Syzygy ideals for determinantal ideals and the syzygetic Castelnuovo lemma

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If I is an ideal in a (local, or graded) ring S and s is a generator of the module of i^{th} syzygies of I, then the *syzygy ideal* of s, recently studied by Schreyer and others (see Ehbauer [1994]), is roughly speaking the smallest ideal I' inside I such that s is the image of an i^{th} syzygy of I'. The syzygy ideal can be defined in terms of the $\mathrm{Ext}^\bullet(k,k)$-module structure on $\mathrm{Tor}_\bullet(S/I,k)$, where k is the residue field. In this note we will give an explicit computation of this module structure, and thus of the syzygy ideals, in the case where I is generated by the maximal minors of an $f \times g$ matrix ϕ of linear forms over a polynomial ring, and I has "expected" codimension $f-g+1$.

In Section 3 we conjecture that the behavior of syzygy ideals characterizes the *1-genericity* (Eisenbud [1988]) of ϕ in the sense that ϕ is 1-generic if and only if the syzygy ideal of each $(f-g)^{\text{th}}$ syzygy of I is all of I. We prove the "if" statement of this conjecture, and we prove the "only if" statement in the case where $g=2$. As an application we give a new, direct proof of a scheme-theoretic generalization of Green's "syzygetic Castelnuovo Lemma" (Green [1984]) proved independently by Yanagawa [1994] and Ehbauer [1994]: A finite subscheme $\Gamma \subset \mathbf{P}^r$ of length $\geq r+3$ lies on a smooth rational normal curve iff Γ contains a subscheme of length $r+3$ in linearly general position, and $\mathrm{Tor}_{r-2}(I_\Gamma,k)_r \neq 0$.

Here is a typical example in which syzygy ideals arise. Let C be a smooth curve of genus g, canonically embedded in $\mathbf{P}^{g-1}$. Let $S = k[x_0,\dots x_{g-1}]$ be the homogeneous coordinate ring of $\mathbf{P}^{g-1}$, and let I be the homogeneous ideal of C. It is known (see Saint-Donat [1973]) that if $\mathrm{Tor}_{g-4}(I,k)_{g-2} \neq 0$, then the curve C is trigonal and lies on a 2-dimensional rational normal scroll, whose ideal J satisfies

$$\mathrm{Tor}_{g-4}(J,k)_{g-2} = \mathrm{Tor}_{g-4}(I,k)_{g-2}.$$

The first author is grateful to the NSF and the second author to the DFG and the NSF for support during the preparation of this work.

A consequence of Theorem 3.1 below is that J is the syzygy ideal of any syzygy $\tau \in \mathrm{Tor}_{g-4}(I,k)_{g-2}$.

We are grateful to Frank-Olaf Schreyer for introducing us to syzygy varieties and their applications.

1 Syzygy submodules and ideals

To begin, we give a definition of syzygy ideals somewhat more general than the one given in Ehbauer [1994]. Suppose that k is a field, $S = k[x_0, \dots, x_n]$ is a polynomial ring over k, and $\mathbf{m}$ is the ideal generated by the variables of S. Let M be a finitely generated graded S-module, perhaps an ideal of S. The graded vector space $\mathrm{Tor}_i(M,k)$ may be identified with the space of generators of the i^{th} free module in the minimal free resolution of M, and we refer to elements of $\mathrm{Tor}_i(M,k)$ as i^{th} syzgies of M. Let $\tau \in \mathrm{Tor}_i(M,k)$ be a syzygy of degree j. The image in $M \otimes k$ of $\tau \otimes \mathrm{Ext}^i(k,k)$ under the multiplication map

$$\mathrm{Tor}_i(M,k) \otimes \mathrm{Ext}^i(k,k) \longrightarrow \mathrm{Tor}_0(M,k) = M \otimes k$$

is a subspace denoted in the sequel $\tau\mathrm{Ext}^i(k,k)$. There is a natural identification $\mathrm{Ext}^\bullet(k,k) = \wedge^\bullet W^*$, where $W := S_1$ denotes the vector space of linear forms in S, and W^* is its k-dual. Thus $\mathrm{Ext}^\bullet(k,k)$ is concentrated in degree i and $\tau\mathrm{Ext}^i(k,k) \subset (M \otimes k)_{j-i}$. Lifting this space to a space in M_{j-i} we get a subspace that is well defined modulo the generators of M of *lower* degree. Therefore, by adding $M_{<j-i}$ we get a well defined module of M.

Definition The *syzygy submodule* of τ is the module generated by representatives of $\tau\mathrm{Ext}^i(k,k)$ together with $M_{<j-i}$.

The syzygy submodule was defined in Ehbauer [1994] only in the case where M is an ideal containing no linear forms in the polynomial ring S, and τ has degree $i+2$: Let $K^\bullet$ be the Koszul complex of S. We can identify $\mathrm{Tor}_i(M,k)_{i+2}$ with $\mathrm{Tor}_{i+1}(S/M,k)_{i+2}$, and thus with the Koszul homology $(\mathrm{H}_{i+1}(S/M \otimes K^\bullet))_{i+2}$. The element τ is represented by a cycle $\tau' \in \wedge^{i+1}W \otimes (S/M)_1 = \wedge^{i+1}W \otimes W$, where now the $\otimes$ is taken over k. Since τ' is a cycle, its image in $\wedge^i W \otimes (S/M)_2$ is 0, and thus its image $d\tau'$ under the differential d of K lies in $\wedge^i W \otimes M_2 \subset \wedge^i W \otimes S_2$. We may regard $d\tau'$ as a map $d\tau' : \wedge^i W^* \longrightarrow M_2$; the image of this map is the syzygy submodule of τ described above. The same method works whenever M is a submodule of a free S-module.

The pairing between $\mathrm{Ext}^\bullet(k,k)$ and $\mathrm{Tor}_\bullet(M,k)$ can be understood directly from the Yoneda description of Ext. Given an extension

$$\epsilon : \quad 0 \longrightarrow k \longrightarrow A_j \longrightarrow A_{j-1} \longrightarrow \dots \longrightarrow k \longrightarrow 0,$$

we break it into a succession of short exact sequences

$$0 \longrightarrow B_{l+1} \longrightarrow A_l \longrightarrow B_l \longrightarrow 0,$$

with $B_1 = B_{j+1} = k$. Each of these sequences induces a long exact sequence in $\mathrm{Tor}_\bullet(M, -)$ and in particular a connecting homomorphism

$$\mathrm{Tor}_{i-l}(M, B_l) \longrightarrow \mathrm{Tor}_{i-l-1}(M, B_{l+1}).$$

The composition of these maps is the map

$$\mathrm{Tor}_i(M, k) \longrightarrow \mathrm{Tor}_{i-j}(M, k)$$

that is multiplication by ϵ. To actually compute this map, we use the description of $\mathrm{Ext}^\bullet(k, k)$ via the Koszul complex $K^\bullet$, which is a free resolution of k, to represent an element $\epsilon \in \mathrm{Ext}^j(k, k)$ as a map $\wedge^j W \longrightarrow k$. Let

$$\ldots \longrightarrow G_i \longrightarrow \ldots \longrightarrow G_0 \longrightarrow M \longrightarrow 0$$

be a minimal free resolution of M. An i^{th} syzygy τ of M may be represented by an element of G_i, and thus by a map $\tilde{\tau} : S \longrightarrow G_i$. The image of $G_{i-1}^* \longrightarrow G_i^*$ is in $\mathbf{m}G_i$, and thus the composite $G_{i-1}^* \longrightarrow G_i^* \xrightarrow{\tilde{\tau}^*} S$ lifts along the first map of the Koszul complex to a map $G_{i-1}^* \xrightarrow{\tilde{\tau}_1^*} W \otimes S$. Continuing the lifting we get a commutative diagram

$$\begin{tikzcd}
G_i^* \arrow[d, "\tilde{\tau}^*"'] & G_{i-1}^* \arrow[l] \arrow[d, "\tilde{\tau}_1^*"] & \ldots \arrow[l] & G_{i-j}^* \arrow[l] \arrow[d, "\tilde{\tau}_j^*"] \\
S & W \otimes S \arrow[l] & \ldots \arrow[l] & \wedge^j W \otimes S \arrow[l]
\end{tikzcd}$$

Dualizing $\tilde{\tau}_j^*$, tensoring with k, and composing with ϵ^*, we get a composite map

$$k \xrightarrow{\epsilon^*} \wedge^j W^* \xrightarrow{\tilde{\tau}_j \otimes k} G_{i-j} \otimes k = \mathrm{Tor}_{i-j}(M, k).$$

The image of $1 \in k$ is the class $\tau\epsilon$.

If we work with $G_i \otimes K^\bullet$ in place of $K^\bullet$ we may do this for all the i^{th} syzygies at once and we obtain:

Lemma 1.1 *Let $\ldots \to G_i \to \ldots \to G_0 \to M \to 0$ be a minimal free resolution of M. If*

$$\begin{tikzcd}
G_i^* \arrow[d, "m_{i,0}"'] & G_{i-1}^* \arrow[l] \arrow[d, "m_{i,1}"'] & \ldots \arrow[l] & G_{i-j}^* \arrow[l] \arrow[d, "m_{i,j}"'] \\
G_i^* \otimes S & G_i^* \otimes W \arrow[l] & \ldots \arrow[l] & G_i^* \otimes \wedge^j W \arrow[l]
\end{tikzcd}$$

is a lifting of the natural identification $m_{i,0} : G_i^* \to G_i^* \otimes S$, *then the multiplication map*

$$\mu_{i,j} : \mathrm{Tor}_i(M,k) \otimes \mathrm{Ext}^j(k,k) \to \mathrm{Tor}_{i-j}(M,k)$$

is the composition

$$\mathrm{Tor}_i(M,k) \otimes \mathrm{Ext}^j(k,k) = G_i \otimes \wedge^j W^* \xrightarrow{m_{i,j}^* \otimes k} G_{i-j} \otimes k \qquad \blacksquare$$

Below we shall carry this computation through in general when M is a determinantal ideal and the resolution is the Eagon-Northcott complex. Here is a simple special case:

Example 1.2 Let I be the ideal of three points in the plane, $(xy, xz, yz) \subset S := k[x,y,z]$. The ideal I has free resolution

$$0 \longrightarrow S^2(-3) \xrightarrow{\phi} S^3(-2) \longrightarrow I \longrightarrow 0.$$

Choosing bases e_1, e_2 for $S^2(-3)$ and f_1, f_2, f_3 for $S^3(-2)$ we may write ϕ as a matrix

$$\phi = \begin{pmatrix} z & 0 \\ -y & -y \\ 0 & x \end{pmatrix}.$$

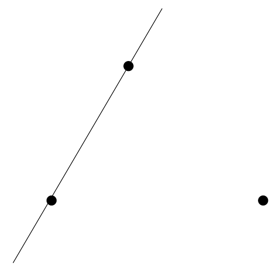
Note that I is the ideal of 2×2 minors of ϕ, and the resolution is the Eagon-Northcott complex. From the commutativity of the diagram

$$\begin{tikzcd}[column sep=huge, row sep=huge]
S^2(-3) \arrow[r, "\begin{pmatrix} z & 0 \\ -y & -y \\ 0 & x \end{pmatrix}"] & S^3(-2) \arrow[r, "\begin{pmatrix} xy & xz & yz \end{pmatrix}"] & S \\
S(-3) \arrow[u, "\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}"] \arrow[r] & S(-2) \otimes W^* \arrow[u, "\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}"'] &
\end{tikzcd}$$

and the computation above we see that the syzygy ideal of e_1 is the image of the composite map

$$\begin{pmatrix} xy & xz & yz \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & xz & yz \end{pmatrix}.$$

Thus the syzygy ideal of e_1 is the ideal (xz, yz) corresponding to a line and a point:



The dots shown are the three points defined by the ideal I.

Similarly, the syzygy ideal of the second generator is (xz, yz); the syzygy ideal of the difference of these two generators is (xy, yz); and the syzygy ideal of the sum (or for any other linear combination) is I itself. We shall see that this variety is possible only because of the zeros in the matrix ϕ.

2 The module structure of the Eagon-Northcott complex

Recall (for example from Eisenbud [1995, Appendix 2.6]) that if F, G are free modules of ranks $f \geq g$ over S and $\phi : F \to G$ is a map whose ideal $I := I_g(\phi)$ of $g \times g$-minors has the "expected" codimension $f - g + 1$, then the free resolution of I is given by the Eagon-Northcott complex

$$0 \longrightarrow (\mathrm{Sym}_{f-g} G)^* \otimes \wedge^f F \longrightarrow \ldots \longrightarrow G^* \otimes \wedge^{g+1} F \longrightarrow \wedge^g F \longrightarrow I \longrightarrow 0.$$

Henceforward we suppose that ϕ is represented by a matrix of linear forms. In more invariant terms, we can think of ϕ as a map of vector spaces $\phi' : \bar{F} \otimes \bar{G}^* \longrightarrow S_1 = W$, where $\bar{X}$ denotes $k \otimes_S X$ and all other tensor products are taken over k.

We will explicitly compute the multiplication maps

$$\mu_{i,j} : \mathrm{Tor}_i(I, k) \otimes \mathrm{Ext}^j(k, k) \longrightarrow \mathrm{Tor}_{i-j}(I, k).$$

To make this explicit we identify $\mathrm{Tor}_i(I, k) = (\mathrm{Sym}_i \bar{G})^* \otimes \wedge^{g+i} \bar{F}$, and $\mathrm{Ext}^j(k, k) = \wedge^j W^*$.

To describe the $\mu_{i,j}$ we use two additional multilinear constructions: First, the *Cauchy decomposition* of $\wedge^j(\bar{F}^* \otimes \bar{G})$ (see for example Fulton-Harris [1991]) yields maps $c_j : \wedge^j(\bar{F}^* \otimes \bar{G}) \longrightarrow (\wedge^j \bar{F}^*) \otimes (\mathrm{Sym}_j \bar{G})$, that can be conveniently defined by the formula

$$c_j : \quad (f_1 \otimes g_1) \wedge \ldots \wedge (f_j \otimes g_j) \mapsto (f_1 \wedge \ldots \wedge f_j) \otimes (g_1 \cdots g_j).$$

Second, we use the module structures (Boubakese: inner products)

$$m : \wedge^j \bar{F}^* \otimes \wedge^{g+i} \bar{F} \longrightarrow \wedge^{g+i-j} \bar{F}$$
$$n : (\mathrm{Sym}_i \bar{G})^* \otimes \mathrm{Sym}_j \bar{G} \longrightarrow (\mathrm{Sym}_{i-j} \bar{G})^*$$

For these see for example Eisenbud [1995, Appendix 2.4].

Theorem 2.1 *With notation and identifications as above, the map*

$$\mu_{i,j}:\ \mathrm{Tor}_i(I,k)\otimes \mathrm{Ext}^j(k,k) \longrightarrow \mathrm{Tor}_{i-j}(I,k)$$

is the composite

$$\begin{aligned}(\mathrm{Sym}_i\bar G)^*\otimes \wedge^{g+i}\bar F\otimes \wedge^j W^* &\xrightarrow{1\otimes 1\otimes \wedge^j\phi'^*} (\mathrm{Sym}_i\bar G)^*\otimes \wedge^{g+i}\bar F\otimes \wedge^j(\bar F^*\otimes \bar G)\\ &\xrightarrow{1\otimes 1\otimes c_j} (\mathrm{Sym}_i\bar G)^*\otimes \wedge^{g+i}\bar F\otimes \wedge^j\bar F^*\otimes \mathrm{Sym}_j\bar G\\ &\xrightarrow{n\otimes m} (\mathrm{Sym}_{i-j}\bar G)^*\otimes \wedge^{g+i-j}\bar F.\end{aligned}$$

Proof Let $\bar m_{i,j}$ be the composite map defined in the statement of Theorem 2.1, and let $m_{i,j}$ be the map obtained by tensoring $\bar m_{i,j}$ with S. By Lemma 1.1 it suffices to show that, for each fixed i, the maps $m_{i,j}$ form a map of complexes from the tensor product of $(\mathrm{Sym}_i\bar G)^*\otimes\wedge^{g+i}\bar F$ with the Koszul complex to the Eagon-Northcott complex. For this it suffices to show that the maps $\bar m_{i,j}$ themselves yield commutative diagrams of vector spaces:

$$\begin{tikzcd}(\mathrm{Sym}_{i-j+1}\bar G)^*\otimes\wedge^{g+i-j+1}\bar F \arrow[r, "e"] & (\mathrm{Sym}_{i-j}\bar G)^*\otimes\wedge^{g+i-j}\bar F\otimes W\\ (\mathrm{Sym}_i\bar G)^*\otimes\wedge^{g+i}\bar F\otimes\wedge^{j-1}W^* \arrow[u, "\bar m_{i,j-1}"] \arrow[r, "d"'] & (\mathrm{Sym}_i\bar G)^*\otimes\wedge^{g+i}\bar F\otimes\wedge^{j}W^*\otimes W \arrow[u, "\bar m_{i,j}\otimes W"']\end{tikzcd}$$

in which the maps e come from the differentials of the Eagon-Northcott complex and the maps d come from the differentials in the Koszul complex. Given the definitions above this is a straightforward computation. ∎

3 Syzygy ideals of determinantal ideals

Recall that a map

$$\phi: F\to G$$

of free modules over a polynomial ring $S=\mathrm{Sym}_\bullet(W)$ which is represented by a matrix of linear forms corresponds to a map of vector spaces $\bar F\longrightarrow \bar G\otimes W$, or equivalently to a pairing

$$\phi':\quad \bar F\otimes\bar G^*\longrightarrow W.$$

We call ϕ or ϕ' *1-generic* if for every "pure" element $0\neq a\otimes b\in\bar F\otimes\bar G^*$ we have $\phi'(a\otimes b)\neq 0$. The maximal minors of a 1-generic map necessarily generate an ideal of the expected codimension (see Eisenbud [1988]).

Theorem 3.1 *Suppose that S is a polynomial ring over a field k, and that $\phi : F \to G$ is a map of free S-modules of ranks f and g respectively, $f \geq g$, represented by a matrix of linear forms. Suppose further that the determinantal ideal $I := I_g(\phi)$ has codimension $f-g+1$. If the syzygy ideal of each nonzero element of* $\operatorname{Tor}_{f-g}(I,k)$ *is I, then the map ϕ is 1-generic; if $g = 2$, then the converse also holds.*

Proof By Theorem 2.1, we must show that the map ϕ is 1-generic if the pairing

$$\mu_{f-g,f-g} : \ (\operatorname{Sym}_{f-g}\bar{G})^* \otimes \wedge^{f-g} W^* \longrightarrow \wedge^g \bar{F}$$

takes $\tau \otimes \wedge^{f-g} W^*$ onto $\wedge^g \bar{F}$ for every $\tau \in (\operatorname{Sym}_{f-g}\bar{G})^*$. Choosing a generator for $\wedge^f \bar{F}^*$ we may identify $\wedge^g \bar{F}$ with $\wedge^{f-g}\bar{F}^*$. Rearranging the tensor factors, $\mu_{f-g,f-g}$ yields a map $\wedge^{f-g}\bar{F} \otimes (\operatorname{Sym}_{f-g}\bar{G})^* \longrightarrow \wedge^{f-g} W$ which is easily seen to be a composite

$$\wedge^{f-g}\bar{F} \otimes (\operatorname{Sym}_{f-g}\bar{G})^* \longrightarrow \wedge^{f-g}(\bar{F} \otimes \bar{G}^*) \longrightarrow \wedge^{f-g} W,$$

where the first map is the part of the Cauchy decomposition dual to the one described in Section 3 and the second is the $(f-g)^{\text{th}}$ exterior power of the pairing $\phi' : \ \bar{F} \otimes \bar{G}^* \longrightarrow W$ corresponding to ϕ. Simplifying the notation somewhat and generalizing the situation by dropping the dependence of the power $f-g$ on the ranks of F and G, the desired result follows from:

Theorem 3.2 *Let $\alpha : \ A \otimes B^* \longrightarrow C$ be a pairing of vector spaces over a field k. Let*

$$\alpha_n : \quad \wedge^n A \otimes (\operatorname{Sym}_n B)^* \longrightarrow \wedge^n C$$

be the composition of the dual Cauchy map $\wedge^n A \otimes (\operatorname{Sym}_n B)^* \longrightarrow \wedge^n (A \otimes B^*)$ with the n^{th} exterior power of α.

a) *If α_n is 1-generic for some n between 1 and the rank of A, then $\alpha = \alpha_1$ is 1-generic.*

b) *If* $\operatorname{rank} B \leq 2$, *then the converse also holds.*

Proof Identify $(\operatorname{Sym}_n B)^*$ with the n^{th} divided power of B^*, and write the n^{th} divided power of an element $b \in B^*$ as $b^{(n)}$. One checks first that the restriction of α_n to the subspace $\wedge^n A \otimes b^{(n)}$ is the n^{th} exterior power of the map which is the restriction of α to $A \otimes b$.

To prove part a), suppose that α is not 1-generic, so that some nonzero vector $a \otimes b$ goes to 0. Then $\alpha_n(a \wedge a_2 \wedge \cdots \wedge a_n \otimes b^{(n)}) = 0$ for any elements $a_2, \ldots, a_n$ and thus part a) follows.

Conversely, suppose that α is 1-generic, and $\operatorname{rank} B = 2$. We may harmlessly replace C by the image of α. From the classification theory of 1-generic $2 \times m$ matrices (see Eisenbud-Harris [1987], Harris [1992], or directly from the Kronecker-Weierstrass theory of matrix pencils, see Gantmacher [1959,1986]) it follows that A may be identified as a direct sum of the form

$$A = \mathrm{Sym}_{d_1}(B^*) \oplus \ldots \oplus \mathrm{Sym}_{d_m}(B^*),$$

and C may be identified as the direct sum

$$C = \mathrm{Sym}_{d_1+1}(B^*) \oplus \ldots \oplus \mathrm{Sym}_{d_m+1}(B^*),$$

in such a way that the map α is the direct sum of the multiplication maps

$$\mathrm{Sym}_{d_i}(B^*) \otimes B^* \longrightarrow \mathrm{Sym}_{d_1+1}(B^*).$$

To prove part b), we use a degeneration argument: Suppose that the nonzero pure vector $a \otimes b$ goes to zero under α_n. We may without loss of generality suppose that the ground field k is infinite, so with a general choice of generators s, t of B^* we may suppose that $b = s^{(d)} +$ (lower order) involves a pure divided power of s. Taking the limit of a suitable one-parameter subgroup of $GL(2,k)$, we may actually suppose that $b = s^{(d)}$. The remark at the beginning of the proof finishes the argument. ■

We conjecture that the "converse" part of Theorem 3.1 holds without restriction on g, and consequently that Theorem 3.2 is true without restriction on the rank of B. We can prove this for the syzygies coming from "pure" vectors in $\mathrm{Tor}_{f-g}(I,k) = (\mathrm{Sym}_{f-g}G)^*$. One way to formulate the missing step is this:

Conjecture 3.3 *Let ψ be a $p \times q$-matrix of linear forms over a polynomial ring T and let n denote the minimal dimension of the span of any generalized row of ψ. If $k \leq q-n+1$ and $k \leq p$, then $\binom{n+k-1}{k}$ is the minimal dimension of the span of any generalized row of $\wedge^k\psi$.*

It is not hard to reduce Conjecture 3.3 to the following special case: If every generalized row of ψ consists of linear forms that span the space of linear forms of T, then every generalized row of $\wedge^k\psi$, with k as above, spans the k^{th} power of the maximal ideal of T. The case of two variables is easy, following the ideas above; we do not have a proof in the case of three variables.

The proof of Theorem 3.1 just given is similar to something implicit in Green's proof [1996] of the linear syzygy conjecture. The syzygy variety is the image of an exterior minors map in the sense used in that paper.

4 Syzygetic Castelnuovo Lemma

Recall that a finite subscheme $\Gamma \subset \mathbf{P}^r$ is said to be in *linearly general position* if for every hyperplane H of $\mathbf{P}^r$ the scheme $H \cap \Gamma$ has length $\leq r$.

In Eisenbud-Harris [1992] it was shown that if k is an algebraically closed field and the length of Γ is $\leq r+3$, then Γ lies on a rational normal curve of degree r iff Γ is in linearly general position. (The case when Γ is reduced is the original "Castelnuovo Lemma".) See Eisenbud-Popescu [1998] for a version that works without algebraic closure. For large numbers of points one needs an additional condition, which can be expressed in terms of syzygies:

Theorem 4.1 *(Syzygetic Castelnuovo Lemma for schemes). A finite subscheme $\Gamma \subset \mathbf{P}^r_k$ of length $\geq r+3$ over an algebraically closed field k lies on a smooth rational normal curve iff*

a) *Γ contains a subscheme of length $r+3$ in linearly general position; and*

b) $\operatorname{Tor}_{r-2}(I_\Gamma, k)_r \neq 0$.

Theorem 4.1 was proved for reduced Γ by Green [1984], and Yanagawa remarked in [1994] that the scheme theoretic version follows, given the results of Eisenbud-Harris [1992]. The full statement also follows from the results of Ehbauer [1994]. Here we give a direct proof, using a technique related to that of Green's recent proof of the "Linear Syzygy Conjecture" [1996]. In Green's terms, we identify a certain syzygy ideal with an ideal of exterior minors.

In fact if we assume that Γ contains a (locally) Gorenstein subscheme of degree $r+3$ in linearly general position, then one can replace the reference to Eisenbud-Harris in the proof below with a reference to Eisenbud-Popescu [1998], and omit the hypothesis of algebraic closure. One should compare the result with the much easier result describing when a projective scheme has $\operatorname{Tor}_{r-1}(I_\Gamma, k)_r \neq 0$. See also Cavaliere, Rossi, Valla [1995] for some variations.

Proposition 4.2 *Let I be any ideal in the polynomial ring in $r+1$ variables $S = k[x_0, \ldots, x_r]$. We have $\operatorname{Tor}_{r-1}(I, k)_r \neq 0$ iff I contains the ideal of 2×2 minors of a rank 2 matrix of the form*

$$\begin{pmatrix} x_0 & x_1 & \ldots & x_r \\ l_0 & l_1 & \ldots & l_r \end{pmatrix}$$

where the l_i are linear forms, and the row of l_i is not a scalar multiple of the row of x_i. In this case, if k is algebraically closed, the scheme associated to I lies on the union of nontrivial linear subspaces L_1 and L_2 with $\operatorname{codim} L_1 + \operatorname{codim} L_2 = r+1$. *In particular, the forms in I do not vanish on any set of $r+2$ points in linearly general position.*

Proof The first statement of Proposition 4.2 follows directly from the computation of Koszul cohomology. To deduce the second statement, note that the matrix is too large to be 1-generic given that k is algebraically closed and the number of variables is only $r+1$ (Eisenbud [1988]). Thus after row and column operations and a linear change of variables we may suppose that for some $i < r$ the forms $l_0, \ldots, l_i$ are linearly independent while $l_{i+1} = \ldots = l_r = 0$. It follows that the scheme associated to I is contained in the union of the $(r-i-1)$-plane $l_0 = \ldots = l_i = 0$ and the i-plane $x_{i+1} = \ldots = x_r = 0$. ■

For the proof of Theorem 4.1 we use the following remark:

Lemma 4.3 *Let $M \subset N$ be graded modules over a graded ring $S = k \oplus W \oplus \ldots$. If the lowest degree of a generator of N is d, then for any $i \geq 0$ the natural map $\mathrm{Tor}_i(M,k)_{d+i} \longrightarrow \mathrm{Tor}_i(N,k)_{d+i}$ is an inclusion, and if $\tau \in \mathrm{Tor}_i(M,k)_{d+i}$ is any element, then the syzygy submodule of τ in M is equal to the syzygy submodule of τ in N.* ■

Proof of Theorem 4.1 The ideal of a rational normal curve C in $\mathbf{P}^r$ is in suitable coordinates generated by the ideal of 2×2 minors of the 1-generic matrix

$$\begin{pmatrix} x_0 & x_1 & \ldots & x_{r-1} \\ x_1 & x_2 & \ldots & x_r \end{pmatrix}$$

(see for example Harris [1992]). Its resolution is given by the Eagon-Northcott complex; in particular, the i^{th} syzygies of I_C have degree $i+2$. If Γ is a subscheme of C of length at least $r+1$, then I_Γ contains no linear forms. By Lemma 4.3 the natural morphisms $\mathrm{Tor}_i(I_C,k) \longrightarrow \mathrm{Tor}_i(I_\Gamma,k)$ are monomorphisms, and thus in particular $\mathrm{Tor}_{r-2}(I_\Gamma,k)_r \neq 0$.

For the converse, suppose Λ is a subscheme of length $r+3$ of Γ, and that Λ is in linearly general position. By Eisenbud-Harris [1992] there is a rational normal curve C containing Λ. It is easy to compute syzygies for any subscheme of a rational normal curve (see for example Eisenbud [1995 Appendix A2.21].) In the case of a subscheme of length $r+3$, it follows that $\mathrm{Tor}_{r-2}(I_\Lambda,k)_r = \mathrm{Tor}_{r-2}(I_C,k)_r$. Since I_Λ does not contain any linear forms, we have by Lemma 4.3 that $\mathrm{Tor}_{r-2}(I_\Gamma,k)_r \subset \mathrm{Tor}_{r-2}(I_\Lambda,k)_r$. If $\tau \in \mathrm{Tor}_{r-2}(I_\Gamma,k)_r$, then we may regard τ as a syzygy of I_Λ and then of I_C. As a syzygy of I_C, its syzygy ideal is all of I_C, by Theorem 3.1. By the Lemma 4.3, its syzygy ideal as a syzygy of I_Λ or of I_Γ is the same. Thus I_Γ contains I_C. ■

We close this note with an application to another well-known result:

Proposition 4.4 *Let $\Gamma \subset \mathbf{P}^r$ be a finite subscheme of degree $r+2$. If Γ is in linearly general position, then the homogeneous coordinate ring S_Γ is Gorenstein.*

Proof Write $S = k[x_0, \ldots, x_r]$ for the homogeneous coordinate ring of $\mathbf{P}^r$. The ideal of Γ is easily be seen to be 2-regular, and since it contains no linear forms the generators of the canonical module $\omega_\Gamma = \mathrm{Ext}^r(S_\Gamma, S(-r-1))$ can only be in degrees -1 and 0. If ω_Γ had a generator in degree 0, then we would have $\mathrm{Tor}_{r-1}(I,k)_r \neq 0$; applying Proposition 4.2 we derive a contradiction from the fact that Γ is in linearly general position. Thus ω_Γ has all its generators in degree -1. Computing Hilbert functions we see that there is just one generator in this degree, so S_Γ is Gorenstein as required. ∎

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