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Hilbert Functions, Residual Intersections, and Residually S_2 Ideals

MARC CHARDIN[1], DAVID EISENBUD[2]⋆ and BERND ULRICH[3]⋆

[1]*Institut de Mathématiques, CNRS & Université Paris 6, 4, place Jussieu, F–75252 Paris Cedex 05, France. E-mail: chardin@math.jussieu.fr*

[2]*Mathematical Sciences Research Institute, 1000 Centennial Drive, Berkeley, CA 94720, U.S.A. e-mail: de@msri.org*

[3]*Department of Mathematics, Michigan State University, East Lansing, MI 48824, U.S.A. e-mail: ulrich@math.msu.edu*

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Abstract. Let R be a homogeneous ring over an infinite field, $I \subset R$ a homogeneous ideal, and $\mathfrak{a} \subset I$ an ideal generated by s forms of degrees $d_1, \ldots, d_s$ so that $\operatorname{codim}(\mathfrak{a} : I) \geqslant s$. We give broad conditions for when the Hilbert function of $R/\mathfrak{a}$ or of $R/(\mathfrak{a} : I)$ is determined by I and the degrees $d_1, \ldots, d_s$. These conditions are expressed in terms of residual intersections of I, culminating in the notion of residually S_2 ideals. We prove that the residually S_2 property is implied by the vanishing of certain Ext modules and deduce that generic projections tend to produce ideals with this property.

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Introduction

Let I be a homogeneous ideal in a polynomial ring $R = k[x_1, \ldots, x_n]$ over an infinite field k. Choose degrees $d_1, \ldots, d_s$, and consider the family of ideals $\mathfrak{a}$ generated by elements $a_1, \ldots, a_s \in I$ of degrees $d_1, \ldots, d_s$. By semicontinuity there are open sets in this family consisting of ideals $\mathfrak{a}$ such that the Hilbert functions of $R/\mathfrak{a}$ and of the residual intersection $R/(\mathfrak{a} : I)$ are constant. One might ask:

(A) What do these open sets look like?
(B) What are these generic Hilbert functions?

If $I = (x_1, \ldots, x_n)$ and $s \leqslant n$, then the answers are well known: The open sets of question (A) each consist precisely of the ideals $\mathfrak{a}$ of codimension s, and the generic Hilbert series of question (B) are $\prod(1 - t^{d_i})/(1 - t)^n$ except in case $s = n$, where

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$R/(\mathfrak{a}:I)$ is $R/\mathfrak{a}$ modulo its socle, and has Hilbert series

$$\frac{\prod(1-t^{d_i})}{(1-t)^n} - t^{\sum(d_i-1)}.$$

Even for this I, both questions are open when $s > n$ (see [24], [9], [1] and [13] for partial results).

In this paper we will take up question (A), always in the case $s \leqslant n$, but for more general ideals I. Our main results are that, in a wide range of cases, codimension conditions define the open sets in question. In a later paper [8] we will compute the generic Hilbert functions of question (B) under stronger hypotheses on I. For the purpose of describing our results, we say that the ideal I satisfies condition

(A1) if the Hilbert function of $R/\mathfrak{a}$ is constant on the open set of ideals $\mathfrak{a}$ generated by s forms of the given degrees such that $\operatorname{codim}(\mathfrak{a}:I) \geqslant s$; and
(A2) if the Hilbert function of $R/(\mathfrak{a}:I)$ is constant on this set.

Both conditions are satisfied automatically for $s \leqslant g := \operatorname{codim} I$ (the case $s = g$ is the theory of linkage). Thus we will focus on the cases $s > g$.

Our first results concern the case $s = g+1$ and the case of ideals of small codimension (see Theorem 1.1 and Corollary 2.2):

THEOREM 0.1. *With notation as above, both conditions (A1) and (A2) are satisfied if*

- $s = g+1$ *and, for (A2), I is a complete intersection locally in codimension g; or*
- $\operatorname{codim} I = 2$ *and R/I is Cohen–Macaulay; or*
- $\operatorname{codim} I = 3$ *and R/I is Gorenstein.*

Before stating further results we motivate the codimension condition in (A1) and (A2). Recall from Artin and Nagata [3] that an ideal J in any ring satisfies the condition G_s if, for each prime ideal $\mathfrak{p}$ containing J with $\operatorname{codim} \mathfrak{p} \leqslant s-1$, the minimal number of generators $\mu(J_\mathfrak{p})$ is at most $\operatorname{codim} \mathfrak{p}$. If I satisfies G_s then $\operatorname{codim}(\mathfrak{a}:I) \geqslant s$ when $\mathfrak{a}$ is generated by s generic forms of sufficiently large degrees, and thus the codimension condition is necessary for $\mathfrak{a}$ to be in the desired open set. On the other hand, many interesting ideals satisfy G_s. For example any smooth (or even locally complete intersection) projective variety can be defined (scheme-theoretically) by an ideal satisfying G_∞ – that is, G_s for every s. Most of our results give cases where an ideal satisfying G_{s-1} has property (A1) and cases where an ideal satisfying G_s has property (A2).

The following gives the flavor of what we can prove. A special case of a combination of Theorem 2.1, Proposition 3.1, and Corollary 4.3 (a) says that I satisfies (A1) or (A2) if it satisfies G_{s-1} or G_s, respectively, and if the depths of the rings R/I^j are not too small for $j \leqslant s-g-1$. In fact we only need to assume the vanishing of a single local cohomology module of each of these rings:

THEOREM 0.2. *Let R be a polynomial ring over a field with irrelevant maximal ideal $\mathfrak{m}$, let $I \subset R$ be a homogeneous ideal of codimension g and dimension d, and assume that $H_{\mathfrak{m}}^{d-j}(R/I^j) = 0$ for $1 \leqslant j \leqslant s-g-1$. If I satisfies G_{s-1} or G_s, then (A1) or (A2), respectively, hold.*

For example, the vanishing condition is satisfied for $s = g+2$ if R/I is Cohen–Macaulay.

The distinction between ideals that satisfy the conditions and those that don't is sometimes rather subtle. For example the ideal of a Veronese surface $\mathbb{P}^2 \hookrightarrow \mathbb{P}^5$ (2 by 2 minors of a generic symmetric 3 by 3 matrix) and that of a two-dimensional rational normal scroll in $\mathbb{P}^5$ (2 by 2 minors of a sufficiently general 2 by 4 matrix of linear forms) behave differently: The ideal of the Veronese satisfies (A1) and (A2) for all s (Theorem 2.1, Proposition 3.1, and [12, 2.3 and 3.3]), whereas the ideal of the scroll satisfies these properties for $s \leqslant 5 = g+2$ (Corollary 4.6) but does not satisfy (A1) for $s = 6$ (Remark 6.5).

To explain these results, we introduce our main definitions:

DEFINITION. Let R be a graded ring, and let $I \subset R$ be a homogeneous ideal.

(a) We say that I is *s-parsimonious* if the following holds for each $0 \leqslant i \leqslant s$: for every i-generated homogeneous ideal $\mathfrak{b} \subset I$, and every homogeneous element $a \in I$ such that

$$\operatorname{codim}(\mathfrak{b} : I) \geqslant i \quad \text{and} \quad \operatorname{codim}((\mathfrak{b}, a) : I) \geqslant i+1$$

we have

$$\mathfrak{b} : I = \mathfrak{b} : a.$$

(b) We say that I is *s-thrifty* if the following holds for each $0 \leqslant i \leqslant s$: for every i-generated homogeneous ideal $\mathfrak{b} \subset I$, and every homogeneous element $a \in I$ such that

$$\operatorname{codim}(\mathfrak{b} : I) \geqslant i \quad \text{and} \quad \operatorname{codim}((\mathfrak{b} : I), a) \geqslant i+1$$

we have

$$(\mathfrak{b} : I) \cap I = \mathfrak{b} \text{ and } a \text{ is a nonzerodivisor on } R/(\mathfrak{b} : I).$$

The definition applies to every ring (for example to localizations of graded rings) since we regard otherwise ungraded rings as being trivially graded (every element has degree 0). The idea behind the names is that if $\mathfrak{b}$ and a are sufficiently general, so that the codimension conditions hold, then there are no 'extra' elements that annihilate a (as compared with I) modulo $\mathfrak{b}$.

Theorem 2.1 states that if I is G_{s-1} and $(s-1)$-parsimonious, then (A1) is satisfied, and that if I is G_s and $(s-1)$-thrifty then (A2) holds. The actual result also includes the case where we know the hypotheses only 'locally in codimension $\leqslant r$' and yields only some of the coefficients of the Hilbert polynomial – thus it can be used for projective varieties, where the hypotheses are fulfilled locally on the variety. Theorem 2.3 refines this statement, and computes the changes in Hilbert functions as the degrees d_i are changed.

Given these results it is interesting to have conditions under which an ideal is s-parsimonious or s-thrifty. The rest of the paper is devoted to such conditions. We show (Proposition 3.1) that parsimony and thrift can be deduced from the condition that certain residual intersections satisfy the property S_2 of Serre. This condition of being residually S_2 has other applications – for example to d-sequences, and to bounding the codimension of annihilator ideals – see Corollary 3.6.

Of course we gain nothing by replacing the assumptions of parsimony or thrift by the condition residually S_2 unless we can check the latter condition more easily! In Theorem 4.1, the hardest result of the paper, we give a sufficient condition for residually S_2, essentially in terms of the vanishing of local cohomology – for example in the case of smooth projective varieties, this condition becomes the vanishing of certain cohomology of some powers of the conormal bundle, as in Theorem 0.2 above.

In Section 5, we give an interesting class of residually S_2 ideals by proving that if $X \subset \mathbb{P}^n_k$ is an isomorphic projection of a reduced complete intersection, then the defining ideal I of X satisfies conditions (A1) and (A2) for $s \leqslant n$. More generally, for any projection, we give conditions in terms of the codimension of the conductor – see Theorem 5.3.

In Section 6 we give a number of examples showing that our cohomological condition for parsimony is not too far from being sharp in interesting cases.

A problem related to the one above is as follows: Given a homogeneous ideal of a polynomial ring in n variables $I \subset R$, we again choose s elements of I of given degrees d_i and consider

$$J := (a_1, \ldots, a_s) : I^\infty = \cup_j \big((a_1, \ldots, a_s) : I^j\big).$$

It is easy to see that if the d_i are large enough then for generic choices of the a_i the ideal J will have codimension $\geqslant s$, and that for any a_i such that J has codimension $\geqslant s$, the ideal J is unmixed of codimension exactly s (or the unit ideal). As before, there is a 'generic' Hilbert function. Now suppose only that the codimension of J is s. Under what circumstances can we conclude that the Hilbert function (or Hilbert polynomial or ... or the degree) is equal to the generic one? For example, it was discovered in the 19th century by Chasles, Halphen, Schubert (and subsequently proved by Kleiman, [18]) that if I is the ideal of the Veronese surface in the projective space of plane conics, $s = 5$, and the a_i are the sextic equations that say that a conic is tangent to 5 given general conics, then J is the reduced ideal of 3264 points (see [19] for the history of this problem and for references). Such sextic equations actually span the symbolic square of I up to an irrelevant component. If we just assume that the a_i are 5 sextics in the symbolic square general enough so that J has codimension 5, is it still true that the degree of R/J is 3264? Does this condition determine the Hilbert function of R/J? Answers to these questions might be interesting geometrically.

1. Ideals of Small Codimension

In this section we take up the ideals of small codimension mentioned in Theorem 0.1. We do not need the G_s conditions that will appear in many other results of this paper, and we also prove the stronger result that the Hilbert functions of the rings $R/\mathfrak{a}$ and $R/(\mathfrak{a} : I)$ are determined by that of R/I – one does not need to know deeper invariants. The proof could easily be made into a computation of the new Hilbert functions from the old one; we hope to return to such results in a subsequent paper.

THEOREM 1.1. *Let R be a polynomial ring over a field, let $I \subset R$ be a homogeneous ideal and let $\mathfrak{a} \subset I$ be a homogeneous s-generated ideal with* $\operatorname{codim}(\mathfrak{a} : I) \geqslant s$. *If I is perfect of codimension 2 or Gorenstein of codimension 3, then the Hilbert functions of $R/\mathfrak{a}$ and $R/(\mathfrak{a} : I)$ are determined by the Hilbert function of R/I and the degrees of the s homogeneous generators of $\mathfrak{a}$.*

Proof. We may assume that the ground field k is infinite and that $s \geqslant 2$.

First consider the case where I is perfect of codimension 2. In this case I has a homogeneous minimal resolution of the form

$$0 \to G \oplus F_1 \xrightarrow{\phi} G \oplus F_0 \longrightarrow I \longrightarrow 0$$

where G is the largest free summand the two free modules have in common. Writing the Hilbert series of R/I in the form $p(t)/(1-t)^{\dim R}$ we see that giving the Hilbert series is equivalent to giving F_1 and F_0.

Let $d_1, \dots, d_s$ be the degrees of the s generators of $\mathfrak{a}$, and write $H = \oplus_{i=1}^{s} R(-d_i)$. We need to show that the Hilbert functions of $I/\mathfrak{a}$ and of $R/(\mathfrak{a} : I)$ are determined by the modules F_0, F_1, H.

The R-module $I/\mathfrak{a}$ has a homogeneous presentation

$$G \oplus F_1 \oplus H \xrightarrow{\psi} G \oplus F_0 \longrightarrow I/\mathfrak{a} \longrightarrow 0,$$

where $\psi = (\varphi | *)$ has size n by $n-1+s$, say. Now $\mathfrak{a} : I = \operatorname{ann}(I/\mathfrak{a}) \subset \sqrt{I_n(\psi)}$, hence codim $I_n(\psi) \geqslant$ $\operatorname{codim}(\mathfrak{a} : I) \geqslant s = (n-1+s) - n + 1 \geqslant 2$. Thus by [7, 3.1], $\mathfrak{a} : I = I_n(\psi)$. Furthermore, the Buchsbaum–Rim and Eagon–Northcott complexes associated to ψ yield homogeneous free resolutions of $I/\mathfrak{a} =$ coker ψ and of $R/(\mathfrak{a} : I) = R/I_n(\psi)$, respectively. These resolutions show that the Hilbert functions of coker ψ and of $R/I_n(\psi)$ are determined by the graded modules F_0, F_1, H, G, as long as codim $I_n(\psi) \geqslant s$.

To show the independence from G, consider the n by $n-1+s$ matrix $\eta = \begin{pmatrix} 1_G & 0 \\ 0 & 0 \end{pmatrix}$ and the family of homogeneous matrices $\psi_\lambda = \psi + \lambda\eta = \begin{pmatrix} \lambda 1_G + \varepsilon & * \\ * & * \end{pmatrix}$, $\lambda \in k$. For λ general, codim $I_n(\psi_\lambda) \geqslant s$, and we may replace ψ by ψ_λ without changing F_0, F_1, H, G. But since $\lambda 1_G + \varepsilon$ is invertible for general λ, coker ψ_λ has a homogeneous presentation of the form

$$F_1 \oplus H \longrightarrow F_0 \longrightarrow \operatorname{coker} \psi_\lambda \longrightarrow 0,$$

eliminating the dependence on G.

Next, consider the case where I is Gorenstein of codimension 3. By [6, 2.1], I has a homogeneous minimal resolution of the form

$$0 \longrightarrow R(-e) \longrightarrow G^*(-e) \oplus F^*(-e) \xrightarrow{\varphi} G \oplus F \longrightarrow I \longrightarrow 0,$$

where φ is an alternating matrix, $G \cong G^*(-e)$ has even rank, and F, e are determined by the Hilbert function of R/I. Now one proceeds as above, replacing [7, 3.1] by [20, 10.5 (a) and its proof], and the complexes of Buchsbaum–Rim and Eagon–Northcott by the complexes of [20, 10.5 (b),(c)]. □

2. Parsimonious Ideals

By a 'homogeneous ring' (over k) we shall mean an $\mathbb{N}$-graded Noetherian ring R where $R = R_0[R_1]$ and $R_0 = k$ is a field. Write $[[M]]$ for the Hilbert series of a finitely generated graded module M over a homogeneous ring. We shall say that the Hilbert series of two finitely generated graded modules M, N over a homogeneous ring R are *r-equivalent* and write $[[M]] \underset{r}{\equiv} [[N]]$, if they differ by the Hilbert series of R-modules whose annihilators have codimension at least $r+1$. Note that if dim $R = r$ then this means that the Hilbert series are equal; while for smaller values of r, it implies that the terms of the Hilbert polynomials of M and N of degrees dim $R-1, \dots,$ dim $R-1-r$ coincide. We write $M \underset{r}{\cong} N$ if there is a sequence of homogeneous R-linear maps between M and N that are isomorphisms locally in codimension $\leqslant r$.

THEOREM 2.1. *Let R be a homogeneous ring over an infinite field, let $I \subset R$ be a homogeneous ideal, and let $\mathfrak{a} \subset I$ be a homogeneous s-generated ideal with* codim$(\mathfrak{a} : I) \geqslant s$.

(a) *If I satisfies G_{s-1} and is $(s-1)$-parsimonious locally in codimension $\leqslant r$, then the Hilbert series of $R/\mathfrak{a}$ is determined, up to r-equivalence, by I and the degrees of the s homogeneous generators of $\mathfrak{a}$.*

(b) *If I satisfies G_s and is $(s-1)$-thrifty locally in codimension $\leqslant r$, then the Hilbert series of $R/(\mathfrak{a}:I)$ is determined, up to r-equivalence, by I and the degrees of the s homogeneous generators of $\mathfrak{a}$.*

Theorem 2.1 will be deduced from Theorem 2.3 below.

COROLLARY 2.2. *Let R be a homogeneous Cohen–Macaulay ring, let $I \subset R$ be a homogeneous ideal of codimension g, and let $\mathfrak{a} \subset I$ be a homogeneous $(g+1)$-generated ideal with* $\operatorname{codim}(\mathfrak{a}:I) \geqslant g+1$. *The Hilbert function of $R/\mathfrak{a}$ is determined by I and the degrees of the $g+1$ homogeneous generators of $\mathfrak{a}$. If moreover I is G_{g+1} (i.e., a complete intersection locally in codimension g) then the same is true for $R/(\mathfrak{a}:I)$.*

Proof. Any ideal of codimension g in a Cohen–Macaulay ring is g-parsimonious and g-thrifty. Thus after an extension to make the ground field infinite we may apply Theorem 2.1 (a) or (b), respectively. □

Here is a more explicit version of Theorem 2.1 that allows one to pass from one $\mathfrak{a}$ to another. This can actually be used to compute Hilbert functions, as will be illustrated in the last section.

THEOREM 2.3. *Let R be a homogeneous ring over an infinite field, let $I \subset R$ be a homogeneous ideal, and let $\mathfrak{a} \subset I$ be an ideal generated by $s \leqslant \dim R$ forms of degrees $d_1, \ldots, d_s$ with* $\operatorname{codim}(\mathfrak{a}:I) \geqslant s$. *Let $c_1, \ldots, c_s$ be forms of degrees $e_1, \ldots, e_s$ contained in I and for each $\xi = \{i_1, \ldots, i_k\} \subset [s]$ write $\mathfrak{c}_\xi = (c_{i_1}, \ldots, c_{i_k})$. Assume that* $\operatorname{codim}(\mathfrak{c}_\xi : I) \geqslant |\xi|$ *for every ξ.*

(a) *If I satisfies G_{s-1} and is $(s-1)$-parsimonious locally in codimension $\leqslant r$, then*

$$[[R/\mathfrak{a}]] \underset{r}{\equiv} t^{d_1+\cdots+d_s-e_1\cdots-e_s} \sum_{k=0}^{s} (-1)^{s-k} \sum_{\substack{\xi\subset[s] \\ |\xi|=k}} \prod_{j\notin\xi} (1-t^{e_j-d_j})[[R/\mathfrak{c}_\xi]].$$

(b) *If I is $(s-1)$-thrifty locally in codimension $\leqslant r$, and* $\operatorname{codim}(I+(\mathfrak{c}_\xi:I)) \geqslant |\xi|+1$ *for every ξ such that $|\xi| \leqslant s-1$, then*

$$[[R/(\mathfrak{a}:I)]] \underset{r}{\equiv} t^{d_1+\cdots+d_s-e_1\cdots-e_s} \sum_{k=0}^{s} (-1)^{s-k} \sum_{\substack{\xi\subset[s] \\ |\xi|=k}} \prod_{j\notin\xi} (1-t^{e_j-d_j})[[R/(\mathfrak{c}_\xi:I)]].$$

For the proof of Theorem 2.3 we will need the following observation about thrift:

LEMMA 2.4. *Let R, I, $\mathfrak{b}$ and a be as in the definition of thrift and let '$\overline{}$' denote images in $\overline{R} := R/(\mathfrak{b}:I)$. If I is s-thrifty, then:*

(a) $\mathfrak{b}:I = \mathfrak{b}:a$.

(b) $((\mathfrak{b},a):I)/(\mathfrak{b}:I) \cong \overline{(a)}:\overline{I}$.

Proof. For (a), since the element a is a non zerodivisor modulo $\mathfrak{b} : I$, the inclusions

$$a(\mathfrak{b} : a) \subset \mathfrak{b} \subset \mathfrak{b} : I$$

yield $\mathfrak{b} : a \subset \mathfrak{b} : I$, hence $\mathfrak{b} : a = \mathfrak{b} : I$.

Part (b) holds because

$$((\mathfrak{b} : I), a) : I = ((\mathfrak{b} : I) \cap I, a) : I = (\mathfrak{b}, a) : I. \qquad \square$$

We shall make frequent use of a general position lemma:

LEMMA 2.5. *Let R be a homogeneous ring over an infinite field, and let $I \subset R$ be a homogeneous ideal satisfying G_{s-1}. Let $\mathfrak{a} \subset I$ be an ideal generated by forms $a_1, \ldots, a_s$ of degrees $d_1, \ldots, d_s$, and assume that* $\operatorname{codim}(\mathfrak{a} : I) \geqslant s$.

(a) *If $d_s = \min\{d_i\}$, then the generators $a_1, \ldots, a_s$ can be chosen so that*

$$\operatorname{codim}((a_1, \ldots, a_{s-1}) : I) \geqslant s - 1.$$

(a′) *If I satisfies G_s and $d_s = \min\{d_i\}$, then the generators $a_1, \ldots, a_s$ can be chosen so that*

$$\operatorname{codim}(((a_1, \ldots, a_{s-1}) : I), a_s) \geqslant s.$$

(b) *For every $e \geqslant \max\{d_i\}$, there exist forms $c_1, \ldots, c_s$ of degree e in I such that*

$$\operatorname{codim}(\mathfrak{c}_\xi : I) \geqslant |\xi|, \qquad \text{for every } \xi \subset [s],$$
$$\operatorname{codim}(I + (\mathfrak{c}_\xi : I)) \geqslant |\xi| + 1, \quad \text{for every } \xi \subset [s] \text{ with } |\xi| \leqslant s - 2.$$

(b′) *If I satisfies G_s, then for every $e \geqslant \max\{d_i\}$, there exist forms $c_1, \ldots, c_s$ of degree e in I such that*

$$\operatorname{codim}(\mathfrak{c}_\xi : I) \geqslant |\xi|, \qquad \text{for every } \xi \subset [s],$$
$$\operatorname{codim}(I + (\mathfrak{c}_\xi : I)) \geqslant |\xi| + 1, \quad \text{for every } \xi \subset [s] \text{ with } |\xi| \leqslant s - 1.$$

Now assume that $d_1 = \cdots = d_s = d$. Let $\mathfrak{g} \subset I$ be another ideal generated by forms $g_1, \ldots, g_s$ of degree d, so that $\operatorname{codim}(\mathfrak{g} : I) \geqslant s$.

(c) *The generators $a_1, \ldots, a_s$ and $g_1, \ldots, g_s$ can be chosen so that for $\mathfrak{b}_i = (a_1, \ldots, a_{i-1}, g_{i+1}, \ldots, g_s)$, $1 \leqslant i \leqslant s$, we have*

$$\operatorname{codim}(\mathfrak{b}_i : I) \geqslant s - 1, \ \operatorname{codim}((\mathfrak{b}_i, a_i) : I) \geqslant s, \ \operatorname{codim}((\mathfrak{b}_i, g_i) : I) \geqslant s.$$

(c′) *If I satisfies G_s, then the generators $a_1, \ldots, a_s$ and $g_1, \ldots, g_s$ can be chosen so that for $\mathfrak{b}_i = (a_1, \ldots, a_{i-1}, g_{i+1}, \ldots, g_s)$, $1 \leqslant i \leqslant s$, we have*

$$\operatorname{codim}(\mathfrak{b}_i : I) \geqslant s - 1, \ \operatorname{codim}((\mathfrak{b}_i : I), a_i) \geqslant s, \ \operatorname{codim}((\mathfrak{b}_i : I), g_i) \geqslant s.$$

Proof. The proof is very similar to that given in [26, the proof of 1.4]. □

Proof of Theorem 2.1. By Lemma 2.5 (b) or (b′), there exist forms $c_1, \dots, c_s$ as in Theorem 2.3 (a) or (b), respectively. Now this theorem implies that the Hilbert series of $R/\mathfrak{a}$ or $R/(\mathfrak{a} : I)$, respectively, are determined, up to r-equivalence, by $c_1, \dots, c_s$, by I, and by the degrees of the s homogeneous generators of $\mathfrak{a}$. □

Proof of Theorem 2.3. We may assume that $s \geqslant 1$. Notice that I satisfies G_s in part (b). Choose $d \geqslant \max\{d_i, e_i \mid i\}$ and set $t = \#\{i \mid d_i \neq d \text{ or } e_i \neq d\}$. We are going to prove both statements by induction on t.

If $t = 0$, then $d = d_1 = \cdots = d_s = e_1 = \cdots = e_s$, and the assertion is that $[\![R/\mathfrak{a}]\!] \underset{r}{\equiv} [\![R/\mathfrak{c}]\!]$ or $[\![R/(\mathfrak{a} : I)]\!] \underset{r}{\equiv} [\![R/(\mathfrak{c} : I)]\!]$, respectively. We may choose $a_1, \dots, a_s$ and $c_1, \dots, c_s$ as in Lemma 2.5 (c) or (c′), respectively. (It is possible that we lose some of the hypotheses on the subset ideals $\mathfrak{c}_\xi$, but it does not matter since the assertion is simpler in this special case.) It suffices to show that setting $\mathfrak{b}_i = (a_1, \dots, a_{i-1}, c_{i+1}, \dots, c_s)$ for $1 \leqslant i \leqslant s$, we have $[\![(\mathfrak{b}_i, a_i)]\!] \underset{r}{\equiv} [\![(\mathfrak{b}_i, c_i)]\!]$ for (a) and $[\![(\mathfrak{b}_i, a_i) : I]\!] \underset{r}{\equiv} [\![(\mathfrak{b}_i, c_i) : I]\!]$ for (b). But indeed the definition of parsimony gives $(\mathfrak{b}_i, a_i)/\mathfrak{b}_i \underset{r}{\cong} (R/(\mathfrak{b}_i : I))(-d) \underset{r}{\cong} (\mathfrak{b}_i, c_i)/\mathfrak{b}_i$. On the other hand by Lemma 2.4 (b), thrift implies $((\mathfrak{b}_i, a_i) : I)/(\mathfrak{b}_i : I) \underset{r}{\cong} ((\mathfrak{b}_i, c_i) : I)/(\mathfrak{b}_i : I)$, since a_i and c_i are non zerodivisors modulo $\mathfrak{b}_i : I$ and have the same degree.

Next assume that $t > 0$. We may suppose that $d_s = \min\{d_i\}$, and that $d_s < d$ or $e_s < d$. Choose $a_1, \dots, a_s$ as in Lemma 2.5 (a) or (a′), respectively. Write $\mathfrak{a}' = (a_1, \dots, a_{s-1})$. Let x be a linear form of R which is not in any of the finitely many primes of codimension $s - 1$ containing $\mathfrak{a}' : I$, of codimension $|\xi|$ containing $\mathfrak{c}_\xi : I$, $|\xi| \leqslant s - 1$, and, for (b), of codimension $|\xi| + 1$ containing $I + (\mathfrak{c}_\xi : I)$, $|\xi| \leqslant s - 2$. Write $y = x^{d-d_s}$ and $z = x^{d-e_s}$. Now ya_s and zc_s are forms of degree d. Moreover, $\operatorname{codim}((\mathfrak{a}', ya_s) : I) \geqslant s$ or $\operatorname{codim}((\mathfrak{a}' : I), ya_s) \geqslant s$, respectively, and the sequence $c_1, \dots, c_{s-1}, zc_s$ has the same codimension properties as $c_1, \dots, c_s$.

We first treat part (a). By the definition of parsimony we have

$$\mathfrak{a}/\mathfrak{a}' \underset{r}{\cong} (R/(\mathfrak{a}' : I))(-d_s) \underset{r}{\cong} ((\mathfrak{a}', ya_s)/\mathfrak{a}')(d - d_s),$$

so that

$$[\![R/\mathfrak{a}]\!] \underset{r}{\equiv} t^{d_s-d}[\![R/(\mathfrak{a}', ya_s)]\!] + (1 - t^{d_s-d})[\![R/\mathfrak{a}']\!]. \tag{2.6}$$

Now, from the induction hypothesis, and setting $D_i = d_1 + \cdots + d_i$ and $E_i = e_1 + \cdots + e_i$, we obtain

$$[\![R/(\mathfrak{a}', ya_s)]\!] \underset{r}{\equiv} t^{D_{s-1}-E_{s-1}} \sum_{k=0}^{s} (-1)^{s-k} \sum_{\substack{|\xi|=k \\ s\in\xi}} \prod_{j\notin\xi} (1 - t^{e_j-d_j})[\![R/(\mathfrak{c}_{\xi-\{s\}}, zc_s)]\!].$$

Formula (2.6) with $\mathfrak{c}_\xi$ in place of $\mathfrak{a}$ shows that

$$t^{e_s-d}[[R/(\mathfrak{c}_{\xi-\{s\}}, zc_s)]] \underset{r}{\equiv} [[R/\mathfrak{c}_\xi]] - (1 - t^{e_s-d})[[R/\mathfrak{c}_{\xi-\{s\}}]],$$

and substituting into the previous formula we get

$$\begin{aligned} &t^{d_s-d}[[R/(\mathfrak{a}', ya_s)]] \\ &\underset{r}{\equiv} t^{D_s-E_s}\sum_{k=1}^{s}(-1)^{s-k}\sum_{\substack{|\xi|=k\\ s\in\xi}}\prod_{j\notin\xi}(1-t^{e_j-d_j})([[R/\mathfrak{c}_\xi]] - (1-t^{e_s-d})[[R/\mathfrak{c}_{\xi-\{s\}}]]) \\ &= t^{D_s-E_s}\left[\sum_{k=1}^{s}(-1)^{s-k}\sum_{\substack{|\xi|=k\\ s\in\xi}}\prod_{j\notin\xi}(1-t^{e_j-d_j})[[R/\mathfrak{c}_\xi]] + \right. \\ &\qquad \left. + \sum_{k=0}^{s-1}(-1)^{s-k}\sum_{\substack{|\xi|=k\\ s\notin\xi}}(1-t^{e_s-d})\prod_{\substack{j\notin\xi\\ j\neq s}}(1-t^{e_j-d_j})[[R/\mathfrak{c}_\xi]]\right]. \end{aligned}$$

Also by induction hypothesis,

$$(1-t^{d_s-d})[[R/\mathfrak{a}']] \underset{r}{\equiv} t^{D_s-E_s}\sum_{k=0}^{s-1}(-1)^{s-k}\sum_{\substack{|\xi|=k\\ s\notin\xi}}(t^{e_s-d} - t^{e_s-d_s})\prod_{\substack{j\notin\xi\\ j\neq s}}(1-t^{e_j-d_j})[[R/\mathfrak{c}_\xi]].$$

Taking the sum, using (2.6), and noticing that $(1-t^{e_s-d}) + (t^{e_s-d} - t^{e_s-d_s}) = 1 - t^{e_s-d_s}$, we obtain

$$\begin{aligned} [[R/\mathfrak{a}]] &\underset{r}{\equiv} t^{D_s-E_s}\left[\sum_{k=1}^{s}(-1)^{s-k}\sum_{\substack{|\xi|=k\\ s\in\xi}}\prod_{j\notin\xi}(1-t^{e_j-d_j})[[R/\mathfrak{c}_\xi]] + \right. \\ &\qquad \left. + \sum_{k=0}^{s-1}(-1)^{s-k}\sum_{\substack{|\xi|=k\\ s\notin\xi}}\prod_{j\notin\xi}(1-t^{e_j-d_j})[[R/\mathfrak{c}_\xi]]\right]. \end{aligned}$$

This is the formula asserted in (a).

We now turn to the proof of (b). By Lemma 2.4 (b),

$$(\mathfrak{a} : I)/(\mathfrak{a}' : I) \underset{r}{\cong} (((\mathfrak{a}', ya_s) : I)/(\mathfrak{a}' : I))(d - d_s),$$

which yields,

$$[[R/(\mathfrak{a} : I)]] \underset{r}{\equiv} t^{d_s - d}[[R/((\mathfrak{a}', ya_s) : I)]] + (1 - t^{d_s - d})[[R/(\mathfrak{a}' : I)]]. \tag{2.7}$$

One can now proceed as above, formally replacing (2.6) by (2.7). □

3. Residually S_2 Ideals Are Parsimonious and Thrifty

We know no useful characterization of s-parsimonious and s-thrifty ideals, but parsimony and thrift are implied by the Artin–Nagata property studied in [26]. Here we give better sufficient conditions. In this section we define the notion 's-residually S_2' and exhibit some of its properties. The advantage of the condition 's-residually S_2' is that it can be checked from homological properties of I and its powers. Such a criterion is given in the next section.

DEFINITION. Let R be a Noetherian ring, let $I \subset R$ be an ideal of codimension g, let $K \subset R$ be a proper ideal, and let $s \geqslant g$ be an integer.

(a) K is called an *s-residual intersection* of I if there exists an s-generated ideal $\mathfrak{a} \subset I$ such that $K = \mathfrak{a} : I$ and codim $K \geqslant s$.
(b) K is called a *geometric s-residual intersection* of I if K is an s-residual intersection of I and if in addition codim$(I + K) \geqslant s + 1$.

We shall often write '$K = \mathfrak{a} : I$ is an s-residual intersection of I' to indicate that the conditions of the above definition are satisfied.

DEFINITION. Let R be a Noetherian ring, let $I \subset R$ be an ideal of codimension g, and let s be an integer.

(a) I is said to be *s-residually S_2* if for every i with $g \leqslant i \leqslant s$ and every i-residual intersection K of I, R/K is S_2.
(b) I is said to be *weakly s-residually S_2* if for every i with $g \leqslant i \leqslant s$ and every geometric i-residual intersection K of I, R/K is S_2.

These properties are weaker versions of the properties AN_s and AN_s^- of [26], where the residual intersections are required to be Cohen–Macaulay instead of merely S_2. Notice that if $s \leqslant g - 1$ then I is automatically s-residually S_2.

We can now state the main result of this section:

PROPOSITION 3.1. *Let R be a graded Cohen–Macaulay ring, and let $I \subset R$ be a homogeneous ideal. If I satisfies G_s and is weakly $(s-1)$-residually S_2, then I is s-parsimonious and s-thrifty.*

Proof. The statement follows from Corollary 3.6 (a). □

SOURCES OF EXAMPLES. The hypotheses of Proposition 3.1 are fulfilled if I satisfies G_s and if moreover, after localizing, I has the sliding depth property or, more restrictively, is strongly Cohen–Macaulay (which means that for every i, the i-th Koszul homology H_i of a generating set $h_1, \ldots, h_n$ of I satisfies depth $H_i \geqslant \dim R - n + i$ or is Cohen–Macaulay, respectively) ([12, 3.3], [17, 3.1]). The latter condition always holds if I is a Cohen–Macaulay almost complete intersection or a Cohen–Macaulay deviation 2 ideal of a Gorenstein ring ([4, p.259]). It is also satisfied for any ideal in the linkage class of a complete intersection ([16, 1.11]). Standard examples include perfect ideals of codimension 2 ([2], [10]) and perfect Gorenstein ideals of codimension 3 ([27]).

We shall often use the following remark on general position:

LEMMA 3.2 (e.g. [3, 2.3] or [26, 1.6 (a)]). *Let R be a Noetherian local ring, let $\mathfrak{a} \subset I \subset R$ be ideals, and assume that $\mathfrak{a}$ is s-generated with* $\operatorname{codim}(\mathfrak{a} : I) \geqslant s$ *and that I satisfies G_s. Then there exists a generating sequence $a_1, \ldots, a_s$ of $\mathfrak{a}$ such that with $\mathfrak{a}_i = (a_1, \ldots, a_i)$ and $K_i = \mathfrak{a}_i : I$,* $\operatorname{codim} K_i \geqslant i$ *and* $\operatorname{codim}(I + K_i) \geqslant i + 1$ *whenever* $0 \leqslant i \leqslant s - 1$.

The next result contains basic facts about parsimony:

PROPOSITION 3.3. *Let R be a Noetherian graded ring, and let $I \subset R$ be a homogeneous ideal.*

(a) *I is s-parsimonious if and only if for every i, $\mathfrak{b}$, as in the definition of parsimony, $\mathfrak{b} : I$ is unmixed of height i (or $\mathfrak{b} : I = R$).*
(b) *If I is s-parsimonious then for every a, $\mathfrak{b}$ as in the definition of thrift, a is a non zerodivisor on $R/(\mathfrak{b} : I)$.*

Proof. It suffices to prove that if I is s-parsimonious and $\mathfrak{b} : I \neq R$, then $\mathfrak{b} : I$ is unmixed of height i. The converse is obvious and part (b) follows immediately from (a).

Thus, let $\mathfrak{p} \subset R$ be a prime of height $\geqslant i + 1$ that contains $\mathfrak{b} : I$. Choose an element $x \in \mathfrak{p}$ not in any minimal prime of $\mathfrak{b} : I$ of height i. Now $\operatorname{codim}((\mathfrak{b}, ax) : I) \geqslant i + 1$, hence by the parsimony of I,

$$\mathfrak{b} : I = \mathfrak{b} : (ax) = (\mathfrak{b} : a) : x \supset (\mathfrak{b} : I) : x.$$

Hence $x \in \mathfrak{p}$ is a nonzerodivisor on $R/(\mathfrak{b} : I)$, showing that $\mathfrak{p}$ cannot be an associated prime of $\mathfrak{b} : I$. $\square$

The next result shows that the unmixedness condition of the previous proposition is always satisfied by weakly $(s-1)$-residually S_2 ideals. This generalizes [26, 1.7] (which, in turn, extends parts of [17, 3.1]).

PROPOSITION 3.4. *Let R be a Cohen–Macaulay ring, let $I \subset R$ be an ideal, and assume that I is G_s and weakly $(s-1)$-residually S_2. For every s-residual intersection $K = \mathfrak{a} : I$ of I one has:*

(a) *K is unmixed of codimension s.*
(b) *The associated primes of $\mathfrak{a}$ have codimension at most s.*
(c) *If K is a geometric residual intersection, then $K \cap I = \mathfrak{a}$.*

Proof. After localizing the proof is the same as that of [26, 1.7]. □

COROLLARY 3.5. *Let R be a Cohen–Macaulay ring, let $I \subset R$ be an ideal satisfying G_s, and let $I' \subset I$ be an ideal that agrees with I locally up to codimension s. If I is s-residually S_2 (respectively weakly s-residually S_2) then so is I'.*

Proof. Proposition 3.4 (a) implies that for codim $I \leqslant i \leqslant s$, every i-residual intersection of I' is also an i-residual intersection of I, and one is geometric iff the other is. □

COROLLARY 3.6. *Let R be a Cohen–Macaulay ring, and let $I \subset R$ be an ideal satisfying G_s.*

(a) *If I is weakly $(s-1)$-residually S_2, then I is s-parsimonious and s-thrifty.*
(b) *Suppose that R is local and I is weakly $(s-2)$-residually S_2. If $\mathfrak{a} \subset I$ is an s-generated ideal with* $\operatorname{codim}(\mathfrak{a} : I) \geqslant s$, *and $a_1, \ldots, a_s$ satisfy the conditions of Lemma 3.2, then $a_1, \ldots, a_s$ is a d-sequence relative to I (that is, $((a_1, \ldots, a_i) : a_{i+1}) \cap I = (a_1, \ldots, a_i)$ for $0 \leqslant i \leqslant s-1$).*
(c) *Suppose that I is weakly $(s-1)$-residually S_2. If $\mathfrak{a} \subsetneq I$ is an ideal generated by s elements, then* $\operatorname{codim}(\mathfrak{a} : I) \leqslant s$.

Proof. (a) The ideal I is s-parsimonious by Propositions 3.3 (a) and 3.4 (a), and then I is s-thrifty by Propositions 3.3 (b) and 3.4 (c).
(b) Let $0 \leqslant i \leqslant s-1$ and write $\mathfrak{a}_i = (a_1, \ldots, a_i)$. Notice that $\operatorname{codim}((\mathfrak{a}_i : I), a_{i+1}) \geqslant i+1$. By part (a) of this corollary, I is $(s-1)$-parsimonious and $(s-1)$-thrifty. Therefore $(\mathfrak{a}_i : a_{i+1}) \cap I = (\mathfrak{a}_i : I) \cap I = \mathfrak{a}_i$.
(c) We may suppose that $\operatorname{codim}(\mathfrak{a} : I) \geqslant s$, and then use Proposition 3.4 (a). □

4. A Sufficient Condition For Residually S_2

We now turn to the main technical result of this paper: A sufficient condition for an ideal I of codimension g to be s-residually S_2. This condition is more general than the one of [26, 2.9] (which is a sufficient condition for the stronger Artin–Nagata property) as it only requires the vanishing of $s-g+1$ local cohomology modules.

The condition involves the vanishing of certain $\operatorname{Ext}^i_R(R/I^n, R)$. Occasionally this vanishing holds not for I^n but for an ideal equal to I^n up to a certain codimension, and this is sometimes enough. To formalize this possibility, we make a definition:

DEFINITION. Let R be a Noetherian ring, and let $I \subset R$ be an ideal of codimension g. We will say that ideals $I_1, \dots, I_r$ are *good approximations* of the first r powers of I if the following two conditions are satisfied:

(a) I_j coincides with II_{j-1} locally up to codimension $g+j-1$ whenever $1 \leqslant j \leqslant r$.
(b) I_j coincides with I^j locally up to codimension $g+j-1$ whenever $2 \leqslant j \leqslant r-1$.

For convenience we set $I_j = I^j = R$ if $j \leqslant 0$. Note that I_j need not contain I_{j+1} (but in practice they often do).

For example, one may choose I_j to be

(a) I^j; or
(b) $(I^j)^{\leqslant g+r-1}$ (here $J^{\leqslant i}$ denotes the intersection of all primary components of codimension at most i, for $J \subset R$ any ideal); or
(c) $(I^j)^{\leqslant \min\{g+j, g+r-1\}}$; or
(d) $(I^j)^{\leqslant g+j-1}$ in case I^j has no associated primes of codimension $g+j$ for $1 \leqslant j \leqslant r-1$; or
(e) $I^{(j)}$ in case I^j has no embedded associated primes of codimension at most $\min\{g+j, g+r-1\}$ for $1 \leqslant j \leqslant r$ (here $I^{(j)}$ denotes the j-th symbolic power).

THEOREM 4.1. *Let R be a Gorenstein ring, and let $I \subset R$ be an ideal of codimension g satisfying G_s for some $s \geqslant g$. Suppose that $I_1, \dots, I_{s-g+1}$ are good approximations of the first $s-g+1$ powers of I locally at every maximal ideal containing I. If $\mathrm{Ext}_R^{g+j}(R/I_j, R) = 0$ for $1 \leqslant j \leqslant s-g+1$, then I is s-residually S_2.*

COROLLARY 4.2. *Let R be a local Gorenstein ring, and let $I \subset R$ be an ideal of codimension g satisfying G_s for some $s \geqslant g$. Let $I_1, \dots, I_{s-g+1}$ be ideals such that I_j coincides with I^j locally up to codimension* $\min\{g+j, s\}$. *If* $\mathrm{Ext}_R^{g+j}(R/I_j, R) = 0$ *for $1 \leqslant j \leqslant s-g+1$, then I is s-residually S_2.*

For applications to projective varieties it is convenient to reformulate Theorem 4.1 in terms of local cohomology:

COROLLARY 4.3. *Let $(R, \mathfrak{m})$ be a local Gorenstein ring, and let $I \subset R$ be an ideal of codimension g satisfying G_s for some $s \geqslant g$. Set $d = \dim R/I$. Suppose that $I_1, \dots, I_{s-g+1}$ are good approximations of the first $s-g+1$ powers of I. The ideal I is s-residually S_2 if either*

(a) *$H_{\mathfrak{m}}^{d-j}(R/I_j) = 0$ for $1 \leqslant j \leqslant s-g+1$; or*
(b) *we have containments $I_1 \supset \cdots \supset I_{s-g+1}$ and $H_{\mathfrak{m}}^{d-i}(I_{j-1}/I_j) = 0$ for $1 \leqslant j \leqslant i \leqslant s-g+1$.*

The conditions of Corollary 4.3 are satisfied in particular if depth $R/I^j \geqslant d-j+1$ for $1 \leqslant j \leqslant s-g+1$, which in turn holds if I is strongly Cohen–Macaulay (assuming that I satisfies G_s) ([11, the proof of 5.1]).

COROLLARY 4.4. (cf. also [23,4.1]). *Let* $(R, \mathfrak{m})$ *be a local Gorenstein ring, let* $I \subset R$ *be an ideal of codimension g, and let* $I_1 = I^{\leqslant g}$. *If* $H_{\mathfrak{m}}^{d-1}(R/I_1) = 0$ *where* $d = \dim R/I$, *then I is g-residually* S_2.

COROLLARY 4.5. *Let* $(R, \mathfrak{m})$ *be a regular local ring which is essentially of finite type over a perfect field k, and let* $I \subset R$ *be an ideal of codimension g. Write* $I_1 = I^{\leqslant g+1}$, $A = R/I_1$, *and* $d = \dim A$. *Assume that A is reduced and a complete intersection locally in codimension 1 (in A). If* $H_{\mathfrak{m}}^{d-1}(A) = H_{\mathfrak{m}}^{d-2}(A) = 0$ *and* $H_{\mathfrak{m}}^{d-3}(\Omega_k(A)) = 0$, *then I is* $(g+1)$*-residually* S_2.

Proof. The standard exact sequence

$$0 \longrightarrow I_1/I_1^{(2)} \longrightarrow \Omega_k(R) \otimes_R A \cong \oplus A \longrightarrow \Omega_k(A) \longrightarrow 0$$

shows that we may apply Corollary 4.3 (b) taking $I_1, I_1^{(2)}$ as I_1, I_2. □

Here are the consequences for Hilbert functions:

COROLLARY 4.6. *Let R be a homogeneous Gorenstein ring, let* $I \subset R$ *be a homogeneous ideal of codimension g satisfying* G_{g+1}, *and let* $\mathfrak{a} \subset I$ *be a homogeneous* $(g+2)$*-generated ideal with* $\operatorname{codim}(\mathfrak{a} : I) \geqslant g+2$. *Let* $I_1 = I^{\leqslant g}$, *let* $\mathfrak{m}$ *be the irrelevant maximal ideal of R, and write* $d = \dim R/I$. *If* $H_{\mathfrak{m}}^{d-1}(R/I_1) = 0$, *then the Hilbert function of* $R/\mathfrak{a}$ *is determined by I and the degrees of the* $g+2$ *homogeneous generators of* $\mathfrak{a}$. *If moreover I satisfies* G_{g+2}, *the same is true for the Hilbert function of* $R/(\mathfrak{a} : I)$.

Proof. The assertion follows from Corollary 4.4, Proposition 3.1, and Theorem 2.1. □

COROLLARY 4.7. *Let R be a polynomial ring over a perfect field k, let* $I \subset R$ *be a homogeneous ideal of codimension g, and let* $\mathfrak{a} \subset I$ *be a homogeneous* $(g+3)$*-generated ideal with* $\operatorname{codim}(\mathfrak{a} : I) \geqslant g+3$. Let $I_1 = I^{\leqslant g+1}$, *let* $\mathfrak{m}$ *be the irrelevant maximal ideal of R, and write* $A = R/I_1$ *and* $d = \dim A$. *Assume that A is reduced and a complete intersection locally in codimension 1. If* $H_{\mathfrak{m}}^{d-1}(A) = H_{\mathfrak{m}}^{d-2}(A) = 0$ *and* $H_{\mathfrak{m}}^{d-3}(\Omega_k(A)) = 0$, *then the Hilbert function of* $R/\mathfrak{a}$ *is determined by I and the degrees of the* $g+3$ *homogeneous generators of* $\mathfrak{a}$. *If moreover I satisfies* G_{g+3}, *the same is true for the Hilbert function of* $R/(\mathfrak{a} : I)$.

Proof. One uses Corollary 4.5, Proposition 3.1, and Theorem 2.1. □

We next turn to some lemmas necessary for the proof of Theorem 4.1. The first is an easy consequence of the change–of–rings spectral sequence (or one can simply argue using injective resolutions):

LEMMA 4.8. *Let* $R \to S$ *be a homomorphism of rings, and let M be an R-module with* $\operatorname{Ext}_R^j(S, M) = 0$ *for* $j < g$. *Then for every integer n and every S-module N, there is a* *natural homomorphism*

$$\mathrm{Ext}^n_S(N, \mathrm{Ext}^g_R(S, M)) \longrightarrow \mathrm{Ext}^{n+g}_R(N, M),$$

which is an isomorphism if $n = 0$ *or if* $\mathrm{Ext}^j_R(S, M) = 0$ *for* $j \neq g$.

The following provides a crucial step in the proof of Theorem 4.1:

LEMMA 4.9. (cf. also [26, 2.1]). *Let R be a Noetherian local ring satisfying S_2, assume that R has a canonical module $\omega = \omega_R$, and write* $—^\vee = \mathrm{Hom}(—, \omega)$. *Let $I \subset R$ be an ideal, let a be an R-regular element contained in I, let $K = (a) : I \neq R$, and write $\bar{R} = R/K$. We have*

$$\mathrm{Ext}^1_R(\bar{R}, \omega) \cong (I\omega)^{\vee\vee}/a\omega.$$

Proof. There are natural isomorphisms

$$\begin{aligned}(a) : I &\cong \mathrm{Hom}(I, (a))\\ &\cong \mathrm{Hom}(I, R)a\\ &\cong \mathrm{Hom}(I, \mathrm{Hom}(\omega, \omega))a\\ &\cong \mathrm{Hom}(I \otimes_R \omega, \omega)a\\ &\cong \mathrm{Hom}(I\omega, \omega)a,\end{aligned}$$

which yield $K^\vee = a^{-1}(I\omega)^{\vee\vee}$.

Now applying $—^\vee$ to the exact sequence

$$0 \to K \to R \to \bar{R} \to 0$$

we get an exact sequence

$$0 \to \omega \to a^{-1}(I\omega)^{\vee\vee} \to \mathrm{Ext}^1_R(\bar{R}, \omega) \to 0.$$

Thus $\mathrm{Ext}^1_R(\bar{R}, \omega) \cong a^{-1}(I\omega)^{\vee\vee}/\omega$, and the desired result follows upon multiplication by a. □

Proof of Theorem 4.1. Let $K = \mathfrak{a} : I$ be any s-residual intersection of I. We may assume that R is local. Let K_i be ideals as in Lemma 3.2 and write $R_i = R/K_i$. The theorem is a consequence of the following assertions, which we shall prove by induction on i:

(a) R_i satisfies S_2 for $0 \leqslant i \leqslant s$;
(b) $\mathrm{Ext}^{i+1}_R(I_{i-g+2}R_i, R) = 0$ for $0 \leqslant i \leqslant s-1$;
(c) $\omega_{R_i} \cong (I_{i-g+1}R_i)^{\vee\vee}$ for $0 \leqslant i \leqslant s-1$, where $—^\vee = \mathrm{Hom}(—, \omega_{R_i})$; note that this notation implicitly uses the value of i.

We first show that if $s > 0$ (as will be the case in the proof of parts (b) and (c)) then $I_jR_0 \cong I_j$ for all $j \geqslant 1$. Equivalently, $I_j \cap (0 : I) = 0$ for all $j \geqslant 1$. Indeed, since R is unmixed it suffices to prove this after localizing at a prime $\mathfrak{p}$ of R such that dim $R_\mathfrak{p} = 0$. Since $s > 0$, the ideal I satisfies G_1, so $I_\mathfrak{p}$ is 0 or the unit ideal. By the definition of good approximations, we have $(I_j)_\mathfrak{p} \subset I_\mathfrak{p}$, and the formula follows.

To prove (a), (b), (c), first let $i = 0$. In this case our assertion is only nontrivial if $g = 0$. So let $g = 0$ and write $—^* = \mathrm{Hom}(—, R)$. As for (a), notice that $0 : I = 0 : I_1 \cong (R/I_1)^*$. Since $\mathrm{Ext}^1_R(R/I_1, R) = 0$, applying $—^*$ to a free R-resolution $\cdots \to F_2 \to F_1 \to F_0 = R$ of R/I_1, yields an exact sequence

$$0 \to 0 : I \to R \to F_1^* \to F_2^*,$$

which shows that $R_0 = R/(0 : I)$ satisfies S_2.

As for (b) and (c), notice that $s > 0$, so $I_2 R_0 \cong I_2$, which gives (b). Furthermore $I_1 R_0 \cong I_1$, thus $(I_1 R_0)^{\vee\vee} \cong (I_1 R_0)^{**} \cong I_1^{**}$. As I_1 is unmixed locally in codimension one, I_1 and $0 : (0 : I_1)$ coincide locally in codimension one. Hence $I_1^{**} \cong (0 : (0 : I_1))^{**}$. But $(0 : (0 : I_1))^{**} \cong 0 : (0 : I_1) = 0 : (0 : I)$, and the latter module is ω_{R_0}.

We now perform the induction step from $i \geqslant 0$ to $i + 1$.

- For (a) we may suppose $i \leqslant s - 1$. By the induction hypothesis, I is i-residually S_2. Propositions 3.4 (a) and 3.1 and Lemma 2.4 (b) imply that R_i satisfies S_2 and is equidimensional of codimension i in R, a_{i+1} is regular on R_i, $K_{i+1}R_i = a_{i+1}R_i : IR_i \cong \mathrm{Hom}(IR_i, R_i)$, and $\mathrm{depth}_I(\omega_{R_i}) > 0$. Since II_{i-g+1} and I_{i-g+2} coincide locally in codimension $i + 1$, part (c) for i shows that $(I\omega_{R_i})^\vee \cong (I_{i-g+2}R_i)^\vee$. Putting this together, one obtains natural isomorphisms,

$$\begin{aligned} K_{i+1}R_i &\cong \mathrm{Hom}(IR_i, R_i) \\ &\cong \mathrm{Hom}(IR_i \otimes_{R_i} \omega_{R_i}, \omega_{R_i}) && \text{(as } R_i \text{ is } S_2) \\ &\cong \mathrm{Hom}(I\omega_{R_i}, \omega_{R_i}) && \text{(as } \mathrm{depth}_I(\omega_{R_i}) > 0) \\ &\cong \mathrm{Hom}(I_{i-g+2}R_i, \omega_{R_i}) && \text{(by the remark above)} \\ &\cong \mathrm{Ext}^i_R(I_{i-g+2}R_i, R) && \text{(by Lemma 4.8).} \end{aligned}$$

  By part (b) for i, $\mathrm{Ext}^{i+1}_R(I_{i-g+2}R_i, R) = 0$. On the other hand since R_i has codimension i, $\mathrm{Ext}^\ell_R(I_{i-g+2}R_i, R) = 0$ whenever $\ell \leqslant i - 1$. Thus, dualizing a free R-resolution of $I_{i-g+2}R_i$ into R, one sees that the R_i-module $\mathrm{Ext}^i_R(I_{i-g+2}R_i, R)$ satisfies S_3. Therefore $K_{i+1}R_i$ is S_3, and hence $R_{i+1} \cong R_i/K_{i+1}R_i$ satisfies S_2. This concludes the proof of (a) for $i + 1$.

- For (b) and (c) we may suppose $i + 1 \leqslant s - 1$. Recall that by Corollary 3.6 (b) and Proposition 3.1, $a_1, \ldots, a_{i+1}$ form a d-sequence and a_k is regular on R_{k-1} whenever $1 \leqslant k \leqslant i + 1$. Thus for $1 \leqslant k \leqslant i + 1$ and $j \leqslant s - g + 1$ there are complexes

$$\mathcal{C}_{kj}: \quad 0 \to I_{j-1}R_{k-1} \xrightarrow{a_k} I_j R_{k-1} + a_k I_{j-1} R_{k-1} \longrightarrow I_j R_k \longrightarrow 0$$

having nontrivial homology at most in the middle. Call this homology H and notice that

$$H = (K_k \cap (I_j + a_k I_{j-1}))/((K_{k-1} \cap (I_j + a_k I_{j-1})) + a_k I_{j-1}).$$

We first claim that

$$\operatorname{Ext}_R^{\ell}(H, R) = 0 \quad \text{whenever } \ell \leqslant \min\{g + j - 1, i + 1\} \tag{4.10}$$

or equivalently

$$H_{\mathfrak{p}} = 0 \quad \text{whenever } \dim R_{\mathfrak{p}} \leqslant \min\{g + j - 1, i + 1\}.$$

To see this let $\mathfrak{p} \in \operatorname{Spec}(R)$ with $\dim R_{\mathfrak{p}} \leqslant \min\{g + j - 1, i + 1\}$. Notice that $(II_{j-1})_{\mathfrak{p}} = (I_j)_{\mathfrak{p}} = (I^j)_{\mathfrak{p}}$. If $I \not\subset \mathfrak{p}$, then $I_{j-1} \not\subset \mathfrak{p}$, $I_j \not\subset \mathfrak{p}$, and $(K_k)_{\mathfrak{p}} = (a_1, \ldots, a_k)_{\mathfrak{p}}$. Hence $H_{\mathfrak{p}} = 0$ in this case. Next assume that $I \subset \mathfrak{p}$. Then $j \geqslant 1$ since $g \leqslant \dim R_{\mathfrak{p}} \leqslant g + j - 1$, and $I_{\mathfrak{p}} = (a_1, \ldots, a_{i+1})_{\mathfrak{p}}$ since $\dim R_{\mathfrak{p}} \leqslant i + 1 \leqslant s - 1$. Now $(K_k \cap (I_j + a_k I_{j-1}))_{\mathfrak{p}} = (K_k \cap I^j)_{\mathfrak{p}}$. Since $j \geqslant 1$ and $a_1, \ldots, a_k, \ldots, a_{i+1}$ form a d-sequence generating $I_{\mathfrak{p}}$, we conclude that

$$\begin{aligned}
(K_k \cap I^j)_{\mathfrak{p}} &= ((a_1, \ldots, a_k) \cap I^j)_{\mathfrak{p}} \\
&= ((a_1, \ldots, a_k) I^{j-1})_{\mathfrak{p}} \qquad\qquad \text{(cf. [15, Theorem 2.1])} \\
&\subset ((K_{k-1} \cap I^j) + a_k I^{j-1})_{\mathfrak{p}}. \\
&= ((K_{k-1} \cap I_j) + a_k I^{j-1})_{\mathfrak{p}}.
\end{aligned}$$

The vanishing of $H_{\mathfrak{p}}$ will follow once we have shown that $(I^{j-1})_{\mathfrak{p}} \subset (I_{j-1})_{\mathfrak{p}}$, for which we may assume $j \geqslant 2$. By assumption I^{j-1} and I_{j-1} coincide locally in codimension $g + j - 2$. Since $\operatorname{Ext}_R^{g+j-1}(R/I_{j-1}, R) = 0$, the ideal I_{j-1} has no associated primes of codimension $g + j - 1$. Therefore $(I^{j-1})_{\mathfrak{p}} \subset (I_{j-1})_{\mathfrak{p}}$. This concludes the proof of (4.10).

- We now turn to the proof of (b) for $i + 1 \leqslant s - 1$. We show, by induction on k, $0 \leqslant k \leqslant i + 1$, that $\operatorname{Ext}_R^{g+j-1}(I_j R_k, R) = 0$ whenever $k - g + 2 \leqslant j \leqslant i - g + 3$. Statement (b) follows if we set $k = i + 1$.
First suppose $k = 0$.
– If $j \leqslant 0$, then $g \geqslant 2$, hence $R_0 = R$ and $I_j R_0 = R R_0 = R$.
– If $j \geqslant 1$, then $I_j R_0 \cong I_j$ by the remark at the beginning of the proof. Now the assertion follows because $\operatorname{Ext}_R^{g+j-1}(I_j, R) = 0$ for $-g + 2 \leqslant j \leqslant s - g + 1$.
Next suppose $1 \leqslant k \leqslant i + 1$. Assuming $k - g + 2 \leqslant j \leqslant i - g + 3$, as in the desired formula, we have
– $\operatorname{Ext}_R^{g+j-2}(H, R) = 0$ by (4.10),
– $\operatorname{Ext}_R^{g+j-2}(I_{j-1} R_{k-1}, R) = \operatorname{Ext}_R^{g+j-1}(I_j R_{k-1}, R) = 0$ by induction hypothesis,
– $\operatorname{Ext}_R^{g+j-1}(I_j R_{k-1} + a_k I_{j-1} R_{k-1}, R)$ embeds into $\operatorname{Ext}_R^{g+j-1}(I_j R_{k-1}, R)$, since $a_k \in I$ and locally in codimension $g + j - 1$, I_j and II_{j-1} coincide.
Now using the complex $\mathcal{C}_{kj}$ we derive $\operatorname{Ext}_R^{g+j-1}(I_j R_k, R) = 0$, proving (b).

- Finally, we prove (c) for $i+1 \leqslant s-1$. By part (a) and Proposition 3.4 (a), we know that R_i and R_{i+1} satisfy S_2 and are equidimensional of codimension i and $i+1$ in R. Also, by Proposition 3.1 and Lemma 2.4 (b), $K_{i+1}R_i = a_{i+1}R_i : IR_i$ and a_{i+1} is regular on R_i. Write $—^* = \mathrm{Hom}(—, \omega_{R_i})$ and $—^\vee = \mathrm{Hom}(—, \omega_{R_{i+1}})$.

Consider the complex

$$\mathcal{C}_{i+1,i-g+2}: \quad 0 \longrightarrow I_{i-g+1}R_i \xrightarrow{a_{i+1}} I_{i-g+2}R_i + a_{i+1}I_{i-g+1}R_i \longrightarrow I_{i-g+2}R_{i+1} \longrightarrow 0,$$

introduced above, which has nontrivial homology H at most in the middle. By (4.10), $\mathrm{Ext}^\ell_R(H, R)=0$ for $\ell \leqslant i+1$, and by part (b), $\mathrm{Ext}^{i+1}_R(I_{i-g+2}R_i, R) = 0$. Notice that, since $a_{i+1} \in I$ and I_{i-g+2} coincides with II_{i-g+1} locally in codimension $i+1$, the map

$$\mathrm{Ext}^\ell_R(I_{i-g+2}R_i + a_{i+1}I_{i-g+1}R_i, R) \longrightarrow \mathrm{Ext}^\ell_R(I_{i-g+2}R_i, R)$$

is a monomorphism for $\ell \leqslant i+1$ and is an isomorphism for $\ell \leqslant i$. Further, $\mathrm{Ext}^i_R(I_{i-g+2}R_{i+1}, R) = 0$. Thus the above complex induces an exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Ext}^i_R(I_{i-g+2}R_i, R) & \longrightarrow & \mathrm{Ext}^i_R(I_{i-g+1}R_i, R) & \longrightarrow & \mathrm{Ext}^{i+1}_R(I_{i-g+2}R_{i+1}, R) & \longrightarrow & 0, \\ & & \| \wr & & \| \wr & & \| \wr & & \\ & & (I_{i-g+2}R_i)^* & & (I_{i-g+1}R_i)^* & & (I_{i-g+2}R_{i+1})^\vee & & \end{array}$$

where the various identifications are special cases of Lemma 4.8. Dualizing again, we obtain an exact sequence

$$0 \longrightarrow (I_{i-g+1}R_i)^{**} \xrightarrow{a_{i+1}} (I_{i-g+2}R_i)^{**} \longrightarrow (I_{i-g+2}R_{i+1})^{\vee\vee} \longrightarrow \mathrm{Ext}^{i+1}_R((I_{i-g+1}R_i)^*, R).$$

This shows that

$$(I_{i-g+2}R_i)^{**}/a_{i+1}(I_{i-g+1}R_i)^{**}$$

is isomorphic to an R_{i+1}-submodule of $(I_{i-g+2}R_{i+1})^{\vee\vee}$, and that both modules coincide locally in codimension one in R_{i+1}. Thus, since R_{i+1} is S_2, $(I_{i-g+2}R_{i+1})^{\vee\vee} \cong ((I_{i-g+2}R_i)^{**}/a_{i+1}(I_{i-g+1}R_i)^{**})^{\vee\vee}$, and we must show that the latter module is isomorphic to $\omega_{R_{i+1}}$.

Now I_{i-g+2} and II_{i-g+1} coincide locally in codimension $i+1$ and R_i satisfies S_2, hence by our induction hypothesis, $(I_{i-g+2}R_i)^{**}/a_{i+1}(I_{i-g+1}R_i)^{**} \cong (I\omega_{R_i})^{**}/a_{i+1}\omega_{R_i}$. But Lemma 4.9 shows the latter module is isomorphic to $\mathrm{Ext}^1_{R_i}(R_{i+1}, \omega_{R_i})$. Thus it remains to prove that $\mathrm{Ext}^1_{R_i}(R_{i+1}, \omega_{R_i})^{\vee\vee} \cong \omega_{R_{i+1}}$. Now Lemma 4.8 yields a natural map $\mathrm{Ext}^1_{R_i}(R_{i+1}, \omega_{R_i}) \to \omega_{R_{i+1}}$, which is an isomorphism locally in codimension one in R_{i+1} because R_i satisfies S_2. Thus, since R_{i+1} is S_2, we get $\mathrm{Ext}^1_{R_i}(R_{i+1}, \omega_{R_i})^{\vee\vee} \cong (\omega_{R_{i+1}})^{\vee\vee} \cong \omega_{R_{i+1}}$. □

5. A Class of Residually S_2 Ideals

In this section we illustrate how general projections can produce an abundance of ideals that are s-residually S_2, but in general fail to satisfy the stronger conditions AN_s or AN_s^-.

THEOREM 5.1. *Let k be an algebraically closed field, and let $Y \subset \mathbb{P}_k^{n+t}$ be a reduced complete intersection of dimension d. Consider a general projection $X \subset \mathbb{P}_k^n$ of Y, and let $I \subset k[X_0, \ldots, X_n]$ be the (saturated) ideal defining X. Let d_e be the dimension of the closed subset $\{y \in Y \mid \operatorname{edim} \mathcal{O}_{Y,y} \geqslant e\}$, and write $s = 2n - 3 - \max_e\{n-1, 2d, d_e + e - 1\}$ (using the convention that $\dim \varnothing = -\infty$). Then I is s-residually S_2. In particular, if Y is nonsingular (or, more generally, if $\operatorname{edim} B_{\mathfrak{q}} \leqslant 2\dim B_{\mathfrak{q}} + 1$ for every nonmaximal homogeneous prime $\mathfrak{q}$ of the homogeneous coordinate ring B of Y), then I is $\min\{n-2, 2n-2d-3\}$-residually S_2.*

Proof. Let A, B be the homogeneous coordinate rings of X, Y respectively. The geometric condition implies that the conductor of $A \subset B$ has codimension at least $\min_e\{d+1, n-d, n+d-d_e-e+1\}$. The theorem thus follows from the more general result in Theorem 5.3. □

Remark 5.2. If in the setting of Theorem 5.1, $Y \subset \mathbb{P}_k^{n+t}$ is nondegenerate and nonsingular with $t \geqslant 1, d \geqslant 1$ and $n-d \geqslant 2$, then I does not satisfy AN_{n-d}^-.

Proof. Let A, B be the homogeneous coordinate rings of X, Y. Since the conductor of the extension $A \subset B$ has codimension $\geqslant \min\{d+1, n-d\} \geqslant 2$ and $A \neq B$, it follows that $A = k[X_0, \ldots, X_n]/I$ cannot be Cohen–Macaulay. As I is an unmixed radical ideal of codimension $n-d$, there has to exist a geometric link of I that is not Cohen–Macaulay. □

To state our more general result we replace the complete intersection above by an ideal J whose conormal module has symmetric powers with sufficiently high depths. This condition is automatically satisfied if J is an unmixed locally strongly Cohen–Macaulay ideal satisfying certain conditions on the local numbers of generators ([11, the proof of 5.1]).

THEOREM 5.3. *Let k be a perfect field, let $R \subset S$ be regular domains that are k-algebras essentially of finite type, and set $t = \operatorname{trdeg}_R S$. Let J be an ideal of codimension g in S satisfying G_{s+t+1}. Set $I = J \cap R$, consider $A = R/I \subset B = S/J$, and let $\mathfrak{C} = A :_A B$ denote the conductor. Assume that B is reduced and a complete intersection locally in codimension 1 (in B) and that $\operatorname{codim}_A \mathfrak{C} \geqslant s + t - g + 3$. If $\operatorname{projdim}_S \operatorname{Sym}_j^B(J/J^2) \leqslant g + j$ for $0 \leqslant j \leqslant s + t - g$, then I is s-residually S_2.*

Proof. We induct on s, the assertion being trivial for $s \leqslant -1$.

Notice that I is the intersection of the contractions of all minimal primes of J. Now, replacing R, S, A, B by affine domains and computing dimensions, one easily sees that codim $I \geqslant g-t$. Thus nothing is to be shown if $s < g-t$, and we may from now on assume that $s \geqslant g-t$.

But then by our assumption, $\operatorname{codim}_A \mathfrak{C} > 0$ and hence $\mathfrak{C}$ contains a non zerodivisor of A, which implies that B is a finite A-module. Since $s \geqslant g-t$, B is a perfect S-module of grade g by our assumption, and therefore every minimal prime of J has the same codimension g. Localizing R we may suppose that $(R, \mathfrak{m})$ is a local ring. Moreover, if $\mathfrak{p} \in V(I)$ then there exists $\mathfrak{q} \in V(J)$ with $\mathfrak{q} \cap R = \mathfrak{p}$, and for every such $\mathfrak{q}$ the dimension formula yields $\dim S_{\mathfrak{q}} = \dim R_{\mathfrak{p}} + t$ since the residue field extension of $R_{\mathfrak{p}} \to S_{\mathfrak{q}}$ is algebraic. Applying this equality to any minimal prime $\mathfrak{p}$ of I one sees that $\dim R_{\mathfrak{p}} = \dim S_{\mathfrak{q}} - t = (\dim S_{\mathfrak{q}} - \dim B_{\mathfrak{q}}) - t = g - t$. Thus every minimal prime of I has the same codimension $g-t$. On the other hand, if we choose $\mathfrak{q}$ to be the preimage of any maximal ideal $\mathfrak{n}$ of B, then $\dim B_{\mathfrak{n}} = \dim S_{\mathfrak{q}} - \dim J_{\mathfrak{q}} = (\dim R + t) - g = \dim R - \operatorname{codim} I = \dim A$, or equivalently, the codimension of every maximal ideal of B is $\dim A$. Finally, the preimage of $\mathfrak{C}$ in R has codimension at least $(s+t-g+3) + \operatorname{codim} I = s+3$, showing that the R-module B/A vanishes locally in codimension $\leqslant s+2$.

Let M and N be finitely generated R-modules. We write $M \underset{r}{\cong} N$ if there is an R-linear map between M and N that is an isomorphism locally in codimension $\leqslant r$ in R. Notice that if $M \underset{r}{\cong} N$ then $H^i_{\mathfrak{m}}(M) \cong H^i_{\mathfrak{m}}(N)$ as long as $i \geqslant \dim R - r + 1$.

After these preparatory remarks we are now going to prove that I satisfies G_{s+1}. To this end let $\mathfrak{p} \in V(I)$ with $\dim R_{\mathfrak{p}} \leqslant s$. Choose $\mathfrak{q} \in V(J)$ with $\mathfrak{q} \cap R = \mathfrak{p}$. Since $\dim S_{\mathfrak{q}} = \dim R_{\mathfrak{p}} + t \leqslant s + t$, we have $\mu(J_{\mathfrak{q}}) \leqslant \dim S_{\mathfrak{q}}$, or equivalently, the deviation $d(B_{\mathfrak{q}})$ of $B_{\mathfrak{q}}$ is at most $\dim B_{\mathfrak{q}}$. Now the equality $A_{\mathfrak{p}} = B_{\mathfrak{q}}$ yields $\mu(I_{\mathfrak{p}}) = \operatorname{codim} I_{\mathfrak{p}} + d(A_{\mathfrak{p}}) = (g-t) + d(B_{\mathfrak{q}}) \leqslant (g-t) + \dim B_{\mathfrak{q}} = (\operatorname{codim} J_{\mathfrak{q}} + \dim B_{\mathfrak{q}}) - t = \dim S_{\mathfrak{q}} - t = \dim R_{\mathfrak{p}}$. Thus I satisfies G_{s+1}.

Write $d = \dim A$. Having established the property G_s, we know from Corollary 4.3 (b) that I is s-residually S_2 once we show

$$H^{d-i}_{\mathfrak{m}}(I^{j-1}/I^j) = 0 \quad \text{for} \quad 1 \leqslant j \leqslant i \leqslant s+t-g+1. \tag{5.4}$$

So let $1 \leqslant j \leqslant s+t-g+1$ and set $M = \operatorname{Sym}^B_{j-1}(J/J^2)$. For every maximal ideal $\mathfrak{n}$ of B, our assumption on J implies $\operatorname{depth}_{B_{\mathfrak{n}}} M_{\mathfrak{n}} \geqslant \dim B_{\mathfrak{n}} - j + 1 = d - j + 1$. Thus the $\operatorname{Rad}(B)$-depth of the B-module M is at least $d-j+1$, which yields $H^{d-i}_{\operatorname{Rad}(B)}(M) = 0$ as long as $i \geqslant j$. This shows that

$$H^{d-i}_{\mathfrak{m}}(\operatorname{Sym}^B_{j-1}(J/J^2)) = 0 \quad \text{for} \quad 1 \leqslant j \leqslant i \leqslant s+t-g+1. \tag{5.5}$$

Since $A \underset{s+2}{\cong} B$ and $d-1 = \dim R - g + t - 1 \geqslant \dim R - (s+2) + 1$, we have $H^{d-1}_{\mathfrak{m}}(A) \cong H^{d-1}_{\mathfrak{m}}(B)$, and hence (5.4) follows from (5.5) if $s = g-t$. Thus we may from now on assume that $s \geqslant g-t+1$.

But then by our assumption, $\operatorname{projdim}_S J/J^2 \leqslant g+1$. Since J is a complete intersection locally in codimension $g+1$ in S, we conclude that the B-module

J/J^2 is torsionfree. The Zariski Sequence associated to the homomorphisms $k \to S \to B$ yields an exact sequence of B-modules

$$T_1(S/k, B) = 0 \longrightarrow T_1(B/k, B) \longrightarrow T_1(B/S, B) = J/J^2$$

$$\longrightarrow T_0(S/k, B) = \Omega_k(S) \otimes_S B \longrightarrow T_0(B/k, B) = \Omega_k(B) \longrightarrow 0,$$

where $T_0(S/k, B)$ is projective and hence free ([21, 2.3.5 and 3.1.5]). Since B is reduced, the B-module $T_1(B/k, B)$ is torsion ([21, 2.3.4 and 3.1.5]), and hence $T_1(B/k, B) = 0$ by the torsionfreeness of J/J^2. In particular J/J^2 is a first syzygy module in a free resolution of $T_0(B/k, B)$.

Likewise, the morphisms $k \to R \to A$ give rise to two short exact sequences of B-modules

$$0 \longrightarrow T_1(A/k, B) \longrightarrow T_1(A/R, B) = I/I^2 \otimes_A B \longrightarrow U \longrightarrow 0$$
$$0 \longrightarrow U \longrightarrow T_0(R/k, B) \longrightarrow T_0(A/k, B) \longrightarrow 0,$$

where $T_0(R/k, B)$ is free. Notice that U is a first syzygy module in a free resolution of $T_0(A/k, B)$.

Finally, from the homomorphisms $k \to A \to B$ we obtain the Zariski Sequence

$$T_{i+1}(B/A, B) \longrightarrow T_i(A/k, B) \longrightarrow T_i(B/k, B) \longrightarrow T_i(B/A, B).$$

If $\mathfrak{p} \in \operatorname{Spec}(R)$ with $\dim R_{\mathfrak{p}} \leqslant s + 2$, then $T_i(B/A, B)_{\mathfrak{p}} \cong T_i(B_{\mathfrak{p}}/A_{\mathfrak{p}}, B_{\mathfrak{p}}) = T_i(A_{\mathfrak{p}}/A_{\mathfrak{p}}, B_{\mathfrak{p}}) = 0$ for every i ([21, 2.3.3 and 3.1.1]). Thus $T_i(B/A, B) \underset{s+2}{\cong} 0$, which implies $T_i(A/k, B) \underset{s+2}{\cong} T_i(B/k, B)$.

We first make use of the identification $T_0(A/k, B) \underset{s+2}{\cong} T_0(B/k, B)$. From it we obtain two short exact sequences

$$0 \longrightarrow K \longrightarrow T_0(A/k, B) \longrightarrow V \longrightarrow 0$$
$$0 \longrightarrow V \longrightarrow T_0(B/k, B) \longrightarrow L \longrightarrow 0,$$

where $K \underset{s+2}{\cong} 0 \underset{s+2}{\cong} L$. Comparing first syzygy modules in free B-resolutions, we conclude that U and $\operatorname{syz}_1(K) \oplus \operatorname{syz}_1(V)$ are stably isomorphic, and likewise for J/J^2 and $\operatorname{syz}_1(V) \oplus \operatorname{syz}_1(L)$. Thus $U \oplus \operatorname{syz}_1(L)$ and $J/J^2 \oplus \operatorname{syz}_1(K)$ are stably isomorphic, which allows us to assume that U is a direct summand of $J/J^2 \oplus \operatorname{syz}_1(K)$. But $\operatorname{syz}_1(K) \underset{s+2}{\cong} F$ for some free B-module F, and hence

$$\operatorname{Sym}^B_{j-1}(U) \overset{\oplus}{\hookrightarrow} \operatorname{Sym}^B_{j-1}(J/J^2 \oplus \operatorname{syz}_1(K)) \underset{s+2}{\cong} \operatorname{Sym}^B_{j-1}(J/J^2 \oplus F)$$
$$\cong \oplus_{v=1}^{j} \operatorname{Sym}^B_{v-1}(J/J^2) \otimes_B \operatorname{Sym}^B_{j-v}(F).$$

Now (5.5) implies

$$H^{d-i}_{\mathfrak{m}}(\operatorname{Sym}^B_{j-1}(U)) = 0 \quad \text{for} \quad 1 \leqslant j \leqslant i \leqslant s + t - g + 1. \tag{5.6}$$

Next we make use of the fact that $T_1(A/k, B) \underset{s+2}{\cong} T_1(B/k, B) = 0$. From it we conclude $I/I^2 \otimes_A B \underset{s+2}{\cong} U$, and thus

$$\mathrm{Sym}^A_{j-1}(I/I^2) \underset{s+2}{\cong} \mathrm{Sym}^B_{j-1}(I/I^2 \otimes_A B) \underset{s+2}{\cong} \mathrm{Sym}^B_{j-1}(U).$$

Hence by (5.6),

$$H^{d-i}_{\mathfrak{m}}(\mathrm{Sym}^A_{j-1}(I/I^2)) = 0 \quad \text{for} \quad 1 \leqslant j \leqslant i \leqslant s + t - g + 1. \tag{5.7}$$

We saw that I satisfies G_{s+1}, and by our induction hypothesis, I is $(s-1)$-residually S_2. Now Corollary 3.6 (b) shows that locally in codimension s, I can be generated by a d-sequence. But then in the natural exact sequence

$$0 \longrightarrow N \longrightarrow \mathrm{Sym}^A_{j-1}(I/I^2) \longrightarrow I^{j-1}/I^j \longrightarrow 0,$$

we have $N \underset{s}{\cong} 0$ ([14, Theorem 3.1]). Now (5.7) implies (5.4). □

6. Examples

First we wish to illustrate how the formulas of Theorem 2.3 can be used for explicit computations of Hilbert functions.

EXAMPLE 6.1. Let k be a field, let $R = k[X_1, \dots, X_6]$, and let $I \subset R$ be the defining ideal of the Veronese surface in $\mathbb{P}^5_k$, that is, the ideal generated by the 2 by 2 minors of the generic symmetric matrix

$$\begin{pmatrix} X_1 & X_2 & X_3 \\ X_2 & X_4 & X_5 \\ X_3 & X_5 & X_6 \end{pmatrix}.$$

We wish to compute $[\![R/(\mathfrak{a} : I)]\!]$, where $\mathfrak{a} \subset I$ is an arbitrary ideal generated by 5 forms of degree $d = e + 2$ so that $\mathrm{ht}(\mathfrak{a} : I) \geqslant 5$. For this we may assume that k is infinite. Let $c_1, \dots, c_6$ be quadrics contained in I that satisfy the conditions of Lemma 2.5 (b'), and write $\mathfrak{c}_i = (c_1, \dots, c_i)$ for $0 \leqslant i \leqslant 5$. Recall that I is (weakly) 6-residually S_2 by [12, 2.3 and 3.3]. Thus Proposition 3.1 and Theorem 2.1 (b) imply that, for every $0 \leqslant i \leqslant 5$ and every $\xi \subset [5]$ with $|\xi| = i$, $[\![R/(\mathfrak{c}_\xi : I)]\!] = [\![R/(\mathfrak{c}_i : I)]\!]$. Furthermore, again by Proposition 3.1, $\mathfrak{c}_i : I = \mathfrak{c}_i : c_{i+1}$, which yields $[\![R/(\mathfrak{c}_i : I)]\!] = [\![R/(\mathfrak{c}_i : c_{i+1})]\!] = [\![\mathfrak{c}_{i+1}/\mathfrak{c}_i]\!]t^{-2}$. Finally, by Proposition 3.4 (a) (or [12, 3.3]), $\mathrm{ht}(\mathfrak{c}_i : I) = i$. Now for $0 \leqslant i \leqslant 2$, $\mathfrak{c}_i : I = \mathfrak{c}_i$ is a complete intersection, whereas by linkage theory, $[\![R/(\mathfrak{c}_3 : I)]\!] = \frac{1+3t}{(1-t)^3}$. Since I has a linear presentation matrix, $\mathfrak{c}_5 : I$ is a complete intersection of 5 linear forms, hence $[\![R/(\mathfrak{c}_5 : I)]\!] = \frac{1}{(1-t)}$. Finally, $[\![R/(\mathfrak{c}_4 : I)]\!] = [\![\mathfrak{c}_5/\mathfrak{c}_4]\!]t^{-2} = \big([\![R/\mathfrak{c}_3]\!] - [\![\mathfrak{c}_4/\mathfrak{c}_3]\!] - [\![I/\mathfrak{c}_5]\!] - [\![R/I]\!]\big)t^{-2} = [\![R/\mathfrak{c}_3]\!]t^{-2} - [\![R/(\mathfrak{c}_3 : I)]\!] - [\![R/(\mathfrak{c}_5 : I)]\!] - [\![R/I]\!]t^{-2} = \frac{1+t}{(1-t)^2}$.

Now using this information in the formula of Theorem 2.3 (b) yields,

$$
\begin{aligned}
[[R/(\mathfrak{a}:I)]] &= \sum_{k=0}^{5}\binom{5}{k} t^{ke}(1-t^e)^{5-k}[[R/(\mathfrak{c}_k:I)]] \\
&= \frac{t^{5e}}{(1-t)} + 5\frac{t^{4e}(1-t^e)(1+t)}{(1-t)^2} + 10\frac{t^{3e}(1-t^e)^2(1+3t)}{(1-t)^3} \\
&\quad + 10\frac{t^{2e}(1-t^e)^3(1+t)^2}{(1-t)^4} + 5\frac{t^{e}(1-t^e)^4(1+t)}{(1-t)^5} + \frac{(1-t^e)^5}{(1-t)^6} \\
&= \frac{1+5t+15t^2+\cdots+6t^{5(d-2)}}{(1-t)} \qquad (\text{for } d \geqslant 3).
\end{aligned}
$$

In particular one can see that the regularity of $R/(\mathfrak{a}:I)$ is $5(d-2)$ and the initial degree of $\mathfrak{a}:I$ is d for $d \geqslant 3$. Also the degree of $R/(\mathfrak{a}:I)$ is $1+10e+40e^2+40e^3+10e^4+e^5 = d^5-40d^2+90d-51$.

Using the corresponding formula for $[[R/\mathfrak{a}]]$ one can also check that the minimal degree of an element in $\mathfrak{a}:I$ that is not in $\mathfrak{a}$ is $3d-4$ for $d \geqslant 3$ (and 1 for $d=2$).

Our main theorems say that under various hypotheses the Hilbert functions of $R/\mathfrak{a}$ or $R/(\mathfrak{a}:I)$ are determined by I and the degrees of the generators of $\mathfrak{a}$. In a few cases (Theorem 1.1) we have seen that a knowledge of the Hilbert function of I alone suffices. Here is an example that shows this is not true in general:

EXAMPLE 6.2. Let k be a field, let $R = k[X_1,\dots,X_6]$, and again let $I \subset R$ be the defining ideal of the Veronese surface in $\mathbb{P}^5_k$, that is, the ideal generated by the 2 by 2 minors of the generic symmetric matrix. Let I' be the defining ideal of the generic rational normal scroll of degree 4 in $\mathbb{P}^5_k$, that is, the ideal of 2 by 2 minors of the matrix

$$
\begin{pmatrix} X_1 & X_2 & X_4 & X_5 \\ X_2 & X_3 & X_5 & X_6 \end{pmatrix}.
$$

Notice that R/I and R/I' have the same Hilbert function. Let $\mathfrak{a} \subset I$ and $\mathfrak{a}' \subset I'$ be ideals generated by 4 quadrics so that $\operatorname{codim}(\mathfrak{a}:I) \geqslant 4$ and $\operatorname{codim}(\mathfrak{a}':I') \geqslant 4$.

By Corollary 2.2 the Hilbert function of $R/(\mathfrak{a}':I')$ is determined by I'. Thus we may perform a direct computation on a particular choice of $\mathfrak{a}'$ to obtain

$$
[[R/(\mathfrak{a}':I')]] = \frac{1+2t-t^2}{(1-t)^2}.
$$

On the other hand, by Example 6.1 we have $[[R/(\mathfrak{a}:I)]] = (1+t)/\big((1-t)^2\big)$.

Next we illustrate the fact that our results about Hilbert functions do not hold in general without parsimony condition or some residual S_2 assumption.

Remark 6.3. Let R be a homogeneous ring, and let $\mathfrak{a} \subset I$ be homogeneous ideals of R. Assume that $\mu(\mathfrak{a}) < s \leqslant \operatorname{codim}(\mathfrak{a} : I)$ and that $0 \neq [\mathfrak{a}]_d \neq [I]_d$ for some d. If $0 \neq a_s \in [\mathfrak{a}]_d$ and $a'_s \in [I]_d \setminus [\mathfrak{a}]_d$, then $\mathfrak{a} = (\mathfrak{a}, a_s)$ and $\mathfrak{a}' = (\mathfrak{a}, a'_s)$ are both generated by s forms in I of the same degrees d_i, and $\operatorname{codim}(\mathfrak{a} : I) \geqslant s, \operatorname{codim}(\mathfrak{a}' : I) \geqslant s$. Nevertheless, $R/\mathfrak{a}$ and $R/\mathfrak{a}'$ do not have the same Hilbert function since $\mathfrak{a} \subsetneq \mathfrak{a}'$.

EXAMPLE 6.4. Let k be an algebraically closed field, let $I \subset k[X_0, \ldots, X_3]$ be the defining ideal of a monomial arithmetically Buchsbaum curve that is not arithmetically Cohen–Macaulay (for instance the rational quartic) in $\mathbb{P}^3_k$. There exists a homogeneous ideal $\mathfrak{a} \subset I$ with $0 \neq [\mathfrak{a}]_d \neq [I]_d$ for some d so that $\mu(\mathfrak{a}) = 3$ and $\mathfrak{a} : I = (X_0, \ldots, X_3)$, as can be easily seen from [5, Theorem 3]. Applying Remark 6.3 with $s = 4 = \operatorname{codim} I + 2$, one sees that Corollary 4.6 can fail if the depth condition on R/I is dropped.

Let R be a homogeneous ring over an infinite field k and let $I \subset R$ be a homogeneous ideal. A homogeneous ideal $\mathfrak{a}$ contained in I is called a *homogeneous reduction* of I if $I^{r+1} = \mathfrak{a} I^r$ for some $r \geqslant 0$. For any homogeneous reduction, $\mu(\mathfrak{a}) \geqslant \ell(I)$, where $\ell(I) = \dim gr_I(R) \otimes_R k$ denotes the analytic spread of I, and if the equality $\mu(\mathfrak{a}) = \ell(I)$ holds we call $\mathfrak{a}$ a *homogeneous minimal reduction* of I. Not every homogeneous ideal I has a homogeneous minimal reduction (consider $I = (X^2, XY, Y^3) \subset k[X, Y]$), but such reductions always exist if I is generated by forms of the same degree. Now in the situation of Remark 6.3, the inequality $\mu(\mathfrak{a}) < s \leqslant \operatorname{codim}(\mathfrak{a} : I)$ implies that $\mathfrak{a}$ is necessarily a reduction of I and hence $\ell(I) < s$ (at least if R is equidimensional) ([25, Proposition 3], which is based on [22, 4.1]). But in fact, also the converse holds, which yields many instances where Remark 6.3 applies:

Remark 6.5. Let R be a homogeneous reduced Cohen–Macaulay ring over an infinite field, let $I \subset R$ be a homogeneous ideal with $\mu(I) \neq \ell(I)$, and let s be an integer with $s > \ell(I)$. If I is G_s and weakly $(s-3)$-residually S_2 (for instance, I is a complete intersection) locally in codimension $s-1$ and if I has a homogeneous minimal reduction $\mathfrak{a}$ (for instance, I is generated by forms of the same degree), then Remark 6.3 applies to $\mathfrak{a} \subset I$.

Proof. Let $\mathfrak{p} \in \operatorname{Spec}(R)$ with $\dim R_{\mathfrak{p}} \leqslant s-1$. If $I \not\subset \mathfrak{p}$, then $\mathfrak{a} \not\subset \mathfrak{p}$ since $\sqrt{I} = \sqrt{\mathfrak{a}}$. If however $I \subset \mathfrak{p}$, then $I_{\mathfrak{p}}$ can be generated by a d-sequence (cf. Corollary 3.6 (b)) and thus has no proper reduction (cf. [15, Theorem 2.2]). Hence in either case $I_{\mathfrak{p}} = \mathfrak{a}_{\mathfrak{p}}$, which gives $\operatorname{codim}(\mathfrak{a} : I) \geqslant s$.

Furthermore, $\mathfrak{a}$ being a reduction of I and R being reduced, it follows that $\mathfrak{a} \neq 0$ has the same initial degree as I. Thus $0 \neq [\mathfrak{a}]_d \neq [I]_d$ for some d since depth $R > 0$. □

So far our counterexamples were largely based on the fact that one of the s elements generating $\mathfrak{a}$ was redundant. We are now going to present an example where both ideals $\mathfrak{a}$ and $\mathfrak{a}'$ are minimally generated by s elements. This example also shows that Corollary 4.7 fails even for Cohen–Macaulay ideals without further assumptions on the square of the ideal.

EXAMPLE 6.6. Let k be a field, let $R = k[X_1, \ldots, X_6]$, let L denote the ideal

$$L = (X_1^3, X_1^2X_2, X_2^3, X_3^3, X_4^3, X_5^3, X_6^3),$$

and let φ be a 2 by 4 matrix whose entries are forms of degree 3 in R and generate L. Let Δ_{ij} be the 2 by 2 minor of φ involving columns i and j, set

$$u = (X_1X_3X_4X_5X_6)^2, \quad v = (X_2X_3X_4X_5X_6)^2, \quad \mathfrak{b} = (\Delta_{12} + \Delta_{34}, \Delta_{13}, \Delta_{14}, \Delta_{23}, \Delta_{24}),$$
$$\mathfrak{a} = (\mathfrak{b}, u\Delta_{34}), \ \mathfrak{a}' = (\mathfrak{b}, v\Delta_{34}), \ \text{and} \quad I = I_2(\varphi).$$

Assume that codim $I \geqslant 3$ (for instance, take

$$\varphi = \begin{pmatrix} X_1^3 & X_2^3 & X_3^3 & X_4^3 \\ X_5^3 & X_3^3 & X_6^3 & X_1^2X_2 \end{pmatrix};$$

in fact if k is infinite and the entries of φ are chosen to be general elements in L then R/I is an isolated singularity of dimension 3). We claim I is a perfect ideal of codimension 3 that is a complete intersection locally on the punctured spectrum, $\operatorname{codim}(\mathfrak{a} : I) \geqslant 6$ and $\operatorname{codim}(\mathfrak{a}' : I) \geqslant 6$, $\mathfrak{a}$ and $\mathfrak{a}'$ are both minimally generated by 5 sextics and one form of degree 16, but $R/\mathfrak{a}$ and $R/\mathfrak{a}'$, and $R/(\mathfrak{a} : I)$ and $R/(\mathfrak{a}' : I)$, respectively, do not have the same Hilbert function.

Proof. Since $\sqrt{I_1(\varphi)} = \sqrt{L} = (X_1, \ldots, X_6)$, the ideal I is a complete intersection locally on the punctured spectrum. Furthermore, from the presentation matrix of the determinantal ideal I it follows that $\mathfrak{b} : I = \mathfrak{b} : (\Delta_{34}) = I_1(\varphi) = L$. Thus $\operatorname{codim}(\mathfrak{a} : I) \geqslant 6$ and $\operatorname{codim}(\mathfrak{a}' : I) \geqslant 6$, and $I/\mathfrak{b} \cong (R/L)(-6)$. Write '$\bar{\ }$' for images in $\bar{R} = R/L$ and notice that $\mathfrak{a}/\mathfrak{b} \cong (\bar{R}\bar{u})(-6)$, $R/(\mathfrak{a} : I) \cong \bar{R}/(\bar{u})$, $\mathfrak{a}'/\mathfrak{b} \cong (\bar{R}\bar{v})(-6)$, $R/(\mathfrak{a}' : I) \cong \bar{R}/(\bar{v})$. Now $\bar{u} \neq 0 \neq \bar{v}$, which already gives $\mu(\mathfrak{a}) = 6 = \mu(\mathfrak{a}')$. Furthermore $\bar{u} \in \operatorname{socle}(\bar{R})$ and $\bar{v} \notin \operatorname{socle}(\bar{R})$ (in order to find two such elements in the same degree, we had to choose L so that $\operatorname{socle}(\bar{R})$ is not pure). But then $\bar{R}\bar{u}$ and $\bar{R}\bar{v}$ have different Hilbert functions, hence the same holds for $R/\mathfrak{a}$ and $R/\mathfrak{a}'$, and for $R/(\mathfrak{a} : I)$ and $R/(\mathfrak{a}' : I)$, respectively. $\square$

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