November 15, 1995

Evolutions, Symbolic Squares, and Fitting Ideals

by

David Eisenbud and **Barry Mazur**

Abstract

Given a reduced local algebra T over a suitable ring or field k we study the question of whether there are nontrivial algebra surjections $R \to T$ which induce isomorphisms $\Omega_{R/k} \otimes T \to \Omega_{T/k}$. Such maps, called evolutions, arise naturally in the study of Hecke algebras, as they implicitly do in the recent work of Wiles, Taylor-Wiles, and Flach. We show that the existence of non-trivial evolutions of an algebra T can be characterized in terms of the symbolic square of an ideal defining T. We give a characterization of the symbolic square in terms of Fitting ideals. Using this and other techniques we show that certain classes of reduced algebras — codimension 2 Cohen-Macaulay, Codimension 3 Gorenstein, licci algebras in general, and some others — admit no nontrivial evolutions. On the other hand we give examples showing that non-trivial evolutions of reduced Cohen-Macaulay algebras of codimension 3 do exist in every positive characteristic.

Introduction

Definition. Let Λ be a ring and let T be a local Λ-algebra essentially of finite type, that is, a localization of a finitely generated Λ algebra. An *evolution* of T over Λ is a local Λ-algebra R essentially of finite type and a surjection $R \to T$ of Λ-algebras inducing an isomorphism $\Omega_{R/\Lambda_R} \otimes T \to \Omega_{T/\Lambda}$. The evolution is *trivial* if $R \to T$ is an isomorphism.

This notion was first formulated (in slightly different language) by Scheja and Storch [1970] and Böger [1971]. In their terminology a pair of ideals $J \subseteq I$ in a polynomial ring P define an evolution $P/J \to P/I$ if I is *differentially dependent* on J, and P/I admits no nontrivial evolutions (that are homomorphic images of P) if I is *differentially basic*. The idea was studied in a number-theoretic context in Mazur [1994].

For example if $f \in P := \Lambda[x_1, \ldots, x_n]_{(x_1,\ldots,x_n)}$ and I is the ideal generated by the derivatives f, then P/I is an evolution of $P/(I, f)$, nontrivial when f is not contained in I (in characteristic zero this means roughly that f is not quasihomogeneous — see Saito [1971]). This may lead one to think that non-trivial evolutions are everywhere. On the other hand, if T is a reduced complete intersection, generically separable over Λ, then every evolution of T is trivial (this well-known result is generalized below). Moreover, we have been unable to find any nontrivial evolution of any reduced algebra in equi-characteristic zero, or of any reduced algebra which is flat over a discrete valuation ring of mixed characteristic.

In this paper we study the general question of existence of nontrivial evolutions. First we exhibit an elementary characterization, found by H. Lenstra, of those algebras that admit a nontrivial evolution in terms of their cotangent sequences. We use this to connect the existence of evolutions with the following question: given a prime ideal (or more generally an unmixed ideal I in a local ring $(P, \mathcal{M})$, when is it true that the symbolic square $I^{(2)}$ is contained in the product $I\mathcal{M}$? (Definitions are given below). We characterize symbolic squares in terms of the behavior of certain Fitting ideals, and use this characterization to show that $I^{(2)} \subseteq I\mathcal{M}$ in the case of perfect ideals of codimension 2. We generalize this relation, using a result of Buchweitz, to the case of licci ideals. We then show that the same relation (and in some cases an even stronger one) holds for several other classes of ideals. We exhibit a result of E. Kunz showing that this is the case for unmixed almost complete intersections. We show that it also holds for unmixed quasihomogeneous ideals in characteristic 0. On the negative side, we give examples of unmixed quasihomogeneous prime ideals in every positive characteristic with non-trivial evolutions; these were found through an analysis of an example by E. Kunz, with the help of Sorin Popescu.

We may pose a question which has a somewhat elementary appearance, but for which an affirmative answer is equivalent to the non-existence of nontrivial evolutions of reduced algebras in equi-characteristic zero.

Problem. Suppose $f \in \mathbf{C}[[x_1, \ldots, x_n]]$ is a power series without constant term over the complex numbers, and I is the ideal of the reduced singular locus of f, that is, I is the radical of the ideal generated by the partial derivatives of f. Does it follow that $f \in (x_1, \ldots, x_n)I$?

It is evident that the answer is "yes" if f has an isolated singularity. Thus the first interesting case occurs when the singular locus of f is a curve in $\mathbf{C}^3$. The results of this paper suffice to prove that the answer is still "yes" in this case, or more generally whenever the embedding dimension of the reduced singular locus of f is less than 4, or in the case of embedding dimension 4 when the reduced singular locus of f is Gorenstein, or licci. In particular, the answer is "yes" for power series in at most 3 variables. We do not know any example where the answer is "no" in characteristic 0 (or for the analogous question in mixed characteristic).

In characteristic 0, when f is quasihomogeneous, one can express f itself as a linear combination of the derivatives of f, with coefficients in $(x_1, \ldots, x_n)$; thus in characteristic 0 the answer to the question above is "yes" for any quasihomogeneous polynomial. In contrast, the answer to the question above is "no" for the following quasihomogenous polynomial in four variables over a field of characteristic p, from the example treated at the end of this paper:

$$f = x_1^{p+1}x_2 - x_2^{p+1} - x_1x_3^p + x_4^p.$$

When T is a complete Λ-algebra that is a homomorphic image of a power series ring of the form $\Lambda[[x_1, ..., x_n]]$, we could define evolutions similarly, using the module of universally finite derivations. We then could extend the notion to the non-local case by requiring the condition above after localization or completion. All the results of this paper could be cast in that general setting. But to simplify our presentation, we shall work with local algebras that are essentially of finite type, as indicated.

We began to study evolutions because of a situation that occurs in the deformation theory of Galois representations. In this theory one deals with a "universal deformation ring" R, about which almost nothing is known beyond the fact that it is a complete noetherian local ring, and one also has a particular quotient of that ring, T, a completion of a Hecke algebra, which is somewhat more accessible. From first order infinitesmal information about deformations, which may be computed by cohomology, one can sometimes show that the mapping $R \to T$ is an evolution; such arguments occur for example in Mazur [1994], which is an exegesis of the work of Flach. At this point if, by virtue of some special properties of T, one knows that it admits no nontrivial evolutions (e.g., if T is a reduced complete intersection) then one may deduce that T is the universal deformation ring. Such an assertian implies that certain Galois representations are modular, and is interesting for that reason.

The work of Wiles and Taylor-Wiles uses other methods to establish that the map $R \to T$ is an isomorphism for the rings R and T which they treat. Their methods show, in addition, that their rings are reduced complete intersections. This implies, of course, that T has no nontrivial evolutions, and gives an a posteriori "explanation" why $R = T$. We would like to have a clearer understanding of evolutions as a tool to be used in this type of argument, even in cases where the rings in question are not complete intersections, or are not known to be.

We are grateful to E. Kunz and H. Lenstra for sharing their work with us and allowing us to include some of their results and examples, to S. Popescu for his help, mathematical and computer-related, with the examples, and to C. Huneke, and M. Hochster with whom we had helpful discussions of the material.

1. Criteria for the existence of evolutions

We shall use a criterion for the existence of evolutions suggested by Lenstra. If T is a ring and $\phi : M \to N$ is an epimorphism of T-modules, then ϕ is *minimal* if there is no proper submodule $M' \subset M$ such that $\phi(M') = N$.

Proposition 1 (Lenstra). *Let Λ be a Noetherian ring and let T be a local Λ-algebra, essentially of finite type over Λ. Every evolution of T is trivial iff for some (equivalently all) presentations $T = P/I$, where P is a localization of a polynomial ring over Λ, the map*

$$d_{T/P/\Lambda} : I/I^2 \to \ker(T \otimes_P \Omega_{P/\Lambda} \to \Omega_{T/\Lambda})$$

induced by the universal derivation is minimal.

Proof. Let $T = P/I$ be any presentation of T where P is a localization of a polynomial ring in finitely many indeterminates over Λ. If J is an ideal of P with $J \subset I$ then an obvious diagram chase shows that the natural surjection $R := P/J \to P/I = T$ is an evolution iff the differential $d_{T/P/\Lambda}$ carries $(J+I^2)/I^2$ onto the same image as I/I^2. Using Nakayama's Lemma we see that $J = I$ iff $(J+I^2)/I^2 = I/I^2$, so $d_{T/P/\Lambda}$ is minimal iff T has no nontrivial evolution of the form P/J.

It remains to show that the condition that $d_{T/P/\Lambda}$ be minimal is independent of the presentation $T = P/I$ chosen. Since the family of presentations is filtered, it is enough to show that if $T = P/I$ is a presentation, P' is a localization of a polynomial ring in one variable x over P, and $T = P'/I'$ is a presentation extending $T = P/I$ in an obvious sense, then $d_{T/P'/\Lambda}$ is minimal iff $d_{T/P/\Lambda}$ is minimal.

Let $g \in P$ be an element with the same image in T as x, so that $x - g \in I'$. Replacing x with the new "variable" $x - g$, we may assume for simplicity that g is 0. We then have $I'/I'^2 = I/I^2 \oplus Tx$ and $\ker(d_{T/P'/\Lambda}) = Tdx \oplus \ker(d_{T/P/\Lambda})$. The required equivalence follows by Nakayama's Lemma. ■

Here is the relation of the problem given in the introduction to the existence of evolutions in characteristic 0:

Corollary 2. *There exists a reduced local* $\mathbf{C}$*-algebra* T *of finite type whose localization at the origin has a nontrivial evolution iff there exists a power series* $f \in \mathbf{C}[[x_1, \ldots, x_n]]$ *without constant term such that*

$$f \notin (x_1, \ldots, x_n)\sqrt{(f, df/dx_1, \ldots, df/dx_n)}.$$

Proof. Set $I := \sqrt{(f, df/dx_1, \ldots, df/dx_n)}$ (since we are in characteristic 0 we could leave out f without changing this ideal). First suppose that $f \notin (x_1, \ldots, x_n)I$. Replacing f by an approximation to very high order, we may suppose that f is a polynomial. Since $f \notin (x_1, \ldots, x_n)I$. we may choose an ideal $J \subset I$ generated by polynomials such that $J \neq I$ but $(J, f) = I$. Writing P for the localization at the origin of the polynomial ring in the x_i it follows from the definition that $P/J \to P/I$ is a nontrivial evolution.

Conversely, suppose $P/J \to P/I$ is a nontrivial evolution, where I is a radical ideal, and P is the localization of a polynomial ring over $\mathbf{C}$ at the origin. Let f' be a minimal generator of I that is not contained in J. By definition, $df' \in \Omega_{P/\mathbf{C}}$ goes to zero in $(\Omega_{P/\mathbf{C}} \otimes P/I)/dJ$. Thus there is are elements $g_i \in J$ and $p_i \in P$ such that $d(f') - \sum r_i d(g_i))$ is in $I\Omega_{P/\mathbf{C}}$. Let $f := f' - \sum r_i g_i$, and note that f is again a minimal generator of I which is not in J. Since $g_i \in J \subset I$ we have

$$df = df' - \sum r_i d(g_i) - \sum g_i d(r_i) \in I\Omega_{P/\mathbf{C}}$$

Thus the partial derivatives of f are all contained in I. Since I is radical and $f = \notin \mathcal{M}I$ we obtain $f \notin \mathcal{M}\sqrt{(f, df/dx_1, \ldots, df/dx_n)}$ as required. ■

In special cases we can identify the kernel of the map $d_{T/P/\Lambda}$ defined in Proposition 1. If I is an ideal in a ring P we define the nth *symbolic power* of I to be the ideal

$$I^{(n)} = \{f \in P | f \in I_Q^n \subset P_Q \text{ for all minimal primes } Q \text{ of } I\}.$$

Note that the symbolic powers depend only on the isolated ($\equiv$ non-embedded) primary components of I, and in particular the first symbolic power is the intersection of the isolated primary components of I.

Theorem 3. *Let Λ be a Noetherian regular ring. Let $P, \mathcal{M}$ be a localization of a polynomial ring in finitely many variables over Λ and let I be an ideal of P. If $T := P/I$ is reduced and generically separable over Λ, then the kernel of $d : I/I^2 \to T \otimes_\Lambda \Omega_{P/\Lambda}$ is $I^{(2)}/I^2$. Thus every evolution of T is trivial iff $I^{(2)} \subset \mathcal{M}I$.*

Proof. Let Q be a minimal prime of I. Since Λ is regular the prime $Q_Q \subset P_Q$ is a complete intersection. By hypothesis P/Q is separable over Λ and thus $d : (I/I^2)_Q \to \Omega_{P_Q/\Lambda}$ is an injection. Since $T \otimes_\Lambda \Omega_{P/\Lambda}$ is free over T the map

$$T \otimes_\Lambda \Omega_{P/\Lambda} \to \oplus_Q T \otimes_\Lambda \Omega_{P_Q/\Lambda}$$

is also a monomorphism, where the direct sum is taken over the minimal primes of I. Thus the kernel of $d : I/I^2 \to T \otimes_\Lambda \Omega_{P/\Lambda}$ is the same as the kernel of $I/I^2 \to \oplus_Q (I/I^2)_Q$, which is $I^{(2)}/I^2$ as required. ∎

This result shows that to prove that every evolution of a reduced, generically separable algebra of finite type is trivial it suffices to show that the symbolic square of the defining ideal I is contained in I times the maximal ideal. The rest of this paper is concerned with results of this type.

Before turning to these results we mention a different approach, suggested by Huneke. Scheja and Storch [1970] and Böger [1971] prove under some additional hypotheses that the kernel of the map $d : I/I^2 \to T \otimes_\Lambda \Omega_{P/\Lambda}$ is I'/I^2, where I' is the integral closure of I^2. Rather than studying symbolic squares, one might study integral closures, and try to prove that for some class of ideals I we have $(I^2)' \subset \mathcal{M}I$. The disadvantage of this approach is that in general the integral closure is much smaller than the symbolic square, so presumably $(I^2)' \subset \mathcal{M}I$ holds less frequently than $I^{(2)} \subset \mathcal{M}I$. The potential advantage is that questions of integral closure can often be reduced to questions about zero-dimensional ideals, which is generally not the case for questions about the symbolic square.

2. Symbolic powers and Fitting ideals

In this section we establish a simple connection between Fitting invariants and symbolic powers, and use it to analyze symbolic powers in certain special cases. First some definitions:

We shall say that I is *unmixed* if all the associated primes of I are isolated. (Caution: The word unmixed is sometimes used to denote an ideal all of whose associated primes have the same dimension — a stronger condition.)

Fitting invariants are defined as follows: Suppose M is a finitely generated module over a commutative ring R, with a presentation of the form $M \cong R^n/K$. For each integer $i \geq 0$ the ith Fitting ideal of M is defined to be $F_i(M) = I_{n-i}(K)$, the ideal generated by the $(n-1) \times (n-i)$ minors that can be formed from the elements of K, regarded as vectors of length n. The Fitting invariants are independent of the presentation chosen, and commute with base change.

We recall two easy facts about Fitting invariants: Let M be a finitely generated R-module. If $M' \subset M$ is a submodule, then $F_i(M) \subset F_i(M/M')$. This is because given a presentation of M we may choose a presentation of M/M' with the same generators and more relations. Also, if $I \subset R$ is an ideal then $F_i(M/IM) \subset F_i(M)+I$. This is because we can derive a presentation for M/IM from one for M by adding only the relations saying that elements of I times the generators are 0.

Theorem 4. *If I is an unmixed ideal of depth $\geq c$ in a Noetherian ring and $x \in I^{(2)}$ then*

$$F_{c-1}(I/(x)) \subset I,$$

If I is generically a complete intersection of codimension c then the converse holds as well.

Proof. First, suppose that $x \in I^{(2)}$. Since I is unmixed, it suffices to prove that $F_{c-1}(I/(x)) \subset I$ after localizing at a minimal prime of I, so we may assume that $x \in I^2$. Thus $I/(x)$ surjects onto $I/I^2 = I \otimes R/I$, so $F_{c-1}(I/(x)) \subset F_{c-1}(I/I^2) \subset F_{c-1}(I) + I$ and it suffices to show that $F_{c-1}(I) \subset I$. This follows from the "structure theorem" of Buchsbaum-Eisenbud [1974].

Suppose that I is generically a complete intersection of depth c and $F_{c-1}(I/(x))$ is contained in I. Since the associated primes of $I^{(2)}$ are all minimal primes of I, we may begin by localizing at one such and suppose that $I = (f_1, ..., f_c)$ is a complete intersection. Under these circumstances we have $I^2 = I^{(2)}$, and we shall prove that $x \in I^2$.

The ideal I is generated by the c elements f_i. We may take the same generators for $I/(x)$. One of the relations on $I/(x)$ may be represented as a column vector whose entries g_i satisfy $x = \sum g_i f_i$. From the definition we see that $F_{c-1}(I/(x))$ contains each g_i so we have $g_i \in I$ and $x = \sum g_i f_i \in I^2$. ■

Remark. In the case where I is generically a complete intersection, the idea used in the second part gives the following more general version:

Theorem $4'$. *If I is an unmixed ideal of depth $\geq c$ in a Noetherian ring and I is generically a complete intersection then $x \in I^{(d+1)}$ iff*

$$F_{N-1}(I^{(d)}/(x)) \subset I,$$

where $N = \binom{c+d-1}{d}$. ■

As an application we can show that the symbolic square of a grade 2 perfect ideal in a ring R (that is, an ideal that contains an R-sequence of length 2 and that has projective dimension 1 as an R-module) cannot be too big if the ideal is generically a complete intersection. By the Hilbert-Burch theorem any such ideal can be represented as the minors of an $n \times (n-1)$ matrix.

Corollary 5. *Suppose that an ideal I in a ring R is the ideal of $(n-1) \times (n-1)$ minors of an $n \times (n-1)$ matrix M, that the depth of I on R is two, and that I is generically a complete intersection in R. If J is the ideal generated by the entries of any column of the matrix M, then $I^{(2)} \subset IJ$.*

Proof. By the Hilbert-Burch Theorem, a free resolution of I can be written in the form

$$0 \to R^{n-1} \to R^n \to I \to 0$$

where the left hand map is M. The generator of I coming from the jth generator of R^n is the minor of M involving all but the jth row. Considering the expansion of a determinant along a column, we see that every element $x \in I$ can be written as the determinant of an $n \times n$ matrix N whose first $n-1$ columns are the columns of M. The matrix N is also the presentation matrix of $I/(x)$. By Theorem 4, $x \in I^{(2)}$ iff $F_1(I/(x)) \subset I$, that is, iff the $(n-1) \times (n-1)$ minors of N are contained in I. Expanding the determinant of N along a column of M regarded as a column of N, we see that x, the determinant, is in IJ as claimed. ∎

In the next section we shall use a theorem of Buchweitz to extend this result to any licci ideal that is generically a complete intersection.

Suppose that I is an ideal of depth c in a ring R, and let $r \in F_c(I)$. In any localization of $R[r^{-1}]$ at a prime ideal we can generate I by c elements, so $I' := IR[r^{-1}]$ is locally a complete intersection, and $I'^{(d)} = I'^{d}$ for all d. This shows that a power of r, and thus also a power of $F_c(I)$, annihilates $I^{(d)}/I^d$ for all d. Using Theorem 4 we may sharpen this result for $d = 2$:

Theorem 6. *Let R be a Noetherian ring. If $I \subset R$ is an ideal that is generically a complete intersection of codimension c, then $F_c(I)$ annihilates $I^{(2)}/I^2$.*

Proof. Let $f_1, ..., f_n$ generate I, and let $M : R^m \to R^n$ be the matrix of relations on the f_i, that is, the image of M is the kernel of the map $f : R^n \to R$ defined by the f_i. Let $\gamma = (g_1, \ldots, g_n) \in R^n$, and write $g = f(\gamma) = \sum g_i f_i$ for the image of γ in I. Theorem 4 shows that g is in $I^{(2)}$ iff the matrix M' obtained from M by adjoining a column with entries g_i has all its $(n-c) \times (n-c)$ minors in I. Since the $(n-c) \times (n-c)$ minors of M are already in I by the theorem of Buchsbaum-Eisenbud [1974], this condition may be expressed by saying that the minors of M' containing the column of g's are in I.

We may express this condition invariantly as follows. The map $\wedge^{n-c-1}M$ corresponds to an element

$$M_{n-c-1} \in \wedge^{n-c-1}R^n \otimes \wedge^{n-c-1}R^{m*},$$

which can be written in terms of the minors of order $n-c-1$ of M. The wedge product with this element defines a map

$$m : R^n \longrightarrow \wedge^{n-c}R^n \otimes \wedge^{n-c-1}R^{m*}.$$

This map takes the element γ to the vector whose entries are the minors of M' of order $n-c$ that involve the column of g_i. Thus $g = f(\gamma) \in I^{(2)}$ iff

$$m(\gamma) \in I \wedge^{n-c} R^n \otimes \wedge^{n-c-1}R^{m*}.$$

Equivalently, the kernel of $R/I \otimes m$ maps by f onto $I^{(2)}/I$.

If γ is in the image of M then $f(\gamma) = 0 \in I^{(2)}$, so $m(\gamma)$ has entries in I. Thus we obtain a complex

$$(R/I)^m \longrightarrow (R/I)^n \longrightarrow \wedge^{n-c}(R/I)^n \otimes \wedge^{n-c-1}(R/I)^{m*}.$$

where the left hand map is $R/I \otimes M$. The cokernel of $R/I \otimes M$ is I/I^2. Thus the discussion above may be summarized by saying that the homology of this complex is $I^{(2)}/I^2$.

We shall conclude the proof by showing that if M is any $n \times m$ matrix over a ring S such that the $(n-c) \times (n-c)$ minors of M are zero, then the complex derived from M as in the display above, replacing R/I by S, has homology annihilated by the ideal J of $(n-c-1) \times (n-c-1)$ minors of M; in fact we shall show that multiplication by any such minor is homotopic to 0 on this complex. Let M' be an $n \times (m-c-1)$ submatrix of M; it suffices to show that multiplication by any maximal minor of M' is homotopic to 0 on the subcomplex

$$S^{m-c-1} \longrightarrow S^n \longrightarrow \wedge^{n-c}S^n \otimes \wedge^{n-c-1}S^{m-c-1*}.$$

The dual of this complex is the right hand end of the complex of Buchsbaum-Rim [1964]. In the case where M' is a matrix whose entries are distinct indeterminates, the Buchsbaum-Rim complex is a free resolution of the cokernel of M'^*. The ideal J then annihilates the cokernel of M' (and is in fact exactly the annihilator of *coker*(M'^*)). Thus multiplication by an $(n-c) \times (n-c)$ minor is homotopic to 0 on the Buchsbaum-Rim complex in this case, and by specialization this holds in every case. (Buchsbaum and Rim actually construct homotopies for these minors explicitly). ∎

It would be interesting to know more about the annihilators of $I^{(d)}/I^d$. One simple case might be that where I is an unmixed codimension 2 homogeneous ideal in a polynomial ring in 3 variables. A preliminary study of examples produced by the program Macaulay of Bayer and Stillman [1981] suggests the following:

Conjecture. If I is the ideal of minors of a "generic" (that is, random) 2×3 matrix of linear forms in 3 variables, then the annihilator of $I^{(d)}/I^d$ is $F_1(I)^e$, where e is the greatest integer $\leq d/2$.

3. Special cases

A. Monomial ideals

Proposition 7. *Suppose that I is a monomial ideal in a polynomial ring $k[x_1, \ldots, x_n]$. If P is a monomial prime ideal containing I then for any $d \geq 0$ we have $I^{(d)} \subset PI^{(d-1)}$.*

Proof. Suppose that $I = \cap Q_j$ is a primary decomposition. A monomial ideal Q_i is primary, say to the monomial prime $P_i = (x_{i_1}, \ldots x_{i_s})$, iff Q_i contains a power of each of the variables x_{i_t} and the minimal generators of Q_i do not involve any variables other than $x_{i_1}, \ldots x_{i_s}$. Thus any power of a primary monomial ideal is again primary, and we have $I^{(d)} = \cap Q_j^d$.

Now suppose that a monomial m is in $I^{(d)}$. By the argument above, for each index j we may write $m = r_j m_{j,1} \cdots m_{j,d}$ with $m_{j,i} \in Q_j$. Since $m \in P$, some variable $x_t \in P$ divides m, and thus divides (at least) one of $r_j, m_{j,1}, \ldots, m_{j,d}$. It follows that m/x_t may be written as a product with at least $d-1$ factors in Q_j, so $m/x_t \in \cap Q_j^{d-1} = I^{(d-1)}$. Thus $m \in PI^{(d-1)}$ as claimed. ∎

B. Licci ideals

Recall that a perfect ideal I in a regular ring R is called **licci** if it is in the linkage class of a complete intersection; see for example Huneke-Ulrich [1987,1992] for some of the marvelous properties of these ideals. Licci ideals include for example all perfect ideals of codimension 2 and all Gorenstein ideals of codimension 3. Thus the following result greatly generalizes Corollary 5.

Theorem 8. *Suppose that R is a regular local ring, and that $I \subset R$ is a perfect ideal that is generically a complete intersection. Let J be the ideal generated by the elements in a row of some presentation matrix (over R or over R/I) of the canonical module $\omega_{R/I}$ of R. If I is licci, then $I^{(2)} \subset JI$.*

Proof. An ideal J as in the Theorem is just the annihilator of some cyclic quotient module $R/J \cong \omega_{R/I}/\omega'$ for some submodule ω' of ω. In particular, such an ideal J must contain I. By a theorem of Buchweitz (Thesis, [1981]; see also Buchweitz-Ulrich [1983]), the fact that I is licci implies that the module $\omega_{R/I} \otimes I/I^2$ is a Cohen-Macaulay module over R/I.

Thus the map

$$\omega_{R/I} \otimes I^{(2)}/I^2 \to \omega_{R/I} \otimes I/I^2$$

is zero. The natural map $I^{(2)} \to I \to I/IJ$ can be factored as

$$I^{(2)} \to R/J \otimes I^{(2)}/I^2 \to R/J \otimes I/I^2 = I/IJ$$

and is thus 0, whence $I^{(2)} \subset IJ$. ∎

C. Almost complete intersections (Kunz)

Another case in which symbolic squares behave well is for almost complete intersections. The following result and its proof was communicated to us by E. Kunz (in a slightly different form). We are grateful to him for allowing us to include it.

Theorem (E. Kunz) 9. *Let P be a regular local ring with maximal ideal be a $\mathcal{M}$, and let $I \subset P$ be an ideal that is generically a complete intersection. If I can be generated by $codim(I) + 1$ elements, then $I^{(2)} \subseteq \mathcal{M}I$.*

Proof. Set $n := codim(I)$. If $I^{(2)} \not\subset \mathcal{M}I$, then $(I/I^{(2)})$ can be generated by n elements. Since $I/I^{(2)}$ is free of rank n locally at any minimal prime of I, we see that $I/I^{(2)} \cong (P/I)^n$, and hence $I/I^2 \cong (P/I)^n \oplus N$ where N is a nonzero module. By the proposition of Vasconcelos [1968] I contains a regular sequence of length greater than n, a contradiction. ∎

D. Quasi-homogeneous ideals

Proposition 10. *Let $P = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k, and let $\mathcal{M} = (x_1, \ldots, x_n)$. Suppose that $I \subset P$ is an unmixed ideal which is quasihomogeneous (that is, homogeneous with respect to some system of strictly positive integer weights of the variables). If $f \in I^{(d)}$ is a quasihomogeneous element, then $f \in deg(f) \cdot \mathcal{M}I^{(d-1)}$. In particular, if $char(k) = 0$ then $I^{(d)} \subseteq \mathcal{M}I^{(d-1)}$, and if in addition I is a radical ideal then every evolution of P/I is trivial.*

Proof. If $f \in I^{(d)}$ then by definition there is a polynomial h, not contained in any of the minimal primes of I, such that $hf \in I^d$. Differentiating we see that for each index j we have $hdf/dx_j + fdh/dx_j \in I^{d-1}$. Since $f \in I^{(d-1)}$ we deduce $hdf/dx_j \in I^{(d-1)}$, whence $df/dx_j \in I^{(d-1)}$. Euler's relation $deg(f) \cdot f = \sum deg(x_j) \cdot x_j df/dx_j$ now shows that $f \in deg(f) \cdot \mathcal{M}I^{(d-1)}$ as required. The last assertion follows from Proposition 1. ∎

In positive characteristic, by contrast, there are evolutions of quasihomogeneous rings, even of 1-dimensional quasihomogeneous domains. The first example of which we are aware was communicated to us by E. Kunz. It is the (localization at the quasihomogeneous maximal ideal of the) domain

$$k[t^{14}, t^{20}, t^{25}, t^{30}, t^{91}],$$

where k is a field of characteristic 2. A careful analysis of Kunz' example led us to a series of simpler examples, in all positive characteristics. We would like to thank Sorin Popescu, who helped us greatly, both with the computation (using Maple and the program Macaulay of Bayer and Stillman [1981–]) and the mathematical aspect of the examples below.

These examples are in certain senses minimal: It follows from the results above that any example of a reduced algebra with nontrivial evolution must have embedding codimension at least 3, and our examples have embedding codimension exactly 3. Also, any local Cohen-Macaulay ring with embedding codimension at least 3 has multiplicity at least 4. Our examples are one-dimensional domains, thus Cohen-Macaulay, and the example in characteristic 2 below has multiplicity exactly 4.

Example Let k be a field of characteristic $p > 0$, and let I be the kernel of the map

$$\begin{aligned} &k[x_1, \ldots, x_4] \to k[t] \\ &x_1, x_2, x_3, x_4 \mapsto t^{p^2}, t^{p(p+1)}, t^{p^2+p+1}, t^{(p+1)^2} \end{aligned}$$

(or the localization of this ideal at the maximal ideal $(x_1, \ldots, x_4)$). Let

$$f = x_1^{p+1}x_2 - x_2^{p+1} - x_1x_3^p + x_4^p$$

We claim that f is a minimal generator of I but f is contained in $I^{(2)}$. (It is easy to see that the derivatives of f are contained in I, but this is not quite equivalent in characteristic p.)

To show that $f \in I^{(2)}$, consider the polynomials

$$\begin{aligned} g_1 &= x_1^{p+1} - x_2^p \\ g_2 &= x_1x_4 - x_2x_3 \\ g_3 &= x_1^p x_2 - x_3^p. \end{aligned}$$

One checks immediately by applying the homomorphism of rings above that $f, g_1, \ldots, g_3 \in I$. Since I is prime, the relations $x_1^p f = g_1g_3 + g_2^p$ and $x_1 \notin I$ show that $f \in I^{(2)}$.

To prove that f is a minimal generator of I it suffices to show that no element of I has a term of the form x_4^a with $0 < a < p$. Since I is generated by binomials, it suffices to show that there is no binomial of the form $x_4^a - x_3^b x_2^c x_1^d$ in I, or equivalently that the equation

$$a(p+1)^2 = b(p^2+p+1) + cp(p+1) + dp^2$$

cannot be satisfied by nonnegative integers a, b, c, d with $0 < a < p$.

This is elementary: Suppose $0 < a < p$. Reducing modulo p we see that $a \equiv b \pmod{p}$. If $b = a + np$ for some $n \geq 1$ then subtracting $a(p+1)^2$ from both sides of the displayed relation we get $(p-1)p \geq ap \geq np(p^2+p+1)$, which is impossible. Thus $b = a$. Subtracting $a(p^2+p+1)$ from both sides of the displayed relation we get $p(p-1) \geq ap = cp(p+1) + dp^2$ and the right hand side is either 0 or $\geq p^2$, a contradiction.

References

D. Bayer and M. Stillman. Macaulay, a computer algebra system. Available free from the authors. (1981-).

E. Böger. Differentielle und ganz-algebraische Abhangigkeit bei Idealen analytischer Algebren. *Math. Z.* **121** (1971) 188–189.

D. A. Buchsbaum and D. Eisenbud. Some structure theorems for finite free resolutions. *Adv. in Math.* **12** (1974), 84–139.

D. A. Buchsbaum and D. S. Rim, A generalized Koszul complex. II. Depth and multiplicity. *Trans. Amer. Math. Soc.* **111** (1964) 197–224.

R.-O. Buchweitz: Contributions à la théorie des singularités: 1. Déformations de Diagrammes, Déploiements et Singularités très rigides, Liaison algébriques.
2. On Zariski's Criterion for Equisingularity and Non-Smoothable Monomial Curves Thèse, Paris VII (1981).

R.-O. Buchweitz, B. Ulrich: Homological Properties which are invariant under Linkage. preprint Purdue (1983).

D. Eisenbud and M. Hochster. A Nullstellensatz with nilpotents, and Zariski's Main Lemma on holomorphic functions. *J. Alg.* **58** (1979) 157–161.

M. Flach. A finiteness theorem for the symmetric square of an elliptic curve. *Invent. Math.* **109** (1992) 307–327.

J. Herzog. Ein Cohen-Macaulay Criterium mit Anwendungen auf den Konormelenmodul und den Differentialmodul. *Math. Z.* **163** (1978) 149–162.

C. Huneke, and B. Ulrich. The structure of linkage. *Ann. of Math.* (2) **126** (1987) 277–334.

C. Huneke, and B. Ulrich. Local properties of licci ideals. *Math. Z.* **211** (1992) 129–154.

A. Kustin and B. Ulrich. A family of complexes associated to an almost alternating map, with applications to residual intersections. *Mem. of the Amer. Math. Soc.* **461** (1992).

B. Mazur. Deformations of Galois representations and Hecke Algebras. Harvard Course Notes, available by request from the author (1994).

M. Nagata. **Local Rings**. Wiley Interscience, New York 1962.

K. Saito. Quasihomogene isolierte Singularitäten von Hyperflächen. *Invent. Math.* **14** (1971) 123–142.

G. Scheja and U. Storch. Über differentielle Abhängigkeit bei Idealen analytischer Algebren. *Math. Z.* **114** (1970) 101–112.

W. Vasconcelos. A note on normality and module of differentials. *Math. Z.* **105** (1968) 291–293.

Dept. of Math., Brandeis Univ., Waltham MA 02254, eisenbud@math.brandeis.edu

Dept. of Math., Harvard Univ., Cambridge MA 20138, mazur@math.harvard.edu