# 8. Computation of Cohomology

By **David Eisenbud**

The problem considered here is: Given a finitely generated graded $S = k[x_0,\dots,x_r]$–module $M$, compute the cohomology of the sheaf $\widetilde{M}$ on $\mathbb{P}^r$. A desirable refinement: starting from a bigraded module, compute the cohomology of the corresponding sheaf on $\mathbb{P}^r \times \mathbb{P}^s$ (this should be much easier to do directly than to first take the Segre embedding, since then one has $(r+1)(s+1)$ variables instead of $r+s+2$ variables). Of course we could also ask for the multigraded case, and more generally the case of local cohomology with arbitrary supports.

What does it mean to compute the cohomology? Depending on circumstances, it might mean anything from finding the dimension $h^i(\widetilde{M}(n))$ of one cohomology vector space $H^i\widetilde{M}(n)$ to finding the cohomology modules $\sum_n H^i\widetilde{M}(n)$, as modules over the ring $S = \sum_n H^0(\mathcal{O}_{\mathbb{P}^r}(n))$.

We will deal with three related methods, appropriate in different circumstances. Here is a summary of them, and their relative virtues and defects:

- **'Eyeballing' a free resolution of $M$:**
  - Often the cheapest in really simple cases.
  - Only computes $H^0$ up to an extension.
  - Really a naive version of the following method.
  - Does not work well in the bigraded case (unless a certain spectral sequence degenerates).

1. **Local Duality:** For $i \geq 1$ we have

$$\sum_n H^i\widetilde{M}(n) \simeq \operatorname{Ext}_S^{r-i}(M,\omega)^\vee,$$

where $\omega$ denotes $S(-r-1)$ and $L^\vee$, for a graded module $L = \sum_n L_n$, denotes $\operatorname{Hom}_{\text{graded } k}(L,k) = \sum_n \operatorname{Hom}(L_n,k)$, the graded vector space dual of the graded module $L$, as a module over $S$. For $i = 0$ we have the exact sequence

$$0 \to \operatorname{Ext}_S^{r+1}(M,\omega)^\vee \longrightarrow M \longrightarrow \sum_n H^0\widetilde{M}(n) \longrightarrow \operatorname{Ext}_S^r(M,\omega)^\vee \to 0.$$

- This form of (1) is better suited for automation, not so suited for "eyeballing".

- It's hard to use this to get the module structures, since $N^\vee$ is hard to compute as a module (it is not even finitely generated in general).
- The method of choice for getting information about

$$\sum_n H^d(\mathcal{O}_X(n)) \text{ or } \omega_X = (\sum_n H^d(\mathcal{O}_X(n)))^\vee$$

when $X$ is arithmetically Cohen–Macaulay subvariety of $\mathbb{P}^r$.
- Does not work in the bigraded case.

2. **Approximation:** For all $i \geq 0$ and any $b \in \mathbb{Z}$ we have

$$\sum_{n \geq b} H^i \widetilde{M}(n) \simeq \operatorname{Ext}^i_S(J, M)_{n \geq b},$$

$J$ is any homogeneous ideal primary to $(x_0, \ldots, x_r)$ and that is contained in

$$(x_0, \ldots, x_r)^a,$$

and where $a = a(M)$ is the maximum of the degrees of syzygies of $M$, diminished by $r + b$; that is, if

$$0 \to F_m \longrightarrow \cdots \longrightarrow F_0 \longrightarrow M \to 0$$

is a minimal free resolution of $M$, and $F_i = \sum S[-a_{ij}]$, then we may take

$$a = \max_{ij}\{\, a_{ij} \,\} - r - b.$$

We think of this as approximation because the modules $\operatorname{Ext}^i_S(J, M)$ "converge" to $\sum_n H^i \widetilde{M}(n)$ as $J$ moves into a higher and higher power of the maximal ideal, for example for the sequence of $J$'s given by $J_a = (x_0^a, \ldots, x_r^a)$.
- Well suited for automatic computation of the structure of the cohomology module in interesting cases.
- Can be extended to a multigraded setting, and in fact to the computation of local cohomology with given supports.
- Can be relatively expensive computationally, but seems best if one wants to understand $\sum_n H^i \widetilde{M}(n)$ as a module over $S$.

## 8.1 Eyeballing

The idea is to compare $M$ through exact sequences with modules whose cohomology are easy to understand—for example the graded free modules, which are sums of modules of the form $S(m)$. Of course $\widetilde{S}(m) = \mathcal{O}_{\mathbb{P}^r}(m)$, and we have

$$\sum_n H^i(\mathcal{O}_{\mathbb{P}^r}(n)) = \begin{cases} S & \text{if } i = 0 \\ 0 & \text{if } 0 < i < r \text{ or } i > r \\ S(-r-1)^\vee = \omega^\vee & \text{if } i = r, \end{cases}$$

(This last formula is somewhat misleading, since it might lead one incorrectly to think that $H^r$ was a contravariant functor! It really should be exhibited functorially, saying that for any graded free module $G$ we have

$$H^r(\widetilde{G}) = \mathrm{Hom}_S(G, \omega)^\vee.$$

If $G = S$ we the get the formula above.)

An example will probably be more helpful here than a theorem:

### Cohomology of the Smooth Rational Quartic in $\mathbb{P}^3$

Let $X \subset \mathbb{P}^3$ be the image of $\mathbb{P}^1$ under the map $(s,t) \mapsto (s^4, s^3t, st^3, t^4)$. Using *Macaulay*, we may easily compute the ideal of $X$ and its free resolution:

$$\begin{array}{cccc} 1 & - & - & - \\ - & 1 & - & - \\ - & 3 & 4 & 1 \end{array}$$

in the notation used by *Macaulay* "betti" command. The last map in the resolution is given by the matrix $(x_0, x_1, x_2, x_3)$. This information in the diagram suffices to compute some cohomology. For example, if $I$ is the ideal sheaf of $X$ and $\mathcal{K}$ is the first syzygy sheaf of $I$, then

$$\begin{aligned} H^1(O_X) &= H^2(I) \\ &\quad \text{since } H^i(O_{\mathbb{P}^3}(m)) = 0 \text{ for } i = 1, 2, \text{all } m \\ &= H^3(\mathcal{K}) \\ &\quad \text{since } H^i(O_{\mathbb{P}^3}(-2)) \oplus H^i(O_{\mathbb{P}^3}(-3)^3) = 0 \text{ for } i = 2, 3 \\ &= \text{coker } H^3(O_{\mathbb{P}^3}(-5)) \longrightarrow H^3(O_{\mathbb{P}^3}(-4)^4) \\ &\quad \text{since } H^4(O_{\mathbb{P}^3}(-5)) = 0 \\ &= \text{coker } S_1^\vee \longrightarrow (S_0^4)^\vee \\ &\quad \ \text{under the map induced by } (x_0, \dots, x_3), \end{aligned}$$

by the computation of the cohomology $H^i O_{\mathbb{P}^3}(n)$ above, where we have written $S_m$ for the vector space of forms of degree $m$ in $S$.

Now the map induced by $(x_0, \dots, x_3)$ is none other than the dual, in the appropriate degree, of the map $S^4 \longrightarrow S(1)$ induced by $(x_0, \dots, x_3)$. Since the map is a monomorphism (in fact an isomorphism) in degree 0, the dual is an epimorphism in degree 0; that is,

$$H^1(O_X) = \text{coker } S_1^\vee \longrightarrow (S_0^4)^\vee = 0.$$

This is indeed correct, since $X$ is a rational curve!

A more subtle example is furnished by the computation of $H^0(O_X(1))$. From the resolution we see that $X$ lies on no hyperplanes, that is $H^0(I(1)) = 0$, and we deduce the exact sequence

$$0 = H^0(I(1)) \to H^0(\mathcal{O}_{\mathbb{P}^3}(1)) \to H^0(\mathcal{O}_X(1)) \to H^1(I(1)) \to H^1(\mathcal{O}_{\mathbb{P}^3}(1)) = 0.$$

Since we know that $H^0(\mathcal{O}_{\mathbb{P}^3}(1)) = S_1$, we can at least see that

$$h^0(\mathcal{O}_X(1)) = 4 + h^1(I(1)).$$

But as above

$$\begin{aligned} H^1(I(1)) &= H^2(\mathcal{K}(1)) \\ &= \ker\,(H^3(\mathcal{O}_X(-4)) \longrightarrow H^3(\mathcal{O}_X(-3)^4) = 0) \\ &= S_0^\vee \end{aligned}$$

so $h^1(I(1)) = 1$, (corresponding to the fact that $X$ is the projection of a curve in $\mathbb{P}^4$). What makes this example subtle is that we were not able to compute $H^0(\mathcal{O}_{\mathbb{P}^3}(1))$ except as an extension of computable objects. This suffices for dimension computations, but would not suffice for finding the module structure (let alone the ring structure) on $\sum_n H^0(\mathcal{O}_X(n))$.

## 8.2 Local Duality

**Theorem 8.2.1.** *For $i \geq 1$ we have*

$$\sum_n H^i\widetilde{M}(n) \simeq \operatorname{Ext}_S^{r-i}(M,\omega)^\vee,$$

*where $\omega$ denotes $S(-r-1)$ and $N^\vee$ denotes* $\operatorname{Hom}_{\text{graded } k}(N,k)$ *the graded vector space dual of the graded module $N$ as a module over $S$. For $i = 0$ we have only the exact sequence*

$$0 \to \operatorname{Ext}_S^{r+1}(M,\omega)^\vee \longrightarrow M \longrightarrow \sum_n H^0\widetilde{M}(n) \longrightarrow \operatorname{Ext}_S^r(M,\omega)^\vee \to 0.$$

This theorem is really just a systematization of the method above. Here is a quick proof which makes this clear and which also shows how the method breaks down in the case of multigradings. We use the language of spectral sequences, but the reader with a distaste for this (suites spectrales $\equiv$ spectral sweets $\equiv$ ghost candy, and who wants to be a ghost?) may easily provide a proof by chasing long exact sequences, following the procedure in the example of "eyeballing" above.

*Proof.* The basis of the proof is the fact that cohomology may be defined from the Čech complex. Recall that the Čech complex of $S$ is:

$$\mathfrak{F}: \quad 0 \to \prod_i S[x_i^{-1}] \to \prod_{i<j} S[x_i^{-1}x_j^{-1}] \to \cdots \to S[x_0^{-1}\cdots x_r^{-1}] \to 0.$$

$\mathfrak{F}$ is easily seen to be the direct limit of the (truncated) Koszul complexes

$$\mathfrak{F}_m: \qquad 0 \to S^n(m) \to \wedge^2 S^n(2m) \to \cdots \to \wedge^n S^n(nm) \to 0,$$

of the ideals $(x_0^m, \ldots, x_r^m)$, (where for compactness we have written $n$ for $r+1$), under the maps

$$f\colon \mathfrak{F}_m \longrightarrow \mathfrak{F}_{m+1}$$

sending

$$\wedge^k S^n(km) \longrightarrow \wedge^k S^n(k(m+1))$$

by the $k$th exterior power of the diagonal map with entries $(x_0, \ldots, x_r)$. From this description, and a knowledge of the homology of the Koszul complexes, the cohomology of the Čech complex is obvious: Its value is given in the table for the cohomology of $\mathcal{O}_{\mathbb{P}^r}$ given above. By definition the cohomology module $\sum_n H^i \widetilde{M}(n)$, for any graded module $M$, is just the cohomology (actually homology, in the usual way of looking at things) of the complex $\mathfrak{F} \otimes M$ at the term

$$\prod_{j_0 < \cdots < j_i} S[x_{j_0}^{-1} \cdots x_{j_i}^{-1}] \otimes M.$$

Now given a free resolution $F$ of $M$, we get a double complex $\mathfrak{F} \otimes F$. As usual, two spectral sequences converge to the total homology of this complex. Since the complexes

$$\prod_{j_o < \cdots < j_i} S[x_{j_0}^{-1} \cdots x_{j_i}^{-1}] \otimes F$$

have as their only homology

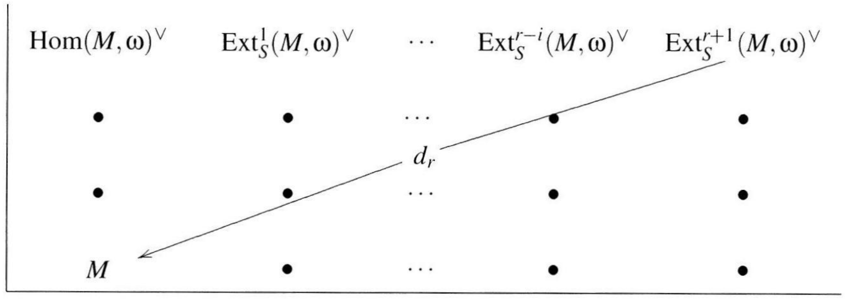
$$\prod_{j_o < \cdots < j_i} S[x_{j_0}^{-1} \cdots x_{j_i}^{-1}] \otimes M,$$

one of these spectral sequences degenerates and shows that the homology modules of the total complex $\mathfrak{F} \otimes F$ are precisely the cohomology modules of $\widetilde{M}$. Thus the other spectral sequence gives

$$H_q(H^p(\widetilde{F})) \Rightarrow H^{p-q}(\widetilde{M}),$$

where $H_q(H^p(\widetilde{M}))$ denotes the homology of the complex made from $F$ by sheafifying and replacing each term by its $p$th cohomology module.

Now $F$ is a complex of free modules, so $H^p(\widetilde{F}) = 0$ unless $p = 0$ or $p = r$. Further, the complex $H^0(\widetilde{F})$ is just the original complex $F$, and thus has homology only at the 0th place. Thus the $E_2$ term of the spectral sequence has nonzero entries in only two rows: in one of these we get only the module $M$, in the 0th position, and in the other, $r$ rows above, we get the homology of the complex $H^r(\widetilde{F})$. Now for any graded module $G$, $H^r(\widetilde{G})$ is $\mathrm{Hom}(G, \omega)^\vee$, and $^\vee$ is an exact functor, so the cohomology of $H^r(\widetilde{F})$ is $\mathrm{Ext}_S^{r-i}(M, \omega)^\vee$. The spectral sequence can thus be represented as follows:




where we have drawn the only non vanishing higher differential. Now the $i$th cohomology of $M$ is computed from the terms on the $(r-i)$th diagonal of this diagram, so we see that for $i>0$ we have $\sum_n H^i\widetilde{M}(n) \simeq \mathrm{Ext}^{r-i}_S(M,\omega)^\vee$ as required by the theorem, whereas for $i=0$ we get a filtration of $\sum_n H^i\widetilde{M}(n)$ consisting of a submodule which is the cokernel of $d_r$, and a quotient module which is $\mathrm{Ext}^r_S(M,\omega)^\vee$; that is, we get an exact sequence of the form required for the theorem ($d_r$ is a monomorphism because its kernel is $\sum_n H^{r+1}\widetilde{M}(n)$ which is zero since we are on $\mathbb{P}^r$.) □

The spectral sequence argument works just as well in the case of a product of projective spaces. The difference is that there are 4 nonzero cohomology modules of a sheaf on $\mathbb{P}^r \times \mathbb{P}^s$ associated to a bigraded module (computed from the Künneth formula) instead of 2. Thus the spectral sequence has 4 nontrivial rows and lots of nontrivial differentials, so methods 1 and 2 cannot be applied in general in the bigraded (and even less in the multigraded) case.

## 8.3 Approximation

**Theorem 8.3.1.** *For all $i \geq 0$ and any $b \in \mathbb{Z}$ we have*

$$\sum_{n\geq b} H^i\widetilde{M}(n) \simeq \mathrm{Ext}^i_S(J,M)_{n\geq b},$$

*where $J$ is any homogeneous ideal primary to $(x_0,\dots,x_r)$ contained in $(x_0,\dots,x_r)^a$, where $a=a(M)$ is the maximum of the degrees of the syzygies of $M$, diminished by $r+b$; that is, if*

$$0 \to F_m \longrightarrow \cdots \longrightarrow F_0 \longrightarrow M \to 0$$

*is a free resolution of $M$, and $F_i = \sum S(-a_{ij})$, then we may take*

$$a = (\max_{ij} a_{ij}) - r - b.$$

*Proof.* Because both cohomology and Ext fit into long exact sequences, it is enough to check that the formula holds for the modules $M = S(m)$, where it is trivial, and use the short exact sequences that make up the resolution for $M$. □

Since the homology of the truncated Koszul complex $\mathfrak{F}_m$ is $\mathrm{Ext}^i_S(J_m,M)$, and since homology commutes with direct limits, we get

$$\sum_n H^i\widetilde{M}(n) = \lim_m \mathrm{Ext}^i_S(J_m,M)$$

from the definition above of Čech cohomology. Since the $J_m$ are cofinal with any sequence of ideals $I_m$ such that $I_m$ is primary to $(x_0,\dots,x_r)$ and $I_m \subset (x_0,\dots,x_r)^m$, we can replace the $J_m$ in this formula with $I_m$, and the formula will remain true. The result above is simply a quantitative version giving "uniform convergence on compact subsets": in the degrees above any given degree, equality already holds at some finite stage, which is explicitly estimated in terms of the twists in the resolution of $M$. This is really the best one can do: the cohomology modules $\sum_n H^i\widetilde{M}(n)$ themselves are often not even finitely generated modules, whereas each $\mathrm{Ext}^i_S(J_m,M)$ is finitely generated! On the other hand the given estimate for $a$, though sometimes sharp, is not always so; it would be quite interesting to have other, more subtle results along these lines. For example, is it better computationally to use the ideals $J_m$ above or to use the powers $I_m = (x_0,\dots,x_r)^m$?

In one case there is a sharper convergence result, which seems worth mentioning:

**Theorem 8.3.2.** *If $i$ is the smallest integer $> 0$ for which $\sum_n H^i\widetilde{M}(n) \neq 0$, then the natural map*

$$\mathrm{Ext}^i_S(J,M) \longrightarrow \sum_n H^i\widetilde{M}(n)$$

*is injective, with image the submodule of all elements annihilated by $J$.*

*Proof.* We may rephrase (a stronger version of) this in terms of local cohomology: if $i$ is the smallest integer such that the local cohomology module $H^{i+1}_{\mathfrak{m}}(M) \neq 0$, then the natural map

$$\mathrm{Ext}^{i+1}_S(S/J,M) \longrightarrow H^{i+1}_{\mathfrak{m}}(M)$$

is injective with image the submodule annihilated by $J$. This is obvious if $i+1=0$. If $i+1>0$ then $M$ has depth $>0$, and we may finish by an induction using a nonzero divisor $x$ on $M$ and the associated exact sequence

$$0 \to M \xrightarrow{\cdot x} M \longrightarrow M/xM \to 0.$$

In general however, the maps $\mathrm{Ext}^i_S(J,M) \to \sum_n H^i\widetilde{M}(n)$ are not well–behaved away from the range given in Theorem 8.3.1.

The adaptation of this method to the bigraded case is the following:

**Theorem 8.3.3.** *If $M$ is a bigraded module on $S = k[x_0,\dots,x_r,y_0,\dots,y_s]$ and $\widetilde{M}$ is the corresponding sheaf on $\mathbb{P}^r \times \mathbb{P}^s$, then for all $i \geq 0$ and any $(b_1,b_2) \in \mathbb{Z}^2$ we have*

$$\sum_{(m,n)\geq(b_1,b_2)} H^i\widetilde{M}(m,n) \simeq \mathrm{Ext}^i_S(J,M)_{\geq(b_1,b_2)},$$

*where $J = J_{b_1} J_{b_2}$ is any product of homogeneous ideals*

$$J_{b_1} \subset k[x_0,\dots,x_r],\ J_{b_2} \subset k[y_0,\dots,y_s]$$

*primary to $(x_0,\dots,x_r)$ and $(y_0,\dots,y_s)$, and contained in $(x_0,\dots,x_r)^{a_1}$, $(y_0,\dots,y_s)^{a_2}$ respectively, where $(a_1,a_2)$ is the maximum of the degrees of the syzygies of $M$, diminished by $(r+b_1, s+b_2)$.*

*Proof.* Because of the existence of bigraded free resolutions, it suffices as before to prove that the theorem holds in case $M = S(m,n)$. But both $\mathrm{Ext}^i_S(J,M)$ and

$$\sum_{(m,n)} H^i \widetilde{M}(m,n)$$

satisfy a Künneth formula, so we reduce everything to Theorem 8.3.1. □

**Problem 8.3.1.** It might be more convenient if we could use arbitrary ideals contained in higher and higher powers of the ideal of bilinear forms

$$(x_0,\dots,x_r)(y_0,\dots,y_s),$$

but this is not the case—if we take ideals with embedded components corresponding to the prime ideal $(x_0,\dots,x_r,y_0,\dots,y_s)$ then counterexamples can be manufactured. However, one suspects that it is enough to take ideals primary to $(x_0,\dots,x_r, y_0,\dots,y_s)$ and contained in large powers of it; is this true? Are there any particularly simple such ideals (for example, with few, nice, generators), to make it really worthwhile computationally?