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Chains of maps between indecomposable modules

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We shall call a sequence of homomorphisms of modules (or objects in some other Abelian category)

$$\varepsilon : M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{s-1}} M_s$$

a *Harada-Sai sequence* if each M_i is an indecomposable R-module of finite length, no f_i is an isomorphism, and the composite map $M_1 \to M_s$ is nonzero. We write $\lambda(M_i)$ for the length of M_i. Suppose that all the $\lambda(M_i)$ are bounded by a number b. Fitting's lemma shows that if all the M_i were equal and all the f_i were equal, then the length s of the sequence ε is at most $b-1$. A significant generalization, the Harada-Sai Lemma [4], says that even with distinct M_i and f_i the length s is bounded, this time by $2^b - 1$. For typical applications see Bongartz [1], Gabriel [3], and Ringel [5].

The aim of this note is to sharpen the Harada-Sai Lemma by showing exactly which *length sequences* $(\lambda(M_1), \ldots, \lambda(M_s))$ are possible, and what the length of the image of the composite map $M_1 \to M_s$, which we call the *composite rank* of ε, can be. Here are some simple examples:

• There is a Harada-Sai sequence with all $\lambda(M_i) = b$ iff $s \leqq 2^{b-1}$, just over half the maximal length of Harada-Sai sequence where all the $\lambda(M_i) \leqq b$.

• There is no Harada-Sai sequence with length sequence (2, 3, 3, 3, 4, 4).

• There are Harada-Sai sequences with length sequence (3, 2, 3, 3, 3), but none with the permuted sequences (3, 3, 2, 3, 3) or (2, 3, 3, 3, 3).

• There are Harada-Sai sequences with length sequence (4, 4, 2, 4, 3, 4) and composite rank 1 and 2, but if we *increase* the length 2 to length 3, producing a length sequence (4, 4, 3, 4, 3, 4), then the largest composite rank *decreases* to 1.

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To state the full result we introduce what will be the universal sequences $\lambda^{(b)}$ of lengths: Set $\lambda^{(1)} = (1)$ and, inductively, take $\lambda^{(b)}$ to be the result of adding b at the beginning of, at the end of, and in between every pair of elements of, $\lambda^{(b-1)}$. Thus for example

$$\lambda^{(1)} = (1),$$

$$\lambda^{(2)} = (2,1,2),$$

$$\lambda^{(3)} = (3,2,3,1,3,2,3),$$

$$\lambda^{(4)} = (4,3,4,2,4,3,4,1,4,3,4,2,4,3,4).$$

Note that $\lambda^{(b)}$ has $2^b - 1$ elements.

We say that a sequence $\lambda \in \mathbb{N}^m$ is *embeddable* in $\mu \in \mathbb{N}^n$ if there is a strictly increasing function $\sigma : \{1, \dots, m\} \to \{1, \dots, n\}$ such that $\lambda_i = \mu_{\sigma(i)}$ for all $1 \leqq i \leqq m$ (we write $\sigma : \lambda \to \mu$).

Theorem 1 (Embedding). *There is a Harada-Sai sequence with length sequence λ iff there is an embedding $\sigma : \lambda \to \lambda^{(b)}$ for some b.*

The classic Harada-Sai Lemma follows from the "only if" statement, since the length of the sequence $\lambda^{(b)}$ is $2^b - 1$. The proof of the embedding theorem given at the beginning of the next section, seems to us simpler than the previously known proof of the Harada-Sai Lemma (Ringel [5]). A further consequence of the "if" statement of the embedding theorem is that in any Harada-Sai sequence of maximal lengths the composite ranks of all subsequences are determined:

Corollary 2. *If ε as above is a Harada-Sai sequence of length $s = 2^b - 1$ with $\lambda(M_i) \leqq b$ for all i, then the composite rank of any subsequence $M_{i_1} \to \cdots \to M_{i_t}$ is precisely* $\min\{\lambda_j^{(b)} \mid i_1 \leqq j \leqq i_t\}$.

From the embedding theorem we also get a sharper Harada-Sai bound, taking into account the structure of the ring:

Corollary 3. *Let R be an Artinian algebra and let $l = l(R)$ be the maximum, over all simple left R-modules N, of the minimum of the lengths of the projective cover and the injective hull of N. The length s of any Harada-Sai sequence ε of left modules with lengths $\leqq b$ is bounded by $s \leqq 2^{b-l+1}(2^{l-1} - 1) + 1$.*

For example, if R is the ring of $n \times n$ lower-triangular matrices, then $l(R) = \lceil n/2 \rceil$.

In view of Corollary 2, we define the *rank* of an embedding $\sigma : \lambda = (\lambda_1 \dots \lambda_s) \to \lambda^{(b)}$ to be $\operatorname{rank}(\sigma) = \min\{\lambda_j^{(b)} \mid \sigma(1) \leqq j \leqq \sigma(s)\}$. The following more precise version of Theorem 1 is the main result of this paper.

Theorem 4 (Rank). *There is a Harada-Sai sequence with length sequence λ and composite rank r iff there is an embedding $\sigma : \lambda \to \lambda^{(b)}$ such that $r = \operatorname{rank}(\sigma)$.*

It is easy to decide whether or not a sequence $\lambda = (\lambda_1, \ldots, \lambda_s)$ is embeddable with given rank in $\lambda^{(b)}$ (of course we may take $b = \max\{\lambda_i\}$): We define the "optimal" embedding inductively by

$$\sigma_{\mathrm{opt}}(i) = \min\{j \,|\, j > \sigma_{\mathrm{opt}}(i-1) \text{ and } \lambda_i = \lambda_j^{(b)}\}$$

as long as such an index j exists.

Proposition 5. *A length sequence $\lambda = (\lambda_1, \ldots, \lambda_s)$ is embeddable in $\lambda^{(b)}$, iff $\sigma_{\mathrm{opt}}(i)$ is defined for all $i \leqq s$. There exists a rank r embedding of λ in $\lambda^{(b)}$ iff $r \leqq \mathrm{rank}(\sigma_{\mathrm{opt}})$.*

Here is an open problem: Find a generalization of Theorem 1 with hypothesis depending on the maps f_i and not the modules M_i in ε. For example, one might want to assume that no composition of the maps f_i is an isomorphism, and no image of such a composition is a direct summand. Does this suffice to bound the length of the sequence?

Proofs

Proof of Theorem 1. We begin by showing that the length sequence $\lambda = (\lambda_1, \ldots, \lambda_s)$ of a Harada-Sai sequence ε as above is embeddable in $\lambda^{(b)}$, where $b = \max\{\lambda_i\}$. We do this by induction on b, the case $b = 1$ being trivial.

If $\lambda_i = \lambda_{i+1} = b$ then we can interpolate some indecomposable summand of the image of M_i in M_{i+1} between M_i and M_{i+1} and get a new Harada-Sai sequence one step longer than the original. Repeating this process, we may assume that no two consecutive λ_i are equal to b.

Any composition $g := f_j f_{j-1} \cdots f_i : M_i \to M_j$ of consecutive maps in a Harada-Sai sequence is a non-isomorphism. For if g were an isomorphism then $\lambda(M_{i+1}) > \lambda(M_i)$ to avoid f_i being an isomorphism, and thus $M_i \to M_{i+1} \to M_j \xrightarrow{g^{-1}} M_i$ exhibits M_i as a proper direct summand of M_{i+1}, contradicting the definition.

Let $\mu = (\mu_1, \ldots, \mu_t)$ be the subsequence of λ consisting of those terms that are $< b$. By induction there is an embedding $\sigma' : \mu \to \lambda^{(b-1)}$. Since every second element of $\lambda^{(b)}$ is b, and the remaining elements of $\lambda^{(b)}$ make up $\lambda^{(b-1)}$, this embedding can be extended to the desired embedding of λ in $\lambda^{(b)}$, completing the proof of embeddability.

The other implication of Theorem 1 follows from the existence of a maximal Harada-Sai sequence, constructed below. □

Proof of Corollary 2. It is clear that the given rank is an upper bound. If the actual rank were less, we could form a new Harada-Sai sequence of the form

$$M_1 \to M_2 \to \cdots \to M_{i_1} \to N \to M_{i_t} \to M_{i_t+1} \to M_s$$

where N is an indecomposable summand of $\mathrm{Im}\, M_{i_1} \to M_{i_t}$, and the length sequence of this new sequence would not be embeddable. □

Proof of Corollary 3. Let ε be a Harada-Sai sequence of R-modules of lengths bounded by b, and consider the projective cover $P(M_1) \to M_1$. There is an indecomposable direct summand P' of $P(M_1)$ such that the composition $P' \to M_1 \to M_2 \to \cdots \to M_s$ is not zero. Let P be the image of P' in M_1. Let N be the simple module which is the top of P. The module N must occur in the image of the composite map $P \to M_1 \to M_s$, so we can find a map $M_s \to I'$, where $I' = I(N)$ is the injective hull of N, so that the composite $P \to M_1 \to M_s \to I'$ is nonzero. We let I be the image of M_s in I'. Both P and I have lengths $\leqq \min(b, l)$. By the embeddability statement of Theorem 1, the sequence $(\lambda(P), \lambda(M_1), \ldots, \lambda(M_s), \lambda(I))$ is embeddable in $\lambda^{(b)}$. But the lengths of the subsequences of $\lambda^{(b)}$ starting and ending with numbers $\leqq l$ are bounded by $2^{b-l+1}(2^{l-1}-1)+1$. □

Proof of Proposition 5. The first statement is immediate, and it is easy to see that no embedding has rank $> \tau_{\text{opt}} := \operatorname{rank}(\sigma_{\text{opt}})$. If $0 < r < r_{\text{opt}}$ then we construct a new embedding $\tau: \lambda \to \lambda^{(b)}$ such that $r = \operatorname{rank}(\tau)$. Let s, t, u, v be the smallest integers such that $\lambda_s^{(b)} = r_{\text{opt}}$, $\lambda_t^{(b)} = r_{\text{opt}} - 1$, $\lambda_u^{(b)} = r$, $\lambda_v^{(b)} = r - 1$. We set $\tau(i) = \sigma(i)$ if $\sigma(i) \leqq t$. There is an embedding γ identifying that part of the sequence $\lambda^{(b)}$ between s and t with that part between u and v. We define $\tau(i) = \gamma\sigma(i)$ for i such that $\sigma(i) > t$. □

Proof of Theorem 4. Let ε be a Harada-Sai sequence and let σ_{opt} be its optimal embedding into $\lambda^{(b)}$. We now proceed by induction on the length s. Set

$$m = \min\{\lambda_i^{(b)} \mid \sigma_{\text{opt}}(1) \leqq i \leqq \sigma_{\text{opt}}(s)\},$$

the desired bound. If any λ_i is equal to m we are done, so we assume $\lambda_i > m$ for all i. Let k_0 be the smallest integer k such that $\lambda_{k_0}^{(b)} = m$. Since the embedding σ_{opt} is optimal, we have $\sigma_{\text{opt}}(1) < k_0 < \sigma_{\text{opt}}(s)$. On the other hand, if $k_0 < \sigma_{\text{opt}}(s-1)$ we are done by induction, so we may assume that $\sigma_{\text{opt}}(s-1) < k_0 < \sigma_{\text{opt}}(s)$.

Suppose that for some $i < j$ we have $\lambda_i = \lambda_j = m+1$. As in the proof of Theorem 1 we see that the rank of the composite $M_i \to M_j$, and with it the composite rank of ε, is $\leqq m$ as required. Thus we may further assume that at most one λ_i is equal to $m+1$.

The cases $s \leqq 2$ are trivial. Assuming that $s \geqq 3$ we will show that for some integer $1 < t < s$ the compositions $g: M_1 \to M_t$ and $h: M_t \to M_s$ both have rank $\leqq m+1$, while M_t has length $> m+1$. It follows that the rank of the composite $hg: M_1 \to M_s$ has rank at most $m+1$; and if its rank is equal to $m+1$, then as above the maps

$$\operatorname{Im} g \to M_t \to \operatorname{Im} hg \cong \operatorname{Im} g$$

show that the image of g is a proper summand of M_t, contradicting the definition. Thus it suffices for the rank statement to produce a t with the properties above.

Since every number greater than m comes before m in the sequence $\lambda^{(b)}$ there is a number $k < k_0$ with $\sigma_{\text{opt}}(1) \leqq k \leqq \sigma_{\text{opt}}(s)$ and $\lambda_k^{(b)} = m+1$. Let k_1 be the minimal such number, and choose t' minimal such that $\sigma_{\text{opt}}(t') \geqq k_1$. If $\lambda_{t'} > m+1$ then we choose $t = t'$, while if $\lambda_{t'} = m+1$ then we choose $t = t'+1$. We check the properties of t as follows, using repeatedly the remark that every number between b and $\lambda_i^{(b)}$ appears before $\lambda_i^{(b)}$ in the sequence $\lambda^{(b)}$:

• $t > 1$: The case $t = 1$ would require $k_1 < \sigma_{\text{opt}}(1) = \sigma_{\text{opt}}(t) = \sigma_{\text{opt}}(t') < k_0$. As the subsequence of $\lambda^{(b)}$ between $m+1 = \lambda_{k_1}^{(b)}$ and $m = \lambda_{k_0}^{(b)}$ is a repeat of the part of the sequence before $\lambda_{k_1}^{(b)}$, this contradicts the optimality of σ_{opt}.

• $t < s$: If $\sigma_{\text{opt}}(s-1) < k_1$ then since every number $\geqq m+1$ occurs between $\lambda_{k_1}^{(b)}$ and $\lambda_{k_0}^{(b)}$, we would have $\sigma_{\text{opt}}(s) < k_0$, a contradiction. If actually $\sigma_{\text{opt}}(s-1) = k_1$, so that $\lambda_{s-1} = m+1$, then the same contradiction would occur unless $\lambda_s = m+1$, contradicting our assumption that at most one of the λ_k is equal to $m+1$. Thus $\sigma_{\text{opt}}(s-1) > k_1$. It follows that $t \leqq s-1$ unless $t' = s-1$, $\lambda_{s-1} = m+1$. This last case again contradicts the optimality of σ_{opt}, since λ_s could be found strictly between $\lambda_{t'}^{(b)} = m+1$ and $\lambda_{k_0}^{(b)} = m$.

• $\operatorname{rank}(g : M_1 \to M_t) \leqq m+1$: This follows from the induction, since

$$\sigma_{\text{opt}}(1) \leqq k_1 \leqq \sigma_{\text{opt}}(t) .$$

• $\operatorname{rank}(h : M_t \to M_s) \leqq m+1$: The sequence $(\lambda_1^{(b)}, \ldots, \lambda_{k_1-1}^{(b)})$ is the same as the sequence $(\lambda_{k_1+1}^{(b)}, \ldots, \lambda_{k_0-1}^{(b)})$; it follows that if τ is the optimal embedding of the length sequence $(\mu_1 := \lambda_t, \ldots, \mu_{s-t+1} := \lambda_s)$ then $\tau(1) \leqq k_1 \leqq \tau(s-t+1)$. The desired conclusion then follows by our induction.

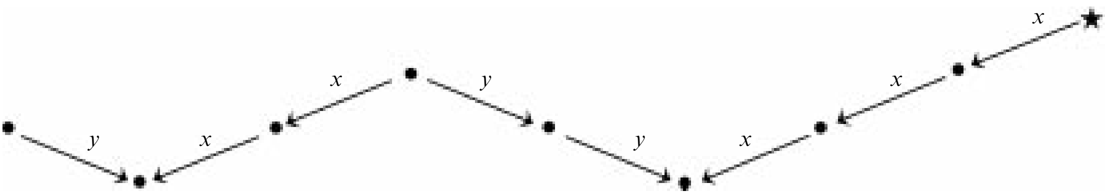
• $\operatorname{length}(M_t) > m+1$: In the sequence $\lambda^{(b)}$ there is an occurence of m between the first two occurences of $m+1$. The first occurence of $m+1$ is in the k_1 place, that of m is in the k_0 place. Since $k_1 < \sigma_{\text{opt}}(t) < k_0$ we thus have $\sigma_{\text{opt}}(t) \neq m+1$, and it follows that $\sigma_{\text{opt}}(t) > m+1$. This completes the proof of the rank condition.

Construction of examples

By Corollary 2 the existence statements of Theorems 1 and 4 follow from the existence, for each b, of a Harada-Sai sequence with length sequence $\lambda^{(b)}$. By Corollary 3 there is no ring of finite length over which all such possibilities exist. One of the simplest rings over which they could all exist is the ring $R = K[x,y]/(xy)$, where K is a field and $K[x,y]$ denotes the commutative polynomial ring; we shall construct them there. (Harada-Sai sequences of maximum length have been constructed over certain noncommutative rings of finite representation type by Bongartz [1].)

For any (noncommutative) word w in x and y we construct an indecomposable R-module $M(w)$ of length one more than the number of letters in the word. Each module is constructed with a particular basis, and we will only consider maps that take basis elements to basis elements. Among the basis elements, one is distinguished; we call it ★.

Rather than give a formal definition we describe the module $M(w)$ for the word $w = yx^2y^2x^3$: In this case $M(w)$ has a K-basis of $8 = (1+2+2+3)$ elements labelled ● and one element labelled ★, with multiplication table given by



Note that $y\star = 0$ and $\star \notin xM(w)$; equivalently, $\star$ appears at the right hand side of the diagram above.

To construct the Harada-Sai sequences we use three functors:

◇ $D = \mathrm{Hom}_K(-, K) : \mathrm{mod}_R \to \mathrm{mod}_R$ is the duality functor; the distinguished basis vector $\star$ of $\mathrm{Hom}_K(M, K)$ is taken to be the dual basis vector to the distinguished basis vector $\star$ of M.

◇ $E : \mathrm{mod}_R \to \mathrm{mod}_R$ the functor induced by the ring automorphism

$$R \to R, \quad x \mapsto y, \quad y \mapsto x\,.$$

Observe that $EM(w) = M(\hat{w})$, where $\hat{w}$ is obtained from w by exchanging x and y. If $f\colon M(w_1) \to M(w_2)$ satisfies $f(\star) = \star$, then $E(f) : M(\hat{w}_2) \to M(\hat{w}_1)$ satisfies $E(f)(\star) = \star$.

◇ $F\big(M(w)\big) := M(wx)$. Note that $M(w)$ is naturally a submodule of $M(wx)$; we take the new $\star$ to be the newly added basis vector. If $f\colon M(v) \to M(w)$ is a map preserving $\star$, then we take $F(f) : M(vx) \to M(wx)$ to be f extended in such a way as to preserve the new $\star$.

For each $b \in \mathbb{N}$, we construct a Harada-Sai sequence $\varepsilon^{(b)}$ in mod_R, having length sequence $\lambda^{(b)}$. We proceed by induction on b, taking $\varepsilon^{(1)}$ to be the sequence whose only terms is the one-dimensional module $M(1)$, where 1 represents the empty word.

Suppose that $\varepsilon^{(b)}$ has the form

$$\varepsilon^{(b)} : M(w_1) \longrightarrow M(w_2) \longrightarrow \cdots \longrightarrow M(w_{2^{b-1}}) \longrightarrow \cdots \longrightarrow M(w_{2^b - 1})$$

and all maps take $\star$ to $\star$. Note that $\lambda^{(b)}_{2^{b-1}} = 1$, so $w_{2^{b-1}} = 1$. We define $\varepsilon^{(b+1)}$ to be the sequence with left segment

$$\begin{aligned} \alpha : F\big(M(w_1)\big) &\longrightarrow F\big(M(w_2)\big) \longrightarrow \cdots \\ &\longrightarrow F\big(M(w_{2^{b-1}})\big) = M(x) = FE\big(M(w_{2^{b-1}})\big) \longrightarrow \cdots \\ &\longrightarrow FE\big(M(w_{2^b - 1})\big)\,. \end{aligned}$$

In all these modules $\star$ is outside the radical. Thus we may continue the sequence α with the map $f\colon FE\big(M(w_{2^b - 1})\big) \to M(1)$ sending $\star$ to $\star$ and sending all the other basis vectors to 0. We define the right-hand segment of $\varepsilon^{(b+1)}$ to be $D(\alpha)$. The whole sequence $\varepsilon^{(b+1)}$

consists of these two together with the module $M(1)$ in the middle. We may represent this symbolically as

$$\varepsilon^{(b+1)}: \alpha \xrightarrow{f} M(1) \xrightarrow{D(f)} D(\alpha)\,.$$

Since the composition of all the maps in the sequence sends ★ to ★ it is nonzero, and we are done. □

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