F_0/\mathfrak{m}^2F_0 \\ \Big\downarrow{\scriptstyle \overline{\alpha}_1} & & \Big\downarrow \\ G_1/\mathfrak{m}G_1 & \longrightarrow & \mathfrak{m}G_0/\mathfrak{m}^2G_0 \end{array}$$

whose vertical maps are induced by $\alpha_\bullet$ and whose horizontal maps are induced by the differentials of $F_\bullet$ and $G_\bullet$.

Since α_0 makes F_0 a summand of G_0, the right-hand vertical map is a monomorphism. Because $F_\bullet$ is irredundant, the top map is a monomorphism. It follows that the map $\overline{\alpha}_1 : F_1/\mathfrak{m}F_1 \to G_1/\mathfrak{m}G_1$ induced by α_1 is a monomorphism, and since G_1 is free, this implies that α_1 is split. □

Lemma 2.3 may be applied to give a lower bound on Betti numbers.

PROPOSITION 2.4
Suppose that $\alpha_\bullet : F_\bullet \to G_\bullet$ is a map from a standard linear complex to a minimal free resolution. There exists a map $\beta_\bullet : F_\bullet \to G_\bullet$ which is homotopic to $\alpha_\bullet$ and such that $\beta_\bullet$ factors through the maximal irredundant quotient $F'_\bullet$ of $F_\bullet$. Further, if $\alpha_0 : F_0 \to G_0$ is a split monomorphism, then the rank of G_i is at least as big as the rank of F'_i for all i.

Proof
Let $\gamma_\bullet : F_\bullet \to F'_\bullet$ be the natural map to the maximal irredundant quotient. The image

of F_1' in $F_0' = F_0$ is contained in that of F_1, so there is a map $\beta_1 : F_1' \to G_1$ lifting $\alpha_0 : F_0' \to G_0$. Since $G_\bullet$ is acyclic, we may continue to lift, and so inductively we get a map of complexes $\beta_\bullet' : F_\bullet' \to G_\bullet$. We take $\beta_\bullet = \beta_\bullet' \gamma_\bullet$, and thus we have $\beta_0 = \alpha_0$. Since $F_\bullet$ is a free complex and $G_\bullet$ is acyclic, this implies that $\alpha_\bullet$ is homotopic to $\beta_\bullet$.

The second statement follows by applying Lemma 2.3 to the map $\beta_\bullet'$. □

In [EP1] a rigidity result for irredundancy is required. The proof given there is just a reference to the result on the rigidity of Tor proved by M. Auslander and D. Buchsbaum in [AB]. The exterior method yields a novel proof of this result (which we do not need in the sequel).

PROPOSITION 2.5
Let R be a graded commutative or anticommutative ring, and let M be a graded R-module that is generated by M_0. If $R' \subset R$ is a graded subring such that $R_1' \cdot M_0 = M_1$, then M is generated by M_0 as an R'-module also.

Proof
We show by induction on n that $R_1' \cdot M_n = M_{n+1}$, the initial case $n = 0$ being the hypothesis. If $R_1' \cdot M_{n-1} = M_n$, then $R_1' \cdot M_n = R_1' \cdot (R_1 M_{n-1}) = R_1 \cdot (R_1' M_{n-1}) = R_1 \cdot M_n = M_{n+1}$, where the second equality holds by the (anti)commutativity assumption and the third by the induction hypothesis. □

COROLLARY 2.6 (Linear rigidity)
Let $S = k[x_0, \ldots, x_r]$ be a polynomial ring, and let

$$F_\bullet : \quad \cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0$$

be an irredundant standard linear free complex. Let $S' = S/I$ be a graded quotient of S. The complex $G_\bullet := S' \otimes_S F_\bullet$ is irredundant if and only if $\mathrm{H}_1(G_\bullet)$ is zero in degree 1.

Proof
The irredundancy of $G_\bullet$ involves information about only S_0' and S_1', so we may assume that S' is a polynomial ring $S' = \mathrm{Sym}(V')$ with $V' = V/I_1$. Write $E' = \wedge(V'^*) \subset E = \wedge(V^*)$ for the corresponding exterior algebras. If $F_\bullet = \mathbf{L}(P)$, then the complex $G_\bullet$ over S' corresponds to the same graded vector space P, regarded as an E'-module by restriction of scalars. Consider now the truncated complex

$$K_\bullet : \quad \cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow G_1 \longrightarrow G_0.$$

The hypothesis $\mathrm{H}_1(G_\bullet)_1 = 0$ implies that $\mathrm{H}_1(K_\bullet)_1 = 0$ and hence, by Proposition 2.2, that $P_1^* = V'^* \cdot P_0^*$. On the other hand, P^* is generated as an E-module by P_0^* since $F_\bullet$ is an irredundant standard linear complex. So, by Proposition 2.5, we also have $P^* = E' \cdot P_0^*$. By Proposition 2.2, this suffices. □

Remark 2.7
A similar "linear rigidity" result for complexes over the exterior algebra can also be deduced from Proposition 2.5 applied this time for R a polynomial ring.

3. The complexes $F_\bullet(\mu)$
We apply Proposition 2.2 and Lemma 2.3 to a family of linear complexes $F_\bullet(\mu)$ (which were called $E^{-1}(\mu)$ in [EP1]). It is convenient here to define them in terms of the functor **L**.

Let U, V, and W be finite-dimensional vector spaces of dimensions u, v, and w, respectively. Let $A := \sum_l \wedge^l W^* \otimes \mathrm{Sym}_l(U^*)$, and let $Q := \sum_l \wedge^{l+1} W^* \otimes \mathrm{Sym}_l(U^*)$. Then A is an anticommutative graded algebra, and Q is a graded A-module. If $\mu : W \otimes U \to V$ is a pairing, then the dual $\mu^* : V^* \to W^* \otimes U^*$ extends to a map of algebras $\tilde{\mu} : E = \wedge(V^*) \to A$. We regard Q as an E-module via $\tilde{\mu}$, and we consider the corresponding linear free complex $F_\bullet(\mu) := \mathbf{L}(Q^*)$ over $S = \mathrm{Sym}(V)$:

$$\begin{aligned} F_\bullet(\mu) : \quad & 0 \to \wedge^w W \otimes D_{w-1}(U) \otimes S(-w+1) \\ & \to \cdots \to \wedge^2 W \otimes U \otimes S(-1) \to W \otimes S, \end{aligned}$$

where $D_m(U)$ denotes the m-graded piece of the divided power algebra.

Though we do not need the formula, it is easy to give the differentials explicitly: $\delta_l(\mu) : F_l(\mu) \to F_{l-1}(\mu)$ is the composite of the tensor product of the diagonal maps of the exterior and divided powers algebras,

$$\wedge^{l+1} W \otimes D_l U \otimes S(-l) \longrightarrow \wedge^l W \otimes W \otimes D_{l-1} U \otimes U \otimes S(-l),$$

and the map induced by the pairing μ,

$$\wedge^l W \otimes W \otimes D_{l-1} U \otimes U \otimes S(-l) \longrightarrow \wedge^l W \otimes D_{l-1} U \otimes S(-l+1).$$

Note that if $V = W \otimes U$ and μ is the identity map, then Q is generated as an E-module by Q_0, so the complex $F_\bullet(\mu)$ is irredundant by Proposition 2.2. The proof in [EP1] shows that $F_\bullet(\mu)$ remains exact when μ is specialized to a certain pairing coming from the canonical module of a generic set of γ points in $\mathbb{P}^r$ for suitable γ, r. Here we are instead interested in knowing whether $F_\bullet(\mu)$ has an irredundant quotient complex with the same first and last terms as $F_\bullet(\mu)$. This condition turns out to be a lot easier to analyze!

Following Eisenbud [E], we say that $\mu : W \otimes U \to V$ is 1-*generic* if $\mu(a \otimes b) \neq 0$ for all nonzero vectors $a \in W$ and $b \in U$. We say that μ is *geometrically* 1-*generic* if the induced pairing $\mu \otimes \overline{k}$ is 1-generic, where $\overline{k}$ is the algebraic closure of k. If μ is geometrically 1-generic, then by a classic and elementary result of Hopf the dimensions u, v, w of the three vector spaces satisfy the inequality $v \geq u + w - 1$.

The main result of this section is the following.

THEOREM 3.1
If the pairing $\mu : W \otimes U \to V$ is geometrically 1-generic and if $W \neq 0$ and $U \neq 0$, then $F_\bullet(\mu)$ and its maximal irredundant quotient $F'_\bullet(\mu)$ have the same last term $F_{w-1} = F'_{w-1}$; that is,

$$F'_\bullet(\mu) : \quad \wedge^w W \otimes D_{w-1}(U) \otimes S(-w+1) \longrightarrow F'_{w-2} \longrightarrow \cdots \longrightarrow F'_1 \longrightarrow W \otimes S.$$

Proof
We use notation as above. Let Q' be the submodule of Q generated by Q_0, and set $P = Q^*$, $P' = Q'^*$, so that $F_\bullet = \mathbf{L}(P)$ and $F'_\bullet = \mathbf{L}(P')$. By Proposition 2.2, the conclusion of the theorem is equivalent to the statement that the E-multiplication map

$$\begin{array}{ccc} m : E_{w-1} \otimes_k P_0 & \longrightarrow & P_{-w+1} \\ \| & & \| \\ \wedge^{w-1} V^* \otimes W^* & \longrightarrow & \wedge^w W^* \otimes \operatorname{Sym}_{w-1}(U^*) \end{array}$$

is surjective. Via the natural identification $W^* \otimes \wedge^w W \cong \wedge^{w-1} W$, this is equivalent to the map

$$m' : \quad \wedge^{w-1} V^* \otimes_k \wedge^{w-1} W \longrightarrow \operatorname{Sym}_{w-1}(U^*)$$

induced by μ being surjective. A straightforward computation shows that the image of m' is the space generated by the $((w-1) \times (w-1))$-minors of the linear map of free $\operatorname{Sym}(U^*)$-modules

$$\bar{\mu} : \quad W \otimes \operatorname{Sym}(U^*) \longrightarrow V \otimes \operatorname{Sym}(U^*)(1)$$

associated to μ, and the next lemma shows that under the 1-genericity assumption these minors span $\operatorname{Sym}_{w-1}(U^*)$. This proves the theorem. □

LEMMA 3.2
If $\mu : W \otimes U \to V$ is a geometrically 1-generic pairing of finite-dimensional vector spaces and if d is an integer such that $d \leq \dim W$, then the $(d \times d)$-minors of the associated linear map $\bar{\mu} : W \otimes \operatorname{Sym}(U^*) \to V \otimes \operatorname{Sym}(U^*)(1)$ of free $\operatorname{Sym}(U^*)$-modules span $\operatorname{Sym}_d(U^*)$.

Proof
It is enough to prove the lemma for an algebraically closed base field k, so that the Nullstellensatz holds. Also, u, v, w continue to be the dimensions of the three vector spaces.

We first show that, for an arbitrary subspace $W' \subset W$ of dimension d, the $(d \times d)$-minors of the restricted map

$$\bar{\mu}' : \quad W' \otimes \mathrm{Sym}(U^*) \longrightarrow V \otimes \mathrm{Sym}(U^*)(1),$$

regarded as polynomial functions on U, have a common zero only at the origin $0 \in U$. For $\bar{\mu}'$ is essentially a $(v \times d)$-matrix whose entries are linear functions on U, so its $(d \times d)$-minors have a common zero at a point $b \in U$ if and only if the matrix's columns, when evaluated at b, are not linearly independent. But this means that there is a $0 \neq a \in W'$ such that $\mu(a \otimes b) = 0$, which gives $b = 0$ by 1-genericity.

A general position argument shows that, for a general projection $V \twoheadrightarrow V'$ with $\dim V' = d + u - 1$, the only common zero of the $(d \times d)$-minors of the composite map

$$\bar{\mu}'' : \quad W' \otimes \mathrm{Sym}(U^*) \longrightarrow V' \otimes \mathrm{Sym}(U^*)(1)$$

is still $0 \in U$. (One may argue that the dual of $\bar{\mu}'$ may be regarded as a surjective map $\mathcal{O}^v \twoheadrightarrow \mathcal{O}^d(1)$ of vector bundles on $\mathbb{P}^{u-1}$, and a rank d vector bundle generated by global sections on a $(u-1)$-dimensional space can always be generated by just $d + u - 1$ of them (see, e.g., Eisenbud and E. Evans [EE] or J.-P. Serre [S]).)

The expected codimension of the locus defined by the $(d \times d)$-minors of the $((d+u-1) \times d)$-matrix representing $\bar{\mu}''$ is u, which is equal to the actual codimension. Thus the Eagon-Northcott complex resolving the minors of $\bar{\mu}''$ is exact, and it follows that there are $\binom{d+u-1}{d}$ linearly independent such minors. Since this is also the dimension of $\mathrm{Sym}_d(U^*)$, we see that the minors span $\mathrm{Sym}_d(U^*)$, as required. □

We say that a map of complexes $\alpha_\bullet : F_\bullet \to G_\bullet$ is a *degreewise split injection* if each $\alpha_i : F_i \to G_i$ is a split injection (this is weaker than being a split injection of complexes); in this case, we say that $F_\bullet$ is a *degreewise direct summand* of $G_\bullet$.

The irredundant complexes $F'_\bullet(\mu)$ arise in geometric situations as follows.

THEOREM 3.3
Let L, L' be two line bundles on a scheme X over a field k, and let $L'' := L \otimes L'$. Let $W \subset \mathrm{H}^0(L)$, $U \subset \mathrm{H}^0(L')$ be nonzero finite-dimensional linear series such that the multiplication

$$\mu : \quad W \otimes U \xrightarrow{0} (L'')$$

is geometrically 1-generic. Let $V \subset \mathrm{H}^0(L'')$ be a finite-dimensional linear series containing $W \cdot U$. Let $S := \mathrm{Sym}(V)$, and let $M \subset \bigoplus_{n \geq 0} \mathrm{H}^0(L \otimes L''^{\otimes n})$ be a finitely *generated graded S-submodule such that $W \subset M_0$. Then the maximal irredundant quotient*

$$F'_\bullet(\mu): \quad \wedge^w W \otimes D_{w-1}(U) \otimes S(-w+1) \longrightarrow F'_{w-2} \longrightarrow \cdots \longrightarrow F'_1 \longrightarrow W \otimes S$$

of the linear free complex $F_\bullet(\mu)$ injects as a degreewise direct summand of the minimal free resolution of M. (Here, as above, $w = \dim W$.)

Proof
Let $G_\bullet$ be the minimal free resolution of M over S. The hypotheses on M imply that $G_0 = (W \otimes S) \oplus L$ with L a free S-module. We can construct inductively a morphism $\alpha_\bullet: F_\bullet(\mu) \to G_\bullet$ such that $\overline{\alpha_0}: F_0(\mu)/\mathfrak{m}F_0(\mu) \to G_0/\mathfrak{m}G_0$ is a monomorphism, as a lifting

$$\begin{array}{ccccccc}
\cdots \longrightarrow & \wedge^3 W \otimes D_2 U \otimes S(-2) & \longrightarrow & \wedge^2 W \otimes U \otimes S(-1) & \longrightarrow & W \otimes S & \\
 & \vdots\downarrow & & \alpha_1 \vdots\downarrow & & \cap\downarrow & \\
\cdots \longrightarrow & G_2 & \longrightarrow & G_1 & \longrightarrow & (W \otimes S) \oplus L & \twoheadrightarrow M
\end{array}$$

For instance, α_1 exists since the composition $\wedge^2 W \otimes U \otimes S(-1) \to W \otimes S \to M$ vanishes. So by Lemma 2.3, $\alpha_\bullet$ induces a degreewise split inclusion $F'_\bullet(\mu) \hookrightarrow G_\bullet$ with $F'_\bullet(\mu)$ having the form given in Theorem 3.1 (since μ is geometrically 1-generic). □

It is easy to see that if X is a geometrically reduced and irreducible scheme, then any pairing $\mu: W \otimes U \longrightarrow V$ induced by multiplication of sections as in Theorem 3.3 is geometrically 1-generic. The hypothesis sometimes holds for reducible or nonreduced schemes too, as in the case we use for the proof of Theorem 1.1.

The following example shows that the complexes $F_\bullet(\mu)$ and $F'_\bullet(\mu)$ are not always the same.

Example 3.4
If we take $U = W = \mathrm{H}^0(\mathscr{O}_{\mathbb{P}^1}(2))$ with μ the multiplication map to $V = \mathrm{H}^0(\mathscr{O}_{\mathbb{P}^1}(4))$, that is,

$$\mu: \quad k[s,t]_2 \otimes k[s,t]_2 = W \otimes U \longrightarrow V = k[s,t]_4,$$

then μ is geometrically 1-generic because $k[s,t]$ is a domain and remains so over the algebraic closure of k. The module $M = \bigoplus_{n \geq 0} \mathrm{H}^0(\mathscr{O}_{\mathbb{P}^1}(2) \otimes \mathscr{O}_{\mathbb{P}^1}(4n))$ of Theorem 3.3 is the graded module associated to the line bundle $\mathscr{O}_{\mathbb{P}^1}(2 \text{ points})$ on $\mathbb{P}^1$, regarded as a module over the homogeneous coordinate ring S of $\mathbb{P}^4$ via the embedding of $\mathbb{P}^1$ as the rational normal quartic C. This line bundle is $\omega_C(1)$, the twist of the canonical line bundle by the hyperplane line bundle, so the minimal resolution of

M is the dual of the minimal resolution of the homogeneous coordinate ring of C, suitably shifted. It has the form

$$G_\bullet: \quad 0 \longrightarrow S(-4) \longrightarrow S^6(-2) \longrightarrow S^8(-1) \longrightarrow S^3 \longrightarrow M \longrightarrow 0.$$

On the other hand, $F_\bullet$ has the form

$$F_\bullet(\mu): \quad 0 \longrightarrow S^6(-2) \longrightarrow (S^3 \otimes S^3)(-1) \xrightarrow{\delta_1(\mu)} S^3,$$

and $F_1 = S^9(-1)$ cannot inject into $G_1 = S^8(-1)$. The columns of the matrix of $\delta_1(\mu)$ have exactly one k-linear dependence relation, so the maximal irredundant quotient of $F_\bullet$ has the form

$$F'_\bullet: \quad 0 \longrightarrow S^6(-2) \longrightarrow S^8(-1) \longrightarrow S^3.$$

So, in particular, $F'_2 = F_2$, in accordance with Theorem 3.1. By Lemma 2.3, any map from $F_\bullet$ to $G_\bullet$ lifting the identity on F_0 is a degreewise split monomorphism on $F'_\bullet$; in fact, $F'_\bullet$ is isomorphic to the linear strand of $G_\bullet$ in this case.

4. Failure of the MRC

Before proving Theorem 1.1, we present the bare bones of Gale duality. The reader who wishes to go further into this rich and beautiful subject can consult Eisenbud and Popescu [EP1], [EP2] or I. Dolgachev and D. Ortland [DO].

Let $\Gamma \subset \mathbb{P}^r$ be a set of $\gamma = r+s+2$ points. The linear forms on $\mathbb{P}^r$ define a linear series $V \subset \mathrm{H}^0(\mathscr{O}_\Gamma(1))$. The orthogonal complement $V^\perp \subset \mathrm{H}^0(\mathscr{O}_\Gamma(1))^*$ is identified by Serre duality with a linear series $(\omega_\Gamma)_{-1} \subset \mathrm{H}^0(K_\Gamma(-1))$ (where K_Γ denotes the canonical sheaf of Γ). Under mild hypotheses on Γ, the linear series $(\omega_\Gamma)_{-1}$ is very ample and gives an embedding whose image $\Gamma' \subset \mathbb{P}^s$ is the *Gale transform* of Γ. As a set it is well defined only modulo linear changes of coordinates on $\mathbb{P}^s$ and $\mathbb{P}^r$.

Concretely, if the columns of the $((r+1) \times \gamma)$-matrix M represent the points of $\Gamma \subset \mathbb{P}^r$ and if N is a $(\gamma \times (s+1))$-matrix whose columns are a basis of $\ker(M)$, then the rows of N represent the points of $\Gamma' \subset \mathbb{P}^s$.

In the proof of Theorem 1.1 we need four facts about the Gale transform, which the reader may easily verify (or find in [EP1], [EP2]).

- *The subset* $\Gamma' \subset \mathbb{P}^s$ *is defined for a general* Γ. In fact, it is defined as long as no subset containing all but two of the points of Γ lies in a hyperplane of $\mathbb{P}^r$.
- *The Gale transform of* Γ' *is* Γ.
- *There is a natural identification of* $V = \mathrm{H}^0(\mathscr{O}_{\mathbb{P}^r}(1))$ *with* $(\omega_{\Gamma'})_{-1}$.
- Γ *is general if and only if* Γ' *is general*. More formally, a $\mathrm{GL}(s+1)$-invariant Zariski open condition on Γ' induces a $\mathrm{GL}(r+1)$-invariant Zariski open condition on Γ.

Proof of Theorem 1.1
Let $s \geq 3$ be an integer, and set $r = \binom{s+1}{2} + \delta$. Suppose that

$$0 \leq \delta \leq \binom{s}{2} - \begin{cases} 1 & \text{if } s \leq 4, \\ 2 & \text{if } s \geq 5, \end{cases}$$

and let $\Gamma \subset \mathbb{P}^r$ be a set of $\gamma(r,s) = r + s + 2$ points, in linearly general position. We show that the MRC fails for Γ. From this statement it is easy to solve for γ in terms of r, obtaining the range given in Theorem 1.1.

Let I_Γ be the homogeneous ideal of $\Gamma \subset \mathbb{P}^r$, and let

$$\omega_\Gamma = \operatorname{Ext}_S^{r-1}\big(I_\Gamma, S(-r-1)\big)$$

be the canonical module. The free resolution of ω_Γ is, up to a shift in degree, the dual of the resolution of S/I_Γ.

We assume that Γ is a general set of points such that the following hold: its Gale transform $\Gamma' \subset \mathbb{P}^s$ is defined and is in linearly general position, Γ imposes independent conditions on quadrics, and Γ' is not contained in any quadric (note that $\binom{s+2}{2} < \gamma(r,s) \leq \binom{r+2}{2}$).

Since $\gamma(r,s) > \binom{s+2}{2}$, it follows that Γ' does not impose independent conditions on quadrics of $\mathbb{P}^s$, and thus $\mathrm{H}^1(\mathscr{I}_{\Gamma'}(2)) \neq 0$. We set

$$U := (\omega_{\Gamma'})_{-2} = \operatorname{Hom}_k\big(\mathrm{H}^1(\mathscr{I}_{\Gamma'}(2)), k\big) \neq 0.$$

(In this setting we have $\dim U = \delta + 1$.) Write $W = \mathrm{H}^0(\mathscr{O}_{\mathbb{P}^s}(1))$, and $V = \mathrm{H}^0(\mathscr{O}_{\mathbb{P}^r}(1))$. Multiplication induces a natural pairing

$$\mu : \quad W \otimes U \longrightarrow V = (\omega_{\Gamma'})_{-1}.$$

We now carry out the checks needed to apply Theorem 3.3. First, $\Gamma' \subset \mathbb{P}^s$ is in linearly general position, so by Kreuzer [K], the pairing μ is 1-generic over any field and thus geometrically 1-generic. Second, $\Gamma \subset \mathbb{P}^r$ imposes independent conditions on quadrics, so $(\omega_\Gamma)_{-d} = 0$ for $d \geq 2$. Hence $\omega_\Gamma(-1)$ is a finitely generated graded S-submodule of $\bigoplus_{n\geq 0} \mathrm{H}^0(K_\Gamma(n-1))$ containing $W = (\omega_\Gamma)_{-1}$ in degree zero.

Therefore the irredundant linear free quotient $F'_\bullet(\mu)$ of the complex $F_\bullet(\mu)$ injects onto a degreewise split direct summand of the minimal free resolution of $\omega_\Gamma(-1)$, which is the dual of the minimal free resolution of $(S/I_\Gamma)(r+2)$. Hence the true graded Betti number $\beta_{(r-s),(r-s+2)}$ for S/I_Γ satisfies

$$\beta_{(r-s),(r-s+2)} \geq \operatorname{rank} F'_s(\mu) = \binom{s+\delta}{\delta} > 0.$$

The corresponding expected graded Betti number $\widetilde{\beta}_{(r-s),(r-s+2)}$ is found in [EP1], or it can be computed by taking the Hilbert series for Γ,

$$\sum_i \dim(S/I_\Gamma)_i \cdot t^i = 1 + (r+1)t + \gamma(r,s) \cdot \frac{t^2}{1-t} = \frac{\sum_j b_j t^j}{(1-t)^{r+1}},$$

and calculating that

$$(-1)^{r-s} b_{r-s+2} = \frac{(2\delta + 4 - s^2 + s)}{(s^2 - s + 2\delta + 4)} \cdot \binom{\binom{s+1}{2} + \delta}{s}.$$

This number is the alternating sum of the graded Betti numbers of degree $r-s+2$, in this case $\beta_{(r-s),(r-s+2)} - \beta_{(r-s+1),(r-s+2)}$. Thus the expected graded Betti number is

$$\widetilde{\beta}_{(r-s),(r-s+2)} = \max\bigl((-1)^{r-s} b_{r-s+2}, 0\bigr).$$

The MRC certainly fails when $\widetilde{\beta}_{(r-s),(r-s+2)} = 0$, which is equivalent to $0 \le \delta \le \binom{s}{2} - 2$. The MRC also fails for $s = 3, 4$ and $\delta = \binom{s}{2} - 1$ because

$$\binom{s+\delta}{\delta} > \widetilde{\beta}_{(r-s),(r-s+2)}$$

in those cases. (For $s = 4$ this is Boij's example.) Straightforward computation shows that there are no other cases where $\binom{s+\delta}{\delta} > \widetilde{\beta}_{(r-s),(r-s+2)}$. □

Schreyer observed around 1983 (unpublished) that the MRC fails for linearly normal curves of degree $d > g^2 - g,\ g \ge 4$. A result of Green and Lazarsfeld [GrL] implies that this bound can be improved to $d > 3g - 3$. We can use the technique developed above to give a new proof of the relevant part of the Green-Lazarsfeld result, recovering the failure of the MRC in a new way.

THEOREM 4.1
Let C be a smooth curve of genus $g \ge 2$, let $L \in \mathrm{Pic}^d(C)$ be a line bundle on C of degree $d \ge 2g+2$, and denote by $\varphi_{|L|} : C \to \mathbb{P}^{d-g}$ the embedding defined by the complete linear system $|L|$.

(a) *If $\mathrm{H}^0(L \otimes \omega_C^{-1}) \neq 0$ (i.e., equivalently, if the curve $\varphi_{|L|}(C) \subset \mathbb{P}^{d-g}$ has a $(d-2g+1)$-secant $\mathbb{P}^{d-2g-1}$), then the maximal irredundant quotient of the free linear complex*

$$F_\bullet := \mathbf{L}\Bigl(\bigoplus_{l \ge 0} \wedge^{l+1} \mathrm{H}^0(\omega_C) \otimes D_l\bigl(\mathrm{H}^0(L \otimes \omega_C^{-1})\bigr)\Bigr)$$

injects as a degreewise direct summand of the minimal free resolution of the $\mathrm{Sym}(\mathrm{H}^0(L))$-module $\bigoplus_{n \ge 0} \mathrm{H}^0(\omega_C \otimes L^n)$. In particular, $\varphi_{|L|}(C) \subset \mathbb{P}^{d-g}$ does not satisfy the property N_{d-2g}.

(b) *If* $g \geq 4$ *and* $d > 3g-3$, *then the MRC fails for* $\varphi_{|L|}(C) \subset \mathbb{P}^{d-g}$.

Remark 4.2
By Green [Gr1], $\varphi_{|L|}(C) \subset \mathbb{P}^{d-g}$ satisfies the property N_{d-2g-1} (i.e., the embedding is projectively normal, the homogeneous ideal $I_{\varphi_{|L|}(C)}$ is generated by quadrics, and all syzygies in the first $(d-2g-2)$ steps in its minimal free resolution are linear). On the other hand, if C is nonhyperelliptic, then property N_{d-2g} fails for $\varphi_{|L|}(C) \subset \mathbb{P}^{d-g}$ if and only if the curve has a $(d-2g+1)$-secant $(d-2g-1)$-plane, by Green and Lazarsfeld [GrL, Th. 2]. Theorem 4.1(a) spells out explicitly this failure.

Proof of Theorem 4.1
By hypothesis, $d \geq 2g+2$, so $\varphi_{|L|}(C) \subset \mathbb{P}^{d-g}$ is arithmetically Cohen-Macaulay; its homogeneous ideal $I_{\varphi_{|L|}(C)}$ is generated by quadrics and is 2-regular. Now Theorem 3.3, applied to the geometrically 1-generic pairing

$$\mu : \quad \mathrm{H}^0(\omega_C) \otimes \mathrm{H}^0(L \otimes \omega_C^{-1}) \to \mathrm{H}^0(L),$$

yields the complex $F_\bullet = F_\bullet(\mu)$ whose maximal irredundant quotient injects as a degreewise direct summand of the minimal free resolution of $\bigoplus_{n \geq 0} \mathrm{H}^0(\omega_C \otimes L^n)$, which is the dual of the minimal free resolution of $S/I_{\varphi_{|L|}(C)}(d-g+1)$. Here and in the sequel, $S = \mathrm{Sym}(\mathrm{H}^0(L))$. It follows that the graded Betti number $\beta_{d-2g,d-2g+2}$ of $S/I_{\varphi_{|L|}(C)}$ satisfies the inequality

$$\beta_{d-2g,d-2g+2} \geq \dim \mathrm{Sym}_{g-1}\big(\mathrm{H}^0(L \otimes \omega_C^{-1})\big) = \binom{d-2g+1}{g-1}.$$

In particular, $\beta_{d-2g,d-2g+2} \geq 1$, and so $\varphi_{|L|}(C) \subset \mathbb{P}^{d-g}$ does not satisfy property N_{d-2g}, which finishes the proof of part (a).

Since the embedding $\varphi_{|L|}(C) \subset \mathbb{P}^{d-g}$ is arithmetically Cohen-Macaulay, its hyperplane section $\Gamma \subset \mathbb{P}^{d-g-1}$ has the same graded Betti numbers as $\varphi_{|L|}(C)$. Thus, as in [EP1] or the proof of Theorem 1.1, we obtain the following for the alternated sum of graded Betti numbers of degree $d-2g+2$ for $S/I_{\varphi_{|L|}(C)}$:

$$\beta_{d-2g,d-2g+2} - \beta_{d-2g+1,d-2g+2} = \frac{(d-g^2+g)}{(d-2g+2)}\binom{d-g-1}{g-1}.$$

If $3g-2 \leq d \leq g^2-g$, the MRC predicts that $\beta_{d-2g,d-2g+2} = 0$, while from (a) we get $\beta_{d-2g,d-2g+2} \geq 1$; hence the MRC fails for d in the above range.

For $d \geq g^2-g+1$, the expected $\beta_{d-2g+1,d-2g+2}$ is zero while the expected $\beta_{d-2g,d-2g+2}$ is always larger than $\binom{d-2g+1}{g-1}$, and so the linear complex in (a) does not account for the failure of the MRC.

For $g \geq 4$, there exist a $g^1_{g-1} = |\mathcal{O}_C(D)|$ on C. The pairing

$$\eta: \quad \mathrm{H}^0\big(\mathcal{O}_C(D)\big) \otimes \mathrm{H}^0\big(L \otimes \mathcal{O}_C(-D)\big) \to \mathrm{H}^0(L)$$

is 1-generic and thus η defines a $(2 \times (d-2g+2))$-matrix with linear entries in $\mathbb{P}^{d-g}$ whose (2×2)-minors vanish in the expected codimension (see [E]). Therefore the Eagon-Northcott complex resolving the (2×2)-minors of this matrix is a linear exact complex of length $d-2g+1$ which injects as a degreewise direct summand in the top strand of the resolution of $I_{\varphi_{|L|}(C)}$. In particular, $\beta_{d-2g+1,d-2g+2} \geq (d-2g+1) > 1$, so the MRC also fails for $\varphi_{|L|}(C)$ when $d \geq g^2 - g + 1$. This concludes the proof of part (b). □

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