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Green's Conjecture:

An Orientation

by

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These notes parallel the introduction to Mark Green's conjecture on the free resolutions of canonical curves (Green [1984] and Green-Lazarsfeld [1985b]) that I presented at the Sundance 90 conference. They have a two-fold purpose: to introduce commutative algebraists with a modest background in algebraic geometry to a formulation of Green's conjecture that is more algebraic than the usual one; and to survey some of the approaches to Green's conjecture that have been tried.

The first section leads up to an algebraic conjecture (somewhat wild) generalizing Green's conjecture. The second section tries to explain the attractiveness of canonical rings of curves (for algebraists; geometers already know this!) and explains the connection between the algebraic conjecture of section 1 and the usual version of Green's conjecture. The third section surveys some promising approaches to Green's conjecture.

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1. Ideals generated by quadrics and 2-linear resolutions

Notation

Let $S = k[x_0, \ldots, x_r]$, and let $R = S/I$ be a homogeneous factor ring of S. If we assume that I contains no linear forms, and the projective dimension of S/I is m, then we may represent the minimal free resolution $\mathcal{F}$ of S/I by a **betti diagram** (as in the "betti" command of the program Macaulay of Bayer and Stillman) of the form

step of resolution

(degree of syzygy) - (step of resolution)		0	1	2	3	...	m
	0	1	-	-	-	...	
	1	-	a_1	a_2	a_3	...	a_m
	2	-	b_1	b_2	b_3	...	b_m
	3	-	...	...	...	...	c_m
	.						.
	.						.
	.						

meaning that $\mathcal{F}$ can be written as

$$0 \leftarrow S/I \leftarrow S \leftarrow S(-2)^{a_1} \oplus S(-3)^{b_1} \oplus \ldots \leftarrow S(-3)^{a_2} \oplus S(-4)^{b_2} \oplus \ldots$$

$$\leftarrow \ldots. \leftarrow S(-(m+1))^{a_m} \oplus S(-(m+2))^{b_m} \oplus S(-(m+3))^{c_m} \oplus \ldots \leftarrow 0.$$

Here the dashes in the betti diagram represent zeros. We define the **2-linear strand** of $\mathcal{F}$ to be the subcomplex

$$0 \leftarrow S/I \leftarrow S \leftarrow S(-2)^{a_1} \leftarrow S(-3)^{a_2} \leftarrow \ldots. \leftarrow S(-(m+1))^{a_m} \leftarrow 0.$$

Note that if some a_i is 0 then, since $\mathcal{F}$ is minimal, all a_j for $j \geq i$ are 0 as well. Thus we define the **length of the 2-linear strand** to be the largest number n such that $a_n \neq 0$. For simplicity we will call this this n the **2-linear projective dimension** and write

$$2LP\,(S/I) = n.$$

The Theme

In general, not too much is known about the linear strand and its length (but see Eisenbud-Koh [1991?a,b] for some conjectures in a slightly different context.) However, the theme of Green's conjecture is that long linear strands in free resolutions of nice ideals have a fairly simple and uniform origin. To understand it consider first a very easy and well-known result:

Lemma 1 . 1: If $I \subset J$ are ideals containing no linear forms, then the 2-linear strand of the minimal free resolution $\mathcal{F}$ of S/I is a summand of the 2-linear strand of the resolution $\mathcal{G}$ of S/J. In particular, the length of the 2-linear strand of $\mathcal{G}$ is $\geq$ that of the 2-linear strand of $\mathcal{F}$.

Proof: Let $\varphi\colon \mathcal{F} \to \mathcal{G}$ be a comparison map covering the canonical projection $S/I \to S/J$. We claim that the map on the i^{th} step of the resolution, $\varphi_i : \mathcal{F}_i \to \mathcal{G}_i$, induces a split monomorphism on the 2-linear part of $\mathcal{F}$.

At the 0^{th} step $\mathcal{F}_0 = \mathcal{G}_0 = S$, and the result is clear. Since J contains no linear forms, the quadratic minimal generators in the ideal I are among the minimal generators of J; this is exactly equivalent to the desired statement. An easy induction completes the proof. //

Thus one kind of acceptable "explanation" for the length (or size...) of the 2-linear part of the resolution of S/I would be that I contains an ideal of some standard form, which is known to have a long (or large...) 2-linear part. This is the form which Green's conjecture takes. Before stating the conjecture, we will describe the ideals of "standard form" that arise:

Examples

If I is the ideal generated by the 2×2 minors of a generic $p\times q$ matrix, then the 2-linear strand is known (Lascoux [1978]) to have length $\geq p+q-3$, with equality in characteristic 0 (and most likely all the time.) This turns out to be true of ideals of 2×2 minors of considerably more general matrices: following Eisenbud [1988] we define a matrix $L = (\lambda_{ij})$ of linear forms to be **1-generic** if no entry λ_{ij} is 0, and none can be made 0 by row and column transformations. We have

Theorem 1 . 2: If L is a 1-generic matrix of linear forms over a polynomial ring S, and I is the ideal generated by the 2×2 minors of L, then the minimal free resolution of S/I has a 2-linear strand of length $\geq p+q-3$.

This result was proved under various extraneous hypotheses by Green and Lazarsfeld (Green [1984;Appendix]); in a more algebraic setting by myself (unpublished); and then, in a simpler way, by Koh and Stillman [1989]. The most familiar case of Theorem 1 . 2 is that in which one of p, q, say p, is 2: Assuming that if the $2\times q$ matrix of linear forms L is 1-generic, it is not hard to show that the ideal of minors of L is of generic codimension ($= q-1$), so the minimal free resolution $\mathcal{F}$ is the the Eagon-Northcott complex, which consists entirely of a linear strand of length exactly $p+q-3 = q-1$. For arbitrary p,q and a generic matrix L, the computation of Lascoux shows that the 2-linear strand has length exactly $p+q-3$; the condition of 1-genericity is sufficient (but is not necessary; it would be nice to know some necessary conditions) to preserve at least one of the required syzygies.

An algebraic conjecture:

The boldest possible conjecture would now be to say that a sort of converse to Theorem 1 should be true: that is, any ideal I such that $2LP(S/I) = n$ should contain an ideal of 2×2 minors of a 1-generic $p\times q$ matrix with $p+q-3 = n$. Alas, this is <u>false</u>.

First of all, it is possible to start with a 1-generic matrix and replace some of its entries with 0's without spoiling the length of the 2-linear part of the resolution of its ideal of 2×2 minors, so that some matrices which are not 1-generic might perform the same function. However, if we assume that I is a prime ideal not containing any linear forms, then of course I cannot contain a nonzero determinant of a 2×2 matrix of linear forms with one entry = 0. Thus I could not contain a nontrivial ideal of 2×2 minors of any matrix of linear forms other than a 1-generic matrix (or a 1-generic matrix expanded with rows and columns of zeros.) So if we assume that I is prime, this objection becomes void.

Second, there are ideals whose 2-linear part is nontrivial but which contain no determinants at all!. For note that any determinant of a 2×2 matrix of linear forms is a quadric of rank ≤ 4 (the rank must be 3 or 4 if the matrix is 1-generic.) Thus the ideal I generated by a single rank s quadric, with $s\geq5$, has a 2-linear part of length 1 but certainly contains no 2×2 minors of interest. The same holds for some quadrics of rank ≤ 4 if the field is not algebraically closed: for example $x^2+y^2+z^2+w^2$ is not a determinant of a matrix of linear forms over the real numbers.

But what about prime ideals generated by quadrics of rank ≤ 4 (or more generally, primes whose degree 2 parts are spanned by quadrics of rank ≤ 4) over an algebraically closed field? There are still counterexamples, due to Schreyer [1986, 199?], in characteristic 2 (analogous cases, in characteristics $\neq 2$, are known not to be counterexamples.) However, one may be optimistic and feel that the theory of ideals with lots of quadratic generators might well be a little different in characteristic 2. (Of course the pessimistic will feel instead that we are being warned.) If we suppose that char k = 0, or even that it is $\neq 2$, I know no further counterexamples, so I rashly make the

Conjecture 1 . 3: Let k be an algebraically closed field of characteristic $\neq 2$, and let $I \subset S = k[x_0, \ldots ,x_r]$ be a prime ideal, containing no linear form, whose quadratic part is spanned by quadrics of rank ≤ 4. If 2LP(S/I) = n, then I contains an ideal of 2×2 minors of a 1-generic $p\times q$ matrix with $p+q-3 = n$.

Green's conjecture, from the algebraic point of view, is just the special case of this where we assume <u>in addition</u>:

a) S/I is normal (= integrally closed)
b) dim S/I = 2
c) S/I is Gorenstein
d) degree S/I = 2r

(conditions c,d could be replaced by the equivalent condtions that

S/I is a Cohen-Macaulay domain such that modulo a maximal regular sequence of linear forms it has Hilbert function 1, γ, γ, 1 for some integer γ, which is r-1 in the notation above.)

There is really no argument connecting any of these four conditions to the conclusion of the conjecture; but there are some geometric techniques (described below) which have lead to the verification of Green's conjecture under these extra hypotheses in many special cases (for example in all cases with $r \le 7$; see Schreyer [1986].)

2. Canonical rings of curves

In the previous section I claimed that the special hypotheses a) - d) might well be irrelevant to the conjecture. In this section I want to explain why geometers nevertheless find a)-d) so entrancing, and describe a geometric reformulation under these conditions.

What is the simplest interesting kind of variety? A curve, of course! (Well, almost of course: rational varieties also have exercised some claims, especially in certain periods, because they are so easy to specify. For example a rational surface can be specified as a plane with some marked points (to blow up) and curves (to blow down.) But for our purposes here, the anwer is certainly "a curve.") It's natural to consider first only nonsingular projective curves; and one finds that the set of isomorphism classes of these curves breaks up by genus into well-behaved algebraic varieties, the "moduli spaces."

Of course any algebraist would rather have a ring than a variety. Do these curves give rings? Not immediately. The simplest and most attractive way for a ring to come from a variety is as the homogeneous coordinate ring of that variety **in some projective embedding**. Thus to get a ring, one wants not only a curve but an embedded curve. The space of embeddings of a curve is not too bad, but there is a simpler way out of this dilemma than studying all embeddings: Leaving aside the so-called **hyperelliptic** curves, which are in any case quite well understood, every curve comes with a uniquely distinguished embedding in projective space, called, for obvious reasons, the **canonical embedding** -- that is, there really is a distinguished **canonical ring**, the homogeneous coordinate ring of the the canonically embedded curve, corresponding to each abstract (non-hyperelliptic) nonsingular curve[2]. With more sophistication the hyperelliptic curves can be included here too, but we will not worry about this point.

Moreover, the canonical rings of curves turn out to be quite simple: they are (precisely) the graded domains which are algebras over a field k and satisfy properties a)-d) from the last section. The number γ introduced there is g-2, where g is the genus of the curve.

2. Other embeddings are interesting too. See for example Eisenbud [transcanonical], and Martens-Schreyer [1986] for some interesting cases, and Green-Lazarsfeld [1985b] for some general conjectures.

The fact that the canonical rings of curves are Cohen-Macaulay is called Noether's theorem (see Arbarello et al [1985]). The Gorenstein property follows easily from this using duality theory. That the quadratic part of the ideals is spanned by quadrics of rank 4 was conjectured by Andreotti and Mayer and proved in general by Green [1984] (see also Smith and Varley [1989].)

Let us now take a look at the free resolution of the canonical ring of a curve of genus g (as a module over the homogenous coordinate ring of the projective space in which the curve is canonically embedded: if the curve has genus g, then this is $S = k[x_0, \ldots, x_r]$, with $r = g-1$.) From the Gorenstein property and property d) it follows that the resolution has betti diagram of the form:

	0	1	2	...	g-4	g-3	g-2
0	1	-	-	...	-	-	-
1	-	a_1	a_2	...	a_{g-4}	a_{g-3}	-
2	-	a_{g-3}	a_{g-4}	...	a_2	a_1	-
3	-	-	-	...	-	-	1

Note the symmetry between the i^{th} row and the $(3-i)^{th}$ row, and that b_i (in our previous diagram) is now given as a_{g-2-i}. Also from the Hilbert function in d) one computes:

$$a_i - a_{g-1-i} = i\binom{g-2}{i+1} - (g-1-i)\binom{g-2}{i-2} \quad \text{for } i = 1 \ldots g-2,$$

so that in fact all the betti numbers are known if we know

$$(*) \qquad a_{\lfloor g/2 \rfloor}, \ldots, a_{g-3}.$$

As we have already mentioned, the vanishing of a_i implies that of a_{i+1}. Similarly if a_i is 1 a_{i+1} vanishes (probably $a_{i+1} \neq 0$ actually implies that a_i is rather large.) For small g and in certain other cases we know a lot about the sequence (*). However, in general, we know little about which sequences of numbers (*) can occur. The same remarks are valid for the free resolution of any homogeneous $R = S/I$ satisfying hypotheses c) and d) from section 1 -- the assumptions that R is 2-dimensional, normal or even a domain, are irrelevant. Of course the possibilities for the sequence (*) may well expand if we weaken the hypotheses in this way. But also in the general case, we do not know which sequences (*) occur. It is not even known whether the sequence can be 0, ... , 0! In fact a leading special case of Green's conjecture (the so-called "generic case") is just this:

Generic Green's Conjecture 2 . 1: There exists a homogeneous Gorenstein ring (respectively a homogeneous Gorenstein normal domain) as above with

$$a_{\lfloor g/2 \rfloor} = 0$$

(and thus $a_i = 0$ for $i \geq [g/2]$.)

We will call the two versions of this conjecture, with and without the "normal domain" condition, the strong and weak forms, respectively. The strong form is known to hold for $g \leq 17$, by virtue of computer work (at least for some characteristic, and a fortiori for almost all characteristics, including characteristic 0) using the approach of Weyman detailed below. It is also known that each of the cases {g odd} and {g even} implies the other (work of Ein, Bayer-Stillman and Weyman...).

To massage Green's conjecture into its usual statement, and to relate it to Conjecture 1 . 3, we quote a (rather easy) result derived from our [1988, pp. 549-552]:

Theorem 2 . 2: Let $X \subset \mathbb{P}^r$ be a reduced irreducible linearly normal nondegenerate curve. There is a 1-generic $p \times q$ matrix of linear forms L whose 2×2 minors vanish on X iff the hyperplane bundle $\mathcal{O}_X(1)$ can be factored as $\mathcal{L}_1 \otimes \mathcal{L}_2$ with $h^0\mathcal{L}_1 \geq p$ and $h^0\mathcal{L}_2 \geq q$. //

Here the conditions "linearly normal nondegenerate" mean that the natural map $H^0(\mathcal{O}_{\mathbb{P}^r}(1)) \to H^0(\mathcal{O}_X(1))$ is an isomorphism.

Recall that the Clifford index of a line bundle $\mathcal{L}$ on a curve C is by definition the number

$$\text{Cliff } \mathcal{L} := g+1 - (h^0(\mathcal{L}) + h^0(\omega\otimes\mathcal{L}^{-1})) = \deg \mathcal{L} - 2(h^0(\mathcal{L})-1)$$

and that the Clifford index of C (in case the genus of C is ≥ 3) is the minimum of all Clifford indices of bundles $\mathcal{L}$ with $h^0\mathcal{L} \geq 2$ and $h^0\omega\otimes\mathcal{L}^{-1} \geq 2$. Putting this together, we get

Corollary 2 . 3: Let C be a curve of Clifford index c canonically embedded in $\mathbb{P}^{g-1}$. If c' is the maximum number such that there is a 1-generic $p\times q$ matrix of linear forms L whose 2×2 minors vanish on X and $p+q-3 = c'$, then $c' = g-2-c$. //

Thus Conjecture 1 . 3 becomes, in the case of canonical curves:

Conjecture 2 . 4 (Green): The length of the 2-linear part of the resolution $\mathcal{F}$ of the canonical ring S/I of a curve of genus g and Clifford index c is

$$2LP\ (S/I) = g-2-c.$$

Equivalently, with notation as in section 1 above, we have

$$b_1 = \ldots. = b_{c-1} = 0, \text{ but } b_c \neq 0.$$

In Green's terminology, this says that a curve of Clifford index c satisfies condition N_{c-1} but not N_c. Note that the fact that the length of the linear part is <u>at least</u> g-1-c (equivalently $b_c \neq 0$) comes "for free" from Theorems 1 . 2 and 2 . 3; this "easy half" was first proved by Green and Lazarsfeld in the appendix to Green's [1984].

The generic curve of genus g is known to have Clifford index $\lfloor (g-1)/2 \rfloor$ -- in the sense that this is the value of the Clifford index taken on by the curves in an open dense set of the moduli space. Thus the generic form of Green's conjecture becomes:

Generic Green's Conjecture (geometric version) 2 . 5: The free resolution of the canonical ring of a generic curve of genus g has

$$a_{\lfloor g/2 \rfloor}, \dots , a_{g-3} = 0, 0, \dots ,0.$$

Since the condition in the conclusion of the conjecture is a Zariski open condition in families of curves, it would be the same to assert that there exists a smooth curve whose canonical ring satisfies this conclusion -- or even a locally Gorenstein, smoothable, one-dimensional, canonically embedded scheme with this property.

3. Special cases and general approaches

In this section we will list the known approaches to Green's conjecture. In order to give something absolutely explicit, we summarize an approach due to Weyman giving generators and relations for certain graded modules of finite length over polynomial rings whose hilbert functions determine the desired betti numbers. We apologize in advance to the expert whose favorite bit we skipped.

I. Degenerations and the strong generic conjecture

This section describes approaches to the "generic" Green's conjecture (1.2 or 1.5 above) using the technique of degenerations.

These approaches to the (strong form of the) generic Green's conjecture lead quite quickly to explicit ideals of canonical curves, one for each genus, which experimentally have the kind of resolution predicted by Green's conjecture for the generic curve. If one could prove that one of these ideals really has the desired resolution, then one would have a proof of the generic Green's conjecture, since each of these examples represents a "degeneration" of a smooth canonical curve of genus g. There is even one version of this, coming from the Ribbon approach of Bayer-Eisenbud, that would lead to a proof that Green's conjecture holds for a curve of each Clifford index.

There is an interesting opportunity for an algebraist here. All that is involved is to find the free resolution of a particular ideal. So far at least, geometric ideas have not been of much use.

Though the problems involved are of the most concrete and specific kind (specific ideals whose minimal resolutions must be calculated -- sometimes with explicitly known non-minimal resolutions, so that the computation comes down to deciding the rank of an explicitly known matrix with integer entries!), people have not written much up.

A. **Cuspidal rational curves and the tangent developable surface.** (Buchweitz, Schreyer, Bayer-Stillman, Weyman, Green, Kempf...)

The following construction was noted by Buchweitz and Schreyer some time around 1983: One way to get a canonical curve of genus g is to take a rational curve with g cusps in $\mathbb{P}^{g-1}$. Such curves turn out to be the hyperplane sections of an arithmetically Cohen-Macaulay surface (actually a degenerate K3 surface, if there are any geometers listening) obtained as the tangent developable surface (≡ the union of the tangent lines) to the rational normal curve in $\mathbb{P}^g$. This is the rational surface with affine parametrization $\mathbb{A}^2 \to \mathbb{A}^g$ given by

$$(x,y) \mapsto (x,x^2,\ldots,x^g) + y(1,2x,3x^2,\ldots,gx^{g-1})$$

$$= (x+1,\ x^2+2xy,\ \ldots\ ).$$

Since these curves are smoothable, a proof of the generic version of Green's conjecture would follow if we could check the conjecture on the resolution of any one of these curves, or, equivalently, on the resolution of the tangent developable surface itself.

We will show below that if we take the rational normal curve to have equations given by the minors $\Delta_{a,b}$ of the matrix

Let $\Delta_{a,b}$ $(0 \le a\ ,\ b \le g-1)$ be the minor involving the columns a and b of the matrix

$$\begin{pmatrix} x_0 & x_1 & \cdots & x_{g-1} \\ x_1 & x_2 & \cdots & x_g \end{pmatrix}$$

with the usual convention that $\Delta_{a,b}=-\Delta_{b,a}$. If we take the rational normal curve to be the curve with equations

$$\Delta_{a,b} = 0 \qquad\qquad (0 \le a\ ,\ b \le g-1),$$

then we will show below that the quadratic equations of the tangent developable surface are

$$\Gamma_{a,b} := \Delta_{a+2,b} - 2\Delta_{a+1,b+1} + \Delta_{a,b+2} = 0 \quad (0 \le a < b \le g-3).$$

(For $g \geq 6$ the quadratic equations generate the ideal of the tangent developable surface, but in any case only the quadratic equations are involved in checking Green's conjecture.) Thus:

To prove the (strong) generic version of Green's conjecture, and indeed to prove the conjecture for generic smooth curves, it is enough to show that the 2-linear part of the minimal resolution of the ideal J generated by the linear combinations

$$\Delta_{a,b+2} - 2\Delta_{a+1,b+1} + \Delta_{a+2,b} \qquad 0 \leq a < b \leq g-3$$

of the minors of the matrix M has length $\leq$ (or, equivalently, $=$) $\lfloor g/2 \rfloor - 1$.

We will now derive these equations for the tangent developable surface. We will then exhibit an explicit non-minimal free resolution for the ideal J in a form discovered by Jerzy Weyman. All this will be done in terms of multilinear algebra and the representation theory of SL_2; the reader without a taste for such constructions may wish to skip to the description of "Weyman's Modules", below:

We begin with an invariant description of the ideal of the rational normal curve. We think of $\mathbb{P}^1$ as $\mathbb{P}(V)$, the space of 1-quotients of a 2-dimensional vector space V, and write Aut $\mathbb{P}^1 = SL(V) = SL_2$. The ambient space $\mathbb{P}^g$ of the rational normal curve is then $\mathbb{P}(S_gV)$, where S_g denotes the g^{th} symmetric power functor, and we think of S_gV as the forms of degree g on $\mathbb{P}^1$. The space of all quadratic forms on this $\mathbb{P}^g$ is $S_2(S_gV)$. The restriction map of these to the rational normal curve ($= \mathbb{P}^1$) is then the natural map

$$S_2S_gV \to S_{2g}V,$$

and the quadratic part of the ideal of the rational normal curve is the kernel of this map. It is convenient to denote a basis of V by $\{1,x\}$, and more generally a basis for S_gV by $\{1,x, \dots ,x^g\}$, in which case the map $S_2S_gV \to S_{2g}V$ above may be written

$$x^a \cdot x^b \mapsto x^{a+b}.$$

In characteristic 0 the kernel is easy to describe and analyze, as follows: We have

$$S_2S_gV = S_{2g}V \oplus S_{2g-4}V \oplus S_{2g-8}V \oplus \dots .$$

The quadratic part of the ideal of the rational normal curve itself consists of all but the first of these summands. It is thus

$$S_2S_{g-2}V = S_{2g-4}V \oplus S_{2g-8}V \oplus \dots ,$$

and (as we shall show) the quadratic part of the ideal of the tangent developable surface consists of all but the first two. It is thus

$$S_2S_{g-4}V = S_{2g-8}V \oplus S_{2g-12}V \oplus \dots .$$

In both cases the inclusion maps are given by "inner multiplication" by

$$\alpha := x^2 \cdot 1 - x \cdot x \in S_2S_2V,$$

where by inner multiplication we mean the equivariant pairing

$$S_2S_aV \otimes S_2S_bV \to S_2S_{a+b}V$$

given by

$$x^u \cdot x^v \otimes x^s \cdot x^t \mapsto x^{u+s} \cdot x^{v+t} + x^{u+t} \cdot x^{v+s}.$$

Unfortunately, this description does not work in arbitrary characteristic, so we adopt another, only a little less simple which does (I am grateful to Jerzy Weyman for discussions of this matter):

We can identify the space Q of quadratic forms in the ideal of the rational normal curve with the representation $\wedge^2 S_{g-1}V$ (which is the same as $S_2S_{g-2}V$ in characteristic 0). To get the inclusion map, note that the kernel of $S_2S_gV \to S_{2g}V$ is spanned by the elements

$$\Delta_{a,b} = x^a x^{b+1} - x^{a+1} x^b \in S_2S_gV \qquad \text{for } 0 \le a < b \le g-1,$$

which are the images of the obvious basis vectors

$$x^a \wedge x^b \in \wedge^2 S_{g-1}V$$

(this is half the image of the product of $(x\otimes 1 - 1\oplus x)$ and $x^a\otimes x^b - x^b \otimes x^a$ in $S_gV\otimes S_gV$, under the map to S_2S_gV; the map is thus equivariant.)

We claim next that the quadratic part of the ideal of the tangent developable surface is given by the representation

$$\wedge^2 S_{g-3}V \subset \wedge^2 S_{g-1}V.$$

As in the characteristic 0 description, this map can be described as an "inner multiplication with the element α", where this time inner multiplication refers to the pairing

$$\wedge^2 S_aV \otimes S_2S_bV \to \wedge^2 S_{a+b}V$$

given by

$$x^u \wedge x^v \otimes x^s \cdot x^t \mapsto x^{u+s} \wedge x^{v+t} + x^{u+t} \wedge x^{v+s}.$$

The image of $x^a \wedge x^b$ in S_2S_gV is now easily computed to be

$$\Gamma_{a,b} = \Delta_{a+2,b} - 2\Delta_{a+1,b+1} + \Delta_{a,b+2} \qquad \text{for } 0 \le a < b \le g-3$$

We now claim that the quadratic part of the ideal of the tangent developable surface is generated by these. It is useful to fall back on the characteristic 0 computation: since $\alpha * S_2S_{g-2}V$ generates a maximal invariant subspace of S_2S_gV, the same will be true of S_2S_{g-4} in S_2S_{g-2}. A direct computation shows that the image of S_2S_{g-4} in S_2S_g is then spanned by the elements

$$\Gamma_{a,b+1} - \Gamma_{a+1,b} \qquad \text{for } 0 \le a \le b \le g-4$$

-- this is the image of $x^a \cdot x^b$. But over $\mathbb{Q}$ these span the same space as the elements $\Gamma_{a,b}$ themselves. Thus over $\mathbb{Q}$ the $\Gamma_{a,b}$ span a maximal invariant subspace of the ideal of the rational normal curve, and it suffices to show that they vanish on the tangent developable surface. By invariance it is enough to show that these forms vanish on a single tangent line to the rational normal curve. For example, the tangent line at the point "x=0", that is, (1,0, ... ,0), is the line consisting of all elements (s, t, 0, ... ,0). This vanishing occurs because the only minor $\Delta_{u,v}$ not vanishing on this line is $\Delta_{0,1}$, and this one never occurs in the expression for $\Gamma_{a,b}$.

We have now shown that the representation $\wedge^2 S_{g-3}V \subset S_2 S_g V$ defines the quadratic part of the ideal of the tangent developable surface in characteristic 0. To complete the proof in all characteristics it suffices to show that, if we take V for a moment to be $\mathbb{Z}^2$, the quotient $S_2S_gV/\{\Gamma_{a,b} \mid 0 \le a < b \le g-3\}$ is torsion free, which amounts to finding a minors of the appropriate size in the inclusion matrix which are relatively prime. This is a straightforward computation.

Because the ideal J of the tangent developable surface is SL_2-invariant, it is natural to try to write down a resolution in terms of representation theory; for example, the Koszul cohomology groups might be written in terms of representations of SL_2. Several people seem to have tried this without success; an invariant but nonminimal resolutions are known, but not how to make any of them minimal! One such construction is detailed in the paper of Bayer and Stillman elsewhere in these proceedings.

Here is such a construction of a nonminimal resolution, from Weyman's private notes (about 1986). It turns out that this is a special case of the construction pursued in Weyman's paper [1989], (where the ideal J of the tangent developable is called J_{g-1}) to which the reader may go for more details.

Let R be the polynomial ring $k[S_gV]$, and let J be the ideal of the tangent developable surface, as above. Weyman's idea is based on a computation of the free resolution of the normalization A of R/J, realized by pushing forward the structure sheaf from the desingularization of the tangent developable surface. He finds:

1) A decomposes as SL(V)-module as

$$A = \oplus_{d\geq 0} S_{(g-1)d}V \otimes S_d V.$$

2) The cokernel C := A/(R/J) decomposes as

$$C = \oplus_{d\geq 1} S_{dg-2}V \otimes \wedge^2 V.$$

3) The minimal free resolution of A over R is

$$0 \leftarrow A \leftarrow S_0V(0) \oplus \wedge^2V\otimes S_{g-2}V(-1) \leftarrow \wedge^2V\otimes\wedge^2(S_{g-2}V)\otimes S_2V(-2)$$

$$\dots \leftarrow \wedge^2V\otimes\wedge^{i+1}(S_{g-2}V)\otimes S_{2i}V(-i-1) \leftarrow \dots$$

(The terms are to be thought of as the free R-modules obtained by tensoring the given representation, over k, with R.) Note that SL(V)-equivariantly the term $\wedge^2 V = k$ could have been suppressed, but has been carried along to make the descriptions of the subsequent maps simpler; it also makes the description GL(V)-equivariant. The differential between the terms of the form $\wedge^2V\otimes\wedge^i(S_{g-2}V)\otimes S_{2i-2}V(-i)$is the composite

$$\begin{array}{l}
\wedge^2V\otimes\wedge^{i+1}(S_{g-2}V)\otimes S_{2i}V(-i-1)\\
\quad\Big\downarrow\ 1\otimes\Delta\otimes\Delta\\
\wedge^2V\otimes\wedge^{i}(S_{g-2}V)\otimes S_{g-2}V\otimes S_2V\otimes S_{2i-2}V(-i-1)\\
\quad\Big\downarrow\ 1\otimes 1\otimes \text{mult}\otimes 1\\
\wedge^2V\otimes\wedge^{i}(S_{g-2}V)\otimes S_gV\otimes S_{2i-2}V(-i-1)\\
\quad\Big\downarrow\ \text{the factor } S_gV \text{ is absorbed into the coefficients}\\
\wedge^2V\otimes\wedge^{i}(S_{g-2}V)\otimes S_{2i-2}V(-i)
\end{array}$$

where Δ represents the diagonal map on the symmetric or exterior algebra, mult is the natural map $S_{g-2}V\otimes S_2V \to S_gV$, and the last map is the one which multiplies the factor S_gV with the "coefficients", in $k[S_gV]$, and raises the degree by 1.

The remaining piece of the differential,

$$S_0V(0) \leftarrow \wedge^2V\otimes\wedge^2(S_{g-2}V)\otimes S_2V(-2)$$

comes from a higher map in the spectral sequence which gives rise to this resolution. Thus we know its definition only on the kernel of the map

$$\wedge^2V\otimes\wedge^2(S_{g-2}V)\otimes S_2V(-2) \to \wedge^2V\otimes S_{g-2}V(-1)$$

which is the part of the differential we have already defined. Since it plays no role in the computations necessary to check Green's conjecture, we will worry about it no further.

4) The minimal free resolution over R of C is

$$0 \leftarrow C \leftarrow \wedge^2V\otimes S_{g-2}V(-1) \leftarrow (\wedge^2V)^{\otimes 2}\otimes S_{g-3}V\otimes S_{g-1}V(-2)$$
$$\ldots \leftarrow (\wedge^2V)^{\otimes i+1}\otimes S_{g-i-2}V\otimes\wedge^i(S_{g-1}V)(-i-1) \leftarrow \ldots$$

The differential is the composite

$$\begin{array}{l}
(\wedge^2V)^{\otimes i+1}\otimes S_{g-i-2}V\otimes\wedge^i(S_{g-1}V)(-i-1) \\
\quad\Big\downarrow\ [\ (\wedge^2V)^{\otimes i+1} = (\wedge^2V)^{\otimes i}\otimes\wedge^2V\ ]\otimes 1\otimes\Delta \\
(\wedge^2V)^{\otimes i}\otimes\wedge^2V\otimes S_{g-i-2}V\otimes\wedge^{i-1}(S_{g-1}V)\otimes S_{g-1}V(-i-1) \\
\quad\Big\downarrow\ 1\otimes\eta\otimes 1\otimes 1 \\
(\wedge^2V)^{\otimes i}\otimes V\otimes S_{g-i-1}V\otimes\wedge^{i-1}(S_{g-1}V)\otimes S_{g-1}V(-i-1) \\
\quad\Big\downarrow\ \text{multiply } V \text{ and } S_{g-1} \text{ to } S_gV \text{ and absorb}
\end{array}$$

$$(\wedge^2V)^{\otimes i}\otimes S_{g-i-1}V\otimes\wedge^{i-1}(S_{g-1}V)(-i),$$

where η: $\wedge^2V\otimes S_{g-i-2}V \to V\otimes S_{g-i-1}V$ is the composite

$$\begin{array}{c}
\wedge^2V\otimes S_{g-i-2}V \\
\Big\downarrow \Delta\otimes 1 \\
V\otimes V\otimes S_{g-i-2}V \\
\Big\downarrow 1\otimes 1\otimes \text{mult} \\
V\otimes S_{g-i-1}V
\end{array}$$

as in the Koszul complex.

5) The resolution of R/J itself is now obtained from a mapping cylinder for a comparison map Θ from the resolution of A to the resolution of C lifting the natural epimorphism $A \to C$. Let us write

$$A_i = \wedge^{i+1}(S_{g-2}V)\otimes S_{2i}V \quad (\text{for } i > 0;\ A_0 \text{ has another term, } = S_0V)$$

$$C_i = S_{g-i-2}V\otimes\wedge^i(S_{g-1}V)$$

for the i^{th} term in the resolutions of A and C respectively, where for simplicity we have suppressed the factors of $\wedge^2V$ and the twist (-i-1). We wish to give an explicit expression for the map Θ on the i^{th} term of the resolution,

$$\Theta_i\colon\ A_i \ \to\ C_i.$$

To this end we denote a basis of each S_mV by powers of x, as above. Let Δ: $S_{2i}V \to (S_2V)^{\otimes i}$ be the diagonal map, and write

$$\Delta(w) = \Sigma_s\, w_s^1 \otimes \dots \otimes w_s^i\ .$$

The map Θ_i is given by the formula

$$\Theta_i(x_{j_1}\wedge\ldots\wedge x_{j_{i+1}}\otimes w)$$

$$= \sum_s \sum_{1\le a_2\le j_2-j_1,\ \ldots\ ,\ 1\le a_{i+1}\le j_{i+1}-j_i} x^{j_1+a_2+\ldots+a_{i+1}-i}\otimes\{\, x^{j_2-a_2}\cdot w_s^1\wedge\ldots\wedge x^{j_{i+1}-a_{i+1}}\cdot w_s^i\,\}$$

for $0\le j_1<\ldots<j_{i+1}\le g-2$ and $w\in S_{2i}V$.

6) We see that R/J has a nonminimal resolution of the form

$$0\leftarrow R/J\leftarrow R\ \leftarrow \ker(\Theta_1)(-2)\oplus \mathrm{coker}(\Theta_2)(-3)$$
$$\ldots\leftarrow \ker(\Theta_i)(-2)\oplus \mathrm{coker}(\Theta_{i+1})(-3)\leftarrow\ldots$$
$$\ldots\leftarrow\ R(-g-1)\leftarrow 0$$

Thus to prove the generic Green's conjecture, it is enough to show that the map Θ_i of vector spaces given explicitly above is onto for $i\le\lfloor g/2\rfloor$.

B. **Weyman's Modules**:

Making use of the representation theory in a different way, and working over a field k of characteristic 0, Jerzy Weyman (1988, unpublished) has defined for each $i\ge 2$ a graded module which I will call W(i), generated in degree 2 and having finite length over the polynomial ring in i+1 variables. It has the following remarkable property: the Hilbert function of W(i) determines the number we called a_{i-1} in section 1 for <u>all</u> the tangent developable surfaces, and thus for the cuspidal rational curves of <u>every</u> genus g. The mixing of the different genera in a single module seems to me quite surprising. I am not aware of any geometric explanation of these modules.

To be specific, suppose we are dealing with a g cuspidal curve, and write [i,j] for the "Newton coefficient" (i+j)!/i!j!. Weyman proves:

$a_i = (2i+1)[g-i-2,i+1] - (i+2)[g-i-1,i+1] + [g-i,i+1] + \dim_k W(i+1)_{g-i}$.

The strong form of the generic Green's conjecture can thus be seen to hold in odd genus $g = 2h+1$ as soon as $W(h+1)_{h+1} = 0$, and in even genus $g = 2h$ as soon as $W(h+1)_{h-1} = 0$.

The simple existence of such modules W(i) has some notable consequences: First, since W(h+1) is generated in degree 2, Weyman concludes that the generic Green's conjecture in the even genus case implies the corresponding conjecture in the odd genus case. (Bayer, Stillman, and Ein report that they have proven the reverse implication, so that the even and odd genus cases are actually equivalent.)

Second, since W(i+1) has finite length, Weyman concludes that for sufficiently large g the general curve of genus g has the "correct" i^{th} syzygy (sharper versions of this have been proven by others -- see below.)

The modules W(i) are quite easy to write down in terms of generators and relations: Let V be a 2-dimensional vector space as above, and write S_m for the m+1-dimensional vector space which is the m^{th} symmetric power of V, considered as a representation of SL(V). There is a (unique up to scalars) inclusion of representations

$$h\colon S_{2i-2} \to \wedge^2 S_i \cong S_{2i-2} \oplus S_{2i-6} \oplus S_{2i-10} \oplus \dots .$$

Of course writing $T = k[S_i]$ for tie polynomial ring on the generators of S_i, we may tensor with T and regard h as a map of free T-modules

$$h\colon T\otimes S_{2i-2} \to T\otimes\wedge^2 S_i.$$

On the other hand, the Koszul complex contains a map of degree 1

$$\kappa\colon T\otimes\wedge^3 S_i \to T\otimes\wedge^2 S_i.$$

The module W(i) is then generated by the elements of $\wedge^2 S_i$ (regarded as having degree 2) subject to the degree 0 relations given by h and the degree 1 relations given by κ; or in other words,

$$W(i) = \operatorname{coker}(h,\kappa) : T\otimes S_{2i-2} \oplus T\otimes\wedge^3 S_i \rightarrow T\otimes\wedge^2 S_i.$$

To make this completely explicit, we give a formula for h. After multiplying each column by a certain integer to avoid denominators (this is a harmless operation in characteristic 0!) we may write h as an integral matrix

$$h = (h_{[p,q],\, j}) \qquad (0 \le p < q \le i, \quad 0 \le j \le 2i-2)$$

defined recursively by

$$h_{[0,1],0} = 1;$$

$$h_{[p,q],j} = (i-p+1)h_{[p-1,q]} + (i-q+1)h_{[p,q-1]}$$

where the terms $h_{[s,t],r}$ with $s<0$ or $t \le s$ are to be interpreted as 0.

Using these formulas and the program Macaulay of Bayer-Stillman, a computer has checked the generic Green's conjecture (strong form) for $g \le 17$ in char 31991 (and thus char 0, and almost every other characteristic.) This is computationally the most effective technique currently known.

Other degenerate canonical curves have also been studied in the hope of getting at the strong generic form of the conjecture. These include:

C. **Nodal rational curves**: Two rational normal curves meeting in g+1 points (Bayer, Stillman -- reported elsewhere in these proceedings.)

Bayer and Stillman have tried to compute the Koszul cohomology groups giving the betti numbers of the canonical embedding of a rational cuspidal (or nodal) curve by regarding the canonical series on the cuspidal curve as an incomplete series on the normalization, and computing Koszul cohomology. In this way one obtains very explicit integer matrices, and the generic form of Green's conjecture would be proved by showing that they have certain ranks. This approach will be described in a separate article by them.

D. **Graph curves** (Bayer, Eisenbud, Park [1991?])
E. **Ribbons** (Bayer, Eisenbud [199?], Fong [199?])

II. Large genus compared to Clifford index.

L. Ein [1987]: an induction formula for the b_j (in the notation of section 1) passing from generic curves of low genus to generic curves of somewhat higher genus.

Schreyer has proved that the conjecutre holds sharply for p-gonal curves as long as p is small compared to the genus (roughly

$g \geq p^2$ + a linear function of p

is required (this effective version of Weyman's result was found independantly.) Ferola (Thesis [????]) has given a better bound, on the order of $p^2/2$.

Voisin (for $g \geq 11$) and Schreyer in general [1988], and even for some singular curves, have proved the conjecture, in a somewhat sharpened form, if Cliff $C \leq 2$.

III. Special cases

Complete results are known -- and even published! -- in a few other cases. Most important, all possibilities for the resolutions of smooth canonical curves in every characteristic are known in case the genus is ≤ 8; this work was done by Schreyer in his Thesis [1986], and extended by him to certain singular curves in his [199?].

Loose has dealt with smooth plane curves of every genus, and also the smooth curves which can be embedded in $\mathbb{P}^3$ as the intersection of a surface and a quadric [1989]. It would be interesting to try to extend this work to some other complete intersections.

IV. Other approaches

A. Locally decomposable sections of Vector bundles:

A very different approach to Green's conjecture has been introduced by Paranjape and Ramanan [1988], who reduce it to some new and remarkably general questions about stable vector bundles on curves. The following remarks are due to them:

Consider a canonically embedded curve $C \subset \mathbb{P}^{g-1}$ (or the canonical map in the case of a hyperelliptic curve C) and let T be the vector bundle on $\mathbb{P}^{g-1}$ given by the obvious exact sequence

$$0 \to \mathcal{O}(-1) \to \mathcal{O}^g \to T \to 0$$

(so that T is the tangent bundle of $\mathbb{P}^{g-1}$ twisted by $\mathcal{O}(-1)$.) Let E be the restriction of T to C, and consider the natural map

$$p_j\colon \ \wedge^j H^0(E) \to H^0(\wedge^j E).$$

Paranjape and Ramanan show that this map is always an injection, and that Green's conjecture is equivalent to showing that it is a surjection for $j \le \mathrm{Cliff}\, C$.

Let $S \subset H^0(\wedge^j E)$ be the cone of "pointwise decomposable" sections: that is, sections σ such that at each point x of C, the element $\sigma_x \in E_x$ may be written as the exterior product of j elements of the vector space E_x. It is clear that the image of p_j, which is the span of all globally decomposable elements, must be contained in the span of S. Paranjape and Ramanan show that the image of p_j is actually equal to the span of S, the largest possible value, iff $j \le \mathrm{Cliff}\, C$.

There remains the problem of when the pointwise decomposable sections span the whole of $H^0(\wedge^j E)$. Paranjape and Ramanan have shown that they do span in the case of hyperelliptic curves, and with Hulek they show that they span if C is trigonal and $j = 2$ or C is plane quintic and j is arbitrary. But this question could be asked for <u>any</u> vector bundle E, generated by global sections, say:

Question: If E is a vector bundle on a curve C, generated by global sections, is it the case that the pointwise decomposable sections span $H^0(\wedge^j E)$ for every j? (The first open case would be $j=2$, rank $E = 4$.)

One would need only a much weaker result to settle Green's conjecture, as the bundle E of primary interest is stable (semi-stable if C is hyperelliptic) and has stronger and stronger stability properties as Cliff C increases.

B. The classification of cubics and the dual socle

Let R be the homogeneous coordinate ring of a canonical curve, and let x,y be general elements of R of degree 1. As we have explained above, R/(x,y) is an Artinian Gorenstein ring with Hilbert function 1, γ, γ, 1, where $\gamma+2$ is the genus of C. We may of course write $R/(x,y) = k[z_1, \dots , z_\gamma]/J$. Such an Artinian Gorenstein factor ring corresponds to a cubic form F, up to scalars, in another set of variables $t_1, \dots t_\gamma$ as follows: If we regard the polynomial ring $k[t_1, \dots t_\gamma]$ as a module over $k[z_1, \dots , z_\gamma]$ by letting z_m act as the partial derivative $\partial/\partial t_m$ (this is in characteristic 0; in positive characteristic one must use divided power algebras...) then J is the annihilator of F and F is a generator of the submodule annihilated by J. (We are using the formulation of "inverse systems" of Macaulay; in modern terms, everything flows from the fact that $k[z_1, \dots , z_\gamma]$ is, with this module structure, the injective envelope of k as a $k[z_1, \dots , z_\gamma]$-module.)

It now makes sense to ask how the graded betti numbers in the free resolution of R/(x,y) as a $k[z_1, \dots , z_\gamma]$-module, which are the same as the betti numbers in Green's conjecture, are reflected in the properties of the cubic form F. Cubic forms are interesting and much studied objects in their own right. In the case $\gamma=3$ (genus 5), which is the first not quite trivial case for Green's conjecture, the cubic forms in question correspond to cubic curves in the projective plane, an especially well-loved case. Michael Stillman and I (unpublished) have investigated this case, and found that the betti numbers of R/(x,y) reflect quite a classical invariant of the cubic curve: ignoring the cases where on the one hand the cubic curve is a cone or on the other hand J contains a linear form, we found that the betti numbers are either of the form

1	-	-	-
-	3	-	-
-	-	3	-
-	-	-	1

(Cliff C = 2) or of the form

1	-	-	-
-	3	2	-
-	2	3	-
-	-	-	1

(Cliff C = 1.)

We proved that the latter case occurs precisely for cubics which are in the closure of the locus of smooth cubics of j-invariant 0 -- either the smooth cubic of j-invariant 0 or a cuspidal rational curve or the union of a conic and one of its tangent lines!

It would be interesting to extend the classification, even just to cubics in 4 variables.

References

E. Arbarello, M. Cornalba, P. A. Griffiths, and J. Harris. Geometry of Algebraic Curves vol I. (1985).

D. Bayer and D. Eisenbud. Ribbons. (in preparation)

D. Bayer and D. Eisenbud, appendix by S.-W. Park. Graph Curves. Appendix: Gonality and homoliferous connectivity of graph curves. Adv. in Math., to appear (1991?)

D. Bayer and M. Stillman, Some matrices related to Green's Conjecture. This Proceedings.

L. Ein. A remark on the syzygies of the generic canonical curves. Journal of Differential Geometry 26 (1987) 361--365.

D. Eisenbud. Linear Sections of Determinantal varieties. Am. J. Math. 110 (1988) 541-575.

D. Eisenbud. Transcanonical embeddings of hyperelliptic curves. J. Pure and Applied Alg. 19 (1980) 77-83.

D. Eisenbud and J. H. Koh. Nets of alternating matrices and the linear syzygy conjecture. Adv. in Math. to appear (1991?a)

D. Eisenbud and J.H. Koh. Some linear syzygy conjectures. Adv. in Math. to appear (1991?b)

F. Ferola. Resolutions of algebraic varieties. Thesis: (in preparation).

L.-Y. Fong Studies on families of curves Brandeis Thesis -- in preparation. (1991)

M. Green. Koszul cohomology and the geometry of projective varieties. (with an appendix by M. Green and R. Lazarsfeld.) J. Diff. Geom. 19 (1984) 125-171.

M. Green. Quadrics of rank four in the ideal of a canonical curve. Invent. Math. 75 (1984) 85--104.

M. Green and R. Lazarsfeld. Some results on the syzygies of finite sets and algebraic curves. Compositio Math. 67 (1988) 301-314.

M. Green and R. Lazarsfeld. A simple proof of Petri's theorem on canonical curves. Geometry today (Rome, 1984) Progr. Math., 60 Birkhauser Boston, Boston, Mass. (1985) 129-142.

M. Green and R. Lazarsfeld. On the projective normality of complete linear series on an algebraic curve. Invent. Math. 83 (1985) 73-90.

J. Koh and M. Stillman. Linear syzygies and line bundles on an algebraic curve. J. Alg. 125 (1989) 120-132.

A. Lascoux. Syzygies des variétés déterminantielles. Adv. in Math. 30 (1978) 202-237.

F. Loose. On the graded Betti numbers of plane algebraic curves. Manuscripta Mathematica 64 (1989) 503--514.

G. Martens and F.-O. Schreyer. Line bundles and syzygies of trigonal curves. Abhandlungen aus dem Mathematischen Seminar der Universitat Hamburg 56 (1986) 169--189.

K. Paranjape and S. Ramanan. On the canonical ring of a curve. Algebraic geometry and commutative algebra, Vol. II Kinokuniya, Tokyo (1988) 503--516.

F.-O. Schreyer. Syzygies of canonical curves and special linear series. Mathematische Annalen 275 (1986) 105--137.

F.-O. Schreyer. A standard basis approach to syzygies of canonical curves. J. Reine Angew., to appear (199?)

F.-O. Schreyer. Green's conjecture for general p-gonal curves of large genus. (to appear) (199?)

R. Smith and R. Varley. Deformations of singular points on theta divisors. Theta functions---Bowdoin1987, Part 1 (Brunswick, ME, 1987) Proc. Sympos. Pure Math., 49, Part 1 Amer. Math. Soc., Providence, RI, (1989) 571--579.

C. Voisin. Courbes tetragonales et cohomologie de Koszul. Journal für die Reine und Angewandte Mathematik 387 (1988) 111--121.

J. Weyman. Tangent developable of the rational normal curve and Mark Green's conjectures. (notes from 1988, available from Weyman, Dept of Math., Northeastern Univ., Boston, MA 02115)

J. Weyman. Equations of strata for binary forms. J.of Alg. 122 (1989) 244-249.