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Gale duality and free resolutions of ideals of points

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What is the shape of the free resolution of the ideal of a general set of points in $\mathbf{P}^r$? This question is central to the programme of connecting the geometry of point sets in projective space with the structure of the free resolutions of their ideals. There is a lower bound for the resolution computable from the (known) Hilbert function, and it seemed natural to conjecture that this lower bound would be achieved. This is the "Minimal Resolution Conjecture" (Lorenzini [1987], [1993]). Although the conjecture has been shown to hold in many cases, three examples discovered computationally by Frank-Olaf Schreyer in 1993 strongly suggested that it would fail in general. In this paper we shall describe a novel structure inside the free resolution of a set of points which accounts for the failure and provides a counterexample in $\mathbf{P}^r$ for every $r \geq 6,\ r \neq 9$.

We begin by reviewing the conjecture and its status. Consider a set of γ points in the projective r-space over a field k, say $\Gamma \subset \mathbf{P}^r_k$. Let $S = k[x_0, \ldots, x_r]$, let I_Γ be the homogeneous ideal of Γ, and let S_Γ denote the homogeneous coordinate ring of Γ. Let

$$F_\bullet \ : \ 0 \longrightarrow F_{r-1} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow I_\Gamma \longrightarrow 0$$

be the minimal free resolution of I_Γ, and define the associated (graded) Betti numbers $\beta_{i,j}$ by the formula

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$$F_i = \bigoplus_j S(-j)^{\beta_{i,j}} \ .$$

The minimal free resolution conjecture (**MRC**) can be formulated as follows:

Minimal Resolution Conjecture. *If Γ is a general set of points in $\mathbf{P}^r_k$ over an infinite field k, then for any integers i, j, at most one of $\beta_{i,j}$ and $\beta_{i+1,j}$ is nonzero.*

Since Γ imposes the largest possible number of conditions on forms of every degree, it is easy to compute its Hilbert function. It also follows that if I_Γ contains forms of degree d, then I_Γ is $(d+1)$-regular, that is, $\beta_{ij} = 0$ for $j > i + d$. From these observations the minimal free resolution conjecture can be translated into an explicit formula for the β_{ij} (see §5 below).

The minimal resolution conjecture is known to be true in $\mathbf{P}^2$ (Gaeta [1951] and [1995], Geramita-Lorenzini [1989]), in $\mathbf{P}^3$ (Ballico-Geramita [1986]), in $\mathbf{P}^4$ (Walter [1995], Lauze [1996]), and in $\mathbf{P}^n$ for $n + 1 \leq \gamma \leq n + 4$, or $\gamma = \binom{n+2}{2} - n$ (Geramita-Lorenzini [1989], Cavaliere-Rossi-Valla [1991], Lorenzini [1993]). Its predictions about $\beta_{r-1,j}$ are known to be true in general (Trung-Valla [1989], Lauze [1997]). Most striking, the conjecture is known to hold whenever the number of points in Γ is sufficiently large compared to r (Hirschowitz-Simpson [1996]); the known bound is $\gamma > 6^{r^3 \log r}$.

On the other hand, Schreyer discovered that random sets of points chosen over a field with p elements for various primes p do not satisfy the conclusion of the MRC in the cases: 11 points in $\mathbf{P}^6$, 12 points in $\mathbf{P}^7$, and 13 points in $\mathbf{P}^8$. A few more such examples were discovered by computer search (Boij [1994], Beck-Kreuzer [1996]), but it was unclear whether they were unique accidents or part of a larger picture. Although the experts considered this very strong evidence for the failure of the MRC, there was no proof that the MRC actually fails in these cases.

In this paper we give a geometric construction proving that these cases really represent counterexamples, and providing infinitely many more:

Theorem 0.1. *For any integer $r \geq 6$, $r \neq 9$, there is an integer $\gamma(r)$ such that the Minimal Resolution Conjecture fails for a general set of $\gamma(r)$ points in $\mathbf{P}^r$. More explicitly, we may define s and k by*

$$r = \binom{s+1}{2} + k, \quad 0 \leq k \leq s \ ,$$

and take

$$\gamma(r) = r + s + 2 = \binom{s+2}{2} + k + 1 \ .$$

We do not know whether the MRC holds in $\mathbf{P}^9$, but computational work of Beck and Kreuzer [1996] shows that there are no counterexamples for 50 or fewer points.

The main new idea in the proof of Theorem 0.1 is a relation between the resolution of the ideal of a set of points $\Gamma \subset \mathbf{P}^r$ and the properties of an associated set of points, the *Gale transform*, in a different projective space. The Gale transform is best thought of as a duality of linear series on a finite Gorenstein scheme derived from Serre duality (see §1), but to introduce the ideas, *we assume for the remainder of this introduction that* Γ *is a general set of* γ *points in* $\mathbf{P}^r$.

With this hypothesis, we can describe the Gale transform as follows:

Let Γ be a general set of $\gamma \geq r+3$ points in $\mathbf{P}^r$, and set $s = \gamma - r - 2$. Choose homogeneous coordinates in $\mathbf{P}^r$ so that the points of Γ are represented by the rows of the $(r+1) \times \gamma$ matrix

$$(I_{r+1} \mid B)$$

where I_{r+1} is the $(r+1) \times (r+1)$ identity matrix and B is an $(r+1) \times (s+1)$ matrix of scalars. The Gale transform of Γ is then the set of γ points $\Gamma' \subset \mathbf{P}^s$ with homogeneous coordinate matrix

$$(B^{\mathrm{T}} \mid I_{s+1}) \ .$$

Since Γ is general, the matrix B will be general, so the set Γ' will be general too. The Gale transform has been studied in various guises starting with Pascal's Magic Hexagram; see Dolgachev-Ortland [1988] and Eisenbud-Popescu [1998] for modern treatments.

Write S_Γ for the homogeneous coordinate ring $S_\Gamma = S/I_\Gamma$ of Γ, and

$$\omega_\Gamma = \mathrm{Ext}_S^{r-1}(I_\Gamma, S(-r-1))$$

for the canonical module. The free resolution of ω_Γ is, up to a shift in degree, the dual of the resolution $F_\bullet$ that appears in the Minimal Resolution Conjecture. The importance of the Gale transform for free resolutions comes about from the fact that the linear part of the presentation matrix for $(\omega_\Gamma)_{\geq -1}$, the part of ω_Γ generated in degrees ≥ -1, is *adjoint* in a simple sense to that of $(\omega_{\Gamma'})_{\geq -1}$ (see Proposition 1.1 below and Eisenbud-Popescu [1998, Proposition 4.1]).

To describe how this relation can be exploited, we suppose for simplicity that $r+3 < \gamma \leq \binom{r+2}{2}$ so that Γ does not lie on a rational normal curve and imposes independent conditions on quadrics. It follows that ω_Γ is generated in degree -1, and has relations generated in degree 0. We further assume that $\gamma > \binom{s+2}{2}$, so that Γ' does NOT impose independent conditions on quadrics of $\mathbf{P}^s$. (These hypotheses are all satisfied in the setting of Theorem 0.1.) It follows that $\mathrm{H}^1(\mathscr{I}_{\Gamma'}(2)) \neq 0$, so its dual

$$U := (\omega_{\Gamma'})_{-2} \neq 0 \ .$$

(In the setting of Theorem 0.1 we have $\dim(U) = k+1$.) Write $W = \mathrm{H}^0(\mathcal{O}_{\mathbf{P}^s}(1))$ for the space of linear forms on $\mathbf{P}^s$. There is a natural pairing induced by multiplication

$$\mu : W \otimes U \longrightarrow (\omega_{\Gamma'})_{-1} \ .$$

Knowledge about this map can be used to gain information about the presentation matrix of $(\omega_{\Gamma'})_{\geq -1}$. For example, if $0 \neq u \in U$, and $y_0, \dots, y_s$ generate the linear forms on $\mathbf{P}^s$, then $y_i u,\ y_j u \in (\omega_{\Gamma'})_{-1}$ satisfy the $\binom{s+1}{2}$ "Koszul" relations $y_i \cdot y_j u - y_j \cdot y_i u = 0$. Linear relations of this type give rise, via the adjointness mentioned above, to special linear relations on the generators of ω_Γ.

We will show that the syzygies on these special linear relations of ω_Γ account for the failure of the Minimal Resolution Conjecture: In §3 we describe a complex of free S-modules $E_\bullet(\mu)$ derived from the special linear relations and admitting a "comparison" map to the free resolution $F_\bullet^\vee$ of ω_Γ. The ranks of the free modules in $E_\bullet(\mu)$ are, in the numerical situation of Theorem 0.1, larger than the ranks predicted for the free resolution $F_\bullet^\vee$ by the Minimal Resolution Conjecture; this numerology is checked in §5. Thus if the comparison map is a monomorphism, the minimal free resolution conjecture fails!

The difficult part is to show in the generality of Theorem 0.1 that the comparison map is a monomorphism. The necessary condition is a non-degeneracy condition on the pairing μ. To see what is involved, let us return to the notation of the preceding paragraph. If the elements $y_j u$, $y_i u$ are both 0, then the relation we produced is actually 0. More generally, if the $y_i u$'s are linearly dependent, then we would get fewer than $\binom{s+1}{2}$ independent Koszul relations.

In the first cases, with $k = \dim(U) \leq 2$, elementary arguments suffice. Kreuzer [1994] has shown that the pairing μ is 1-generic in the sense of Eisenbud [1988], and this implies that the Koszul relations coming from a single element u are indeed independent. In the cases when $\dim(U) = 1$ the complex $E_\bullet(\mu)$ is a Koszul complex, and this 1-genericity suffices. In the next case, when $\dim(U) = 2$, the classification of 1-generic pairings coming from the Kronecker-Weierstrass theory of matrix pencils makes it possible to show directly that 1-generic pairings again have the desired property.

However, when $\dim(U) > 2$ the necessary property of μ does not follow from 1-genericity. It may be expressed by saying that the complex $E_\bullet(\mu)$ is *linearly exact*, a notion developed in §2. It is an "open" property of μ, so we may prove it after specializing the points of Γ', and this is how we shall complete the proof. The possibility of specializing is in marked contrast to the original situation: proving the failure of the minimal resolution conjecture for a specialized set of points is worthless!

The specialization argument is carried out in §4. We first specialize Γ to lie on a curve $C \subset \mathbf{P}^s$ with certain properties, under which we may reformulate the property of μ we need into a cohomological condition on the vector bundle T that is the restriction to C of the cotangent bundle of $\mathbf{P}^s$ (the argument has the flavor of the Koszul cohomology arguments in the papers of Green and Lazarsfeld). We then degenerate the curve C itself to a curve C_1, in such a way that the linear series embedding C in $\mathbf{P}^s$ degenerates, and with it T degenerates into a direct sum of copies of a simpler bundle T_1.

There are two cases, depending on the parity of $s = \gamma - r - 2$; if s is odd, we degenerate so that C_1 is a curve of low gonality, and T_1 is a direct sum of copies of a line bundle giving a low-degree map to $\mathbf{P}^1$; the cohomological condition is then easy to check. This case already gives infinitely many counterexamples. On the other hand, if s is even we take C_1 to be a plane curve. The degeneration in this case is much more delicate, since the complete linear series embedding C specializes to an incomplete linear series. The cohomological condition we must check finally follows from a strong form of the stability of the restricted tangent bundle of the general plane curve. For example, in case $s = 4$, we must show the following: Let C be a general plane curve of degree 5, and let T_2 be the restriction to C of the tangent bundle of the plane. If L is a general line bundle of degree 5 on C, then any "twisted endomorphism" $T_2 \longrightarrow T_2 \otimes L$ is zero. The proof of this strong stability for restricted tangent bundles on general plane curves follows ideas of Raynaud [1982] and completes the argument.

An interesting open problem remains whose solution might lead to further families of counterexamples to the Minimal Resolution Conjecture. Suppose as above that $\Gamma \subset \mathbf{P}^r = \mathbf{P}(V)$ is a general set of γ points, and that $\gamma \le \binom{r+2}{2}$ so that Γ imposes independent conditions on quadrics in $\mathbf{P}^r$, and thus ω_Γ is generated in degree -1 as above. Let Γ' be the Gale transform of Γ in $\mathbf{P}^s = \mathbf{P}(W)$, where $s = \gamma - r - 2$. The case treated in Theorem 0.1 may be described as the case where γ is slightly greater than $\binom{s+2}{2}$, so that Γ' fails to impose independent conditions on quadrics, that is, $\mathrm{H}^1(\mathscr{I}_{\Gamma'}(2)) \neq 0$. The method of proof is to find some special syzygies of ω_Γ associated to a certain pairing $W \otimes U \longrightarrow V$, where $U = (\mathrm{H}^1(\mathscr{I}_{\Gamma'}(2)))^*$. We are able to compute at least the linear part of the free resolution of these syzygies (the complex $E_\bullet(\mu)$), and this suffices.

If we examine the case where γ is slightly larger than $\binom{s+d}{d}$, we find that we can begin a similar construction. Taking U_n to be the dual of $\mathrm{H}^1(\mathscr{I}_{\Gamma'}(n+1))$ we derive in Theorem 1.2 below a pairing $\mathrm{Sym}_{n-1} W \otimes U_n \longrightarrow V$ and we show in Theorem 1.2 how to find special syzygies of ω_Γ associated to this pairing; one may get a better result this way than by looking only at the syzygies associated to $U = U_1$. If one could make a construction analogous to that of §3 to exploit these syzygies, one might construct subcomplexes of the resolution of ω_Γ in other ranges of values of γ, and perhaps these would lead to further counterexamples. The first such range of possible examples corresponds to $\gamma = \binom{s+d}{d} + 1$ and $r = \binom{s+d}{d} - s - 1$.

Specializing to $d = 3$, the first open case, we should consider sets of $s^3 + 6s^2 + 11s + 7$ points in $\mathbf{P}^{s^3+6s^2+10s+6}$. The cases $d = 3$, $s = 3, 4, 5$ correspond to 21 points in $\mathbf{P}^{16}$, 36 points in $\mathbf{P}^{30}$ and 57 points in $\mathbf{P}^{50}$ respectively. In these cases the MRC holds! However, in the case $d = 2$ one had to assume $s \geq 3$ before the counterexamples started, and it may be that one simply has to go to larger values of s in the $d = 3$ case....

To see how the proof of Theorem 0.1 works in the easiest interesting case, let Γ be a general set of $\gamma = 11$ points in $\mathbf{P}^6$, Schreyer's first example. With notation as in Theorem 0.1 we have $s = 3$ and $r = 6$. If we display the $\beta_{i,j}$ in a "Betti diagram" similar to that of the program Macaulay (Bayer-Stillman [1989–1996]), then the Minimal Resolution Conjecture predicts that the shape of $F_\bullet$, the resolution of I_Γ is:

degree						
2	17	46	45	4	–	–
3	–	–	–	25	18	4

Conjectural shape of $F_\bullet$

(Brief explanation of Betti diagrams: A complex whose i^{th}-term is $\oplus_j S(-j)^{\beta_{i,j}}$ is represented by a diagram with $\beta_{i,j}$ in the i^{th}-column (counting from 0) of a row marked as being degree $j - i$. Zeros are suppressed or replaced by dashes. Thus the resolution just displayed is the resolution of a module – in this case an ideal – with 17 generators of degree 2.) Thus if the MRC held, then the resolution of ω_Γ would have the form

degree							
−1	4	18	25	–	–	–	–
0	–	–	4	45	46	17	–
1	–	–	–	–	–	–	1

Conjectural resolution of ω_Γ

It is easy to show that, in fact, ω_Γ is generated by $W = (\omega_\Gamma)_{-1}$, which has dimension 4, just as it would be the case if this resolution were correct.

The set Γ' consists of 11 general points in $\mathbf{P}^3 = \mathbf{P}(W)$. Since the space of quadrics in $\mathbf{P}^3$ is 10-dimensional, Γ' fails by one to impose independent conditions on quadrics; that is, $\dim(U) = 1$. In this special case it is quite easy to deduce the whole resolution of the ideal $I_{\Gamma'}$ and thus of the module $\omega_{\Gamma'}$:

degree			
3	9	12	3
4	–	–	1

Resolution of $\mathscr{I}_{\Gamma'}$

The complex $E_\bullet(\mu)$ in this case takes the form

$$E_\bullet(\mu):\ 0 \longrightarrow \wedge^4 W \otimes \mathcal{O}_{\mathbf{P}^r}(-3) \longrightarrow \wedge^3 W \otimes \mathcal{O}_{\mathbf{P}^r}(-2) \longrightarrow \wedge^2 W \otimes \mathcal{O}_{\mathbf{P}^r}(-1) \longrightarrow W \otimes \mathcal{O}_{\mathbf{P}^r}\ ,$$

and can be identified with (a twist of) part of the Koszul complex of a sequence of four linear forms on $\mathbf{P}^r = \mathbf{P}^6$. (These determine a distinguished plane in $\mathbf{P}^6$ whose significance we shall explain in a moment). The desired nondegeneracy condition on μ becomes the condition that these linear forms are linearly independent, so that the complex $E_\bullet(\mu)$ is exact (in this case exactness and linear exactness coincide).

As we have remarked, the 1-genericity proved in Kreuzer [1994] implies the necessary independence in this case. To introduce the ideas used in the general case, we continue differently. Since the condition on Γ (or, equivalently, on Γ') is open, it suffices to prove the result after degenerating Γ' until it lies on a suitable curve $C \subset \mathbf{P}^s = \mathbf{P}^3$. We take C to be a general sextic curve in $\mathbf{P}^3$ of genus 3. Let H denote its hyperplane class. Such a curve is projectively normal. Using this, we show by a Koszul homology argument that the nondegeneracy condition we need follows from the vanishing condition

$$\mathrm{H}^0(\wedge^2 M_H \otimes \mathcal{O}_C(K_C + \Gamma' - 2H)) = 0\ ,$$

where M_H denotes the rank 3 vector bundle that is the kernel of the evaluation map $\mathrm{H}^0(\mathcal{O}_C(H)) \otimes \mathcal{O}_C \longrightarrow \mathcal{O}_C(H)$, and K_C is the canonical class of C.

We may think of the line bundle $\mathcal{O}_C(L) := \mathcal{O}_C(K_C + \Gamma' - 2H)$ simply as a general line bundle of degree $(2g(C) - 2) + 11 - 2 \cdot 6 = 3$ on C. To prove the required vanishing, we degenerate C to a hyperelliptic curve C_1, and simultaneously let H degenerate to $3H_0$, where H_0 denotes the hyperelliptic involution.

It is easy to see that $\mathcal{O}_{C_1}(3H_0)$ is non-special and that

$$M_{3H_0} \cong \bigoplus_{i=1}^{3} \mathcal{O}_{C_1}(-H_0)\ .$$

Thus this bundle on C_1 is a degeneration of M_H on C. It now suffices to show that

$$\mathrm{H}^0((\wedge^2 M_{3H_0}) \otimes \mathcal{O}_{C_1}(L)) = \bigoplus_{i=1}^{3} \mathrm{H}^0(\mathcal{O}_{C_1}(L-2H_0)) = 0 \ .$$

The degree of $\mathcal{O}_{C_1}(L-2H_0)$ is $3-4=-1$, so the result is immediate.

As shown in Proposition 2.1 and Corollary 3.3, the nondegeneracy of μ implies that the complex $E_\bullet(\mu)(1)$ is a subcomplex of the resolution of ω_Γ. It has Betti display

degree							
-1	4	6	4	1	–	–	–

$E_\bullet(\mu)(1)$

Under these circumstances *each* Betti number in the resolution of ω_Γ must be at least as big as the one in $E_\bullet(\mu)(1)$. Thus a lower bound for the size of the actual resolution of ω_Γ is given by the following Betti diagram (where we have indicated the differences from the MRC with boxes).

degree							
-1	4	18	25	$\boxed{1}$	–	–	–
0	–	–	$\boxed{5}$	45	46	17	–
1	–	–	–	–	–	–	1

Lower bound for the resolution (= actual resolution) of ω_Γ

In this case computation shows that our lower bound gives the actual value of the $\beta_{i,j}$. (A similar fact holds for our example in $\mathbf{P}^7$, but not, for example, in $\mathbf{P}^8$.)

There is a curiosity still to explain in the case $r=6$ just described. As we noted, our construction starts from 11 general points in $\mathbf{P}^6$ and produces a distinguished 2-plane $\Pi \subset \mathbf{P}^6$, defined by the four linear forms that enter in the pairing μ. One can show that the plane Π is spanned by (any) three points which together with the 11 initial ones form an arithmetically Gorenstein set in $\mathbf{P}^6$; see Eisenbud-Popescu [1998]. Charles Walter has pointed out to us that Π could also be interpreted as the unique plane in $\mathbf{P}^6$ such that the projection of the 11 points from this plane into $\mathbf{P}^3$ is equivalent to the Gale transform of the 11 points. (This latter characterization follows directly from the theory in §1.)

A more refined minimal free resolution conjecture can be formulated for general subsets Γ of points in a given subvariety X of $\mathbf{P}^r$. The idea is to give oneself the resolution of the ideal of X, and use it to give a minimal possible form for the Betti numbers of Γ. Some results on this problem have been discovered by Mircea Mustaţa [1998]. He proves (for example) that this refined minimal free resolution conjecture is true for points on a general rational quintic curve in $\mathbf{P}^3$, but fails for a rational quintic curve contained in a quadric surface! (In general, the answer for rational curves depends on the splitting of the restriction of the tangent bundle of $\mathbf{P}^3$ to the curve.)

One may also formulate a minimal resolution conjecture for algebraic sets Γ of higher dimension. It is interesting to compare the case of points with that of curves. The minimal resolution conjecture for complete embeddings of large degree (compared with the genus) general curves was shown to be false by Schreyer [1983], and Green-Lazarsfeld [1988]; the failure comes essentially from the existence of special divisors on the curve, which give rise to rational normal scrolls containing the curve, and is quite different in character from the phenomena exhibited here. By contrast, no counterexamples to the appropriate minimal free resolution conjecture are known for ideals in a polynomial ring which are made from a generic vector space of forms of some degree d plus all the forms of degree $d+1$; these are the ideals that seem to be the most reasonable analogue of ideals of general sets of points. However, the problem is computationally difficult, and not many cases have been examined.

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1. The Gale Transform and canonical modules

We first give a modern definition of the Gale transform, and relate the canonical modules of a set of points and its Gale transform. Next we exhibit a peculiar module which maps to the canonical module of a suitable set of points. This module has a natural interpretation in terms of the Gale transform. In the next section we will exhibit a subcomplex of the resolution of this module that is "responsible" for the failure of the minimal resolution conjecture.

Recall that a linear series on a scheme X is a pair (V, L) consisting of a line bundle L on X and a vector space V of global sections of L. The Gale transform is an involution on the space of linear series on a finite Gorenstein scheme Γ. Of course it is somewhat pedantic to speak of line bundles and global sections on a finite scheme, since any such scheme is affine and every line bundle is trivial, but it has the same virtues as does the distinction between a vector space and its dual: this language will allow us to make definitions without any arbitrary choices.

If Γ is a Gorenstein scheme, finite over a field k, then Serre duality provides a canonical "trace" $\tau : \mathrm{H}^0(K_\Gamma) \longrightarrow k$ with the property that for any line bundle L on Γ the composition

$$\mathrm{H}^0(L) \otimes_k \mathrm{H}^0(K_\Gamma \otimes L^{-1}) \longrightarrow \mathrm{H}^0(K_\Gamma) \longrightarrow k$$

of τ with the multiplication map gives a perfect pairing between $\mathrm{H}^0(L)$ and $\mathrm{H}^0(K_\Gamma \otimes L^{-1})$. If $V \subset \mathrm{H}^0(L)$ is a subspace, then we write $V^\perp \subset \mathrm{H}^0(K_\Gamma \otimes L^{-1})$ for the annihilator.

Definition. *Let k be a field, and let Γ be a Gorenstein scheme finite over k. The Gale transform of a linear series (V, L) on Γ is the linear series $(V^\perp, K_\Gamma \otimes L^{-1})$.*

See Eisenbud-Popescu [1998] for more information and a taste of the rich history of this notion.

We write $S = \mathrm{Sym}(V)$ for the symmetric algebra of V, a polynomial ring over k, and $I_{\Gamma,V} \subset S$ for the homogeneous ideal of the image of Γ in $\mathbf{P}(V)$. Along with the linear series (V, L) we will be interested in its Veronese powers: we write V^n for the image in $\mathrm{H}^0(L^n)$ of $S_n = \mathrm{Sym}_n(V)$. Just as with V we define $(V^n)^\perp \subset \mathrm{H}^0(L^{-n} \otimes K_\Gamma)$ to be the annihilator of V^n.

We now assume for simplicity that Γ is embedded in $\mathbf{P}^r = \mathbf{P}(V)$ by (V, L), and we write

$$\omega_{\Gamma,V} = \mathrm{Ext}_S^{\dim V - 1}(S/I_{\Gamma,V}, S(-\dim(V)))$$

for the canonical module of $S/I_{\Gamma,V}$. (Note that $\omega_{\Gamma,V}$ was called ω_Γ in the introduction.) We will also be interested in the submodule

$$(\omega_{\Gamma,V})_{\geq -1} = \bigoplus_{n \geq -1} (\omega_{\Gamma,V})_n \ .$$

We have

Proposition 1.1. *Let Γ be a finite Gorenstein scheme, and suppose that (V, L) is a very ample linear series on Γ. Then the degree $(-n)$ part of $\omega_{\Gamma,V}$ coincides with $(V^n)^\perp$, and this identification is compatible with multiplication by elements of V. In particular, the degree 0 first order syzygies of $(\omega_{V,\Gamma})_{\geq -1}$ correspond to elements of*

$$N = \ker(V \otimes V^{\perp} \longrightarrow \mathrm{H}^0(K_\Gamma)) \ .$$

If the Gale transform $(W := V^{\perp}, L^{-1} \otimes K_\Gamma)$ defines an embedding of Γ in $\mathbf{P}(W)$, it follows that the same space N gives also the linear relations on the degree (-1) part of $\omega_{W,\Gamma}$. This is the fundamental relation between the resolutions corresponding to the embeddings of Γ by V and by its Gale transform.

Proof of Proposition 1.1. Observe that $\omega_{\Gamma,V}$ is the dual (as graded vector spaces) of $\oplus_n \mathrm{H}^1(\mathscr{I}_{\Gamma,\mathbf{P}(V)}(n))$, and this identification is compatible with multiplication by elements of V. Thus we must prove that

$$(V^n)^{\perp} = \mathrm{H}^1(\mathscr{I}_{\Gamma,\mathbf{P}(V)}(n))^* \ .$$

On the other hand, the exact sequence

$$0 \longrightarrow \mathscr{I}_{\Gamma,\mathbf{P}(V)} \longrightarrow \mathscr{O}_{\mathbf{P}(V)} \longrightarrow \mathscr{O}_\Gamma \longrightarrow 0$$

gives rise to an exact sequence

$$\left[\mathrm{Sym}_n V = \mathrm{H}^0(\mathscr{O}_{\mathbf{P}(V)}(n))\right] \longrightarrow \mathrm{H}^0(L^n) \longrightarrow \mathrm{H}^1(\mathscr{I}_{\Gamma,\mathbf{P}(V)}(n)) \longrightarrow \left[0 = \mathrm{H}^1(\mathscr{O}_{\mathbf{P}(V)}(n))\right] \ ,$$

which yields the required identification. The compatibility with multiplication follows at once.

For the last statement note that $V^{\perp} = (\omega_{\Gamma,V})_{-1}$, so the degree zero first order syzygies of $(\omega_{\Gamma,V})_{\geq -1}$ are by definition the elements of $V \otimes V^{\perp}$ that map to zero via the natural multiplication to

$$(\omega_{\Gamma,V})_0 \subset \mathrm{H}^0(K_\Gamma) \ . \qquad \square$$

Let $W := V^{\perp}$ and set $U_n := (W^n)^{\perp}$. (If, as in the setting of Theorem 0.1, Γ is embedded in $\mathbf{P}(W)$, then U_n is the dual of the vector space $\mathrm{H}^1(\mathscr{I}_{\Gamma,\mathbf{P}(W)}(n)) \neq 0$ as in the proof of Proposition 1.1.) We shall see that the elements of U_n give rise to a family of syzygies on $(\omega_{\Gamma,V})_{\geq -1}$.

To do things invariantly we use the Schur functors

$$\begin{aligned} S_{a,b}(W) :&= \mathrm{Im}(\wedge^{a+1} W \otimes \mathrm{Sym}_{b-1} W \longrightarrow \wedge^a W \otimes \mathrm{Sym}_b W) \\ &= \ker(\wedge^a W \otimes \mathrm{Sym}_b W \longrightarrow \wedge^{a-1} W \otimes \mathrm{Sym}_{b+1} W) \ . \end{aligned}$$

For example, $S_{1,1}(W) = \wedge^2 W$, the inclusion into $\wedge^a W \otimes \mathrm{Sym}_b W = W \otimes W$ being the diagonal map of the exterior algebra.

The following result identifies the syzygies in which we are interested. (For Theorem 0.1 we will use only the case $n = 2$.)

Theorem 1.2. *Let Γ be a Gorenstein scheme, finite over the field k. Let (V, L) and $(W = V^{\perp}, K_{\Gamma} \otimes L^{-1})$ be dual linear series, and set $U_n := (W^n)^{\perp}$, with $n \geq 1$. The natural multiplication*

$$\mu_n : \mathrm{Sym}_{n-1} W \otimes U_n \longrightarrow V$$

induces a map of free $k[V]$-modules

$$\delta_n : S_{1,n-1}(W) \otimes U_n \otimes k[V] \longrightarrow W \otimes k[V](1)$$

whose image lies in the kernel of the natural map

$$W \otimes k[V](1) \longrightarrow \omega_{\Gamma,V} \ .$$

Thus there is a unique map of $k[V]$-modules $(\mathrm{coker}\,\delta_n) \longrightarrow \omega_{\Gamma,V}$ *which extends the inclusion $W = (\omega_{\Gamma,V})_{-1} \subset \omega_{\Gamma,V}$.*

Proof. We define δ_n to be the composite

$$S_{1,n-1}(W) \otimes U_n \otimes k[V] \longrightarrow W \otimes \mathrm{Sym}_{n-1} W \otimes U_n \otimes k[V] \xrightarrow{W \otimes \mu_n} W \otimes k[V](1) \ .$$

As in Proposition 1.1, let N be the kernel of the multiplication map $m : W \otimes V \longrightarrow \mathrm{H}^0(K_{\Gamma})$, the space of degree 0 first order syzygies of $\omega_{\Gamma,V}$. We must show that the image of δ_n is contained in the submodule of $W \otimes k[V](1)$ generated by N, or equivalently that N contains the image of the composite

$$S_{1,n-1}(W) \otimes U_n \longrightarrow W \otimes \mathrm{Sym}_{n-1} W \otimes U_n \xrightarrow{W \otimes \mu_n} W \otimes V \ .$$

Now the natural multiplication map $\mu_n' : \mathrm{Sym}_n W \otimes U_n \longrightarrow \mathrm{H}^0(K_{\Gamma})$ fits in the diagram

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] & W \otimes V \arrow[r, "m"] & W \cdot V \arrow[r, hook] & \mathrm{H}^0(K_{\Gamma}) \\
0 \arrow[r] & S_{1,n-1}(W) \otimes U_n \arrow[r] \arrow[u, dotted] & W \otimes \mathrm{Sym}_{n-1}(W) \otimes U_n \arrow[rr] \arrow[u, "W \otimes \mu_n"'] & & \mathrm{Sym}_n(W) \otimes U_n \arrow[u, "\mu_n'"']
\end{tikzcd}$$

which commutes by the associativity of multiplication, and has exact rows by the definition of N and $S_{1,n-1}$, and this gives the desired inclusion. $\square$

Remarks. 1) It would be possible to make the constructions of this section without assuming that Γ is embedded in $\mathbf{P}(V)$: we would simply work directly with $(V^n)^{\perp}$ in place of $\omega_{\Gamma,V}$.

2) There is a less invariant version of the ideas of this section which is pleasingly direct: Again let Γ be a Gorenstein scheme, finite over a field k, and let $\mathcal{O}_\Gamma$ be the coordinate ring of Γ, a finite dimensional Gorenstein k-algebra. Suppose that Γ is embedded in $\mathbf{P}^r$. If we choose a hyperplane not meeting Γ, we may identify the line bundle $L = \mathcal{O}_\Gamma(1)$ with $\mathcal{O}_\Gamma$, and thus identify the linear series $(V = \mathrm{H}^0(\mathcal{O}_{\mathbf{P}^r}(1)), L)$ with a subspace $V \subset \mathcal{O}_\Gamma$. We also choose an identification of $\mathcal{O}_\Gamma$ with K_Γ (equivalently, we may choose a "trace" functional $\tau : \mathcal{O}_\Gamma \longrightarrow k$ not vanishing on any component of the socle of $\mathcal{O}_\Gamma$) and consider the pairing on $\mathcal{O}_\Gamma$ defined as the composition of this functional with multiplication. We may again define the powers V^n and the spaces $W_n := (V^{-n})^\perp$, but this time they will all be subspaces of $\mathcal{O}_\Gamma$.

2. Linear exactness and linear rigidity

In §3 below we will define a complex of free $k[V]$ modules beginning with the map δ_2, and thus admitting a map to the minimal free resolution of $\omega_{\Gamma,V}$. To prove Theorem 0.1 we will show that this map of complexes is an inclusion in certain circumstances. For this purpose we will use the notion of linear exactness developed in this section.

Let S be a graded ring with $S_0 = k$ a field, and let

$$E_\bullet \; : \; \cdots \longrightarrow E_{i+1} \longrightarrow E_i \longrightarrow \cdots \longrightarrow E_n$$

be a *linear complex* in the sense that each E_i is a free module generated in degree i, so that in particular the differentials are given by matrices of elements of S_1. We shall say that $E_\bullet$ is *linearly exact* if, for all $i > n$, the homology $\mathrm{H}_i(E_\bullet)$ is nonzero only in degrees $> i$, or equivalently if, in any matrix representing a differential of $E_\bullet$, the columns are linearly independent over S_0. (Shifting degrees, we could of course restrict ourselves to the case $n = 0$ without losing anything, but the notation above will be convenient since, in the application below, we will be dealing with resolutions of modules that are not generated in degree 0.)

If M is an S-module generated in degrees $\geq n$ and

$$F_\bullet \; : \; \cdots \longrightarrow F_{i+1} \longrightarrow F_i \longrightarrow \cdots \longrightarrow F_n$$

is the minimal free resolution of M, then F_i is generated in degrees $\geq n$ and we define the *linear part* of $F_\bullet$ to be free the subcomplex $E_\bullet$ such that E_i is the free summand of F_i generated in degree i. It is easy to see that $E_\bullet$ is linearly exact. The following result provides a converse, and gives a criterion for a map from a linearly exact complex to be a monomorphism.

Proposition 2.1. *A linear complex $E_\bullet$ as above is linearly exact iff it is the linear part of a minimal free resolution.*

Further, if $E_\bullet$ is linearly exact, $\alpha : E_\bullet \longrightarrow G_\bullet$ is a map to a free complex with each G_i generated in degrees $\geq i$, and $\alpha_n : E_n \longrightarrow G_n$ is a monomorphism, then α is a split monomorphism.

Proof. We prove the second statement first. Since G_n is generated in degrees $\geq n$, any set of elements of degree n in G_n that are linearly independent over $S_0 = k$ are part of a free basis. Thus E_n maps to a direct summand of G_n. Since $\mathrm{H}_{n+1}(E_\bullet)$ is zero in degree $n+1$, and E_{n+2} is zero in degree $n+1$, we see that the degree $n+1$ part of E_{n+1} maps monomorphically into E_n. Thus it maps monomorphically into G_{n+1}. The second statement of the Proposition now follows by induction.

Again suppose that $E_\bullet$ is linearly exact. Let $F_\bullet$ be the minimal free resolution of the the cokernel of $E_{n+1} \longrightarrow E_n$, so that in particular $F_n = E_n$. Let α be any comparison map that lifts this identity map; by the statement already proved α is a split inclusion of complexes, and we will regard $E_\bullet$ as a subcomplex of $F_\bullet$. We will show it is the linear part.

It suffices to show that E_m contains the whole degree m part of F_m; this is true by definition for $m = n$ and $m = n+1$. Assume by induction that $m \geq n+2$ and that the conclusion is valid for E_{m-1} and E_{m-2}. Since $\mathrm{H}_{m-1}(E_\bullet)$ is zero in degree m, the degree m part of E_m must coincide with the kernel of the restricted differential of $E_\bullet$ from the degree m part of E_{m-1} into that of E_{m-2}, or equivalently into F_{m-2}. Any degree m element of $x \in F_{m-1}$ that is not in E_{m-1} may be taken as a minimal generator of F_{m-1}, so by the minimality of $F_\bullet$ the element x is not in the kernel of $F_{m-1} \longrightarrow F_{m-2}$. Thus the degree m part of the kernel is contained in E_{m-1}, and the degree m part of F_m is equal to that of E_m as asserted. This proves that $E_\bullet$ is the linear part of $F_\bullet$. □

For complexes derived from resolutions it is often sufficient to check linear exactness at the first step:

Lemma 2.2 (Linear Rigidity). *Let $R = k[x_0, \ldots, x_r]$ be a polynomial ring, and let $E_\bullet$ be a linearly exact complex of finitely generated free R-modules starting from E_n with E_i generated in degree i, as above. Let S be a finitely generated positively graded R-algebra, with $S_0 = k$. The complex $G_\bullet := S \otimes_R E_\bullet$ is linearly exact iff $\mathrm{H}_{n+1}(G_\bullet)$ is zero in degree $n+1$.*

Proof. By Lemma 2.1 the complex $E_\bullet$ is the linear part of the free resolution of some R-module M generated in degree n. We must show that if $\mathrm{Tor}_1^R(S, M)_{\leq n+1} = 0$, then $\mathrm{Tor}_i^R(S, M)_{\leq n+i} = 0$ for each $i \geq 1$; a "linear rigidity" theorem for Tor. The proof of Auslander-Buchsbaum [1958] for the rigidity of Tor (reduction to the diagonal plus the rigidity of the Koszul complex) may easily be adapted, using the graded version of Nakayama's Lemma. □

In the theory above it actually suffices to suppose that E_i is generated in degree i just for $i > n$. Thus we may try to apply the theory to the Eagon-Northcott complex and so we see that the complex is linearly exact iff the minors are independent. This leads to the following:

Problem: Under what conditions are the $d \times d$ minors of an $e \times f$ matrix of linear forms linearly independent?

3. The complexes $E_\bullet^m(\mu)$

Given a pairing $\mu : W \otimes U \longrightarrow V$ of vector spaces we define a complex $E_\bullet(\mu)$ of free modules over a polynomial ring $k[V]$ beginning with a map δ which, in the notation of Theorem 1.2, is the map δ_2. (As explained in the introduction, it would be interesting to do something similar for complexes beginning with the map δ_n in general.)

We will show that for the "generic" pairing $\mu : W \otimes U \xrightarrow{=} W \otimes U$ the complex $E_\bullet(\mu)$ is a resolution, so that we can apply Theorem 2.2 to prove that $E_\bullet(\mu)$ is linearly exact for a special μ. In the setting of Theorem 1.2, the complex $E_\bullet(\mu)$ will admit a map to the minimal free resolution of $\omega_{\Gamma,V}$. In the next section we will show that under the hypothesis of Theorem 0.1 the complex $E_\bullet(\mu)$ is linearly exact, and thus that the map is a monomorphism by Proposition 2.1.

To prove that the complex $E_\bullet(\mu)$ is a resolution in the generic case, we use an inductive argument that requires a larger family of complexes, here called $E_\bullet^m(\mu)$; the desired complex $E_\bullet(\mu)$ will appear as $E_\bullet^{-1}(\mu)$.

To construct the complexes we use divided powers. Let U be a finitely generated free module over some ring. We write $D_l U$ for the l^{th}-divided power of U. It is convenient to define $D_l U$ as the dual of the l^{th}-symmetric power of the dual module, that is $D_l U = (\mathrm{Sym}_l(U^*))^*$. What we shall use about $D_l U$ is that it has a "diagonal" map

$$D_{l+1}U \longrightarrow D_l U \otimes U$$

which is the monomorphism dual to the surjective natural multiplication map

$$\mathrm{Sym}_l(U^*) \otimes U^* \longrightarrow \mathrm{Sym}_{l+1}(U^*) \ .$$

See for example Eisenbud [1995, Appendix 2] for the usual definition.

Suppose again that S is a graded ring, with $S_0 = k$ a field, and that U and W are finite dimensional vector spaces over k. Let $\mu : W \otimes U \longrightarrow S_1 = V$ be a homomorphism.

For any integer m, and any integer $l \geq 0$ we define a free module

$$E_l^m(\mu) := \wedge^{l-m} W \otimes D_l U \otimes S(-l)$$

and a map

$$\delta^m_{l+1}(\mu) \,:\, E^m_{l+1}(\mu) \longrightarrow E^m_l(\mu) \ ,$$

which is the composite of the tensor product of the diagonal maps of the exterior and divided powers,

$$\wedge^{l+1-m} W \otimes D_{l+1} U \otimes S(-l-1) \longrightarrow \wedge^{l-m} W \otimes W \otimes D_l U \otimes U \otimes S(-l-1) \ ,$$

and the map induced by μ

$$\wedge^{l-m} W \otimes W \otimes D_l U \otimes U \otimes S(-l-1) \longrightarrow \wedge^{l-m} W \otimes D_l U \otimes S(-l) \ .$$

These maps form complexes of free S-modules

$$E^m_\bullet(\mu) \,:\, \cdots \longrightarrow E^m_{l+1}(\mu) \xrightarrow{\delta^m_{l+1}(\mu)} E^m_l(\mu) \longrightarrow \cdots \longrightarrow E^m_{m_+}(\mu)$$

where the term $E^m_l(\mu)$ is in position l, and m_+, which denotes the positive part of m, is equal to m if $m \geq 0$ and to 0 if $m \leq 0$. We set $E_\bullet(\mu) = E^{-1}_\bullet(\mu)$.

The complexes $E^m_\bullet(\mu)$ are linear in the sense of §2. If u is the rank of U, then $E^{-u}_\bullet(\mu)$ is precisely the linear part of the Eagon-Northcott complex resolving the maximal minors of μ, whence the name E. As with the Eagon-Northcott complex, these complexes may be built up inductively:

Proposition 3.1 (Inductive Construction). *With notation as above, suppose that*

$$0 \longrightarrow W' \longrightarrow W \longrightarrow k \longrightarrow 0$$

is an exact sequence, and let $\mu' : W' \otimes U \longrightarrow S_1$ denote the composition of μ with the inclusion $W' \otimes U \longrightarrow W \otimes U$. There is an exact sequence of complexes

$$0 \longrightarrow E^m_\bullet(\mu') \longrightarrow E^m_\bullet(\mu) \longrightarrow E^{m+1}_\bullet(\mu') \longrightarrow 0 \ .$$

Proof. We use the exact sequence

$$0 \longrightarrow \wedge^{l-m} W' \longrightarrow \wedge^{l-m} W \longrightarrow \wedge^{l-m-1} W' \longrightarrow 0 \ .$$

The commutativity of the necessary diagrams follows by straightforward computation. □

The main result of this section establishes linear exactness:

Theorem 3.2. *Let k be a field, and let W and U be finite dimensional vector spaces over k. Let $S = k[W \otimes U]$ be the symmetric algebra. If $\mu : W \otimes U \longrightarrow S(1)$ is the identity map, then the complex $E_\bullet^m(\mu)$ is linearly exact.*

Proof. We do induction on the rank w of W. If $w = 1$ and $m < 0$ there is nothing to prove. If $w = 1$ and $m \geq 0$, then exactness follows from the fact that the diagonal map $D_{m+1}U \longrightarrow D_m U \otimes U$ is a monomorphism.

Suppose now $w > 1$. Let W' be a codimension 1 subspace of W, so that we have an exact sequence

$$0 \longrightarrow W' \longrightarrow W \longrightarrow k \longrightarrow 0 \ .$$

Using the long exact sequence in homology coming from the inductive construction in Proposition 3.1, everything is clear except the cases where $m \geq 0$ and $i = m + 1$. In this case the exact sequence of complexes has the form

$$\begin{array}{llll}
\textbf{Homological degree:} & & \boldsymbol{m+1} & \boldsymbol{m} \\
E_\bullet^{m+1}(\mu') : & \cdots \longrightarrow & D_{m+1}U \otimes S(-m-1) & \\
\uparrow & & \uparrow & \\
E_\bullet^{m}(\mu) : & \cdots \longrightarrow & W \otimes D_{m+1}U \otimes S(-m-1) \longrightarrow & D_m U \otimes S(-m) \\
\uparrow & & \uparrow & \| \\
E_\bullet^{m}(\mu') : & \cdots \longrightarrow & W' \otimes D_{m+1}U \otimes S(-m-1) \longrightarrow & D_m U \otimes S(-m)
\end{array}$$

and we must show that the connecting homomorphism

$$c : H_{m+1}(E_\bullet^{m+1}(\mu')) \longrightarrow H_m(E_\bullet^m(\mu'))$$

is a monomorphism in degree $m + 1$, which is the lowest degree present in $E_{m+1}^{m+1}(\mu') = D_{m+1}U \otimes S(-m-1)$.

We have $(H_{m+1}(E_\bullet^{m+1}(\mu')))_{m+1} = D_{m+1}(U)$. Let $f \in D_{m+1}(U)$, and write $\sum_i f_i \otimes f_i' \in U \otimes D_m(U)$ for the image of the diagonal map. If we write $W = \langle x \rangle \oplus W'$, and $\bar{f}$ for the class of f in the homology of $E_\bullet^{m+1}(\mu')$, then we see by chasing the diagram that

$$c(\bar{f}) = \sum_i (x \otimes f_i) \otimes f_i' \ \in \ (W \otimes U) \otimes D_m(U) \ .$$

Since the differential of $E_\bullet^m(\mu')$ involves only W', the homology module $\mathrm{H}_m(E_\bullet^m(\mu'))$ surjects onto $(\langle x \rangle \otimes U) \otimes D_m(U) = U \otimes D_m(U)$, and we may recover $\sum_i f_i \otimes f_i'$ as the image of $c(\bar{f})$. Since the diagonal map is a monomorphism on the divided powers, we are done. $\square$

A closer examination of the induction shows that the generic complexes in Theorem 3.2 actually are resolutions if $m \le -w+1$, but not otherwise; for example, if $m = 0$, $w := \dim W = 2$, and $u := \dim U > 2$, then the complex $E^m_\bullet(\mu)$ has the form

$$S^u(-1) \cong \wedge^2 W \otimes U \otimes S(-1) \longrightarrow W \otimes S = S^2 \ ,$$

and the free resolution of which this is the linear part is the Buchsbaum-Rim complex

$$\cdots \longrightarrow W^* \otimes (\wedge^3 S^u)(-3) \longrightarrow S^u(-1) \longrightarrow S^2 \ .$$

The degree 2 relations $W^* \otimes (\wedge^3 S^u)(-3) \longrightarrow S^u(-1)$ are an expression of Cramer's rule. See Eisenbud [1995, Appendix A2.6] for more information.

In our setting the map μ has a geometric origin, and we may use a technique similar to Green's Koszul Homology to check the condition of linear exactness. The following is the result of this section that we shall use in the sequel:

Corollary 3.3. *Let C be a projective scheme over a field k, and let H, L be Cartier divisors on C. Suppose that $\mathcal{O}_C(H)$ is generated by its global sections $W := \mathrm{H}^0(\mathcal{O}_C(H))$, and let M_H be the vector bundle on C which is the kernel of the evaluation map $d_0 : \mathcal{O}_C \otimes W \longrightarrow \mathcal{O}_C(H)$. Set $U := \mathrm{H}^0(\mathcal{O}_C(L))$, and $V := \mathrm{H}^0(\mathcal{O}_C(H+L))$. Let $S = \mathrm{Sym}\, V$ be the polynomial ring, and let $\mu : W \otimes U \longrightarrow V = S_1$ be the multiplication map. The complex $E_\bullet(\mu)$ is linearly exact if and only if* $\mathrm{H}^0(\wedge^2 M_H \otimes \mathcal{O}_C(L)) = 0$.

Proof. By the linear rigidity lemma it is enough to check linear exactness at the first step; that is, we must show that the induced map $\wedge^2 W \otimes U \longrightarrow W \otimes V$ is a monomorphism. For this purpose we use the Koszul complex built on the evaluation map d_0,

$$\begin{aligned} \cdots \xrightarrow{d_3} \wedge^3 W \otimes \mathcal{O}_C(-3H) \xrightarrow{d_2} \wedge^2 W \otimes \mathcal{O}_C(-2H) \\ \xrightarrow{d_1} W \otimes \mathcal{O}_C(-H) \xrightarrow{d_0} \mathcal{O}_C \longrightarrow 0 \ , \end{aligned}$$

tensored with $\mathcal{O}_C(2H+L)$. Now $\ker d_i = \mathrm{Im}\, d_{i+1} = \wedge^{i+1} M_H\ (-(i+1)H)$, for all $i \ge 0$, so the claim of the lemma follows by taking global sections in the short exact sequence

$$0 \longrightarrow \wedge^2 M_H \otimes \mathcal{O}_C(L) \longrightarrow \wedge^2 W \otimes \mathcal{O}_C(L) \xrightarrow{d_1 \otimes \mathcal{O}_C(2H+L)} W \otimes \mathcal{O}_C(H+L) \ . \ \square$$

Remark. We have made the restriction to projective schemes only to ensure the finite dimensionality of the spaces involved. This is actually unnecessary; the complexes $E_\bullet^m$ could have been developed for infinite dimensional spaces. We leave these things to the reader who can find an application...

4. Subcomplexes of the resolution of I_Γ

We prove in this section the main result concerning resolutions of general sets of points $\Gamma \subset \mathbf{P}(V)$ in the range of Theorem 0.1: If $\mu = \mu_2$ is chosen as in §1, then the complex $E_\bullet(\mu) = E_\bullet^{-1}(\mu)$ defined above is linearly exact, and is up to a shift a subcomplex of the minimal free resolution of $\omega_{\Gamma,V}$.

Theorem 4.1. *Let V be an $(r+1)$-dimensional vector space over a field k, and let Γ be a general set of γ points in $\mathbf{P}(V)$. Let $W := V^\perp \subset \mathrm{H}^0(K_\Gamma(-1))$, let $U := (W^2)^\perp \subset \mathrm{H}^0(K_\Gamma^{-1}(2))$, and let $\mu: W \otimes U \longrightarrow V$ be the natural multiplication map. Set*

$$r := \binom{s+1}{2} + t, \quad s \geq 2, \quad 0 \leq t \leq s, \text{ and } \gamma := r + s + 2 \ .$$

If s is even suppose also $\operatorname{char} k \neq 2$. *The complex $E_\bullet(\mu)$ is a direct summand of the minimal free resolution of $\omega_{\Gamma,V}(-1)$.*

Remarks. The given number of points in $\mathbf{P}^r = \mathbf{P}(V)$ is actually the largest number for which the construction is interesting; for smaller numbers there is still a nontrivial complex but it is only sometimes linearly exact. The restriction to characteristic $\neq 2$ is most likely unnecessary, but is not important as our main Theorem 0.1 follows in all characteristics from the characteristic 0 case. The restriction comes only from the use of a theorem of Hartshorne and Gieseker on the semistability of symmetric powers of semistable vector bundles at the very end of the argument.

Proof of Theorem 4.1. We shall show that the complex $E_\bullet(\mu)$ is linearly exact. Since $s \geq 2$ we have $\gamma \leq \binom{r+2}{2}$, so Γ imposes independent conditions on quadrics and thus the homogeneous ideal I_Γ is 3-regular. It follows that, with notation as in Theorem 1.2, $\omega_{\Gamma,V}$ is generated in degrees ≥ -1. The dual of the free resolution of I_Γ is (the beginning of) the minimal free resolution of $\omega_{\Gamma,V}(r+1)$. We will thus deduce Theorem 4.1 from Theorem 1.2 and Proposition 2.1, applied to the complex $E_\bullet(\mu)$ and the minimal free resolution of $\omega_{\Gamma,V}(-1)$.

Our strategy for proving linear exactness is as follows. We wish to apply Corollary 3.3. To do this we must find a scheme C such that W may be interpreted as a space of sections generating a line bundle $\mathcal{O}_C(H)$ and U may be interpreted as the space of all sections of a line bundle $\mathscr{L}$. It is most convenient to regard Γ by its "other" embedding as the Gale transform Γ', since there W is the space of sections of the line bundle responsible for the embedding in $\mathbf{P}^s$, while U may be identified with $\mathrm{H}^1(\mathscr{I}_{\Gamma'}(2))^*$. We cannot take $C = \Gamma'$ itself, however, because U is not a complete linear series. Thus we need some higher-dimensional scheme on which Γ' lies. Since the general set of points Γ' does not (as far as we know) lie on any useful schemes of larger dimension, we will make a degeneration, using the (obvious) openness of the locus, in the space of maps $\mu : W \otimes U \longrightarrow V$, where $E_\bullet(\mu)$ is linearly exact. We shall degenerate Γ' to a set of points, lying on a convenient curve C. In doing this, we must keep the dimensions of W and U constant. Since $V = W^\perp$, the constancy of its dimension is then automatic.

Since Γ' is a general set of $\gamma > \binom{s+2}{2}$ points in $\mathbf{P}^s$, it lies on no quadrics, and this fact determines the dimension of U as $h^1(\mathscr{I}_{\Gamma'}(2))$. We may thus degenerate Γ' to a general subset of a curve C in $\mathbf{P}^s$ that lies on no quadrics (which we will again call Γ').

In order to establish a simple relation between the cohomology of $\mathscr{I}_{\Gamma'}$ and bundles on the curve we will require C to be non-special and quadratically normal. Thus writing H for the hyperplane class on C and setting $d := \deg H,\ g := \text{genus } C$, we need

$$s + 1 = h^0(\mathscr{O}_C(H)) = d + 1 - g$$
$$\binom{s+2}{2} = h^0(\mathscr{O}_C(2H)) = 2d + 1 - g,$$

which in turn yield $d = \binom{s+1}{2}$ and $g = \binom{s}{2}$.

It is easy to compute that the curve defined in $\mathbf{P}^s$ by the vanishing of the 3×3 minors of a general $3 \times (s+1)$ matrix of linear forms M has exactly the invariants required. From the existence of this curve C, and the openness of the desired properties, we see that we may take C to be a general curve of genus $\binom{s}{2}$, embedded by the complete linear series associated to a general divisor H of degree $\binom{s+1}{2}$ in $\mathbf{P}^s$. We will use this freedom to make further degenerations.

The binomial form of the genus formula suggests a plane curve of degree $s+1$, and it is amusing to note that the determinantal curve just defined may be embedded in the plane by the line bundle that is the cokernel of the restriction of M to the curve; in this planar embedding its equation is the determinant of the $(s+1) \times (s+1)$ matrix of linear forms in 3 variables which is adjoint to M. We shall use this construction implicitly later in the proof.

If Γ' is a general divisor of degree γ on a curve C as above, then we can write μ as a map coming from bundles on C as follows: Since $\mathscr{O}_C(H)$ is non-special and the curve C is projectively normal, the cohomology of the short exact sequences

$$0 \longrightarrow \mathscr{I}_C(mH) \longrightarrow \mathscr{I}_{\Gamma'}(mH) \longrightarrow \mathscr{O}_C(mH - \Gamma') \longrightarrow 0 \ ,$$

together with Serre duality yield

$$\begin{aligned} \mathrm{H}^1(\mathscr{I}_{\Gamma'}(mH)) &\cong \mathrm{H}^1(\mathscr{O}_C(mH - \Gamma')) \\ &\cong (\mathrm{H}^0(\mathscr{O}_C(K_C + \Gamma' - mH)))^*, \quad \text{for all } m \geq 1 \ . \end{aligned}$$

We now set $L := K_C + \Gamma' - 2H$, and we have $U = \mathrm{H}^0(\mathscr{O}_C(L))$ as required. Since γ is greater than the genus of C, we may simply describe L as the general divisor of degree $2g - 2 + \gamma - 2d = r - s \leq \binom{s+1}{2}$. With these identifications the pairing $\mu\colon W \otimes U \longrightarrow V$ becomes the multiplication

$$\mu : \mathrm{H}^0(\mathscr{O}_C(H)) \otimes \mathrm{H}^0(\mathscr{O}_C(L)) \longrightarrow \mathrm{H}^0(\mathscr{O}_C(L + H)) \ .$$

By Corollary 3.3 it now suffices to prove for each $s \geq 2$ that $\mathrm{H}^0(\wedge^2 M_H \otimes \mathscr{O}_C(L)) = 0$ where

- C is a general curve of genus $g := \binom{s}{2}$.
- H is a general divisor on C of degree $d := \binom{s+1}{2}$.
- L is a general divisor on C of degree $e \leq \binom{s+1}{2}$.

The standard method of proving such vanishing is by filtration and stability (see Green-Lazarsfeld [1986], Ein-Lazarsfeld [1992]) but it does not yield a strong enough result, and in fact the stability of M_H would not be a strong enough condition, so instead we shall use further degenerations: Depending on the parity of s, we reduce to the case where M_H is a direct sum of line bundles (s odd), or a direct sum of rank 2 vector bundles (s even). The desired vanishing is an open condition on the triples (C, H, L) in any flat family for which the dimension of $\mathrm{H}^0(\mathscr{O}_C(H))$ is constant. Suppose that we can find, for each s, a smooth curve C_0 of genus g, a non-special divisor H of degree d, and some divisor L of degree l on C such that $\mathrm{H}^0(\wedge^2 M_H \otimes \mathscr{O}_C(L)) = 0$. Over a versal deformation of C_0 we may form the space of triples (C, H, L), where H, L are divisors of the given degrees. The base space of this versal deformation maps to the moduli space of curves of genus g and covers an open set therein. Thus the general curve C, with general divisors H and L will have the properties required.

Assume now that s is odd. We let C_0 be a curve of type $(\frac{(s+1)}{2}, s+1)$ on the smooth quadric $Q \subset \mathbf{P}^3$. Let $\mathscr{N}$ be the restriction of $\mathscr{O}_Q(0,1)$ to C_0. Thus $\mathscr{N}$ is a line bundle of degree $(s+1)/2$ generated by global sections, and $\mathscr{N}^{\oplus s}$ is a globally generated vector bundle of rank s and degree $\binom{s+1}{2}$ on C_0. Let W^* be a general $(s+1)$-dimensional subspace of sections $W^* \subset H^0(\mathscr{N}^{\oplus s})$. It is easy to see that W^* generates $\mathscr{N}^{\oplus s}$, and so we define a line bundle $\mathscr{O}_{C_0}(H)$ as the dual of the kernel of the natural evaluation

$$0 \longrightarrow \mathscr{O}_{C_0}(-H) \longrightarrow W^* \otimes \mathscr{O}_{C_0} \longrightarrow \mathscr{N}^{\oplus s} \longrightarrow 0 \ .$$

With these choices $\mathcal{O}_{C_0}(H)$ is a globally generated line bundle of the desired degree $\binom{s+1}{2}$, and W maps naturally to $\mathrm{H}^0(\mathcal{O}_{C_0}(H))$. In fact $\mathcal{O}_{C_0}(H) = \det(\mathcal{N}^{\oplus s}) = \mathcal{N}^{\otimes s} = \mathcal{O}_{C_0}(0,s)$. Furthermore, since $h^1(\mathcal{O}_Q(0,s)) = 0$ and $h^2(\mathcal{O}_Q(-\frac{(s+1)}{2},-1)) = 0$, taking cohomology of the exact sequence

$$0 \longrightarrow \mathcal{O}_Q\left(-\frac{(s+1)}{2},-1\right) \longrightarrow \mathcal{O}_Q(0,s) \longrightarrow \mathcal{O}_{C_0}(H) \longrightarrow 0$$

we get $h^1(\mathcal{O}_{C_0}(H)) = 0$, that is $\mathcal{O}_{C_0}(H)$ is non-special. Thus $h^0(\mathcal{O}_{C_0}(H)) = \chi(\mathcal{O}_{C_0}(H)) = s+1$ and since $h^0(\mathcal{N}^*) = 0$ for degree reasons, we get that $W = \mathrm{H}^0(\mathcal{O}_{C_0}(H))$, whence $M_H \cong (\mathcal{N}^{\oplus s})^*$. To show that $\mathrm{H}^0(\wedge^2 M_H \otimes \mathcal{O}_C(L))$, it is enough to show that $\mathrm{H}^0(\mathcal{N}^{-2} \otimes \mathcal{O}_C(L)) = 0$. This is obviously true for a general L with $\deg L = r-s$, since $\deg(\mathcal{O}_C(L) \otimes \mathcal{N}^{-2}) = r-s-2 \leq \binom{s}{2} - 1 = g(C_0) - 1$ by our initial hypothesis.

Consider the versal deformation of the curve C_0 and over it the space of triples (C,H,L) as above, where H is a divisor of degree d and L is a divisor of degree $r-s$. The locus for which $\mathcal{O}_C(H)$ defines an arithmetically normal embedding in $\mathbf{P}^s$ is open and, as we have seen, non-empty. Furthermore, the vanishing of $\mathrm{H}^0(\wedge^2 M_H \otimes \mathcal{O}_C(L)) = 0$ is an open condition on the collection of triples. Since the vanishing condition is satisfied on C_0, the same follows for the general curve.

Finally, consider the case where s is even. In order to produce a non-special divisor H with the desired properties in this case, we will degenerate further, letting H become special. Thus we must work with incomplete linear series.

Given a divisor H on a curve C and a space of global sections W that generates $\mathcal{O}_C(H)$, we define $M_{W,H}$ to be the kernel of the natural evaluation map:

$$M_{W,H} := \ker(W \otimes \mathcal{O}_C \twoheadrightarrow \mathcal{O}_C(H)) \ .$$

It now suffices, for each even s, to find:

A curve C of genus g, a divisor H of degree d on C, and a space of sections W of dimension $s+1$ of $\mathcal{O}_C(H)$ such that

• $\mathrm{H}^0(\wedge^2 M_{W,H} \otimes \mathcal{O}_C(L)) = 0$ for the general divisor L on C of degree $r-s$, and

• The triple (C,H,W) is a flat limit of triples for which H is non-special (equivalently, where $W = \mathrm{H}^0(\mathcal{O}_C(H))$).

A candidate is constructed for us by the following result:

Proposition 4.2. *Let $s \geq 2$ be even. For any sufficiently general plane curve C_0 of degree $s+1$ there exists a flat irreducible family of smooth plane curves C_t of degree $s+1$, with special fiber C_0 and general fiber C_η, a family of line bundles H_t of degree $\binom{s+1}{2}$, and a family of spaces $W_t \subset \mathrm{H}^0(\mathcal{O}_{C_t}(H_t))$ such that*:

a) *W_t generates $\mathcal{O}_{C_t}(H_t)$,*

b) *H_η is non-special, and*

c) *$M_{H_0} := \ker(W_0 \otimes \mathcal{O}_{C_0} \longrightarrow \mathcal{O}_{C_0}(H_0))$ is the direct sum of $s/2$ copies of the rank 2 vector bundle M which is the kernel of the evaluation map $\mathrm{H}^0(\mathcal{O}_{C_0}(1)) \otimes \mathcal{O}_{C_0} \twoheadrightarrow \mathcal{O}_{C_0}(1)$, where $\mathcal{O}_{C_0}(1)$ induces the planar embedding.*

Proof. We shall construct the family of curves C_t and the family of divisors H_t by constructing the family of vector bundles $\mathcal{E}_t := \ker(W_t \otimes \mathcal{O}_{C_t} \longrightarrow \mathcal{O}_{C_t}(H_t))$. On the generic fiber, we use the following (old) observation: If $B : \mathcal{O}_{\mathbf{P}^2}^{s+1}(-1) \longrightarrow \mathcal{O}_{\mathbf{P}^2}^{s+1}$ is an $(s+1) \times (s+1)$ matrix of linear forms in 3 variables such that $f := \det B \neq 0$, and such that the ideal $I_s(B)$ of $s \times s$-minors of B contains a power of the irrelevant ideal, then $\mathcal{H} := \operatorname{coker} B$ is a line bundle on the degree $s+1$ curve $\{f = 0\}$ with $h^0(\mathcal{H}) = s+1$, $h^1(\mathcal{H}) = 0$ (and thus $\deg \mathcal{H} = \binom{s+1}{2}$ by Riemann-Roch). The vector bundle $M_{\mathcal{H}}$ is of course the image of B.

For the special fiber, we proceed differently. Recall that $\widetilde{M} := \Omega^1_{\mathbf{P}^2}(1)$ is the image of the middle Koszul map

$$\rho : \wedge^2 \mathrm{H}^0(\mathcal{O}_{\mathbf{P}^2}(1)) \otimes \mathcal{O}_{\mathbf{P}^2}(-1) \longrightarrow \mathrm{H}^0(\mathcal{O}_{\mathbf{P}^2}(1)) \otimes \mathcal{O}_{\mathbf{P}^2} \quad ,$$

induced by the 3×3-generic skew-symmetric matrix over the ring $S = \mathrm{Sym}(\mathrm{H}^0(\mathcal{O}_{\mathbf{P}^2}(1))) \cong k[x,y,z]$. If C_0 is any plane curve, then the bundle M defined in the statement of the Proposition is simply $\widetilde{M}|_{C_0}$. We wish to define a general matrix of linear forms whose image is $\widetilde{M}$. For this (and for later purposes) the idea of a "generalized submatrix" of a matrix will be useful: by a generalized $p \times q$ submatrix of a matrix C we mean simply a composition PCQ, where P and Q are scalar matrices, P has p rows, and Q has q columns. Generalized rows or columns of C are generalized submatrices with one row or one column, respectively.

Let now A be a (sufficiently general) generalized $(s+1) \times (s+1)$-submatrix of a $(\frac{3s}{2}) \times (\frac{3s}{2})$-matrix inducing $\rho^{\oplus \frac{s}{2}}$. Notice that $\det A = 0$ since the module $\Omega^1_{\mathbf{P}^2}(1)^{\otimes \frac{s}{2}}$ has only rank s. Let B be a general $(s+1) \times (s+1)$-matrix with linear entries, and set $A_t := A + tB$. We set

$$f_t := \det A_t \quad ,$$

for $t \neq 0$, and for $t = 0$ we take f_t to be the "limit"

$$f_0 := \lim_{t \longrightarrow 0} \frac{\det(A + t \cdot B)}{t} = \sum_{i,j=1}^{s+1} b_{ij} \, |A_{ij}| \quad ,$$

where b_{ij} are the entries of B, while $|A_{ij}|$ denotes the (signed) minor of A obtained by deleting row i and column j.

Proposition 4.3. *Set $m = \frac{s}{2}$. For each generalized $(s+1) \times (s+1)$ submatrix A of $\rho^{\otimes m}$, the ideal of $s \times s$ minors of A may be written in the form $I_s(A) = I_{m-1}(K_1) \cdot I_{m-1}(K_2) \cdot (x, y, z)^2$ for matrices of linear forms K_1, K_2 of sizes $m \times (m-1)$ and $(m-1) \times m$, respectively. Furthermore, each pair of matrices K_1 and K_2 with linear entries arises for some generalized submatrix A. Thus, for a general choice of A, the ideal $I_s(A)$ is a nonsaturated ideal of a reduced set of points, and the equation $\{f = 0\}$ defines a general plane curve C of degree $(s+1)$ containing them.*

Proof of Proposition 4.3. To compute $I_s(A)$ we make use of a special case of the structure theorem for finite free resolutions of Buchsbaum-Eisenbud [1974]. Consider the resolution obtained by taking the direct sum of m copies of the Koszul complex in 3 variables:

$$0 \longrightarrow S^m(-2) \xrightarrow{\kappa^*(-1)^{\oplus m}} S^{3m}(-1) \xrightarrow{\rho^{\oplus m}} S^{3m} \xrightarrow{\kappa^{\oplus m}} S^m(1) \longrightarrow 0 \ ;$$

and to simplify notation, write it as

$$0 \longrightarrow F_3 \xrightarrow{f_3} F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \longrightarrow 0 \ .$$

Let $r_i := \operatorname{rank}(f_i)$, so that in particular $\operatorname{rank}(F_i) = r_i + r_{i+1}$, for all $1 \leq i \leq 3$. The structure theorem asserts the commutativity (up to a sign) of the diagram:

$$\begin{tikzcd}
\wedge^{r_3} F_2^* = \wedge^{r_2} F_2 \arrow[rr, "\wedge^{r_2} f_2"] \arrow[dr, "\wedge^{r_3} f_3^*"'] & & \wedge^{r_2} F_1 = \wedge^{r_1} F_1^* \\
& \wedge^{r_3} F_3^* \cong R \cong \wedge^{r_1} F_0^* \arrow[ur, "\wedge^{r_1} f_1^*"'] &
\end{tikzcd} \qquad (*)$$

In other words, any minor of order r_2 of f_2 may be expressed as the product of complementary minors of orders r_1 and r_3 of f_1 and f_3, respectively.

The choice of the matrix A involves the choice of $s + 1 = 2m + 1$ generalized rows and columns of the matrix defining $\rho^{\otimes m}$, hence the choice of $m - 1$ complementary columns of κ and $m - 1$ complementary rows of κ^*, respectively. We denote by K_1 and K_2 the $m \times (m-1)$ and $(m-1) \times m$-submatrices of κ and $\kappa^*(-1)$ distinguished in this way.

Because of the structure of κ, any $m \times (m-1)$-matrix with linear entries in S can be obtained as K_1 through the appropriate choice of $(m-1)$ generalized columns of κ, and similarly for K_2. In particular, K_i may be chosen to make $I_{m-1}(K_i)$ be the ideal of any sufficiently general set of $\binom{m}{2}$

points in the plane. Diagram $(*)$ expresses the $(2m) \times (2m)$-minors A_{ij} of A as the products of $m \times m$-minors of κ and κ^* that contain K_1 and K_2, respectively. An $m \times m$-minor of κ containing K_1 is a linear combination of the $(m-1) \times (m-1)$ minors of K_1, with coefficients the elements of an arbitrary generalized column of κ. Again because of the structure of κ this column can be taken to be an arbitrary column of linear forms in S. Thus the ideal of $m \times m$-minors of κ containing K_1 is $I_{m-1}(K_1) \cdot (x, y, z)$. As similar remarks hold for K_2, we have proven the first part of the Proposition.

If the choice of the generalized submatrix A is general, then K_1 and K_2 will be general matrices of linear forms, hence their ideals of minors will be reduced ideals of distinct general sets of points in the plane, and the ideal $I_s(A) = I_{m-1}(K_1) \cdot I_{m-1}(K_2) \cdot (x, y, z)^2$ will be a nonsaturated ideal of the union of these two sets of points, as claimed.

Varying the matrix B we obtain for f any form of degree $(s+1)$ in $I_s(A)(x, y, z) = I_{m-1}(K_1) \cdot I_{m-1}(K_2) \cdot (x, y, z)^3$. Since $\binom{s+3}{2} > 2\binom{m}{2}$ the general curve C_0 of degree $s+1$ through two general sets of $\binom{m}{2}$ points in the plane is a general plane curve of degree $s+1$, concluding the argument. □

Completion of the proof of Proposition 4.2. Let C_t be defined by the equation $\{f_t = 0\}$, and let $\mathscr{E}_t$ be the image of the restriction to C_t of the morphism induced by the matrix $A + t \cdot B$. Let $\mathscr{H}_t^{-1}$ be the kernel of the restriction of the matrix $(A + Bt)^*$ to C_t; we write the dual in the form $\mathcal{O}_{C_t}(H_t) = \mathscr{H}_t$, for a family of divisors H_t (defined, for example, by the family of sections that are the images of the first basis vector of the target free module of $A + Bt$). Part a) of the Proposition now follows from the definitions; part b) follows from the remark at the beginning of the proof; and part c) follows from the form of the matrix $A = A + 0 \cdot B$. □

Continuation of the proof of Theorem 4.1. We adopt the notation of Proposition 4.2, but for simplicity we now set $C = C_0$. By Proposition 4.2, C may be chosen to be a general plane curve of degree $s+1$. It suffices to show that $\mathrm{H}^0(\wedge^2(M^{\oplus s/2}) \otimes \mathcal{O}_C(L)) = 0$, where L is a general divisor of degree $r - s$, and for this it is enough to show that both $\mathrm{H}^0(\wedge^2 M \otimes \mathcal{O}_C(L)) = 0$, and $\mathrm{H}^0(M \otimes M \otimes \mathcal{O}_C(L)) = 0$.

Two remarks will make the plausibility of this conclusion clear. First, $r \geq \binom{s+1}{2}$ so $\deg(L) = r - s \geq \binom{s}{2}$, and $g = \binom{s}{2}$ is the genus of C. Thus $\mathcal{O}_C(L)$ is a general line bundle in the Picard variety of C. Second, $\deg(M) = -(s+1)$, so $\chi(\wedge^2 M \otimes \mathcal{O}_C(L))$ and $\chi(M \otimes M \otimes \mathcal{O}_C(L))$ are both ≤ 0. Thus each of the desired vanishings has the form: $\mathrm{H}^0(F \otimes \mathcal{O}_C(L')) = 0$ with F a vector bundle on C with $\chi(F) = 0$, and L' a general divisor of degree ≤ 0. This condition obviously implies that the bundle F must be semistable, and indeed Raynaud [1982] shows that the condition is equivalent to semistability when $\operatorname{rank} F \leq 2$, and also when $\operatorname{rank} F = 3$ on a general curve of a given genus. In fact his argument proves a little more:

Theorem 4.4 (Raynaud). *Let C be a general plane curve of any degree ≥ 3. A vector bundle F of rank ≤ 3 on C with $\chi(F) = 0$ is semistable iff $\mathrm{H}^0(F \otimes \mathcal{O}_C(L')) = 0$ for the general line bundle $\mathcal{O}_C(L')$ of degree 0 on C.*

Discussion of Theorem 4.4. Raynaud [1982, §2] enunciates the result for general curves (not planar). However, his proof shows that if we replace "vector bundle" by "torsion-free sheaf", then the truth of the Theorem for C defines an open set in the moduli of stable curves. Furthermore, his proof shows that this open set includes every irreducible rational curve of arithmetic genus g, having exactly g ordinary nodes. Since the the general map from $\mathbf{P}^1$ into the plane has as image a curve with only ordinary nodes as singularities, these facts imply that the Theorem holds for a general plane curve. □

The first of the necessary vanishings is immediate from the remarks above: Since a general line bundle of degree $\leq g-1$ has no sections $\mathrm{H}^0(\wedge^2 M \otimes \mathcal{O}_C(L)) = \mathrm{H}^0(\mathcal{O}_C(L-N)) = 0$, where N is the divisor of the intersection of C with a line.

For the second vanishing, from the exact sequence

$$0 \longrightarrow \wedge^2 M \longrightarrow M \otimes M \longrightarrow \mathrm{Sym}_2 M \longrightarrow 0 \ ,$$

together with the first vanishing result above, it suffices to show that $\mathrm{H}^0(\mathrm{Sym}_2(M) \otimes \mathcal{O}_C(L)) = 0$, and this puts us in the case of a bundle $F = \mathrm{Sym}_2(M) \otimes \mathcal{O}_C(L)$ of rank 3. An easy degree computation shows that $\chi(F) \leq 0$. We may now invoke Theorem 4.4 to conclude the argument if we can show that F is semistable, and by Hartshorne [1971], Gieseker [1979] (since char $k \neq 2$) it suffices to show that M itself is semistable. As M differs from the restriction to C of the tangent bundle of the projective plane only by twisting by a line bundle, the following elementary result completes now the proof of Theorem 4.1:

Proposition 4.5. *Let $T = T_{\mathbf{P}^2}$ be the tangent bundle of the projective plane. If C is a smooth plane curve of degree $m \geq 3$, then $T|_C$ is stable.*

Proof. Suppose Q is a line bundle quotient of $T|_C$. Since $\deg(T|_C) = 3m$, we must show that $\deg(Q) > \frac{3m}{2}$. But $T(-1)$ is globally generated, so either $Q(-1) = \mathcal{O}_C$, or $Q(-1)$ defines a base point free linear series. Let $e := \deg(Q) - m$ be the degree of $Q(-1)$; we must show that $e > m/2$.

Assume first that $Q(-1) = \mathcal{O}_C$. Restricting the presentation of T to C we obtain maps

$$\mathcal{O}_C(-1) \longrightarrow \mathcal{O}_C^3 \longrightarrow\!\!\!\!\!\rightarrow T|_C(-1) \longrightarrow\!\!\!\!\!\rightarrow Q(-1) = \mathcal{O}_C$$

with composition 0, where the last two maps are surjective. Since the restrictions of linear forms on $\mathbf{P}^2$ are still linearly independent on C, this is a contradiction.

Thus we may suppose that $Q(-1)$ defines a base point free linear series of degree e. It follows at once that $e \geq m-1 > m/2$. For the reader's convenience we give the elementary proof: Since the genus of C is positive, we must have $e \geq 2$. If $m = 3$, then this is the desired result. On the other hand, if $m \geq 4$, then $m-2 < \binom{m-1}{2}$, the genus of C, so if $e \leq m-2$ then $Q(-1)$ is special. In other words the points in a divisor D in the linear series represented by Q impose dependent conditions on the canonical series $\mathcal{O}_C(K_C) = \mathcal{O}_C(m-3)$. But any finite scheme of length $\leq m-2$ in the plane imposes independent conditions on forms of degree $m-3$. $\square$

5. Numerology and failure of the Minimal Resolution Conjecture

Let Γ be a general set of γ points in $\mathbf{P}^r$. In this section we derive the explicit form of the Minimal Resolution Conjecture, which is a lower bound $\tilde{\beta}_{i,j} \leq \beta_{i,j}(I_\Gamma)$ for the graded Betti numbers of the homogeneous ideal I_Γ of Γ. We use this lower bound to prove Theorem 0.1, providing counterexamples to the Minimal Resolution Conjecture. When the lower bound is achieved, we shall say that Γ has *expected Betti numbers*. For the reader's interest we also provide the best upper bound we know.

It is well-known how to do the computation for the precise form of the MRC; it appears explicitly in the Queen's University thesis of Anna Lorenzini as well as in Lorenzini [1987], [1993]), but for the reader's convenience, and because we need details in a certain special case, we spell it out. Since all we use about Γ is its Hilbert function, the same computation would work for any subscheme finite over k imposing "as many conditions as possible" on forms of each degree; we call such a subscheme *sufficiently general.*

We write $\tilde{\beta}_{i,j}$ for the expected dimensions of the Koszul homology, and $\{n\}_+$ for $\max(n,0)$. We set also $\binom{n}{k} = 0$ for $k > n$.

Proposition 5.1. *Let* Γ *be a finite sufficiently general subscheme of* $\mathbf{P}^r$ *having degree* γ, *and define integers* d *and* a *by the conditions*

$$\binom{r+d-1}{d-1} \leq \gamma < \binom{r+d}{d}, \qquad a := \gamma - \binom{r+d-1}{d-1} .$$

The Koszul homology dimensions

$$\beta_{i,j}(I) = \dim_k(\operatorname{Tor}_i^S(I_\Gamma, k)_j)$$

satisfy:

a) $\beta_{i,j} = 0$ *unless* $0 \le i \le r-1,$ *and* $j = i+d$ *or* $j = i+d+1$;

b)

$$\binom{d+i-1}{i}\binom{r+d-1}{d+i} \ge \beta_{i,i+d} \ge \tilde{\beta}_{i,i+d}$$
$$= \left\{\binom{d+i-1}{i}\binom{r+d-1}{d+i} - a\binom{r}{i}\right\}_+$$

$$a\binom{r}{i+1} \ge \beta_{i,i+d+1} \ge \tilde{\beta}_{i,i+d+1} = \left\{a\binom{r}{i+1} - \binom{d+i}{i+1}\binom{r+d-1}{d+i+1}\right\}_+$$

Proof. The following elementary remarks suffice:

a) If S_Γ is the homogeneous coordinate ring of a finite scheme Γ in $\mathbf{P}^r$, and S is the homogeneous coordinate ring of $\mathbf{P}^r$, and x is a linear form not vanishing on any point in the support of Γ, then the graded Betti numbers of S_Γ as an S-module are the same as the graded Betti numbers of S_Γ/xS_Γ as an $R := S/x$-module.

b) Write $R = k[x_1, \ldots, x_r]$, set $m = (x_1, \ldots, x_r)$, and let d be the largest integer such that $\gamma \ge \binom{r+d-1}{d-1}$. In other words, assuming Γ is sufficiently general, d is the smallest degree of a form contained in the homogeneous ideal I_Γ. We may write S_Γ/xS_Γ in the form R/I where $m^{d+1} \subset I \subseteq m^d$.

c) The module R/I may be obtained as a "jump deformation" of the module $R/m^d \oplus m^d/I$. Thus the resolution of R/I has all graded Betti numbers $\ge$ the Betti numbers in the resolution of $R/m^d \oplus m^d/I$.

We get now the desired estimates by putting these things together with a knowledge of the free resolution of R/m^d, which may for example be described as an Eagon-Northcott complex (see Eisenbud [1995]). $\square$

For simplicity, and because it is the case we shall use, we now specialize to the case where $d = 2$, so that the ideal of Γ is generated by quadrics and cubics.

Corollary 5.2. *Let* Γ *be a finite sufficiently general subscheme of* $\mathbf{P}^r$ *having degree* γ *with* $r+1 \le \gamma < \binom{r+2}{2}$. *The expected dimensions of the Koszul homology of* Γ *are*:

$$\tilde{\beta}_{i,i+2} = \left\{(i+1)\binom{r+2}{i+2} - \gamma\binom{r}{i}\right\}_+,$$
$$\tilde{\beta}_{i,i+3} = \left\{\gamma\binom{r}{i+1} - (i+2)\binom{r+2}{i+3}\right\}_+, \quad i = \overline{0, r-1}$$

In particular $\tilde{\beta}_{i,i+2} \neq 0$ *iff* $i < \frac{(r+2)(r+1)}{\gamma} - 2$. *Furthermore* $\tilde{\beta}_{i,i+3} \neq 0$ *iff* $i \geq \frac{(r+2)(r+1)}{\gamma} - 3$.

Proof. Arithmetic, starting from the previous result. □

Thus the "expected" shape of the minimal free resolution of $\mathscr{I}_\Gamma$ is

degree										
0	1	–	–	...	–	–	...	–	–	–
1	–	$*$	$*$	...	$*$	?	...	–	–	–
2				...	?	$*$	...	$*$	$*$	$*$

$\underbrace{\qquad\qquad\qquad}_{\left[\frac{(r+1)(r+2)}{\gamma}\right]-3}$

(where not both of the "?"s in the above display are non-zero!)

As an easy corollary of Theorem 4.1 on linear exactness we obtain now the result announced in the introduction.

Proof of Theorem 0.1. For r and s in the given range the complex $E_\bullet(\mu) = E_\bullet^{-1}(\mu)$ defined at the beginning of Section 4 is linearly exact. Moreover, the twisted complex $E_\bullet(\mu)(r+2)$ maps monomorphically onto a direct summand of the dual of the minimal free resolution of I_Γ. On the other hand, Corollary 5.2 gives as expected graded Betti number $\tilde{\beta}_{(r-s-1),(r-s+2)}$ for I_Γ

$$\tilde{\beta}_{(r-s-1),(r-s+2)} = \left\{ \frac{(2k+4-s^2+s)}{(s^2-s+2k+4)} \cdot \begin{pmatrix} \binom{s+1}{2} + k \\ \binom{s}{2} + k \end{pmatrix} \right\}_+ ,$$

whereas the last (i.e., the s-th) syzygy module in the complex $E_\bullet(\mu)$ has rank

$$\operatorname{rank} E_{s-1}(\mu) = \binom{s+k}{k} .$$

The theorem follows since

$$s^2 - s + 2k + 4 \geq s^2 - s > 0 \ \text{ and } \ 2k + 4 - s^2 + s \leq 3s + 4 - s^2 \leq 0,$$
$$\text{for all } s \geq 4, \ 0 \leq k \leq s, \ \text{while}$$

$$\frac{(2k-2)}{(2k+10)} \cdot \binom{k+6}{k+3} < \binom{k+3}{3}$$

only for $r = k + 6 \in \{6, 7, 8\}$. □

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