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# IDEALS OF MINORS IN FREE RESOLUTIONS

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**Introduction.** Let $R$ be a commutative Noetherian ring, and let

$$\mathscr{F}: \cdots \to F_i \xrightarrow{\phi_i} F_{i-1} \to \cdots \to F_1 \xrightarrow{\phi_1} F_0 \to M \to 0$$

be a free resolution of a finitely generated $R$-module $M$ with annihilator

$$\operatorname{ann} M = J.$$

It is interesting to ask how the invariants of the maps $\phi_i$, such as the ideal $I_j(\phi_i)$ generated by the $j \times j$ minors of $\phi_i$, reflect the properties of $M$. For example, it is not hard to show (see Buchsbaum-Eisenbud [4]) that if the grade of $M$ is $g$ (that is, $g$ is the length of a maximal regular sequence contained in $J$) and $r_i$ is the rank of the map $\phi_i$ (that is, the size of the largest nonvanishing minors of $\phi_i$), then for $i < g$

$$\text{radical } J = \text{radical } I_{r_i}(\phi_i),$$

while at least in finite free resolutions, $I_{r_{g+1}}(\phi_{g+1})$ has strictly larger radical. For the case of $\phi_1$, that is, for a free presentation of $M$, we have sharper estimates. For example, it is well known that

$$(1) \qquad J^{r_1} \subseteq I_{r_1}(\phi_1) \subseteq J.$$

In this paper we will prove some results extending the left-hand inclusion in formula (1), and propose some conjectures which would extend the right-hand inclusion. We also study some related conjectures and results about annihilators of exterior powers of modules.

*Huneke's Conjecture and its extensions.* As is well known, the left-hand inclusion in formula (1) can be sharpened to a chain of inclusions,

$$(2) \qquad JI_j(\phi_1) \subseteq I_{j+1}(\phi_1) \qquad \text{for } j < r_1.$$

(see Lemma 2.1).

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Craig Huneke (in a personal communication) recently conjectured that the same chain of inclusions might hold for all the maps $\phi_i$; that is, we should be able to replace $\phi_1$ and $r_1$ in the formula above with $\phi_i$ and $r_i$. We will prove this formula in characteristic 0. More generally (in the proof of the stronger Theorem 1.1, below) we will exhibit an identity involving the minors of maps in an arbitrary free complex with homotopy that implies the following.

THEOREM 1. *For all $i, j \geqslant 0$,*

$$(r_i - j)JI_j(\phi_i) \subseteq I_{j+1}(\phi_i).$$

The theorem immediately implies the following.

COROLLARY. *For all $i, j \geqslant 0$, with $r_i \geqslant j$*

$$(r_i!/(r_i - j)!)J^j \subseteq I_j(\phi_i).$$

It seems reasonable to hope that the integer coefficients could be removed from these results altogether.

The method of proof is elementary: if $p \in \operatorname{ann}(M)$, then there is a sequence of maps $\psi_i\colon F_{i-1} \to F_i$ which is a homotopy for multiplication by $p$ on $\mathscr{F}$. (The definition is reviewed in Section 1.) It turns out that the results above are simple consequences of multilinear identities concerning the maps $\phi_i$ and $\psi_i$. Easiest and perhaps most interesting among them, we prove in Lemma 1.2 that the characteristic polynomial $\chi_{\phi_i\psi_i}(t)$ of $\phi_i\psi_i$ is $(t-p)^{r_i}t^{r_{i-1}}$. (This implies that for any $p \in J$, the element $\binom{r_i}{j}p^j$ is in $I_j(\phi_i)$—a result that also yields the corollary above, with the numerical coefficient given there.)

*The idealistic Syzygy Conjecture.* The results and conjectures above generalize only the first inclusion in formula (1). What about the second? The result of Buchsbaum-Eisenbud [4] to which we have already referred shows that for all $i \leqslant g$ there must be numbers $n_i$ (possibly depending on $\mathscr{F}$) such that

$$I_{r_i}(\phi_i)^{n_i} \subseteq J.$$

The second inclusion in formula (1) gives a much better statement for $i = 1$, and suggests that much stronger results might hold in general. In fact, for $i = 2$, such a result is known.

THEOREM A (Buchsbaum-Eisenbud [4], Bruns [1]). *If $\mathscr{F}$ as above is the free resolution of an R-module of grade g,*

$$I_{r_2-(g-1)}(\phi_2) \subseteq I_{r_1}(\phi_1) \subseteq J.$$

*as long as $g \geqslant 2$.*

An examination of examples and a comparison with the "Syzygy Conjecture" described below suggests that this might extend as follows.

CONJECTURE 1 (Idealistic Syzygy Conjecture). *If $\mathcal{F}$ as above is the free resolution of an R-module M of grade g, then, setting* $n_i = \binom{g-1}{i-1} - 1$, *we have*

$$I_{r_i - n_i}(\phi_i) \subseteq J$$

*as long as* $g \geqslant i$.

It would follow, simply from the Laplace expansion of minors, that $I_{r_i}(\phi_i) \subseteq J$, and in fact even that

$$I_{r_i}(\phi_i) \subseteq J I_{n_i}(\phi_i). \tag{3}$$

The name we have given this conjecture reflects the fact that it trivially implies the "Syzygy Conjecture" of Eisenbud-Buchsbaum [5] and Horrocks (in Hartshorne's problem list [9]). This conjecture asserts that, if $M$ has grade $g$, then the $i$th syzygy of $M$ has rank at least $\binom{g-1}{i-1}$, or equivalently that $r_i \geqslant \binom{g-1}{i-1}$. We refer the reader to Charalambous and Evans [8] for a recent exposition. The conjecture was originally stated for modules of finite projective dimension, but Hochster and Huneke have pointed out that, since no counterexample is known, it might as well be made in the more general context.

Note that Conjecture 1 really depends only on $M$ and not on the resolution chosen; one could check it for the minimal resolution of $M$ by checking it on some other, nonminimal resolution. This implies, for example, that if Conjecture 1 holds for $M$, then it holds for any localization of $M$. Perhaps for this reason it will be more tractable than the old syzygy conjecture, which is not so stable under localization.

One might ask whether in fact $I_{r_i}(\phi_i) \subseteq J^{p_i}$ for some powers $p_i > 1$ (always assuming $1 < i < g$), and examples suggest that something of this sort might be true. An earlier version of this paper contained the conjecture that $p_i$ could even be taken to be the binomial coefficient $\binom{g-1}{i-1}$, which is suggested by formula (3) if one supposes (perhaps from Huneke's Conjecture) that $I_{n_i}(\phi_i)$ is "roughly" the same as $J^{n_i}$. However, this is false, even for $i = 2$, as was pointed out by Ragnar Buchweitz and Frank Schreyer. Their example is the following.

Let $R$ be the polynomial ring $K[x_1, \ldots, x_5]$ and let $M$ be the $R$-module presented by a sufficiently general $3 \times 8$ matrix $\phi_1$ of linear forms. One may check by direct computation (Buchweitz and Schreyer used the computer algebra system Macaulay of Bayer and Stillman) that the annihilator $J$ of $M$ is the ideal $(x_1, \ldots, x_5)^3$ (which is also equal to $I_{r_1}(\phi_1)$). On the other hand, the syzygies of $M$

are given by an $8 \times 15$ matrix of quadratic forms $\phi_2$. The rank $r_2$ of $\phi_2$ is of course $8 - 3 = 5$. Thus the $5 \times 5$ minors of $\phi_2$ are forms of degree 10, so they are not contained in the fourth power of $J$, contradicting our previous conjecture. (Our current conjecture says in this setting that the $2 \times 2$ minors of $\phi_2$ are contained in $J$, which is obvious.)

We remark that Conjecture 1, like the original syzygy conjecture, reduces by localization to the case where $M$ has finite length. Using this reduction and the easy fact that the conjecture is true if $\mathscr{F}$ is a Koszul complex, we obtain another fragment of favorable evidence.

PROPOSITION 2. *If $R$ is a regular ring and $M = R/P$ is a factor ring of $R$ by a prime ideal $P$, then Conjecture 1 is true for the resolution of $M$.*

*Annihilators of exterior powers of syzygy modules and vector bundles.* Huneke's Conjecture is related to some questions about the annihilators of exterior powers of syzygy modules which we discuss in the last section.

In general little is known about the annihilators of exterior powers of a module except for the elementary results connecting them with Fitting ideals (for the reader's convenience, we reproduce such a result as Lemma 2.1, below) and except in the case of modules of projective dimension 1 (for which, see, for example, [2]). However, if $J$ is the annihilator of a module $M$, then we show that a power of $J$ annihilates certain exterior powers of syzygy modules of $M$. One could hope that in some circumstances these exterior powers would be annihilated by $J$ itself, and if this were true it would provide a strengthening of the conclusion of Huneke's Conjecture, as we show (but this strengthening is not true all the time—there are fairly trivial counterexamples in low characteristic, and also some in characteristic 0).

We conclude with a brief study of the special case of the exterior powers of an ideal $J$ (the first syzygy of the cyclic module $R/J$, which has $J$ as annihilator). The desired strengthening in this case amounts to saying that $J$ annihilates $\bigwedge^2 J$, which is easy seen to be true in characteristic 0 (in general, $2 \cdot J(\bigwedge^2 J) = 0$, and this is sharp). We mention an interesting question about the annihilators of higher exterior powers of ideals, as well.

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**1. A formula for the minors of maps in a complex with homotopy.** Let $R$ be a commutative ring, and let

$$\mathscr{F}\colon \cdots \to F_i \xrightarrow{\phi_i} F_{i-1} \to \cdots \to F_1 \xrightarrow{\phi_1} F_0$$

be a complex of free $R$-modules of finite rank, bounded on the right. We define

$r_i = r_i(\mathscr{F})$ to be the alternating sum of the ranks of the $F_j$ for $j < i$:

$$r_i = \sum_{i=0}^{i-1} (-1)^j \operatorname{rank} F_{i-1-j},$$

with the natural convention that $r_0 = 0$. We will be interested primarily in the case where $\mathscr{F}$ is the free resolution of a module $M$ with nontrivial annihilator, and for this reason we think of $r_i$ as the rank of the map $\phi_i$. Indeed, if we are in this case and the annihilator of $M$ contains a nonzero divisor, then $r_i$ is the rank of $\phi_i$ in the sense that $r_i$ is the size of the largest nonvanishing minor of $\phi_i$.

If $p \in R$, then a *homotopy* $\psi$ *for multiplication by* $p$ *on* $\mathscr{F}$ is a collection of maps

$$\psi_i: F_{i-1} \to F_i$$

such that

$$p \cdot 1_{F_0} = \phi_1 \psi_1$$

$$p \cdot 1_{F_i} = \phi_{i+1}\psi_{i+1} + \psi_i\phi_i \qquad \text{for } i > 0$$

where $1_{F_i}$ is the identity transformation of $F_i$. If $\mathscr{F}$ is a free resolution of an $R$-module $M$, and $p$ annihilates $M$, then it is easy to see that $\mathscr{F}$ has a homotopy for multiplication by $p$ (for example, see Cartan-Eilenberg [7, Ch. V, Prop. 1.2]).

Our main theorem concerns the minors of the maps $\phi_i$ in a complex with a homotopy.

THEOREM 1.1. *Let* $\mathscr{F}$ *be a complex as above. If* $\psi$ *is a homotopy for multiplication by* $p$*, then for every* $i, j \geqslant 0$ *we have*

$$(r_i - j)pI_j(\phi_i) \subset I_{j+1}(\phi_i)I_1(\psi_i).$$

Before proving the result we need to know something about the endomorphisms $\phi_i\psi_i: F_{i-1} \to F_{i-1}$, where $\psi_i$ is part of a homotopy $\psi$ for multiplication by $p$. While these endomorphisms depend on the choice of $\psi$, the result we need shows that their characteristic polynomials do not.

For any endomorphism $\alpha$ of a free module $F$ of finite rank, we write

$$\chi_\alpha(t) = \det(t1_F - \alpha)$$
$$= \sum_{j=0}^{\operatorname{rank} F} (-1)^j \operatorname{trace}(\textstyle\bigwedge^j \alpha)t^{\operatorname{rank} F - j}$$

for the characteristic polynomial of $\alpha$. It is elementary that if

$$F \xrightarrow{\phi} G \xrightarrow{\psi} F$$

are homomorphisms, then

$$\chi_{\phi\psi}(t) = t^{\operatorname{rank} G - \operatorname{rank} F} \chi_{\psi\phi}(t);$$

this follows for example from the fact that

$$\operatorname{trace} \textstyle\bigwedge^j (\phi\psi) = \operatorname{trace}(\bigwedge^j \phi)(\bigwedge^j \psi) = \operatorname{trace}(\bigwedge^j \psi)(\bigwedge^j \phi) = \operatorname{trace} \bigwedge^j (\psi\phi)$$

and the formula for the characteristic polynomial given above.

With this notation, we have the following.

LEMMA 1.2. *Let $\mathscr{F}$ be a complex as above, with homotopy $\psi$ for multiplication by p. For every $i \geqslant 1$ the characteristic polynomial of*

$$\phi_i \psi_i \colon F_{i-1} \to F_{i-1}$$

*is*

$$\chi_{\phi_i \psi_i}(t) = (t - p)^{r_i} t^{r_{i-1}},$$

*and thus*

$$\operatorname{trace}(\textstyle\bigwedge^j \phi_i)(\bigwedge^j \psi_i) = \begin{cases} \binom{r_i}{j} p^j & \text{for } j \leqslant r_i \\ 0 & \text{for } j > r_i. \end{cases}$$

*Remark.* There is at least one special case where the lemma is obvious: if the complex $\mathscr{F}$ is split exact, and $\psi$ is derived in the obvious way from a splitting, then $\phi_i \psi_i$ is simply $p$ times the projection onto the kernel of $\phi_{i-1}$. Thus if, for example, $p$ is a nonzero divisor in $R$, then by inverting $p$ we may reduce the result to the statement that $\chi_{\phi_i \psi_i}(t) = (t - p)^{r_i} t^{r_{i-1}}$ is independent of the choice of homotopy $\psi$.

*Proof.* We do induction on $i$, the case $i = 1$ being obvious from the defining property of the homotopy $\psi$. Because $\phi_{i+1} \psi_{i+1} = p1_{F_i} - \psi_i \phi_i$, we have

$$\begin{aligned} \chi_{\phi_{i+1} \psi_{i+1}}(t) &= \chi_{p1_{F_i} - \psi_i \phi_i}(t) \\ &= \det(t1_{F_i} - (p1_{F_i} - \psi_i \phi_i)) \\ &= \det(-[(p - t)1_{F_i} - \psi_i \phi_i]) \\ &= (-1)^{\operatorname{rank} F_i} \chi_{\psi_i \phi_i}(p - t) \\ &= (-1)^{\operatorname{rank} F_i} (p - t)^{\operatorname{rank} F_i - \operatorname{rank} F_{i-1}} \chi_{\phi_i \psi_i}(p - t) \end{aligned}$$

$$= (-1)^{\operatorname{rank} F_i}(p-t)^{\operatorname{rank} F_i - \operatorname{rank} F_{i-1}}((p-t)-p)^{r_i}(p-t)^{r_{i-1}}$$

$$= (t-p)^{r_{i+1}}t^{r_i}. \quad \square$$

We are now in a position to prove our main result.

*Proof of Theorem 1.1.* Let $F$ be a rank-$j$ summand of $F_i$, and let $E$ be a rank-$j$ summand of $F_{i-1}$. Regarding $E^*$ as a submodule of $F_{i-1}^*$, we write

$$f \in \bigwedge^j F_i$$
$$\varepsilon \in \bigwedge^j F_{i-1}^*$$

for generators of the submodules $\bigwedge^j F$ and $\bigwedge^j E^*$, respectively. If we write $\phi_i(f)$ in place of $\bigwedge^j(\phi_i)(f)$ to simplify the notation, then the element

$$\varepsilon(\phi_i(f))$$

is a $j \times j$ minor of $\phi_i$. Let $\{e_m\}$ and $\{\varepsilon_m\}$ be dual bases of $F_{i-1}$ and $F_{i-1}^*$ respectively, chosen so that the first $j$ elements form bases of $E$ and $E^*$ and $\varepsilon = \varepsilon_1 \wedge \cdots \wedge \varepsilon_j$. With this notation, we will show that

$$(r_i - j)p\varepsilon(\phi_i(f)) = (-1)^j \sum_{m=1}^{\operatorname{rank} F_{i-1}} \varepsilon_m \wedge \varepsilon(\phi_i(f \wedge \psi_i(e_m))).$$

This gives $(r_i - j)p\varepsilon(\phi_i(f))$ as a linear combination of $(j+1) \times (j+1)$ minors of $\phi_i$ with coefficients in $I_1(\psi_i)$, and thus suffices to prove the theorem.

For each $m > j$ we have

$$p\varepsilon(\phi_i(f)) = (-1)^j \varepsilon_m \wedge \varepsilon(\phi_i(f) \wedge pe_m).$$

Substituting the expression $pe_m = \phi_i\psi_i(e_m) + \psi_{i-1}\phi_{i-1}(e_m)$ in the right-hand side of this we obtain

$$p\varepsilon(\phi_i(f)) =$$

$$(I) \qquad (-1)^j \varepsilon_m \wedge \varepsilon(\phi_i(f) \wedge \phi_i\psi_i(e_m))$$

$$(II) \qquad + (-1)^j \varepsilon_m \wedge \varepsilon(\phi_i(f) \wedge \psi_{i-1}\phi_{i-1}(e_m)).$$

The term labelled *(I)* on the right-hand side of this equation is equal to $(-1)^j \varepsilon_m \wedge \varepsilon(\phi_i(f \wedge \psi_i(e_m)))$, which is a linear combination of $(j+1) \times (j+1)$ minors of $\phi_i$, with coefficients in the ideal generated by the $1 \times 1$ minors of $\psi_i$, just as we wanted.

On the other hand, the term labelled $(II)$ may be written as a sum of two terms,

$$(II) =$$

$$(IIa) \qquad \varepsilon(\phi_i(f))\cdot\varepsilon_m(\psi_{i-1}\phi_{i-1}(e_m))$$

$$(IIb) \qquad +\sum_{n=1}^{j} \pm\varepsilon(\phi_i(f_1 \wedge \cdots \wedge \widehat{f_n} \wedge \cdots \wedge f_j \wedge \psi_{i-1}\phi_{i-1}(e_m)))\cdot\varepsilon_m(\phi_i(f_n)).$$

These two terms are not in themselves very promising, but if we sum over $m$ they simplify. The sum of the terms corresponding to $(IIa)$ is

$$\sum_{m=1}^{\operatorname{rank} F_{i-1}} (IIa) = \varepsilon(\phi_i(f))\cdot\operatorname{trace}(\psi_{i-1}\phi_{i-1})$$

$$= \varepsilon(\phi_i(f))\cdot r_{i-1}p$$

by Lemma 1.2. We claim that the sum of the terms corresponding to $(IIb)$ is zero. To see this, reverse the order of summation. The result is the sum over $n$ of the following terms:

$$\sum_{m=1}^{\operatorname{rank} F_{i-1}} \pm\varepsilon(\phi_i(f_1 \wedge \cdots \wedge \widehat{f_n} \wedge \cdots \wedge f_j \wedge \psi_{i-1}\phi_{i-1}(e_m)))\cdot\varepsilon_m(\phi_i(f_n))$$

$$= \pm[(\phi_i(f_1 \wedge \cdots \wedge \widehat{f_n} \wedge \cdots \wedge f_j)(\varepsilon))]$$

$$\cdot\left(\sum_{m=1}^{\operatorname{rank} F_{i-1}} \varepsilon_m(\phi_i(f_n))\cdot\psi_{i-1}\phi_{i-1}(e_m)\right),$$

where the notation $\widehat{f_n}$ indicates that the factor $f_n$ is omitted. But now

$$\sum_{m=1}^{\operatorname{rank} F_{i-1}} \varepsilon_m(\phi_i(f_n))\cdot\psi_{i-1}\phi_{i-1}(e_m) = \psi_{i-1}\phi_{i-1}\phi_i(f_n)$$

$$= 0,$$

as claimed.

Putting these parts together, and remarking that $\varepsilon_m \wedge \varepsilon$ and with it the expression $\varepsilon_m \wedge \varepsilon(\phi_i(f) \wedge pe_m)$ is zero unless $m > j$, we get

$$(\operatorname{rank} F_{i-1} - j)p\varepsilon(\phi_i(f)) = (-1)^j \sum_{m=1}^{\operatorname{rank} F_{i-1}} \varepsilon_m \wedge \varepsilon(\phi_i(f \wedge \psi_i(e_m)) + r_{i-1}p\varepsilon(\phi_i(f))).$$

Taking into account that rank $F_{i-1} - r_{i-1} = r_i$, we get the desired formula. □

**2. Annihilators of exterior powers of syzygy modules and vector bundles.** In this section we will return to the setup of the introduction and consider some possible extensions of Huneke's Conjecture to statements about the annihilators of exterior powers of syzygy modules.

We maintain the following notation: $R$ is a commutative Noetherian ring, and

$$\mathscr{F}: \cdots \to F_i \xrightarrow{\phi_i} F_{i-1} \to \cdots \to F_1 \xrightarrow{\phi_1} F_0 \to M \to 0$$

is a free resolution of a finitely generated $R$-module $M$ with $\operatorname{ann} M = J$. Again we set $M_i = \operatorname{cokernel} \phi_{i+1}$. We define $r_i = \operatorname{rank} M_i$ to be the least number $r$ such that $\bigwedge^{r+1} \phi_i = 0$.

For simplicity, we will assume throughout this section that the annihilator $J$ of $M$ contains a nonzero divisor, that is, $M$ is *torsion*, so that $r_0 := \operatorname{rank} M = 0$. In this case the definition of the $r_i$ just given agrees with the definition given in the previous section. (Proof: after inverting this nonzero divisor, the resolution $\mathscr{F}$ becomes a split exact complex, where the equality is easy to check; and because the element inverted is a nonzero divisor, the same exterior powers of the $\phi_i$ are zero after inverting it as before.) It is a result of Auslander and Buchsbaum that if $M$ is a module of finite projective dimension with nonzero annihilator, then the annihilator does contain a nonzero divisor; so the assumption covers most of the cases in which we might be interested.

We recall two elementary results about the annihilators of exterior powers of a module. (See Buchsbaum-Eisenbud [6] for these and further results in this direction; however, note that there is an error in the direction of the inequality in the statement there of the Theorem 1.2, 2.)

LEMMA 2.1. *If* $N = \operatorname{coker} \phi: F \to G$, *then*
(a) *for all* $0 \leqslant r$,

$$I_{\operatorname{rank} G-r+1}(\phi) \subseteq \operatorname{ann}(\textstyle\bigwedge^r N).$$

(b) *for all* $0 < r$, $0 \leqslant j$, *and* $r + j \leqslant \operatorname{rank} G$,

$$\operatorname{ann}(\textstyle\bigwedge^r N) I_j(\phi) \subseteq I_{j+1}(\phi).$$

*Proof.* (a) This is Corollary 1.4 of Buchsbaum-Eisenbud [6].

(b) The result follows at once from the commutativity of the "obvious" diagram with exact first row

$$\begin{array}{ccccccc}
\bigwedge^{r-1} G \otimes F \otimes \bigwedge^j F & \longrightarrow & \bigwedge^r G \otimes \bigwedge^j F & \longrightarrow & \bigwedge^r N \otimes \bigwedge^j F & \longrightarrow & 0 \\
\downarrow & & \downarrow & & & & \\
\bigwedge^{r-1} G \otimes \bigwedge^{j+1} F & \longrightarrow & \bigwedge^{r+j} G & & & &
\end{array}$$

since the horizontal map on the bottom involves the $j+1 \times j+1$ minors and the right-hand vertical map involves the $(j) \times (j)$ minors. □

Under our hypotheses Lemma 2.1a implies that the exterior power

$$\textstyle\bigwedge^{r_i+1} M_i$$

is torsion; indeed, it is annihilated by $I_{r_{i+1}}(\phi_{i+1})$ which, under our assumptions, contains a nonzero divisor. (Its radical contains $J$, as one sees by inverting an element of $J$ and using the fact that $\mathscr{F}$ becomes split exact.)

It thus makes sense to ask further about the annihilator of this exterior power. Because of the formula in Lemma 2.1b, a positive answer to the following question would give a strengthening of inclusions in Huneke's Conjecture.

*Question 1.* In characteristic 0, does $J$ annihilate $\bigwedge^{r_i+1} M_i$?

The answer to Question 1 is trivially "yes" for $i = 0$, and we will prove that it is "yes" for $i = 1$ in the case where $r_1 = 1$ below. More generally, we will prove a weakening in which $J$ is replaced by $J^{r_i}$. (In the case when $i = 1$, it seems that one could also replace $J$ by $I_{\operatorname{rank} F_0}\phi_1$ without changing the answer.) On the other hand, the answer is actually "no" in some cases with $i = 1$: for example, consider the matrix

$$\phi_1 = \begin{pmatrix} a & 0 & b & c & d \\ 0 & a & e & f & g \end{pmatrix}.$$

One may show by direct computation that if $M = \operatorname{coker} \phi_1$, and $M_1 = \operatorname{image} \phi_1$, its first syzygy, then rank $M_1 = 2$ and

$$\operatorname{ann} M = (a) + I_2(\phi_1)$$

but

$$\operatorname{ann} \textstyle\bigwedge^3 M_1 = (a^2) + I_2(\phi_1) = I_2(\phi_1),$$

so the answer to Question 1 is "no" in this case. It is easy to create other negative examples with $i = 1$ by taking matrices similar to $\phi_1$ above, with different sizes. However, in all the examples we have been able to examine, the answer to Question 1 is "yes" for $i \geqslant 2$, and it seems possible that this is true in general.

Of course if the module $M$ is a graded module of finite length over a polynomial ring, then its syzygies (other than the first) are the modules of sections of vector bundles on projective spaces. The module $M$ itself appears as a cohomology module of such a bundle. Thus the question above suggests the following question.

*Question 2.* In characteristic 0, if $E$ is a vector bundle of rank $r$ on a projective space with homogeneous coordinate ring $R$, and $I$ is the product of the annihilator ideals of the cohomology modules of $E$, is

$$I \subseteq \operatorname{ann}(\textstyle\bigwedge_R^{r+1} \mathrm{H}_*^0(E))?$$

We are grateful to Wolfram Decker for pointing out to us that it is not enough in the above to take $I$ to be the intersection of the annihilators of the cohomology modules of $E$.

Note that the ideal $\operatorname{ann}(\bigwedge_R^{r+1} \mathrm{H}_*^0(E))$ is 0 if $E$ is a sum of line bundles, and, more generally, remains unchanged if $E$ is replaced by $E \oplus$ (line bundle).

We come now to the main result of this section. We define the torsion submodule of a module over a ring $R$ to be the intersection of the kernels of all homomorphisms to $R$.

THEOREM 2.2. *Suppose that the characteristic is 0. For $i$, $r \geqslant 1$ the torsion submodule of $\bigwedge^r M_i$ is annihilated by $J^{r-1}$. Thus in particular*

$$\operatorname{ann}(\textstyle\bigwedge^{r_i+1} M_i) \supseteq J^{r_i}.$$

*Remark.* The proof will show that the torsion submodule of $\bigwedge^r M_i$ is an iterated extension of modules annihilated by $J$. Perhaps it would be possible to analyze the extension classes to show that $\bigwedge^r M_i$ is itself annihilated by $J$, which would give a substantial improvement of Theorem 1.

*Proof.* Consider the sequence of maps

$$\textstyle\bigwedge^r M_i \to M_i^{\otimes r} \to F_{i-1} \otimes M_i^{\otimes(r-1)} \to \cdots \to F_{i-1}^{\otimes(r-1)} \otimes M_i \to F_{i-1}^{\otimes r}.$$

Here the first map is the "diagonal" $\Delta$: $\bigwedge^r M_i \to M_i^{\otimes r}$ defined by

$$\Delta(m_1 \wedge \cdots \wedge m_r) = \sum_{\sigma \in G} (-1)^{\operatorname{sgn}(\sigma)}(m_{\sigma(1)} \wedge \cdots m_{\sigma(r)}),$$

where $G$ is the symmetric group on $r$ letters. The remaining maps are made by tensoring the inclusion $M_i \hookrightarrow F_{i-1}$ with various tensor products of copies of $M_i$ and $F_{i-1}$.

In characteristic 0 the map $\Delta$ is a monomorphism, since if we write $\mu$ for the multiplication map $\mu$: $M_i^{\otimes r} \to \bigwedge^r M_i$, then we have $\mu\Delta = r! \cdot 1_{M_i}$. Also, the map furthest to the right, $F_{i-1}^{\otimes(r-1)} \otimes M_i \to F_{i-1}^{\otimes r}$, is a monomorphism because $M_i \hookrightarrow F_{i-1}$ is and $F_{i-1}^{\otimes(r-1)}$ is free.

On the other hand, for $1 \leqslant j \leqslant r-1$, the kernel of

$$F_{i-1}^{\otimes(j-1)} \otimes M_i^{\otimes(r-j+1)} \to F_{i-1}^{\otimes j} \otimes M_i^{\otimes(r-j)}$$

is

$$\operatorname{Tor}_1^R(F_{i-1}^{\otimes(j-1)} \otimes M_i^{\otimes(r-j)}, M_{i-1}) = \operatorname{Tor}_i^R(F_{i-1}^{\otimes(j-1)} \otimes M_i^{\otimes(r-j)}, M)$$

which is obviously annihilated by $J$.

It follows that the kernel of the composite map

$$\bigwedge^r M_i \to F_{i-1}^{\otimes r}.$$

is annihilated by $J^{r-1}$. Since $F_{i-1}^{\otimes r}$ is free, this kernel contains the torsion submodule, and we are done. □

We deduce a weak version of Theorem 1.

COROLLARY 2.3. *For $i \geq 2$ and $j \leq r_i$*

$$J^{r_{i-1}} I_{j-1}(\phi_i) \subseteq I_j(\phi_i)$$

*in characteristic 0.*

Note that the assumption on the characteristic enters the proof only to show that $\Delta\colon \bigwedge^{r_i+1} M_i \to M_i^{\otimes(r_i+1)}$ is a monomorphism, and the proof actually shows, without any assumption, that the torsion submodule of the image of this map is annihilated by $J^{r-1}$. It seems natural to hope that the annihilator $A$ of this torsion submodule would satisfy

$$A \subseteq I_j(\phi_{i+1}),$$

giving a characteristic-free version of the theorem.

*Example 2.4.* Consider the case where $M = S/J$ is a cyclic module, and $i = 1$, so $M_i = J$. Then the theorem asserts that in characteristic 0, or indeed (analyzing the proof) in any characteristic other than 2, the ideal $J$ annihilates the module $\bigwedge^2 J$. This is elementary: for $a, b, c \in J$, we have

$$a(b \wedge c) = ab \wedge c = b(a \wedge c) = a \wedge bc = c(a \wedge b)$$

$$= ac \wedge b = a(c \wedge b) = -a(b \wedge c),$$

so that $2a(b \wedge c) = 0$. Since $\bigwedge^2 J$ is clearly torsion in any characteristic (we are assuming that $J$ contains a nonzero divisor) it seems interesting to ask what really does annihilate it in characteristic 2.

PROPOSITION 2.5. *Let $K$ be a field of characteristic 2, and let $K[x_1, \ldots, x_g]$ be a polynomial ring over $K$. If $J = (x_1, \ldots, x_g)$, then* ann $\bigwedge^2 J$ *is generated by the squares of elements of $J$ and $J^{g-1}$.*

*Proof.* First, it is easy to see that the given ideal annihilates $\bigwedge^2 J$, and this for any ideal generated by $g$ elements. Namely, if $a, b, c \in J$, then

$$a^2(b \wedge c) = bc(a \wedge a)$$

and similarly all products of the form

$$x_1 \cdots x_{g-1}(x_i \wedge x_j)$$

are zero. To show that these elements actually generate the annihilator, we must simply show that no product of $g - 2$ linear forms annihilates $\bigwedge^2 J$, and by symmetry it is enough to show that

$$x_1 \cdots x_{g-2}(x_{g-1} \wedge x_g) \neq 0.$$

Because the characteristic is 2, the exterior square is the symmetric square modulo the perfect squares, and we may write $\bigwedge^2 J$ as the component of

$$T := K[x_1, \ldots, x_g, t_1, \ldots, t_g]/(t_1^2, \ldots, t_g^2, \{x_i t_j - x_j t_i\}_{1 \leqslant i < j \leqslant g})$$

$$= \frac{K[t_1, \ldots, t_g]}{(t_1^2, \ldots, t_g^2)}[x_1, \ldots, x_g]/(\{x_i t_j - x_j t_i\}_{1 \leqslant i < j \leqslant g})$$

of degree 2 in the new variables $t$. In these terms, we are supposed to show that the element $x_1 \cdots x_{g-2} t_{g-1} t_g$ is nonzero in $T$. But we can map $T$ to $K[t_1, \ldots, t_g]/(t_1^2, \ldots, t_g^2)$ by specializing each $x_i$ to $t_i$, and the element $x_1 \cdots x_{g-2} t_{g-1} t_g$ goes to the nonzero element $t_1 \cdots t_g$ (the generator of the socle). □

Example 2.4 above leads to the natural question (in characteristic 0, for simplicity) of what annihilates the higher exterior powers of an ideal $J$, or equivalently, since $J$ itself annihilates them, what annihilates the exterior powers of the conormal module $J/J^2$. If $J$ can be generated by $\mu$ elements then of course $\bigwedge^{\mu+1} J = 0$, so at least this exterior power is annihilated by the unit idea. It seems thus interesting to ask the following question.

*Question 3.* What is the smallest exterior power of an ideal $J$ which is annihilated by more than $J$ itself?

For a start, we note that if $J = \operatorname{coker}(\phi_1 \colon F \to G)$, and $G$ has rank $\mu$, then $\bigwedge^\mu J$ is annihilated by $I_1(\phi_1)$; in fact,

$$\bigwedge^\mu J = R/I_1(\phi_1).$$

Now $I_1(\phi_1)$ need not in general be any larger than $J$. These two ideals are equal, for example, if $I$ is generated by a regular sequence or if $R$ is a local ring and $J$ is its maximal ideal. However, an old result of Vasconcelos says that if $J$ is of finite projective dimension in a local ring and $I_1(\phi_1) = J$ (equivalently $J/J^2$ is free over $R/J$), then $J$ is generated by a regular sequence. This gives a nice proof of the fact that a ring whose maximal ideal has finite projective dimension must be regular.

The following conjectural answer to Question 3 represents a substantial strengthening of these results.

CONJECTURE 2.6. *If $R$ is a ring of characteristic 0 and $J$ is an ideal of finite projective dimension having codimension $c$, then $J$ is strictly contained in* ann $\bigwedge^{c+1} J$.

It is easy to reduce the conjecture to the case where the ideal $J$ is primary to the maximal ideal of a local ring. Of course it would already be nice to prove the conjecture in the case where $R$ is a regular local ring. To avoid stating too many hypotheses, we will now discuss only what is known in that case: The conjecture is of course true if $J$ is generically a complete intersection. It is also true if $J$ is perfect of codimension $\leqslant 2$, or Gorenstein of codimension 3. In the primary case it seems possible that a finer statement is true: if $P$ is the maximal ideal of $R$, and $J$ is $P$-primary, it seems plausible that $(J : P)$, which is always bigger than $J$, annihilates $\bigwedge^{c+1} J$.

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