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MODULES THAT ARE FINITE BIRATIONAL ALGEBRAS

DAVID EISENBUD AND BERND ULRICH

Let A be a commutative ring and let B be a faithful A-module with a distinguished element $e \in B$. It would be nice to understand in terms of the theory of A-modules whether B supports the structure of an A-algebra with identity element e. In general there is of course nothing unique about such an algebra structure. But there is at most one such structure if B is a *finite birational* A-module in the sense that there is an element $d \in A$, which is a nonzerodivisor on B, such that $dB \subseteq Ae \subseteq B$. In this case, indeed, the algebra structure of B is determined by the fact that it is a subalgebra of $B[d^{-1}] = A[d^{-1}]$.

A number of authors (Catanese [1984], Mond and Pellikaan [1987], de Jong and van Straten [1990], Kleiman and Ulrich [1995]) have given interesting applications of criteria that, under quite special hypotheses, test whether B is an A-algebra in terms of conditions on annihilators of elments of B, or even in terms of a presentation matrix of B as an A-module. It is the purpose of this note to re-examine and generalize these criteria. (For a thorough survey of the history and relations of the criteria, see the introduction to Kleiman and Ulrich [1995].)

Assuming that A is Noetherian, for us the interesting case, the finite birational hypothesis implies that B is a finitely generated A-module (it is contained in $d^{-1}Ae$). If B is an A-algebra, then our hypothesis implies that $\mathrm{End}_A(B) = \mathrm{End}_B(B) = B$, so there is an obvious criterion: B is an A-algebra iff every A-module homomorphism $Ae \to B$ extends to an A-module homomorphism $B \to B$. Equivalently, B is an A-algebra iff the map $B \to \mathrm{Ext}^1_A(B/Ae, B)$, induced by the exact sequence $0 \to Ae \to B \to B/Ae \to 0$ is zero.

We shall write $-^*$ for $\mathrm{Hom}_A(-, A)$. It is easy to see that if B is an A-algebra, then B^{**} is too. In fact, it is not hard to see that B^{**} is an A-algebra iff the composite map $B \to \mathrm{Ext}^1_A(B/Ae, B) \to \mathrm{Ext}^1_A(B/Ae, B^{**})$ is zero. Our first result is that there is a simple alternative criterion in terms of annihilators for determining when this occurs:

THEOREM 1. *Let A be a Noetherian ring, and let B be a birational A-module as above. The following conditions are equivalent*:

(a) *B^{**} is an A-algebra with identity element $e \in B \subseteq B^{**}$.*

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(b) *For every $b \in B$ whose annihilator in A is* 0,

$$\operatorname{ann}(B/Ab) \subseteq \operatorname{ann}(B/Ae).$$

(c) *For some elements $b_i \in B$ that generate B as an A-module, and such that* $\operatorname{ann}(b_i) = 0$, *we have*

$$\operatorname{ann}(B/Ab_i) \subseteq \operatorname{ann}(B/Ae).$$

Example 1. Let k be a field and let $A = k[t^3, t^4, t^5] \subset k[t]$. Set $B = A + At$, the vector space span of $1, t, t^3, t^4, t^5, \ldots$. The A-module B is a finite birational module in the sense above (with $e = 1$). B is obviously not a ring, but it is not hard to see that $B^* = (t^3, t^4, t^5)A$ and thus $B^{**} = k[t]$, which is a ring. Interpreting Theorem 1 in this case, we might for example take $b = t$, and we compute $\operatorname{ann}(B/At) = (t^4, t^5, t^6)A \subset \operatorname{ann}(B/Ae)$, in accordance with condition (b).

What makes Theorem 1 interesting is that condition (c) can easily be deduced from frequently occuring conditions on the minors of a presentation matrix for B. If M is any matrix and k is a non-negative integer, we write $I_k(M)$ for the ideal generated by the $k \times k$ minors of M. In applications, A itself is a factor ring of some larger "ambient" ring R (perhaps a regular ring or a polynomial ring), and we get a stronger result by taking the presentation matrix over R.

THEOREM 2. *Let R be a Noetherian ring, let A be a homomorphic image of R, and let B be a finite birational A-module with distinguished element $e \in B$. Suppose that $M : R^s \to R^t$ is a presentation matrix for B as an R-module whose first row corresponds to the element $e \in B$. Let M_1 be the submatrix of M consisting of all the rows except the first, and let I be the ideal $I_{t-1}(M_1)$. Writing B^{**} for the double dual of B as an A-module, we have*:

(a) *If B^{**} is an A-algebra with identity element e then the radical of I contains $I_{t-1}(M)$.*

(b) *If I contains $I_{t-1}(M)$, and either*

 (b1) *I is a radical ideal*; *or*

 (b2) *I has grade $\geq s - t + 2$ in R,*

 then B^{**} is an A-algebra with identity element e.

Remarks. Here the *grade* of a proper ideal I is defined to be the length of a maximal regular sequence contained in I, or, in another terminology, the depth of I on R. Since B/Ae is a torsion A-module, we must have $s \geq t - 1$. The grade required in (b2) is the maximum possible for $B \neq Ae$. If (b) is satisfied and $s \geq t$ then, by

Buchsbaum-Eisenbud [1977], I is the annihilator of B/Ae, while if $s = t - 1$ then we shall see that $B = Ae$. Similarly, if the grade of $J := I_t(M)$ is $s - t + 1$ and $s \geq t + 1$ then J is the annihilator of A; that is, $A = R/J$.

The proofs show that if A is a graded ring, and B is a graded A-module, then B^{**} is a graded algebra whenever Theorem 1 or 2 shows that B^{**} is an algebra.

Example 1, *continued.* With notation as in Example 1, let $R = k[x, y, z]$, and regard A as a homomorphic image of R by the map sending $x \mapsto t^3$, $y \mapsto t^4$, $z \mapsto t^5$. The module B, as an R-module, has two generators $1, -t$ and presentation matrix

$$\begin{pmatrix} y & z & x^2 \\ x & y & z \end{pmatrix}.$$

The ideal I defined in Theorem 2 is (x, y, z), which satisfies both conditions (b1) and (b2).

We now turn to the proofs. If M is an A-module we write $\operatorname{ann}_A(M)$ or simply $\operatorname{ann}(M)$ for the annihilator $\{a \in A \mid aM = 0\}$ of M in A.

For Theorem 1 we shall use some general remarks (which work in the non-Noetherian case too): For any subsets M, N of an A-algebra C we set

$$(M :_C N) = \{x \in C \mid xN \subseteq M\},$$

and we set

$$M^{-1} = \{x \in C \mid xM \subseteq A1 \subseteq C\}.$$

If B is a subring of C, and M a subgroup, then $(M :_C B)$ is naturally a B-module.

If B is a subring of C, then B^{-1} is a B-module, and thus $BB^{-1} \subset B^{-1}$. The converse fails, as in the example following Theorem 1, but we have:

PROPOSITION 3. *Let C be an A-algebra. If $B \subseteq C$ is an A-module containing* 1, *then $(B^{-1})^{-1}$ is a subring of C iff*

$$BB^{-1} \subseteq B^{-1}.$$

Proof. Note that

$$BB^{-1} \subseteq (B^{-1})^{-1}((B^{-1})^{-1})^{-1}.$$

If $(B^{-1})^{-1}$ is a ring, then $((B^{-1})^{-1})^{-1}$ is a $(B^{-1})^{-1}$-module, so

$$BB^{-1} \subseteq ((B^{-1})^{-1})^{-1} = B^{-1}$$

as required.

Conversely, suppose $BB^{-1} \subseteq B^{-1}$. Since $1 \in B$ we have $BB^{-1} = B^{-1}$ so $(B^{-1})^{-1} = (BB^{-1})^{-1}$. On the other hand, $(BB^{-1})^{-1} = (B^{-1} :_C B^{-1})$ tautologically. In particular $(B^{-1})^{-1}$ is a subring. □

In the main case of interest, where C is the total quotient ring of A, Proposition 3 may be interpreted as a statement about duals as follows:

If A is a subring of C and M and N are A-submodules of C then there is a natural map

$$(M :_C N) \to \mathrm{Hom}_A(N, M); \qquad x \mapsto \{\phi_x : n \mapsto xn\}.$$

If C is a ring of quotients of A and N contains an element a that is invertible in C, then this map is an isomorphism with inverse $\phi \mapsto \phi(a)/a$.

It follows that for any A-submodule B of the total quotient ring K of A that contains a nonzerodivisor of K we have $(A :_K B) = \mathrm{Hom}_A(B, A) =: B^*$, the A-dual of B.

If B is finitely generated as an A-module, then B^{-1} contains a nonzerodivisor (for example the product of the denominators of a finite set of elements that generate B) and thus $(B^{-1})^{-1} = B^{**}$.

PROPOSITION 4. *Suppose that A is a Noetherian ring, that K is a ring of quotients of A, and that M is an A-submodule of K. If M contains a nonzerodivisor of K, then M is generated by nonzerodivisors of K.*

Proof. Without loss of generality we may suppose that $A \subseteq K$ and M is finitely generated. Thus $dM \subseteq A$ for some nonzerodivisor d of A, and we may suppose that M is an ideal of A. Let I be the ideal generated by all the nonzerodivisors of A that are contained in M. If $P_1, \ldots, P_s$ are the associated primes of A, then $M \subseteq I \cup P_1 \ldots \cup P_s$. Since by hypothesis M is not contained in any P_j, the Prime Avoidance Lemma yields $M \subseteq I$, whence $M = I$. □

Example 2. If A contains an infinite field then one can replace K by any Noetherian A-algebra in Proposition 4, but in general this is not possible, as shown by the example

$$A := \mathbf{Z}/2 \subset \mathbf{Z}/2 \times \mathbf{Z}/2 =: B,$$

where B is not generated by nonzerodivisors.

Proof of Theorem 1. Let K be the total quotient ring of A, obtained by inverting all elements that are nonzerodivisors on A. We may regard B as embedded in K, and make the identifications $B^* = B^{-1}$ and $B^{**} = (B^{-1})^{-1}$. If b is any nonzerodivisor of K, then b is invertible in K, and we see directly from the definition that $(Ab :_K B) = bB^{-1}$.

Suppose that B^{**} is a subring of K. It follows by Proposition 3 that $BB^{-1} \subseteq B^{-1}$. Thus if $b \in B$ is invertible in K, then $(Ab :_K B) = bB^{-1} \subseteq B^{-1}$. Thus condition (b) is satisfied.

Condition (b) implies condition (c) by Proposition 4.

Now suppose that condition (c) is satisfied. For each b_i we have immediately $b_iB^{-1} = b_i(A :_K B) \subseteq (Ab_i :_K B)$. On the other hand $(Ab_i :_K B) \subseteq (Ab_i :_K$

$Ab_i) = A$ since b_i has no annihilator in A. Thus $(Ab_i :_K B) = A \cap (Ab_i :_K B) = \operatorname{ann}(B/Ab_i)$ so condition (c) implies $b_i B^{-1} \subseteq B^{-1}$. Since the b_i generate B we have $BB^{-1} \subseteq B^{-1}$. Thus $BB^{-1} \subseteq B^{-1}$, and B^{**} is a ring by Proposition 3. □

In the proof of Theorem 2 we will extend R by adjoining a new indeterminate x. Recall that if R is a local ring with maximal ideal m, then $R(x)$ denotes the local ring $R[x]_{mR[x]}$, which is a localization of the polynomial ring $R[x]$.

LEMMA 5. *Let (R, m) be a Noetherian local ring, let $I := (f_1, \dots, f_n) \subseteq R$ be an ideal, and let $g_1, \dots, g_n$ be any elements of R. If x is a new indeterminate, then the ideal $J := (g_1 + xf_1, \dots, g_n + xf_n) \subseteq R(x)$ satisfies* $\operatorname{grade}(J) \geq \operatorname{grade}(I)$.

Proof. It suffices to show that if all the f_i and g_i are contained in m and the f_i form a regular sequence in R, then the $g_i + xf_i$ form a regular sequence in $R(x)$. Set $y = x^{-1}$. Since x is a unit of $R(x)$, it suffices to see that the elements $h_i := yg_i + f_i$ form a regular sequence. But $R(x) = R(y)$ is a localization of the polynomial ring $R[y]$, in which $y, h_1, \dots, h_n$ obviously form a regular sequence. Thus they also form a regular sequence on the localization $R[y]_{(m,y)}$, where we may permute them without destroying this property. It follows that $h_1, \dots, h_n$ form a regular sequence in the further localization $R(y)$. □

Proof of Theorem 2. The matrix M_1 is a presentation matrix for the module B/Ae. Thus I is the 0th Fitting ideal of B/Ae, and as $I_{t-1}(M)$ is the first Fitting ideal of B, all the conditions of the theorem are independent of the chosen presentation M.

As before, let K be the total quotient ring of quotients of A. We may regard B as a submodule of K. It follows from Proposition 4 above that we can suppose that the generators of B corresponding to the given free generators of R^t are nonzerodivisors in K.

To prove part (a), suppose that B^{**} is an A-algebra. Let b_i be the nonzerodivisor in B that is the image of the ith basis element of R^t, and let M_i be the submatrix of M consisting of all rows of M except the ith. By Theorem 1 and Fitting's Lemma,

$$I_{t-1}(M_i) \subseteq \operatorname{ann}(B/Ab_i) \subseteq \operatorname{ann}(B/Ae) \subseteq \operatorname{Rad}(I).$$

As this is true for every i, condition (a) follows.

Now suppose that I contains $I_{t-1}(M)$ and one of the hypotheses (b1) or (b2) is satisfied. We will show that $\operatorname{ann}(B/Ab_i) \subseteq \operatorname{ann}(B/Ae)$; by Theorem 1 this suffices. First, if I is a radical ideal then I is equal to the annihilator of B/Ae by Fitting's Lemma. Since I is the radical of $I_{t-1}(M)$, another application of Fitting's Lemma shows that I contains the annihilator of each B/Ab_i.

Now suppose (b2) is satisfied. The case $s = t - 1$ is trivial: Here the row of signed minors of M, divided by the determinant of M_1, induces a map $B \to A$ that splits the inclusion $A \to Ae$. Thus A is a summand of B, and since B is birational to A, we have $Ae = B = B^{**}$.

Finally, suppose $s \geq t$. Theorem 1 shows that we may assume R to be local and that we may then replace R by $R(x)$ for a new variable x. Modify the first row of M by adding x times the sum of the other rows. Now by Lemma 5, each of the matrices M_i obtained by omitting one row from M satisfies $\text{grade}(I_{t-1}(M_i)) \geq s - t + 2$. The main theorem of Buchsbaum-Eisenbud [1977] shows that the ideal $I_{t-1}(M_i)$ is the annihilator of B/Ab_i for each i. Since these ideals are all contained in I by hypothesis, we are done. □

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David Eisenbud, Brandeis University, Waltham, MA 02254
eisenbud@math.brandeis.edu

Bernd Ulrich, Department of Mathematics, Michigan State University, Lansing, MI 48824-1027
ulrich@math.msu.edu