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Jet schemes of locally complete intersection canonical singularities*

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Introduction

Let X be a variety defined over an algebraically closed field k of characteristic zero. The mth jet scheme X_m of X is a scheme whose closed points over $x \in X$ are morphisms $\mathcal{O}_{X,x} \longrightarrow k[t]/(t^{m+1})$. When X is a smooth variety, this is an affine bundle over X, of dimension $(m+1)\dim X$. The space of arcs X_∞ of X is the projective limit $X_\infty = \operatorname{proj\,lim}_m X_m$.

Our main result is a proof of the following theorem, which was conjectured by Eisenbud and Frenkel:

Theorem 0.1. *If X is locally a complete intersection variety, then X_m is irreducible for all $m \geq 1$ if and only if X has rational singularities.*

In the appendix, Eisenbud and Frenkel apply this result when X is the nilpotent cone of a simple Lie algebra to extend results of Kostant in the setting of jet schemes.

Note that since X is assumed to be locally complete intersection, hence Gorenstein, a result of Elkik [El] and Flenner [Fl] says that X has rational singularities if and only if it has canonical singularities.

We make also make the following conjecture toward a similar characterization of log canonical singularities.

Conjecture 0.2. If X is locally a complete intersection, normal variety, then X_m is pure dimensional for all $m \geq 1$ if and only if X has log canonical singularities.

We prove the “only if” part of Conjecture 0.2 and show that the “if” part is equivalent to a special case of the Inverse of Adjunction Conjecture due to Shokurov and several other people (see [Kol], Conjecture 7.3).

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* with an appendix by David Eisenbud and Edward Frenkel

One should contrast Theorem 0.1 with the following result of Kolchin.

Theorem 0.3 ([Kln]). *If X is a variety over a field of characteristic zero, then X_∞ is irreducible.*

However, when X is locally a complete intersection and has rational singularities, Theorem 0.1 gives much more information about X_∞ (for example, as we will see, it implies that X_∞ is reduced).

The main technique we use in proving Theorem 0.1 is motivic integration, as developed by Kontsevich, Denef and Loeser, and Batyrev. Here is a brief description of the proof of Theorem 0.1. Consider an embedding $X \subset Y$, of codimension r, where Y is smooth, and an embedded resolution of singularities $\gamma : \widetilde{Y} \longrightarrow Y$ for X. There is a function F_X on Y_∞, defined by $F_X(w) = \mathrm{ord}(w(\mathcal{I}_{X,y}))$, where w is considered as a morphism $w : \mathcal{O}_{Y,y} \longrightarrow k[[t]]$. By integrating the function $f \circ F_X$ on Y_∞, for a convenient function $f : \mathbb{N} \longrightarrow \mathbb{N}$, we get a Laurent series in two variables which encodes information about the dimensions of X_m and the number of irreducible components of maximal dimension.

Applying the change of variable formula in [Ba1] or [DL1], this integral can be expressed as an integral on $\widetilde{Y}_\infty$ and since $\gamma^{-1}(X)$ is a divisor with normal crossings, this can be explicitely computed. If $\gamma^{-1}(X) = \sum_{i=1}^{t} a_i E_i$, where E_1 is the only exceptional divisor dominating X and the discrepancy of γ is $W = \sum_{i=1}^{t} b_i E_i$, then we see that $b_j \geq ra_j$ for all $j \geq 2$ if and only if $\dim X_m = (m+1)\dim X$, and X_m has exactly one component of maximal dimension for all m. When X is locally a complete intersection, this says precisely that X_m is irreducible for all m.

The last step needed is that this numerical condition is equivalent with X having canonical singularities when X is locally a complete intersection. We consider the following construction of γ: let $p : B \longrightarrow Y$ be the blowing-up of Y along X, F the exceptional divisor, and $\tilde{p} : \widetilde{Y} \longrightarrow B$ an embedded resolution of singularities for $F \subset B$. We take $\gamma = p \circ \tilde{p}$ and we show that the numerical condition is equivalent with (B, F) being canonical. By a result of Stevens [St], this is equivalent with F, hence X having canonical singularities.

The computation of motivic integrals gives an analogous condition for a variety which is locally a complete intersection to have pure dimensional jet schemes. The condition is that $b_j \geq ra_j - 1$, for all j. Conjecture 0.2 can therefore be translated into a conjectural analogue of the result of Stevens for log canonical singularities.

The technique we use to describe singularities in terms of jet schemes can be applied also to study pairs (X, D), where X is a smooth variety and D an effective $\mathbb{Q}$-divisor on X. For example, we prove in [Mu] the following characterization of log canonical pairs.

Theorem 0.4. *Let X be a smooth variety and D an effective divisor on X with integral coefficients.*

i) For every positive integer n, the pair $(X, \frac{1}{n}D)$ is log canonical if and only if

$$\dim D_m \leq (m+1)(\dim X - 1/n),$$

for all m.
ii) The log canonical threshold of (X, D) is given by

$$c(X, D) = \dim X - \sup_{m \geq 0} \frac{\dim D_m}{m+1}.$$

In the first section of the paper we give the definition of jet schemes and discuss the irreducibility condition for jets of locally complete intersection varieties. The condition that the jet schemes are irreducible (or pure dimensional) can be formulated in terms of the dimension of the space of jets lying over the singular part.

Starting with equations of X in an affine space $\mathbb{A}^N$, it is easy to give equations for X_m. If X is defined by $(f_\alpha(U)) \subset k[U_1, \ldots, U_N]$, then X_m can be defined in $\mathbb{A}^{(m+1)N}$ by $(f_\alpha^{(j)})_{0 \leq j \leq m}$, where $f_\alpha^{(j)} \in k[U, U', \ldots, U^{(m)}]$ is given by $f_\alpha^{(j)} = D^j(f_\alpha)$, D being the derivation taking $U_i^{(l)}$ to $U_i^{(l+1)}$ for $l \leq m-1$.

Using this explicit description and the irreducibility criterion, one can check directly that some jet schemes are irreducible. We give applications in the last section.

We deduce from the description by equations that if X is locally a complete intersection and X_m is pure dimensional, then it is locally a complete intersection, too. It follows that if X_m is irreducible, then it is reduced. We see also that if X_m is irreducible, then so is X_{m-1}.

In the second section we show that the numerical condition coming from an embedded resolution of singularities of X is equivalent with X having canonical singularities. The third section uses motivic integration as we described.

In the last section we discuss several examples and open questions. We consider first the small dimension case. If X is a singular curve, then X_m is reducible for all m. If X is a surface (and $\operatorname{char} k = 0$), we show that being locally a complete intersection is a necessary condition for the irreducibility of jet schemes. More precisely, if X is a surface, then X_m is irreducible for all m if and only if all the singular points of X are rational double points.

We give an example of a toric variety of dimension 3 which shows that in Theorem 0.1 it is not possible to replace locally complete intersection with Gorenstein. On the other hand, the example of the cone over the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^n$, with $n \geq 2$, shows that the condition of being locally a complete intersection is not necessary in order to have all the jet schemes irreducible.

Our results in the second and the third section, where we used the theory of singularities of pairs and motivic integration, rely on the fact that the characteristic of the ground field is zero. We discuss briefly a possible analogue of Theorem 0.1 in positive characteristic and we end with a characteristic free proof of the fact that if X is a locally complete intersection toric variety, then X_m is irreducible for all m. This is achieved using an inductive description due to Nakajima [Nak] for such varieties in order to describe a desingularization of the “dual” toric variety.

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1. Jet basics

The study of singularities via the space of arcs has gotten a lot of attention recently. Nash initiated this study in [Na]. He suggested that the study of the images $\eta_m(X_\infty) \subseteq X_m$ for all m, where η_m are the canonical projections, should give information about the fibers over the singular points in the desingularizations of X. For more on this approach, see [Le] and [LR]. For applications of spaces of arcs with a different flavour, for example the proof of a geometric analogue of Lang’s Conjecture, see [Bu].

We start by reviewing the definition and the general properties of jet schemes. In the case of locally complete intersection varieties, we give an irreducibility criterion for these schemes and show that under the irreducibility assumption, they are, as well, locally complete intersection varieties.

Let k be an algebraically closed field. If $\mathcal{S}ch/k$ is the category of schemes of finite type over k, consider for every $m \geq 0$ the covariant functor $\mathbf{F} : \mathcal{S}ch/k \longrightarrow \mathcal{S}ch/k$, given by $\mathbf{F}(Y) = Y \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, k[t]/(t^{m+1})$. This functor has a right adjoint $\mathbf{G} : \mathcal{S}ch/k \longrightarrow \mathcal{S}ch/k$, given by $\mathbf{G}(X) = X_m$, and X_m is called the scheme of jets of order m of X.

For an affine scheme $Y = \mathrm{Spec}\, A$, the adjointness relation says that the A-valued points of X_m are in bijection with the $A[t]/(t^{m+1})$-valued points of X. In particular, there are canonical isomorphisms $X_0 \simeq X$ and $X_1 \simeq TX$, where TX is the total tangent space of X.

By adjointness, the canonical embeddings

$$Y \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, k[t]/(t^m) \hookrightarrow Y \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, k[t]/(t^{m+1})$$

induce canonical morphisms $\phi_m^X : X_m \longrightarrow X_{m-1}$, for $m \geq 1$. We will use the notation $\pi_m^X = \phi_1^X \circ \ldots \circ \phi_m^X : X_m \longrightarrow X$, but we will supress the variety X, whenever this leads to no confusion.

The space of arcs of X, denoted by X_∞, is the inverse limit of $\{X_m\}_{m\geq 0}$. This is a scheme over k, in general not of finite type, whose A-valued points are in natural bijection with the $A[[t]]$-valued points of X.

Proposition 1.1. *If $f : X \longrightarrow Y$ is an étale morphism, then $X_m \simeq Y_m \times_Y X$, for all m.*

Proof. The assertion follows by adjointness from the fact that f is also formally étale. $\square$

In particular, the construction of jet schemes is compatible with open immersions. Therefore, in order to describe X_m, we may restrict ourselves to the affine case: suppose $X \subseteq \mathbb{A}^N$, $X = \operatorname{Spec}(R)$ and $R = k[U_1, \dots, U_N]/(f_1, \dots, f_r)$. For every A, an element in $\operatorname{Hom}(\operatorname{Spec} A[t]/(t^{m+1}), X)$ is given by a morphism $\theta : k[U_1, \dots, U_N] \longrightarrow A[t]/(t^{m+1})$ such that $\theta(f_\alpha) = 0$ for all α. The condition $\theta(f_\alpha) = 0$ is equivalent with the vanishing of each of the coefficients of t^i in $\theta(f_\alpha)$, for $0 \le i \le m$ and gives $m+1$ equations. Therefore the map $\theta \longrightarrow (\theta_i^{(j)})$, where $\theta(U_i) = \sum_{j=0}^{m} \theta_i^{(j)} t^j$, induces a closed immersion $X_m \hookrightarrow \mathbb{A}^{(m+1)N}$, such that X_m can be defined by $(m+1)r$ equations.

When $\operatorname{char} k > m$, by normalizing the variables, the equations defining X_m can be written as follows. Let $S_m = k[U_i^{(j)}; 1 \le i \le N, 0 \le j \le m]$ be the coordinate ring of $\mathbb{A}^{(m+1)N}$ and $D : S_m \longrightarrow S_{m+1}$ the unique derivation over k such that $D(U_i^{(j)}) = U_i^{(j+1)}$ for all i and j. If we embed X_m in $\mathbb{A}^{(m+1)N}$ by $\theta \longrightarrow (j!\,\theta_i^{(j)})$, then the ideal of X_m is generated by $f_\alpha^{(j)}$, for all α and all j, $0 \le j \le m$, where $f^{(j)} = D^j(f)$.

We will need later the following lemma.

Lemma 1.2. *For every scheme X and every $u \in X_m$, either $\phi_{m+1}^{-1}(u) = \emptyset$, or $\phi_{m+1}^{-1}(u) \simeq \phi_1^{-1}(x)$, where $x = \pi_m(u)$.*

Proof. If we look at A-valued points, then $\operatorname{Hom}(\operatorname{Spec} A, \phi_{m+1}^{-1}(u))$ is a pseudotorsor over $\operatorname{Hom}(\operatorname{Spec} A, T_x X)$. Indeed, suppose that $v : \mathcal{O}_{X,x} \longrightarrow A[t]/(t^{m+2})$, $v(y) = \sum_{i \le m+1} v_i(y) t^i$ corresponds to an A-valued point of $\phi_{m+1}^{-1}(u)$. Any other such morphism is of the form $v'(y) = v(y) + w(y) t^{m+1}$, where $v_0 + wt$ is an A-valued point of $T_x X$.

If $\phi_{m+1}^{-1}(u) \neq \emptyset$, then for a fixed closed point in $\phi_{m+1}^{-1}(u)$, we get induced A-valued points in $\phi_{m+1}^{-1}(u)$, and therefore an isomorphism $\phi_{m+1}^{-1}(u) \simeq T_x X$. $\square$

It is well-known that if X is a smooth, connected variety of dimension n, then for every m, the morphism π_m is an affine bundle with fiber $\mathbb{A}^{mn}$. Under these circumstances, X_m is smooth, connected, of dimension $(m+1)n$.

We define now a morphism $\Psi_m : \mathbb{A}^1 \times X_m \longrightarrow X_m$ by defining it on A-valued points. If (a, f) corresponds to an A-valued point of $\mathbb{A}^1 \times X_m$, where $a \in A$ and $f : \operatorname{Spec} A[t]/(t^{m+1}) \longrightarrow X$, then $\Psi_m(a, f)$ is given by the composition

$$\operatorname{Spec} A[t]/(t^{m+1}) \xrightarrow{g_a} \operatorname{Spec} A[t]/(t^{m+1}) \xrightarrow{f} X,$$

where g_a corresponds to the morphism of A-algebras which maps t to at.

It is clear that these morphisms are compatible with the projections $\phi_m : X_m \longrightarrow X_{m-1}$ and that the restriction to $k^* \times X_m$ defines an action of k^* on X_m. Notice that there is a canonical section of the projection π_m. This is the morphism $s_m : X \longrightarrow X_m$, which takes an A-valued point of X, $f : \mathrm{Spec}\ A \longrightarrow X$ to the composition $f \circ p_m$, where p_m corresponds to the inclusion $A \hookrightarrow A[t]/(t^{m+1})$. It follows from the definition that $\phi_{m+1} \circ s_{m+1} = s_m$ and $\Psi_m|_{\{0\}\times X_m} = s_m \circ \pi_m$.

We want to study the irreducible components of X_m. The following lemma allows us to relate the irreducible components of X_m and X_{m+1}.

Lemma 1.3. *If $Z \subset X$ is a closed subscheme and S is an irreducible component of $\pi_m^{-1}(Z)$, then $\overline{s_m(\pi_m(S))} \subseteq S$. In particular, $\phi_{m+1}^{-1}(S) \neq \emptyset$.*

Proof. It is enough to notice that since S is an irreducible component of $\pi_m^{-1}(Z)$, we have $\Psi_m(\mathbb{A}^1 \times S) \subseteq S$ and therefore $s_m(\pi_m(S)) \subseteq S$. $\square$

From now on, we will restrict ourselves to the case when X is locally a complete intersection (l.c.i. for short) variety. As usual, a variety is an integral scheme of finite type over k. We denote the smooth part of X by X_{reg} and its complement by X_{sing}.

Proposition 1.4. *Let X be an l.c.i. variety of dimension n and m a positive integer. The scheme X_m is pure dimensional if and only if* $\dim X_m \leq n(m+1)$, *and in this case X_m is locally complete intersection, too. Similarly, X_m is irreducible if and only if* $\dim \pi_m^{-1}(X_{\mathrm{sing}}) < n(m+1)$.

Proof. We have a decomposition

$$X_m = \pi_m^{-1}(X_{\mathrm{sing}}) \cup \overline{\pi_m^{-1}(X_{\mathrm{reg}})}$$

and in general $\overline{\pi_m^{-1}(X_{\mathrm{reg}})}$ is an irreducible component of X_m of dimension $n(m+1)$. Therefore the "only if" part of both assertions is obvious and holds without the l.c.i. hypothesis.

Suppose now that $\dim X_m \leq n(m+1)$. Working locally, we may assume that $X \subset \mathbb{A}^N$ and that X is defined by $N-n$ equations. We have seen that $X_m \subset \mathbb{A}^{N(m+1)}$ is defined by $(N-n)(m+1)$ equations, and therefore every irreducible component of X_m has dimension at least $n(m+1)$. We deduce that X_m is pure dimensional and locally a complete intersection.

If $\dim \pi_m^{-1}(X_{\mathrm{sing}}) < n(m+1)$, this implies that $\dim X_m \leq n(m+1)$, so that X_m is pure dimensional. The above decomposition of X_m shows that X_m is irreducible. $\square$

Proposition 1.5. *If X is an l.c.i. variety and X_m is irreducible for some $m \geq 1$, then X_m is also reduced.*

Proof. By Proposition 1.4, X_m is l.c.i., hence Cohen-Macaulay. Since $\pi_m^{-1}(X_{\mathrm{reg}})$ is smooth, X_m is generically reduced, and we conclude by Macaulay's theorem (see [Ei], Corollary 18.14). $\square$

Proposition 1.6. *If X is an l.c.i. variety of dimension n and $Z \subseteq X$ is a closed subscheme, then* $\dim(\pi_{m+1}^{-1}(Z)) \geq \dim(\pi_m^{-1}(Z)) + n$, *for every* $m \geq 1$. *In particular, if X_{m+1} is irreducible or pure dimensional, then so is X_m.*

Proof. Again, we may assume that $X \subseteq \mathbb{A}^N$ is defined by $N - n$ equations. It follows from the equations of X_{m+1} that we have $\pi_{m+1}^{-1}(Z) \hookrightarrow \pi_m^{-1}(Z) \times \mathbb{A}^N$, such that $\pi_{m+1}^{-1}(Z)$ is defined by $N - n$ equations. The first assertion follows from this once we notice that by Lemma 1.3, for every irreducible component S of $\pi_m^{-1}(Z)$, we have $\phi_{m+1}^{-1}(S) \neq \emptyset$. The last statement is a consequence of Proposition 1.4. □

Proposition 1.7. *If X is an l.c.i. variety and X_m is irreducible for some $m \geq 1$, then X is normal.*

Proof. Since X is in particular Cohen-Macaulay, by Serre's Criterion (see [Ei], Theorem 11.5) it is enough to show that $\mathrm{codim}(X_{\mathrm{sing}}, X) \geq 2$. If X_m is irreducible, by Proposition 1.6, we may assume that $m = 1$. But if $\mathrm{codim}(X_{\mathrm{sing}}, X) = 1$, since for every $x \in X_{\mathrm{sing}}$ we have $\dim \pi_1^{-1}(x) = \dim T_x X \geq \dim X + 1$, it follows that $\dim \pi_1^{-1}(X_{\mathrm{sing}}) \geq 2 \dim X$, contradicting Proposition 1.4. □

2. A criterion for l.c.i. varieties to have canonical singularities

In this section we establish the criterion we will use to check that an l.c.i. variety X has canonical singularities. We embed X in a smooth variety Y and our criterion is in terms of the data coming from an embedded resolution of singularities of $X \subseteq Y$.

We assume that the characteristic of the ground field is zero. For the definitions of singularities of pairs, we refer to [Kol] or to [KM], Chapter 2.3. Let X be a normal l.c.i. variety and we fix an arbitrary embedding $X \hookrightarrow Y$, where Y is a smooth variety. Let r be the codimension of X in Y.

Consider the blowing-up $p : B = Bl_X Y \longrightarrow Y$ of Y along X, and let $F = p^{-1}(X)$ be the exceptional divisor. Since X is locally a complete intersection, F is a projective bundle over X. In particular, F is an integral divisor on B, and is locally a complete intersection. Moreover, F is normal since X is, and therefore B is normal, too.

By Hironaka's embedded resolution of singularities (see [Hir]), there is a morphism $\tilde{p} : \widetilde{Y} \longrightarrow B$ which is proper, an isomorphism over the complement of a proper closed subset of F, and such that $\widetilde{Y}$ is smooth and $\tilde{p}^{-1}(F)$ is a divisor with normal crossings.

Let γ be the composition $p \circ \tilde{p}$. We can write $\gamma^{-1}(X) = \tilde{p}^{-1}(F) = E_1 + \sum_{i=2}^{t} a_i E_i$, where $E_2, \ldots, E_t$ are the exceptional divisors of $\tilde{p}$, and E_1 is the proper transform of F.

The discrepancy W of γ is defined by the formula $K_{\widetilde{Y}} = \gamma^{-1}(K_Y) + W$. We write $W = \sum_{i=1}^{t} b_i E_i$. The following is our criterion for X to have canonical singularities.

Theorem 2.1. *With the above notation, X has canonical singularities if and only if $b_i \geq ra_i$ for every $i \geq 2$.*

Proof. Notice first that we have $K_B = p^{-1}(K_Y) + (r-1)F$. Indeed, in order to compute the coefficient of F, we may restrict to an open subset whose intersection with X is nonempty and smooth, in which case the formula is well-known.

Consider the divisor R on $\widetilde{Y}$, defined by $K_{\widetilde{Y}} = \tilde{p}^{-1}(K_B + F) + R$. We can write $R = -E_1 + \sum_{i=2}^{t} c_i E_i$. But we have

$$R = \gamma^{-1}(K_Y) + W - \tilde{p}^{-1}(p^{-1}(K_Y)) - \tilde{p}^{-1}(rF) = W - \tilde{p}^{-1}(rF).$$

Therefore we have $b_1 = r - 1$ and $c_i = b_i - ra_i$, for all $i \geq 2$. It follows from the definition and from [Kol], Corollary 3.12, that the pair (B, F) is canonical if and only if $c_i \geq 0$ for all $i \geq 2$.

Since B is locally a complete intersection, hence Gorenstein, a result of Stevens [St] (see also [Kol], Theorem 7.9) says that F is canonical if and only if the pair (B, F) is canonical near F. Since $B \setminus F$ is smooth, this means precisely that (B, F) is canonical.

On the other hand, since F is locally a product of X and an affine space, it follows that X is canonical if and only if F is canonical, which completes the proof of the theorem. □

Remark 2.2. Since X is locally a complete intersection, in particular Gorenstein, it is a result of Elkik [El] and Flenner [Fl] (see also [KM], Corollary 5.24) that X has canonical singularities if and only if it has rational singularities.

We give also a necessary condition for X to have log canonical singularities and show that the sufficiency would follow from the Inverse of Adjunction Conjecture ([Kol], Conjecture 7.3).

Theorem 2.3. *With the above notation, if $b_i \geq ra_i - 1$ for every i, then X has log canonical singularities.*

Proof. As in the proof of Theorem 2.1, we have X log canonical if and only if F is log canonical and on the other hand $b_i \geq ra_i - 1$ for all i if and only if the pair (B, F) is log canonical.

It follows from Proposition 7.3.2 in [Kol] that if (B, F) is log canonical, then F is log canonical. □

We conjecture that the converse is also true.

Conjecture 2.4. If X has log canonical singularities, then $b_i \geq ra_i - 1$, for all i.

The argument in the proof of Theorem 2.3 shows that Conjecture 2.4 is implied by the conjecture below. In fact, we will prove in the next section that these two conjectures are equivalent.

Conjecture 2.5 (Inverse of Adjunction). Let X be a normal, l.c.i. variety and D a normal Cartier divisor on X. If D is log canonical, then (X, D) is log canonical around D.

Remark 2.6. In fact, the Inverse of Adjunction Conjecture is more general: it deals with arbitrary normal varieties and with restriction of pairs (see [Kol], Conjecture 7.3). It is known that it is implied by the Log Minimal Model Program (see [K+], Corollary 17.12). In particular, Conjecture 2.4 is true when $\dim Y = 3$.

3. Irreducibility of jet schemes via motivic integration

In this section we use motivic integration to give necessary and sufficient conditions for a variety to have all the jet schemes of the expected dimension and precisely one component of maximal dimension. When the variety is locally a complete intersection, this gives via the results in the previous two sections a proof of Theorem 0.1.

The construction of motivic integrals for smooth spaces is due to Kontsevich [Kon], who used it to prove a conjecture of Batyrev and Dais [BaDa] about the stringy Hodge numbers of varieties with mild Gorenstein singularities (see [Ba1]). An other application is the proof due to Batyrev [Ba2] of a conjecture of Reid on the McKay correspondence (see [Re]). The construction was generalized by Denef and Loeser in [DL1] and [DL2] to singular spaces (see the recent surveys [DL3] and [Lo] for other applications of this idea). We will need only the Hodge realizations of motivic integrals on the space of arcs of a smooth variety. We refer for definitions and proofs to [Ba1] (see also [Cr] for a nice introduction).

From now on, X will be a fixed variety over k, with $\operatorname{char} k = 0$. Unless explicitely mentioned, X is not assumed to be locally complete intersection. We fix an embedding $X \hookrightarrow Y$, where Y is a smooth variety, and an embedded resolution of singularities $\gamma : \widetilde{Y} \longrightarrow Y$ for (Y, X), as in the previous section.

More precisely, we assume that γ is a proper morphism which is an isomorphism over $Y \setminus X$, and $\widetilde{Y}$ is smooth and $\gamma^{-1}(X) = \sum_{i=1}^{t} a_i E_i$ is a divisor with normal crossings. Let $W = \sum_{i=1}^{t} b_i E_i$ be the discrepancy of γ. We set $N = \dim Y$ and $r = \operatorname{codim}(X, Y) \geq 1$. We can further assume that E_1 is the only prime divisor in $\gamma^{-1}(X)$ dominating X and that $a_1 = 1$ and $b_1 = r - 1$. With this notation, we prove the following results.

Theorem 3.1. *The following statements are equivalent:*

i) *For every $i \geq 1$, we have $b_i \geq r a_i - 1$.*
ii) $\dim X_m = (m+1) \dim X$*, for every $m \geq 1$.*
iii) *There is $q \geq 1$, with $a_i | (q+1)$, for all i, such that* $\dim X_q = (q+1) \dim X$.

Theorem 3.2. *The following statements are equivalent:*

i) *For every $i \geq 2$, we have $b_i \geq ra_i$.*
ii) *For every $m \geq 1$, we have* $\dim X_m = (m+1)\dim X$*, and X_m has only one irreducible component of maximal dimension.*
iii) *There is $q \geq 1$, with $a_i|(q+1)$, for all i, such that* $\dim X_q = (q+1)\dim X$*, and such that X_q has only one irreducible component of maximal dimension.*

By combining Theorems 2.1 and 3.2 and Theorems 2.3 and 3.1, we obtain the main results in our paper.

Theorem 3.3. *If X is an l.c.i. variety over k, and* $\mathrm{char}\, k = 0$*, then X_m is irreducible for every m if and only if X has canonical singularities.*

Proof. Notice that by Proposition 1.7 and by the definition of canonical singularities, either condition implies that X is normal, so that Theorem 2.1 applies. Moreover, by Proposition 1.4, if $\dim X_m = (m+1)\dim X$, then X_m is pure dimensional, so that an application of Theorems 2.1 and 3.2 completes the proof. □

Note that because of Remark 2.2, the above result is equivalent with Theorem 0.1.

Theorem 3.4. *If X is a normal, l.c.i. variety over k and* $\mathrm{char}\, k = 0$*, and if X_m is pure dimensional for every $m \geq 1$, then X has log canonical singularities.*

Proof. Again, Proposition 1.4 shows that X_m is pure dimensional if and only if $\dim X_m = (m+1)\dim X$ and we apply Theorems 3.1 and 2.3. □

Conjecture 3.5. If X is an l.c.i. variety over a field of characteristic zero and X has log canonical singularities, then X_m is pure dimensional for every m.

In fact, the above conjecture is equivalent with the conjectures we made in the previous section.

Proposition 3.6. *Conjectures 2.4, 2.5 and 3.5 are equivalent.*

Proof. Theorem 3.3 implies that Conjectures 2.4 and 3.5 are equivalent, and we have seen in the previous section that Conjecture 2.5 implies Conjecture 2.4. It is therefore enough to prove that if Conjecture 3.5 is true for all normal, l.c.i. varieties, then so is Conjecture 2.5.

Using a trick due to Manivel (see [Kol], Lemma 7.1.3), the assertion in Conjecture 2.5 can be reduced to the following: if X is a normal, l.c.i. variety and D is a normal, Cartier divisor on X which is log canonical, then X is log canonical around D.

Applying Conjecture 3.5 to D, we get $\dim D_m = (m+1)(\dim X - 1)$, for all m. We may assume that $D \hookrightarrow X$ is defined by one equation, so that

$D_m \hookrightarrow X_m$ is defined by $m+1$ equations. Therefore, if F is an irreducible component of X_m, with $\dim\ F > (m+1)\dim\ X$, then $F \cap D_m = \emptyset$. Since by Lemma 1.3 we have $s_m(\overline{\pi_m(F)}) \subseteq F$, we deduce that $\overline{\pi_m(F)} \cap D = \emptyset$. We conclude that there is an open neighbourhood U of D such that $\dim\ U_m = (m+1)\dim\ U$. By Theorem 3.1, it is possible to pick m (depending on an embedded resolution of X) such that if $\dim\ U_m = (m+1)\dim\ X$, then U is log-canonical, which finishes the proof of the proposition. $\square$

By Lefschetz's principle, in order to prove Theorems 3.1 and 3.2, we may assume $k = \mathbb{C}$. Before we give the proof of the theorems, we review the basics of motivic integration. For every $m \geq 0$, we have natural projections

$$Y_\infty \xrightarrow{\eta_m} Y_m \xrightarrow{\pi_m} Y,$$

where π_m is an affine bundle with fiber $\mathbb{A}^{mN}$.

The theory provides an algebra $\mathcal{M}$ of subsets of Y_∞, containing the algebra $\mathcal{C}yl$ of cylinders of the form $\eta_m^{-1}(Z)$, where $Z \subseteq Y_m$ is a constructible subset. We consider the ring of Laurent power series in two variables u^{-1} and v^{-1}: $S = \mathbb{Z}[[u^{-1}, v^{-1}]][u, v]$, with the linear topology given by the descending sequence of subgroups $\{\oplus_{i+j\geq l}\mathbb{Z}u^{-i}v^{-j}\}_l$. On $\mathcal{M}$ there is a finitely additive measure μ with values in S, whose restriction to $\mathcal{C}yl$ is defined as follows. If $C = \eta_m^{-1}(Z)$ is a cylinder, then

$$\mu(C) = E(Z; u, v)(uv)^{-(m+1)N} \in S,$$

where $E(Z; u, v)$ is the Hodge–Deligne polynomial of Z. For a variety Z,

$$E(Z; u, v) = \sum_{1\leq p,q\leq \dim Z} \sum_{k\geq 0} (-1)^k h^{p,q}(H_c^k(Z; \mathbb{C}))u^p v^q,$$

where $\{h^{p,q}(H_c^k(Z; \mathbb{C}))\}$ are the Hodge-Deligne numbers of Z. What is important for us is that $E(Z; u, v)$ is a polynomial of degree $2(\dim\ Z)$, and the term of degree $2(\dim\ Z)$ is $c(uv)^{\dim\ Z}$, where c is the number of irreducible components of Z of maximal dimension.

If $T \subseteq Y_\infty$ is a subset such that there is a sequence of cylinders C_i, with $T \subseteq C_i$ and $\mu(C_i) \longrightarrow 0$, then T is in $\mathcal{M}$ and $\mu(T) = 0$.

If $F : Y_\infty \longrightarrow \mathbb{N} \cup \{\infty\}$ is a function such that $F^{-1}(m) \in \mathcal{M}$ for all $0 \leq m \leq \infty$ and $\mu(F^{-1}(\infty)) = 0$, and such that the series

$$\sum_{m\in\mathbb{N}} \mu(F^{-1}(m))(uv)^{-m}$$

is convergent in S, then F is called integrable. In this case, the sum of the above series is called the motivic integral of F and is denoted by $\int_{Y_\infty} e^{-F}$.

In general, a subscheme $Z \subset Y$, with corresponding ideal $\mathcal{I}_Z$, defines a function $F_Z : Y_\infty \longrightarrow \mathbb{N} \cup \{\infty\}$, as follows. If $w \in Y_\infty$ is an arc over

$y \in Y$, then w can be identified with a ring homomorphism $w : \mathcal{O}_{Y,y} \longrightarrow \mathbb{C}[[t]]$ and

$$F_Z(w) = \mathrm{ord}(w(\mathcal{I}_{Z,y})).$$

In [Ba1] and [Cr], the authors consider this function when Z is a divisor in Y. It follows from the definition that $F_Z^{-1}(\infty) = Z_\infty$ and that

$$F_Z^{-1}(m) = \eta_{m-1}^{-1}(Z_{m-1}) \setminus \eta_m^{-1}(Z_m),$$

for any integer $m \geq 0$ (we make the convention that $Z_{-1} = Y$ and $\eta_{-1} = \eta_0$). In particular, $F_Z^{-1}(m)$ is a cylinder.

Lemma 3.7. *If $D \subset Y$ is a divisor and $y \in Y$ is a point such that* $\mathrm{mult}_y D = a$, *then* $\dim\, (\pi_m^D)^{-1}(y) \leq Nm - [m/a]$, *where* $[x]$ *denotes the integral part of x. In particular, for every proper subscheme $Z \subset Y$, we have* $\dim\, Z_m - N(m+1) \longrightarrow -\infty$.

Proof. The second assertion follows from the first one, since working locally we may assume that $Z \subset D$, for some divisor D and if $a = \max_{y \in Y}\{\mathrm{mult}_y D\}$, then $\dim\, Z_m - N(m+1) \leq -[m/a]$.

To prove the first statement, note that the case $a = 0$ is trivial, and therefore we may assume that $y \in D$. It is enough to show that for every $p \geq 1$ we have

$$\dim\, (\pi_{pa}^D)^{-1}(y) \leq paN - p. \tag{3.1}$$

If we pick a regular system of parameters $x_1 \ldots, x_N$ in $\mathcal{O}_{Y,y}$, we get an étale ring homomorphism $w : \mathcal{O}_{\mathbf{A}^N,0} \longrightarrow \mathcal{O}_{Y,y}$. Let $f \in \mathcal{O}_{Y,y}$ be an equation for D at y. In general $f \notin \mathrm{Im}\,(w)$. However, if $g \in \mathcal{O}_{Y,y}$ is such that $g - f \in \underline{m}_{Y,y}^{pa+1}$, then $(\pi_{pa}^D)^{-1}(y) = (\pi_{pa}^{V(g)})^{-1}(y)$. Since w induces an isomorphism of the associated graded rings, we can find g as before such that $g \in \mathrm{Im}\,(w)$. Therefore, in order to prove (3.1), we may assume that $f \in \mathrm{Im}\,(w)$. Since w is étale, by replacing $\mathbf{A}^N$ and Y with suitable open neighbourhoods of 0 and y, respectively, we can apply Lemma 1.1 to reduce to the case when $Y = \mathbf{A}^N$, $y = 0$ and D is defined by a polynomial $f \in k[X] = k[X_1, \ldots, X_N]$.

We use the equations for D_{pa} described in the first section. With the notation $f_0^{(j)} = f^{(j)}(0, X', \ldots, X^{(j)})$ for every j, we have $(\pi_{pa}^D)^{-1}(0) \subseteq Z_p$, where Z_p is defined in Spec $k[X', \ldots, X^{(pa)}]$ by the polynomials $f_0^{(ja)}$, with $1 \leq j \leq p$.

We prove that $\dim\, Z_p \leq paN - p$ by computing the dimension of a deformation of this set. Note that if we put $\deg(X_i^{(j)}) = j$ for every i and j, then each polynomial $f_0^{(j)} \in k[X', \ldots, X^{(pa)}]$ is homogeneous of degree j.

Consider the family $\mathcal{Z}_p$ over Spec $k[t]$ defined in $\mathbf{A}^{paN} \times$ Spec $k[t]$ by the polynomials $(1/t^a) f_0^{(j)}(tX', \ldots, tX^{(j)})$, with $1 \leq j \leq p$. The fiber of $\mathcal{Z}_p$ over every $t_0 \neq 0$ is isomorphic to Z_p, while the fiber over 0 is the corresponding scheme obtained by replacing f with its homogeneous component of degree a. Since all the rings are graded (for the grading we defined above), the semicontinuity theorem for the dimension of the fibers of a morphism (see [Ei], Theorem 14.8) shows that in order to prove that $\dim Z_p \leq paN - p$, we may assume that f is homogeneous of degree a.

Consider now on $k[X, X', \ldots, X^{(pa)}]$ the reverse lexicographic order where we order the variables such that $X_i^{(j)} < X_{i'}^{(j')}$ if $j > j'$ or if $j = j'$ and $i > i'$ (see, for example, [Ei] Chapter 15). Let $m(X) = \mathrm{in}(f)$ be the initial term of f is this order. It is then easy to see that $\mathrm{in}(f_0^{(ja)}) = m(X^{(j)})$, for $1 \leq j \leq p$. Therefore the initial ideal of the ideal defining Z_p has dimension $paN - p$. Since the dimension of an ideal is equal with the dimension of its initial ideal, we deduce that $\dim Z_p = paN - p$, which concludes the proof of the lemma. □

Corollary 3.8. *For every proper subscheme $Z \subset Y$, and every $0 \leq m \leq \infty$, $F_Z^{-1}(m) \in \mathcal{M}$ and $\mu(F_Z^{-1}(\infty)) = 0$.*

Proof. We have already seen that $F_Z^{-1}(m)$ is a cylinder for every integer $m \geq 0$. Moreover, $F_Z^{-1}(\infty) = Z_\infty \subseteq \eta_m^{-1}(Z_m)$ and Lemma 3.7 gives

$$\mu(\eta_m^{-1}(Z_m)) = E(Z_m; u, v)(uv)^{-N(m+1)} \longrightarrow 0.$$

□

In order to prove Theorems 3.1 and 3.2, we will choose a suitable function $f : \mathbb{N} \longrightarrow \mathbb{N}$ (which we extend by $f(\infty) = \infty$) and we will integrate $F = f \circ F_X$ on Y_∞. The change of variable formula (see [Ba1], Theorem 6.27 or [DL1], Lemma 3.3) gives

$$\int_{Y_\infty} e^{-F} = \int_{\widetilde{Y}_\infty} e^{-(F\circ\gamma_\infty + F_W)},$$

in the sense that one integral exists if and only if the other one does, and in this case they are equal. The point is that $F \circ \gamma_\infty = f \circ F_{\gamma^{-1}(X)}$, and since $\gamma^{-1}(X) \cup W$ has normal crossings, the right-hand side integral can be explicitely computed, while for a suitable choice of f, the left-hand side contains the information we need about the dimension of X_m and about the number of its irreducible components of maximal dimension.

Proof of Theorems 3.1 and 3.2. We fix a function $f : \mathbb{N} \longrightarrow \mathbb{N}$, such that for every $m \geq 0$,

$$(\star) \qquad f(m+1) > f(m) + \dim X_m + C(m+1),$$

where $C \in \mathbb{N}$ is a constant with $C > |N - (b_j + 1)/a_j|$, for all j. We extend it by defining $f(\infty) = \infty$. For the proof of the implication *iii*) ⇒ *i*) we will put later an extra condition.

It follows from Corollary 3.8 that if $F = f \circ F_X$, then $F^{-1}(m) \in \mathcal{M}$ for $0 \leq m \leq \infty$, and $\mu(F^{-1}(\infty)) = 0$. Computing the integral of F from the definition, we get $I = \int_{Y_\infty} e^{-F} = S_1 - S_2$, where

$$S_1 = \sum_{m \geq 0} E(X_{m-1}; u, v)(uv)^{-mN - f(m)},$$

$$S_2 = \sum_{m \geq 0} E(X_m; u, v)(uv)^{-(m+1)N - f(m)}.$$

Every monomial which appears in the m^{th} term of S_1, has degree bounded above by $2P_1(m)$ and below by $2P_2(m)$, where

$$P_1(m) = \dim X_{m-1} - mN - f(m),$$

$$P_2(m) = -mN - f(m),$$

for all $m \geq 0$ (recall the convention that $X_{-1} = Y$). Moreover, we always have precisely one monomial of degree $2P_1(m)$, namely $(uv)^{P_1(m)}$, whose coefficient is c_m, the number of irreducible components of maximal dimension of X_{m-1}.

Similarly, every monomial which appears in the m^{th} term of S_2 has degree bounded above by $2Q_1(m)$ and below by $2Q_2(m)$, where

$$Q_1(m) = \dim X_m - (m+1)N - f(m),$$

$$Q_2(m) = -(m+1)N - f(m),$$

for all $m \geq 0$. We always have exactly one monomial of degree $2Q_1(m)$, namely $(uv)^{Q_1(m)}$, whose coefficient is c_{m+1}.

A first consequence of this and Lemma 3.7 is that F is, indeed, integrable. Using condition $(\star)$, it is an easy computation to show that we have $P_1(m+1) < \min\{P_2(m), Q_2(m)\}$, for every $m \geq 0$.

Moreover, Lemma 1.2 gives $\dim X_m \leq \dim X_{m-1} + N$, for every $m \geq 1$, and Lemma 3.7 implies that the inequality is strict for infinitely many m. We deduce that $Q_1(m) \leq P_1(m)$, for every $m \geq 0$ and equality holds if and only if $m \geq 1$ and $\dim X_m = \dim X_{m-1} + N$ (therefore, the inequality is strict for infinitely many m).

We conclude from the above inequalities first that in S_1, the term $(uv)^{P_1(m)}$ appears precisely once for every $m \geq 0$, and has coefficient c_m. Similarly, in S_2, the term $(uv)^{P_1(m)}$ appears at most once. It appears if and only if $m \geq 1$ and $\dim X_m = \dim X_{m-1} + N$, and in this case it has coefficient c_{m+1}.

We use the change of variable formula to compute the integral of F, as

$$I = \int_{Y_\infty} e^{-F} = \int_{\widetilde{Y}_\infty} e^{-(f \circ F_{\gamma^{-1}(X)} + F_W)}.$$

In this form, I can be explicitely computed, since $\gamma^{-1}(X) \cup W$ has normal crossings. For every subset $J \subseteq \{1, \ldots, t\}$, let $E_J^\circ = \cap_{i\in J} E_i \setminus \cup_{i\notin J} E_i$. With this notation, we have

$$\int_{\widetilde{Y}_\infty} e^{-(f\circ F_{\gamma^{-1}(X)}+F_W)} = \sum_{J\subseteq\{1,\ldots t\}} S_J, \text{ where}$$

$$S_J = \sum_{\alpha_i\geq 1, i\in J} E(E_J^\circ; u, v)(uv-1)^{|J|}\cdot (uv)^{-N-\sum_{i\in J}\alpha_i(b_i+1)-f(\sum_{i\in J}a_i\alpha_i)}.$$

We just sketch the proof of this formula, as it is similar to that of Theorem 6.28 in [Ba1], or that of Theorem 1.16 in [Cr].

Since E is additive, we may work locally on $\widetilde{Y}$. In order to compute the part S_J in I which corresponds to arcs over E_J°, we may assume that there is a regular system of parameters $y_1, \ldots, y_N$ on Y, such that E_i is defined by y_i, for all $i \in J$. We have

$$S_J = \sum_{\alpha_i\geq 1, i\in J} \mu(\{u \in \eta_0^{-1}(E_J^\circ) \colon F_{E_i}(u) = \alpha_i, i \in J\})(uv)^{-f(\sum_{i\in J}a_i\alpha_i)-\sum_{i\in J}b_i\alpha_i}.$$

For every $(\alpha_i)_{i\in J}$, if $p \geq \max_{i\in J}\{\alpha_i\}$, then

$$\{u \in \eta_0^{-1}(E_J^\circ) \colon F_{E_i}(u) = \alpha_i, i \in J\} = \eta_p^{-1}(C^p_{\alpha_i, i\in J}),$$

where $C^p_{\alpha_i,i\in J} \hookrightarrow Y_p$ is locally trivial over E_J°, with fiber $(k^*)^{|J|}\times\mathbb{A}^{Np-\sum_{i\in J}\alpha_i}$, and our formula for S_J follows.

Every monomial in the term of S_J corresponding to $(\alpha_i)_{i\in J}$, has degree bounded above by $2R_1(\alpha_i; i \in J)$, and below by $2R_2(\alpha_i; i \in J)$, where

$$R_1(\alpha_i; i \in J) = -\sum_{i\in J}\alpha_i(b_i+1) - f(\sum_{i\in J}a_i\alpha_i)$$

and $R_2(\alpha_i; i \in J) = R_1(\alpha_i; i \in J) - N$. Note that if $J = \emptyset$, then $R_1(\emptyset) = -f(0)$.

We introduce one more piece of notation: for $m \geq 0$, let $\tau(m) = \dim X_m - (m+1)\dim X$. For every $m \geq 0$, we have $\tau_m \geq 0$. We see that for $J \neq \emptyset$, we have

$$R_1(\alpha_i, i \in J) = P_1(\sum_{i\in J}a_i\alpha_i) - \tau(\sum_{i\in J}a_i\alpha_i - 1) - \sum_{i\in J}\alpha_i(b_i+1-ra_i).$$

Moreover, property $(\star)$ implies that if $J \neq \emptyset$, then

$$P_1(\sum_{i\in J}a_i\alpha_i+1) < R_2(\alpha_i; i \in J) < R_1(\alpha_i; i \in J) < \min\{P_2(\sum_{i\in J}a_i\alpha_i-1), Q_2(\sum_{i\in J}a_i\alpha_i-1)\} \tag{3.2}$$

and that $P_1(1) < R_2(\emptyset)$. In particular, this implies that the only monomial of the form $(uv)^{P_1(m)}$ which can appear in the term corresponding to J and $(\alpha_i)_{i\in J}$ is for $m = \sum_{i\in J} a_i\alpha_i$.

To prove the implication $i) \Rightarrow ii)$ in Theorem 3.3, suppose that $b_i \geq ra_i - 1$, for all i and assume that for some $m \geq 1$, we have $\tau(m) > 0$. The above inequalities show that $(uv)^{P_1(m+1)}$ does not appear in the sum S_J, for every J.

As we have seen, this imples that $\dim\, X_{m+1} = \dim\, X_m + N$. In particular, we have $\tau(m+1) > 0$. Continuing in this way, we get $\dim\, X_{p+1} = \dim\, X_p + N$, for every $p \geq m$, a contradiction with Lemma 3.7. Therefore we must have $\tau(m) = 0$, for all m, so $\dim\, X_m = (m+1)\dim\, X$.

Suppose next that $b_i \geq ra_i$, for all $i \geq 2$. The above argument shows that $\tau(m) = 0$, for every $m \geq 0$. In particular, the coefficient of $(uv)^{P_1(m+1)}$ in I is c_{m+1}, for every $m \geq 0$. From the above inequalities, we see that for every $m \geq 0$, the term $(uv)^{P_1(m+1)}$ appears in S_J if and only if $J = \{1\}$ and in this case it has coefficient 1, since $E^\circ_{\{1\}}$ is irreducible. Therefore $c_{m+1} = 1$, for every $m \geq 0$. This proves the implication $i) \Rightarrow ii)$ in Theorems 3.1 and 3.2.

We next turn to the implication $iii) \Rightarrow i)$. Suppose that for some $q \geq 1$, with $a_i|(q+1)$, for all i, we have $\tau(q) = 0$, and that for some $j \leq t$, $b_j < ra_j - 1$.

We pick the function f such that in addition to $(\star)$ it satisfies the following requirement. For every p, consider the set $\mathcal{J}_p$ of all the pairs $(J, (\alpha_i)_{i\in J})$, such that $\sum_{i\in J} a_i\alpha_i = p$. This is clearly a finite set. We require that for every $(J, (\alpha_i)_{i\in J}) \in \mathcal{J}_{q+1}$ and every $(J', (\alpha'_i)_{i\in J'}) \in \mathcal{J}_p$, for some p with $p \leq q$, we have

$$f(q+1) > f(p) - \sum_{i\in J}\alpha_i(b_i+1) + \sum_{i\in J'}\alpha'_i(b_i+1) + N, \tag{3.3}$$

or equivalently, we have $R_1(\alpha_i; i \in J) < R_2(\alpha'_i; i \in J')$.

On the other hand, if $(J', (\alpha'_i)_{i\in J'}) \in \mathcal{J}_p$, for some $p \geq q+2$, then by (3.2) we have

$$P_2(q+1) > R_1(\alpha'_i; i \in J'). \tag{3.4}$$

Note that the top degree monomials which appear in different terms of the sums S_J (for possibly different J) don't cancel each other, because they have positive coefficients. Let d be the highest degree of a monomial which appears in a term corresponding to some $(J, (\alpha_i)_{i\in J}) \in \mathcal{J}_{q+1}$. By the previous remark, the corresponding monomial does not cancel with a monomial in a term corresponding to $(J', (\alpha'_i)_{i\in J'}) \in \mathcal{J}_{q+1}$.

Since $(\{j\}, (q+1)/a_j) \in \mathcal{J}_{q+1}$, our hypothesis implies that $2P_1(q+1) < d$, while from (3.2) we deduce $d < 2\min\{P_2(q), Q_2(q)\}$. Moreover, we deduce from (3.3) and (3.4) that the monomial of degree d does not cancel with monomials in terms corresponding to $(J', (\alpha'_i)_{i\in J'}) \in \mathcal{J}_p$, if $p \neq q+1$. This shows that in I there is indeed a monomial of degree d, where $2P_1(q+1) < d < 2\min\{P_2(q), Q_2(q)\}$, a contradiction.

Suppose next that for some $q \geq 1$, with $a_i|(q+1)$, for all i, we have $\tau(q) = 0$ and $c_{q+1} = 1$, and that for some $j \geq 2$, $b_j < ra_j$. The above argument shows that $b_i \geq ra_i - 1$, for every i. Note that since $a_j|(q+1)$, we have in the expression of I the monomial $(uv)^{P_1(q+1)}$, with coefficient at least 2, once from $S_{\{1\}}$ (with $\alpha_1 = q+1$), and once from $S_{\{j\}}$ (with $\alpha_j = (q+1)/a_j$). This gives a contradiction. $\square$

4. Examples and open problems

In this section, unless explicitly mentioned otherwise, k has arbitrary characteristic.

We consider first the case of curves and show that none of the higher jet schemes of a singular curve can be irreducible. Since we do not assume X to be locally complete intersection, we need first a lemma showing that if the tangent space at a point to a scheme is too big, then so are all the fibers over that point of the higher order jet schemes.

Lemma 4.1. *If X is a scheme and $x \in X$, then* $\dim \pi_{2m}^{-1}(x) \geq \dim_x X + m \dim T_x X$ *and* $\dim \pi_{2m+1}^{-1}(x) \geq (m+1) \dim T_x X$*, for all $m \geq 1$.*

Proof. We prove the first assertion. Let $f : \pi_{2m-1}^{-1}(x) \longrightarrow \pi_{m-1}^{-1}(x)$ be the canonical projection. Using Lemma 1.2, it is enough to show that there is an isomorphism $T_x X^{\oplus m} \simeq f^{-1}(0)$, which maps $C_x X \times T_x X^{\oplus(m-1)} \subseteq T_x X^{\oplus m}$ into $f^{-1}(0) \cap \mathrm{Im}(\phi_{2m})$. Here 0 denotes the image of x by the canonical section of π_{m-1} and $C_x X$ is the tangent cone to X at x.

We give the isomorphism at the level of A-valued points. An A-point of $f^{-1}(0)$ is given by an algebra homomorphism $\theta : \mathcal{O}_{X,x} \longrightarrow A[t]/(t^{2m})$ of the form $\theta(y) = \theta_0(y) + \sum_{i=m}^{2m-1} \theta_i(y) t^i$, where θ_0 corresponds to x. The condition that θ is an algebra homomorphism is equivalent with saying that for every i, with $m \leq i \leq 2m-1$, the morphisms θ_i' mapping y to $\theta_0(y) + \theta_i(y)t$ are A-valued points of $T_x X$.

If θ is a k-valued point of $f^{-1}(0)$, then $\theta \in \mathrm{Im}(\phi_{2m})$ if and only if the morphism $S^2(m_x/m_x^2) \longrightarrow k$, given by $y_1 \cdot y_2 \longrightarrow \theta_m(y_1)\theta_m(y_2)$ factors through m_x^2/m_x^3. This condition is satisfied in particular if θ_m' is a k-valued point of $C_x X$.

The proof of the second assertion is similar, giving for the projection $g : \pi_{2m+1}^{-1}(x) \longrightarrow \pi_m^{-1}(x)$, an isomorphism $g^{-1}(0) \simeq T_x X^{\oplus(m+1)}$. $\square$

Corollary 4.2. *If X is an integral curve, then for any $m \geq 1$, X_m is irreducible if and only if X is nonsingular.*

Proof. Indeed, if $x \in X$ is a singular point, then $\dim T_x X \geq 2$, and by Lemma 4.1, it follows that for every $m \geq 1$, $\dim \pi_m^{-1}(x) \geq m+1$, and therefore $\pi_m^{-1}(x)$ gives an irreducible component of X_m. $\square$

Remark 4.3. If $\operatorname{char} k \neq 2$, it is possible to show that an integral curve X has all the jet schemes pure dimensional if and only if it has at worst nodes as singularities.

We consider next the case of a surface. In this case, if all the jet schemes of X are irreducible, we show that X has to be locally complete intersection, so we can apply our previous theory.

Theorem 4.4. *Assume that* $\operatorname{char} k = 0$ *and let X be a surface. Then the following are equivalent:*
i) X has at worst rational double points as singularities.
ii) X_m is irreducible for every $m \geq 1$.
iii) X_m is irreducible for $m \gg 0$.

Proof. One of the characterizations of rational double points is that they are locally complete intersection rational singularities (see [Du]). Therefore "$i) \Rightarrow ii)$" follows from Theorem 3.3 and the fact that Gorenstein singularities are rational if and only if they are canonical (see Remark 2.2). In order to prove $iii) \Rightarrow i)$, it is enough to show that if X_m is irreducible for some m, then X is locally complete intersection, and then use Proposition 1.6 and Theorem 3.3. But for every $x \in X$, Lemma 4.1 implies that if $\dim T_x X \geq 4$, then $\dim \pi_m^{-1}(x) \geq 2m + 2$, and therefore X_m is not irreducible. We conclude that $\dim T_x X = 3$ at every singular point $x \in X$, in particular that X is locally complete intersection. □

Remark 4.5. It is possible to prove Theorem 4.4 directly, by showing that condition $ii)$ is equivalent to X having the singular points of one of the types in the classification of the rational double points. In fact, that proof shows more, namely that the above conditions are equivalent with having only X_5 irreducible.

Example 4.6. Theorem 3.3 is not true if we replace the condition of X being locally a complete intersection variety, with being Gorenstein. More precisely, it is possible to have a variety with Gorenstein canonical singularities, but such that all its jet schemes are not even pure dimensional.

In fact, it is possible to take X to be a toric variety of dimension 3 (for definitions, basic facts and notations for toric varieties, see [Fu]). If V is a $\mathbb{Q}$-vector space with basis e_i, $1 \leq i \leq 3$, let $N \subset V$ be the lattice spanned by $\{e_1, e_2, e_3, 1/3(e_1 + e_2 + e_3)\}$. If σ is the cone in V spanned by $\{e_i\}_i$, let $X = U_\sigma$ be the associated toric variety. Then X is Gorenstein, since $\sum_i e_i^* \in N^\vee$ (see [Fu], Sections 3.4 and 4.4). By [Fu], Section 3.5, it has rational, hence canonical singularities (see Remark 2.2). Moreover, $\{\sum_i a_i e_i^* \mid a_i \in \mathbb{N}, \sum_i a_i = 3\}$ induce linearly independent elements in $T_x X$, where $x \in X$ is the fixed point under the torus action. Therefore, $\dim T_x X \geq 10$, and by Lemma 4.1 we get for every $m \geq 1$, $\dim \pi_m^{-1}(x) \geq 5m + 3$, and since $\dim X = 3$, X_m is not pure dimensional.

Example 4.7. On the other hand, the condition that X is locally a complete intersection is not necessary in order to have X_m irreducible for every m.

Let $X \subset \mathbb{A}^{2n}$ be the cone over the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^{n-1} \hookrightarrow \mathbb{P}^{2n-1}$. It is defined by the ideal generated by the 2×2 minors of the generic matrix:

$$\begin{pmatrix} U_1 & U_2 & \dots & U_n \\ V_1 & V_2 & \dots & V_n \end{pmatrix}$$

in the ring $S = k[U_1 \dots, U_n, V_1, \dots, V_n]$. Notice that if $n \geq 3$, then X is not a complete intersection, but X_m is irreducible for every $m \geq 1$.

Indeed, since X is defined by degree two homogeneous polynomials, it is easy to see that $\pi_m^{-1}(0) \cong X_{m-2} \times \mathbb{A}^{2n}$, for all $m \geq 1$ (we take X_{-1} to be a point). By induction on m, we get $\pi_m^{-1}(0)$ irreducible, and because $X_{\mathrm{sing}} = \{0\}$, it is enough to find a nonempty subset $U \subset \pi_m^{-1}(0)$ such that $U \subset \overline{\pi_m^{-1}(X_{\mathrm{reg}})}$. We consider the open subset of $\pi_m^{-1}(0)$ given by matrices

$$A = \begin{pmatrix} a_1 & a_2 & \dots & a_n \\ b_1 & b_2 & \dots & b_n \end{pmatrix}$$

with $a_i, b_i \in k[t]/(t^{m+1})$ of order one, such that $a_i b_j = a_j b_i$, for all i and j. There is a unique $p = p_0 + p_1 t + \ldots + p_{m-2} t^{m-2}$, $p_0 \neq 0$, such that for all i, $b_i = p a_i + c_i t^m$, for some $c_i \in k$.

By restricting to a smaller open subset, we may assume that $c_i \neq 0$, for all i. In this case $A \in \overline{\{A_s \mid s \in k^*\}}$, where $A_s \in \pi_m^{-1}(X_{\mathrm{reg}})$ is given by

$$A_s = \begin{pmatrix} a_1 + c_1 s & \dots & a_n + c_n s \\ p_s(a_1 + c_1 s) & \dots & p_s(a_n + c_n s) \end{pmatrix}$$

where $p_s = p + 1/s \cdot t^m$.

Question 4.8. What is the analogue of Theorem 3.3 in positive characteristic? Is it true that if X is an l.c.i. variety, then X_m is irreducible for all m if and only if X has pseudorational singularities? The notion of pseudorational singularities, introduced by Lipman and Teissier in [LT], replaces the notion of rational singularities when a good desingularization theory and Grauert-Riemenschneider theorem are not known. When these results are known (for example, in characteristic zero or for surfaces), the two notions coincide.

A different analogue of rational singularities in positive characteristic, coming from tight closure theory, is that of F-rational singularities (see [Sm1], for definition and relations with the birational geometry). A result of Smith (see [Sm2], Theorem 3.1) says that F-rational singularities are pseudorational. In general, having F-rational singularities is not a necessary condition for having irreducible jet schemes. For example, it follows from Proposition 4.9 below, that $V(X^2 + Y^3 + Z^5)$ has irreducible jet schemes in any characteristic, while it is known that it is not F-rational if $\operatorname{char} k \in \{2, 3, 5\}$ (see [BH], Example 10.3.12).

Proposition 4.9. *For $n \geq 3$, let $F = X_1^{d_1} + \ldots + X_n^{d_n}$ and $Z = V(F) \subset \mathbb{A}^n$, such that there is at most one i, with* $\mathrm{char}\, k | d_i$. *We have Z_m irreducible for all $m \geq 1$ if and only if $\sum_{i=1}^n 1/d_i > 1$.*

Proof. The hypothesis says that $Z \setminus \{0\}$ is smooth; in particular, Z is integral. By Proposition 1.4, Z_m is irreducible for all $m \geq 1$ if and only if $\dim \pi_m^{-1}(0) < (m+1)(n-1)$ for all m, and by Proposition 1.6, it is enough to check this for infinitely many m.

For every $m \geq 1$ and integers $a_1, \ldots, a_n$, with $1 \leq a_i \leq m+1$ for all i, let $V_{a_1,\ldots,a_n}$ be the locally closed subset of $\pi_m^{-1}(0)$ consisting of ring homomorphisms $\phi : k[X_1, \ldots, X_n]/(F) \longrightarrow k[t]/(t^{m+1})$ with $\mathrm{ord}\,(\phi(X_i)) = a_i$ (we make the convention $\mathrm{ord}(0) = m+1$). We obviously have $\pi_m^{-1}(0) = \bigcup_{a_1,\ldots,a_n} V_{a_1,\ldots,a_n}$.

Set $d = \prod_{i=1}^n d_i$ and assume that Z_{d-1} is irreducible. If $a_i = d/d_i$, for every i, then $\phi(X_i)$ can be chosen arbitrarily with order a_i, so that $\dim V_{a_1,\ldots,a_n} = d(n - \sum_i 1/d_i)$. Since $\dim V_{a_1,\ldots,a_n} < d(n-1)$, we get $\sum_i 1/d_i > 1$.

Next suppose that $\sum_i 1/d_i > 1$ and that $d | m+1$. We will show that Z_m is irreducible. Consider first the case when $d_i a_i \geq m+1$, for all i. As above, in this case $\phi(X_i)$ can be chosen arbitrarily with order a_i. Therefore we get

$$\dim V_{a_1,\ldots,a_n} = \sum_i (m - a_i + 1) \leq n(m+1) - (m+1)\sum_i 1/d_i < (m+1)(n-1).$$

On the other hand, if $r = \inf_i \{a_i d_i\} \leq m$, and if $V_{a_1,\ldots,a_n} \neq \emptyset$, then by hypothesis there is i, such that $\mathrm{char}\, k \not| \; d_i$, and $a_i d_i = r$. For every choice of $\phi(X_j)$ with order a_j, $j \neq i$, we have $\phi(X_i) = t^{a_i} u$, where the image of u in $k[t]/(t^{m+1-r})$ can take finitely many values. Moreover, $\phi(X_i)$ is uniquely determined by the image of u in $k[t]/(t^{m+1-a_i})$. Therefore we have

$$\dim V_{a_1,\ldots,a_n} \leq$$

$$r - a_i + \sum_{j \neq i} (m+1-a_j) \leq (n-1)(m+1) + r(1 - \sum_{j=1}^n 1/d_j) < (n-1)(m+1).$$

It follows that $\dim \pi_m^{-1}(0) < (n-1)(m+1)$, so Z_m is irreducible. Since this is true for all m with $d | (m+1)$, it follows by Proposition 1.6 that Z_m is irreducible for all m. □

Question 4.10. We have studied when an l.c.i. variety has irreducible jet schemes. Proposition 1.5 shows that if X_m is irreducible, then X_m is reduced. Is the converse true? We will see below that this the case for $m = 1$.

Question 4.11. Is it true that if X is an l.c.i. variety, then for every $m \geq 1$, $(X_m)_{\mathrm{reg}} = \pi_m^{-1}(X_{\mathrm{reg}})$? A positive answer to this question would give a positive answer to Question 4.10. Indeed, if X_m is reduced, but not irreducible, then any irreducible component contained in $\pi_m^{-1}(X_{\mathrm{sing}})$ has to be generically smooth.

We can give a positive answer when $m = 1$. This is the content of the following proposition.

Proposition 4.12. *If X is an l.c.i. variety, then $(X_1)_{\rm reg} = \pi_1^{-1}(X_{\rm reg})$.*

Proof. We show first that if $u \in \pi_1^{-1}(x)$, then

$$(\dagger) \qquad \dim\, T_u X_1 \geq \dim\, X + \dim\, T_x X.$$

To see this, we may work locally and assume that $X \subset \mathbb{A}^n$ is defined by $\underline{f} = (f_1, \ldots, f_r)$, where $r = n - \dim\, X$. X_1 is defined by $(\underline{f}, \underline{f}')$, and if $u = (x, x') \in X_1 \subset \mathbb{A}^{2n}$, then the Jacobian $J_{(\underline{f}, \underline{f}')}(u)$ is

$$\begin{pmatrix} J_{\underline{f}}(x) & 0 \\ A & J_{\underline{f}}(x) \end{pmatrix}$$

for some $r \times n$ matrix A. Therefore $\operatorname{rk} J_{(\underline{f}, \underline{f}')}(u) \leq r + \operatorname{rk} J_{\underline{f}}(x)$ and $(\dagger)$ follows.

Suppose now that $u \in (X_1)_{\rm reg}$. Consider an open connected neighbourhood $U \subset (X_1)_{\rm reg}$ of u. The inequality $(\dagger)$ for an arbitrary point $u' \in U$ gives

$$\dim\, U \geq \dim\, X + \dim\, \pi_1^{-1}(\pi_1(u')).$$

This implies that $\overline{\pi_1(U)} = X$, and therefore $U \cap \pi_1^{-1}(X_{\rm reg}) \neq \emptyset$, hence $\dim\, U = 2 \dim\, X$. The inequality $(\dagger)$ for u gives now $\dim\, T_x X = \dim\, X$. $\square$

The last case we consider is that of l.c.i. toric varieties. If $\operatorname{char} k = 0$, then X has rational singularities by [Fu], Section 3.5, and it follows by Theorem 3.3 (see, also, Remark 2.2) that X_m is irreducible for every m. We give below a direct argument independent of characteristic, which uses the description due to Nakajima [Nak] of l.c.i. toric varieties. This description was used by Dais, Haase and Ziegler in [DHZ] to show that all such toric varieties have crepant resolutions. The main point in our proof is to exhibit a certain resolution for the "dual" toric variety.

Theorem 4.13. *If X is an l.c.i. toric variety, then X_m is irreducible for all $m \geq 1$.*

Proof. We use notation and results from [Fu]. Since all the semigroups we use are saturated, we make no distinction between the semigroup and the cone it generates.

In general, for two varieties X and Y, we have $(X \times Y)_m \simeq X_m \times Y_m$. Using this, we reduce immediately to the case when $X = U_\sigma$ is affine, where σ is a strongly convex, rational, polyhedral cone of maximal dimension in $N_{\mathbf{R}}$, for some lattice N of rank n. Let $S = \sigma^\vee \cap M$, where $M = N^\vee$ is the dual lattice.

For every face $\tau \prec \sigma$, we have a corresponding orbit O_τ of dimension $n - \dim\, \tau$ and a distinguished point $x_\tau \in O_\tau$ defined by the semigroup morphism $x_\tau : S \longrightarrow k$, $x_\tau(u) = 1$, if $u \in \tau^\perp$ and $x_\tau(u) = 0$, otherwise. By Proposition 1.4, it is enough to show that if $\{0\} \neq \tau \prec \sigma$, then

$$(\star) \qquad \dim \pi_m^{-1}(x_\tau) < mn + \dim\, \tau.$$

We use the inductive description of S, due to Nakajima, for the case when X is locally complete intersection (see [Nak]). There are $r \geq 1$ and $s \geq 0$, with $n = r + s$, such that S can be obtained as follows: take $S_0 = \mathbb{N}^r$, $S \simeq S_s$ and for every i, $1 \leq i \leq s$, there is $x \in S_{i-1} \setminus \{0\}$, such that

$$S_i = S_{i-1} + \mathbb{N}e + \mathbb{N}(x - e) \subset S_{i-1} \oplus \mathbb{Z},$$

where $e = (0, 1)$.

We show by induction on s that if $T \subset S$ is a face of S, then there is a nonsingular fan Δ_s refining S, with rays $v_1, \ldots, v_{r+s}, w_1, \ldots, w_s$ such that:

i) $\{v_1, \ldots, v_{r+s}\}$ span a cone of Δ_s.
ii) For every $i \leq s$, $\{w_i, v_{r+j};\, j \neq i\}$ span a cone of Δ_s.
iii) $\{i \leq r + s | v_i \in T\}$ has at least $\dim\, T$ elements.

The assertion is trivial when $s = 0$. For $1 \leq i \leq s$, let Δ_{i-1} be the refinement corresponding to S_{i-1} and $T \cap S_{i-1}$. Δ_i consists of the cones spanned by $\{C, e\}$ and $\{C, x - e\}$, where C is a cone in Δ_{i-1}. Notice that $\dim(T \cap S_{i-1}) \geq \dim(T \cap S_i) - 1$, with equality if and only if $e \in T$ or $x - e \in T$. If we take v_{r+i} to be e or $x - e$ (we pick the one in T, if possible), and w_i to be the other one, then this refinement satisfies the requirements for S_i.

In order to prove $(\star)$, notice that $\pi_m^{-1}(x_\tau)$ consists of semigroup homomorphisms $\phi : S \longrightarrow k[t]/(t^{m+1})$, such that the composition with the projection onto k is x_τ. Let $T = S \cap \tau^\perp$ and consider the refinemet Δ_s constructed above.

Let $a_1, \ldots, a_{r+s}$ be integers such that $1 \leq a_i \leq m + 1$, if $v_i \notin T$, and $a_i = 0$, if $v_i \in T$. Take $V_{a_1,\ldots,a_{r+s}}$ to be the locally closed subset of $\pi_m^{-1}(x_\tau)$ consisting of morphisms ϕ, with $\operatorname{ord} \phi(v_i) = a_i$. We clearly have $\pi_m^{-1}(x_\tau) = \bigcup_{a_1,\ldots,a_{r+s}} V_{a_1,\ldots,a_{r+s}}$.

Fix $i \leq s$. Since Δ_s is a nonsingular fan, by conditions i) and ii), there is a relation

$$w_i + v_{r+i} = \sum_{j \leq r+s,\, j \neq r+i} b_{ij} v_j.$$

This shows that for $\phi \in V_{a_1,\ldots,a_{r+s}}$, the image of $\phi(w_i)$ in $k[t]/(t^{m+1-a_{r+i}})$ is uniquely determined by the values of ϕ on v_j, $j \leq r + s$. Therefore we get

$$\dim V_{a_1,\dots,a_{r+s}} \leq \sum_{i=1}^{r+s}(m-a_i+1) - \left|\{i \leq r+s | v_i \in T\}\right| + \sum_{i=1}^{s} a_{r+i} \leq$$
$$(m+1)n - \sum_{i=1}^{r} a_i - \dim T < mn + \dim \tau,$$

since $\dim T = n - \dim \tau$, and there is at least one $i \leq r$, with $v_i \notin T$ (as $\tau \neq \{0\}$, we have $T \neq S$, and going downward, we get $T \cap S_0 \neq S_0$). $\square$

Appendix

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Let G be a connected simple algebraic group over an algebraically closed field k of characteristic zero, and $\mathfrak{g}$ its Lie algebra. Denote by $k[\mathfrak{g}]$ the ring of functions on $\mathfrak{g}$. Let $I(\mathfrak{g})$ be the subring of G–invariants (equivalently, $\mathfrak{g}$–invariants) of $k[\mathfrak{g}]$ under the adjoint action. According to a theorem of B. Kostant [Kos], $I(\mathfrak{g}) = k[P_1, ..., P_\ell]$ for some graded elements $P_1, ..., P_\ell$ of $k[\mathfrak{g}]$. Here $\ell = \operatorname{rank} \mathfrak{g}$ and $\deg P_i = d_i$ equals the ith exponent of $\mathfrak{g}$ plus 1. By definition, the nilpotent cone $\mathcal{N}$ of $\mathfrak{g}$ is $\operatorname{Spec} k[\mathfrak{g}]/(I(\mathfrak{g})_+)$, where $(I(\mathfrak{g})_+) = (P_1, ..., P_\ell)$ is the ideal generated by the augmentation ideal $I(\mathfrak{g})_+$ of $I(\mathfrak{g})$.

Kostant [Kos] has proved that $k[\mathfrak{g}]$ is free over $I(\mathfrak{g})$. This important result has numerous applications; in particular, it implies that the universal enveloping algebra $U(\mathfrak{g})$ is free over its center.

Our goal is to formulate and prove an analogue of Kostant's theorem in the setting of jet schemes.

Let G_n be the nth jet scheme of G. This is an algebraic group, which is isomorphic to a semi-direct product of G and a unipotent group. The infinite jet scheme G_∞ of G is a proalgebraic group, which is isomorphic to a semi-direct product of G and a prounipotent group. The Lie algebra of G_n is the nth jet scheme $\mathfrak{g}_n$ of $\mathfrak{g}$. The Lie algebra of G_∞ is the infinite jet scheme $\mathfrak{g}_\infty$ of $\mathfrak{g}$. We have: $\mathfrak{g}_n = \mathfrak{g}[t]/(t^{n+1})$ and $\mathfrak{g}_\infty = \mathfrak{g}[[t]]$. The natural projections $\mathfrak{g}_\infty \to \mathfrak{g}_n$ give us embeddings $\iota_n : k[\mathfrak{g}_n] \hookrightarrow k[\mathfrak{g}_\infty]$.

Let $d = \dim \mathfrak{g}$ and let $x_1, \dots, x_d$ be a basis of $\mathfrak{g}^*$, which we take as a set of generators of $k[\mathfrak{g}]$. We can view them as elements of $k[\mathfrak{g}_n]$ and $k[\mathfrak{g}_\infty]$ using the embeddings ι_n. Define a function $x_i^{(m)}$, $m \geq 0$, on $\mathfrak{g}_n$, $n \geq m$ and on $\mathfrak{g}_\infty$ by the formula

$$x_i^{(m)}(y(t)) = x_i(\partial_t^m y(t)|_{t=0}), \qquad y(t) \in \mathfrak{g}[t]/(t^{n+1}) \text{ or } \mathfrak{g}[[t]].$$

The ring $k[\mathfrak{g}_n]$ (resp., $k[\mathfrak{g}_\infty]$) is the ring of polynomials in $x_i^{(m)}$, $0 \leq m \leq n$ (resp., $m \geq 0$). We introduce a $\mathbb{Z}_{\geq 0}$–gradation on these rings by setting

$\deg x_i^{(m)} = m + 1$. Note that each graded component of any of the above rings is finite-dimensional.

The ring $k[\mathfrak{g}_\infty]$ carries a canonical derivation D of degree 1, defined by the formula $Dx_i^{(m)} = x_i^{(m+1)}$ (it corresponds to the vector field ∂_t on $\mathfrak{g}[[t]]$). Let $P_i^{(m)} = D^m P_i$. Using the injections ι_n, we view $P_i^{(m)}$ as elements of $k[\mathfrak{g}_n], n \geq m$. We have: $\deg P_i^{(m)} = d_i + m$. Since the set $\{P_1, \ldots, P_\ell\}$ is algebraically independent, it follows that the set $\{P_1^{(m)}, \ldots, P_\ell^{(m)}\}_{m\geq 0}$ is also algebraically independent. Because the elements P_i are G–invariant, the element $P_i^{(m)}$ is G_n–invariant for $n \geq m$ and G_∞–invariant.

Denote by $I(\mathfrak{g}_n)$ (resp., $I(\mathfrak{g}_\infty)$) the ring of G_n–invariants (resp., G_∞–invariants) of $k[\mathfrak{g}_n]$ (resp., $k[\mathfrak{g}_\infty]$) under the adjoint action.

Proposition A.1. ([BeDr]) *The ring $I(\mathfrak{g}_n)$ (resp., $I(\mathfrak{g}_\infty)$) is generated by the algebraically independent elements $P_1^{(m)}, \ldots, P_\ell^{(m)}$, $0 \leq m \leq n$ (resp., $m \geq 0$). Thus, $I(\mathfrak{g}_n) = k[P_1^{(m)}, \ldots, P_\ell^{(m)}]_{0\leq m\leq n}$ and $I(\mathfrak{g}_\infty) = k[P_1^{(m)}, \ldots, P_\ell^{(m)}]_{m\geq 0}$.*

Proof. Let $\mathfrak{g}_{\mathrm{reg}}$ be the smooth open subset of $\mathfrak{g}$ consisting of regular elements. It is known that the morphism

$$\chi : \mathfrak{g}_{\mathrm{reg}} \to \mathcal{P} := \operatorname{Spec} k[P_1, \ldots, P_\ell]$$

is smooth and surjective (see [Kos]). Therefore the morphism

$$\chi_n : (\mathfrak{g}_{\mathrm{reg}})_n \to \mathcal{P}_n := \operatorname{Spec} k[P_1^{(m)}, \ldots, P_\ell^{(m)}]_{0\leq m\leq n}$$

is also smooth and surjective.

Consider the map $a : G \times \mathfrak{g}_{\mathrm{reg}} \to \mathfrak{g}_{\mathrm{reg}} \times_{\mathcal{P}} \mathfrak{g}_{\mathrm{reg}}$ defined by the formula $a(g, x) = (x, g \cdot x)$. The map a is smooth, and since G acts transitively along the fibers of χ, it is also surjective. Hence the corresponding map of jet schemes $a_n : G_n \times (\mathfrak{g}_{\mathrm{reg}})_n \to (\mathfrak{g}_{\mathrm{reg}})_n \times_{\mathcal{P}_n} (\mathfrak{g}_{\mathrm{reg}})_n$ is surjective. Given two points y_1, y_2 in the same fiber of χ_n, let (h, y_1) be a point in the (non-empty) fiber $a_n^{-1}(y_1, y_2)$. Then $y_2 = h \cdot y_1$. Hence G_n acts transitively along the fibers of the map χ_n.

Since $k[P_1^{(m)}, \ldots, P_\ell^{(m)}]_{0\leq m\leq n}$ is normal, this implies that the ring of G_n–invariant functions on $(\mathfrak{g}_{\mathrm{reg}})_n$ equals $k[P_1^{(m)}, \ldots, P_\ell^{(m)}]_{0\leq m\leq n}$. Because $\mathfrak{g}_{\mathrm{reg}}$ is a smooth open subset of $\mathfrak{g}$, we obtain that $(\mathfrak{g}_{\mathrm{reg}})_n$ is dense in $\mathfrak{g}_n$, and so any G_n–invariant function on $\mathfrak{g}_n$ is determined by its restriction to $(\mathfrak{g}_{\mathrm{reg}})_n$. This proves the proposition in the case of the finite jet schemes. The same argument works in the case the infinite jet scheme as well. $\square$

Let $I(\mathfrak{g}_n)_+$ be the augmentation ideal of the graded ring $I(\mathfrak{g}_n)$. By Proposition A.1, the ideal $(I(\mathfrak{g}_n)_+)$ in $k[\mathfrak{g}_n]$ generated by $I(\mathfrak{g}_n)_+$ equals $(P_1^{(m)}, \ldots, P_\ell^{(m)})_{0\leq m\leq n}$. Hence we obtain that the nth jet scheme $\mathcal{N}_n$ of the nilpotent cone $\mathcal{N}$ is $\operatorname{Spec} k[\mathfrak{g}_n]/(I(\mathfrak{g}_n)_+)$. Likewise, $\mathcal{N}_\infty = \operatorname{Spec} k[\mathfrak{g}_\infty]/(I(\mathfrak{g}_\infty)_+)$.

According to Kostant's results [Kos] (see [BL], [CG] for a review), the nilpotent cone is a complete intersection, which is irreducible and reduced. Moreover, it is proved in [He] that it has rational (hence canonical) singularities. Therefore we obtain from Theorem 3.3 and Proposition 1.5:

Theorem A.2. *$\mathcal{N}_n$ is irreducible, reduced and a complete intersection.*

Corollary A.3. *The natural map $k[\mathcal{N}_n] \to k[\mathcal{N}_{n+1}]$ is an embedding.*

Proof. Let Y be the open dense G–orbit of regular elements in $\mathcal{N}$. By Theorem A.2, $\mathcal{N}_n$ is irreducible. Hence Y_n is dense in $\mathcal{N}_n$. Since Y is smooth, $Y_{n+1} \to Y_n$ is surjective. Therefore the map $\mathcal{N}_{n+1} \to \mathcal{N}_n$ is dominant. □

Now we can formulate an analogue of the Kostant freeness theorem.

Theorem A.4. (1) *$k[\mathfrak{g}_n]$ is free over $I(\mathfrak{g}_n)$.*
(2) *$k[\mathfrak{g}_\infty]$ is free over $I(\mathfrak{g}_\infty)$.*

Proof. Since $\mathcal{N}_n$ is a complete intersection, it is Cohen-Macaulay. Therefore $k[\mathfrak{g}_n]$ is a flat module over $I(\mathfrak{g}_n) = k[P_1^{(m)}, \dots, P_\ell^{(m)}]_{0\le m\le n}$. Since $I(\mathfrak{g}_n)$ is $\mathbb{Z}_+$–graded with finite-dimensional homogeneous components, flatness implies that $k[\mathfrak{g}_n]$ is free over $I(\mathfrak{g}_n)$ (see, e.g., Ex. 18.18 of [Ei]). This proves part (1).

Since $k[\mathfrak{g}_n]$ is free over $I(\mathfrak{g}_n)$, and both rings are $\mathbb{Z}_+$–graded with finite-dimensional homogeneous components, we obtain that any graded lifting of a k–basis of $k[\mathcal{N}_n]$ to $k[\mathfrak{g}_n]$ is an $I(\mathfrak{g}_n)$–basis of $k[\mathfrak{g}_n]$. Conversely, the image of any $I(\mathfrak{g}_n)$–basis of $k[\mathfrak{g}_n]$ in $k[\mathcal{N}_n]$ is a k–basis of $k[\mathcal{N}_n]$.

Now choose any graded basis S_n of $k[\mathfrak{g}_n]$ over $I(\mathfrak{g}_n)$. Then the image S'_n of S_n in $k[\mathcal{N}_n]$ is a k–basis of $k[\mathcal{N}_n]$. According to Corollary A.3, the image of $S'_n \subset k[\mathcal{N}_n]$ in $k[\mathcal{N}_{n+1}]$ can be extended to a k–basis of $k[\mathcal{N}_{n+1}]$. Hence the image of S_n in $k[\mathfrak{g}_{n+1}]$ (we denote it also by S_n) can be extended to an $I(\mathfrak{g}_{n+1})$–basis S_{n+1} of $k[\mathfrak{g}_{n+1}]$.

Thus, we obtain a directed system $S_n, n \ge 0$, of sets of graded elements of $k[\mathfrak{g}_\infty]$ and embeddings $S_n \hookrightarrow S_m, \forall m \ge n$. Let S be the union of all of the sets $S_n, n \ge 0$. Note that by Proposition A.1, $I(\mathfrak{g}_\infty)$ is the inductive limit of the directed system $I(\mathfrak{g}_n), n \ge 0$. We claim that S is a basis of $k[\mathfrak{g}_\infty]$ over $I(\mathfrak{g}_\infty)$.

Indeed, consider the multiplication map $m : \mathrm{span}(S) \otimes I(\mathfrak{g}_\infty) \to k[\mathfrak{g}_\infty]$. This map is surjective: take any homogeneous element A of $\mathfrak{g}_\infty$ of degree a. Then it already belongs to $k[\mathfrak{g}_a]$. Hence by construction A lies in the image of m. The map m is also injective. Indeed, suppose there is an element B in the kernel of m. Without loss of generality we can assume that B is homogeneous of degree b. But then B belongs to the kernel of the map $\mathrm{span}(S_b) \otimes I(\mathfrak{g}_b) \to k[\mathfrak{g}_b]$, and so $B = 0$ by construction. This completes the proof. □

Remark A.5. Part (1) of Theorem A.4 has been previously proved by Geoffriau [Ge] in the case when $n = 1$ (for arbitrary $\mathfrak{g}$) and in the case when

$\mathfrak{g} = \mathfrak{sl}_2$ (for arbitrary n). We thank M. Duflo for bringing the paper [Ge] to our attention.

Remark A.6. Bernstein and Lunts [BL] have given an alternative proof of the original Kostant freeness theorem, using the isomorphism $I(\mathfrak{g}) \simeq k[\mathfrak{h}]^W$, and the Chevalley theorem which states that $k[\mathfrak{h}]$ is a free over $k[\mathfrak{h}]^W$ (here $\mathfrak{h}$ denotes the Cartan subalgebra of $\mathfrak{g}$, and W denotes the Weyl group of $\mathfrak{g}$). In our case this approach cannot be applied: although there is a natural embedding of $I(\mathfrak{g}_n)$ to $k[\mathfrak{h}_n]$, the ring $k[\mathfrak{h}_n]$ is not free over the image of $I(\mathfrak{g}_n)$. In fact, the latter equals $k[(\mathfrak{h}/W)_n]$, which is strictly smaller than $k[\mathfrak{h}_n]^W$.

Theorem A.2 also implies the following result:

Proposition A.7. *The space of G_n–invariants (resp., G_∞–invariants) of $k[\mathcal{N}_n]$ (resp., $k[\mathcal{N}_\infty]$) consists of constants.*

Proof. The jet scheme Y_n (resp, Y_∞) of the open dense G–orbit Y of $\mathcal{N}$ is an orbit of the group G_n (resp., G_∞) in $\mathcal{N}_n$ (resp., $\mathcal{N}_\infty$). Therefore any invariant function on it is a constant. But according to the proof of Corollary A.3, it is a dense subvariety in $\mathcal{N}_n$ (resp., $\mathcal{N}_\infty$). Hence any invariant function on $\mathcal{N}_n$ (resp., $\mathcal{N}_\infty$) is a constant. □

Remark A.8. According to Theorem A.4, the natural morphisms $\mathfrak{g}_\infty \to \operatorname{Spec} I(\mathfrak{g}_\infty)$ and $\mathfrak{g}_n \to \operatorname{Spec} I(\mathfrak{g}_n)$ are flat. Drinfeld has suggested that these morphisms may be viewed as local counterparts of the Hitchin morphism. More precisely, let X be a smooth projective curve over $\mathbb{C}$, and Bun_G the moduli stack of G–bundles on X. The cotangent space $T^*_{\mathcal{F}} \mathrm{Bun}_G$ to Bun_G at $\mathcal{F} \in \mathrm{Bun}_G$ is isomorphic to $H^0(X, \mathfrak{g}_{\mathcal{F}} \otimes \Omega)$. Here $\mathfrak{g}_{\mathcal{F}} = \mathcal{F} \underset{G}{\times} \mathfrak{g}$, and we identify $\mathfrak{g} \simeq \mathfrak{g}^*$ using the invariant inner product on $\mathfrak{g}$. The Hitchin morphism

$$T^* \mathrm{Bun}_G \to \mathcal{H} = \oplus_{i=1}^{\ell} H^0(X, \Omega^{\otimes d_i})$$

sends $(\mathcal{F}, \omega \in H^0(X, \mathfrak{g}_{\mathcal{F}} \otimes \Omega))$ to $(P_1(\omega), \ldots, P_\ell(\omega)) \in \mathcal{H}$.

Let x be a point of X, and $\widehat{\mathcal{O}}_x$ the completion of the local ring at x. Denote by D_x the formal disc at x, $D_x = \operatorname{Spec} \widehat{\mathcal{O}}_x$. For each $\mathcal{F} \in \mathrm{Bun}_G$ we have a local analogue of the Hitchin map,

$$h_x^{\mathcal{F}} : H^0(D_x, \mathfrak{g}_{\mathcal{F}} \otimes \Omega) \to \mathcal{H}_x = \oplus_{i=1}^{\ell} H^0(D_x, \Omega^{\otimes d_i}).$$

If we trivialize $\mathcal{F}|_{D_x}$ and choose a formal coordinate t at x, the map $h_x^{\mathcal{F}}$ becomes our map $\mathfrak{g}_\infty \to \operatorname{Spec} I(\mathfrak{g}_\infty)$. Actually, for varying $\mathcal{F}$ and $x \in X$, the spaces $H^0(D_x, \mathfrak{g}_{\mathcal{F}} \otimes \Omega)$ and $\mathcal{H}_x$ can be glued together into schemes over $X \times \mathrm{Bun}_G$ equipped with a flat connection along X, and the maps $h_x^{\mathcal{F}}$ can be glued into a morphism between these schemes preserving connections. The Hitchin morphism then appears as the corresponding map of the schemes of horizontal sections.

The flatness of the Hitchin morphism has been proved by Hitchin [Hit] and Faltings [Fa]. Drinfeld has derived the flatness of the morphism

$\mathfrak{g}_n \to \operatorname{Spec} I(\mathfrak{g}_n)$ from the flatness of the Hitchin morphism (private communication). He asked whether one can find a purely “local” argument proving this fact. The above proof answers this question.

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