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# Regularity of Modules over a Koszul Algebra*

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We prove that every module over a commutative homogeneous Koszul algebra has regularity bounded by its regularity over a polynomial ring of which the Koszul algebra is a homomorphic image. From this we derive a result conjectured by George Kempf to the effect that a sufficiently high truncation of any module over a homogeneous Koszul algebra has a linear free resolution. © 1992 Academic Press, Inc.

*Statements.* In this paper $k$ denotes a field; all rings are graded Noetherian $k$-algebras generated in degree 1; and all modules are unital and finitely generated. Recall that the regularity $\operatorname{reg}_R M$ of a module $M$ over such a ring $R$ is defined to be the infimum of the integers $r$ such that for all $i \geqslant 0$,

$$\operatorname{Tor}_i^R(M, k)_{i+s} = 0 \qquad \text{for all} \quad s > r.$$

Note that if $Y$ is a minimal resolution of $k$ by graded free $R$-modules, then $(Y_i)_j = 0$ for $j < i$. Hence if $M$ is generated in degrees $\geqslant -t$ and has regularity $\leqslant r$, then the minimal graded free resolution of $M$ as an $R$-module generated in a strip that extends only $r$ steps above the diagonal and $t$ steps below it.

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In the case where $R$ is the polynomial ring $k[R_1]$, it is clear that $\operatorname{reg}_R M$ takes on a finite value for any $M$, and this value is significant in determining the complexity of computing a free resolution of $M$, as well as in other computational contexts. If $\dim R_1 = s+1$, so that $\operatorname{Proj} R = \mathbb{P}^s$, and $M$ is an $R$-module of the form

$$\bigoplus_{n \geqslant 0} H^0(\mathbb{P}^s, \mathcal{O}_X(n)),$$

for some scheme $X \subset \mathbb{P}^s$, then $\operatorname{reg}_R M$ is the regularity of $X \subset \mathbb{P}^s$ in the sense of Castelnuovo, studied by Mumford [12] and many other authors for theoretical reasons.

However, in the case where $R$ is not a polynomial ring, and $M$ is not a module of finite projective dimension, the regularity seems "often" to be infinite and has not been studied so far as we know, except in one case: the *homogeneous Koszul algebras* introduced (without assuming commutativity) by Priddy [13], which in the context of this paper are called simply Koszul algebras, are characterized by the property that, regarding $k$ as the trivial $R$-module,

$$\operatorname{reg}_R k = 0.$$

That is, Koszul algebras are the algebras over which the resolution of the residue class field is given entirely by linear matrices. Koszul algebras are surprisingly common: they include algebras with quadratic monomial relations [8], the coordinate rings of "Segre–Veronese" embeddings [4], and in fact any algebra with a quadratic straightening law [9], such as the homogeneous coordinate ring of the Segre embedding of $\mathbb{P}^n \times \mathbb{P}^m$ or the homogeneous coordinate ring of the Grassmannian, the homogeneous coordinate rings of canonical curves of Clifford index $>1$ [7] and curves of genus $g$ embedded by a complete series of degree $\geqslant 2g+2$ [5], the homogeneous coordinate rings of an abelian variety embedded by $n$ times an ample divisor with $n \geqslant 4$ [10], and indeed any high Veronese subring of any graded ring [2]. The non-commutative version, studied by Backelin and Fröberg [3], Manin [11], and others includes many more interesting rings, beginning with the exterior algebra.

In this paper we prove:

THEOREM 1. *Let $R$ be a Koszul algebra, and let $Q = k[R_1]$ be the polynomial ring mapping onto $R$. The regularity of any module $M$ over $R$ is finite; in fact,*

$$\operatorname{reg}_R M \leqslant \operatorname{reg}_Q M.$$

Recall that a module is said to have a *linear resolution* (in Kempf's terminology, it is an "awesome" module) if its generators are all in the same degree, say $d$, and all the matrices in its free resolution are matrices of linear forms; that is, $\operatorname{reg}_R M(d)=0$. If it is known that $M$ has a linear resolution, then the effective computation of the minimal resolution of $M$ is greatly simplified by the fact that the ranks of its free modules can be determined from a recurrence relation. Indeed, if $b_i$ is the rank of the $i$th module, then the Poincaré series $\sum_{i\geqslant 0} b_i t^i$ is equal to $H_M(-t)/H_R(-t)$, where

$$H_M(t)=\sum_{j\geqslant 0} \dim_k M_j t^j$$

is the Hilbert series. (To see this, note that taking alternating sums of vector space dimensions in each degree one obtains the equality of formal power series

$$H_M(t)=\sum_{i\geqslant 0} (-1)^i b_i H_R(t)\, t^i = P_M^R(-t)\, H_R(t),$$

where the infinite sum makes sense since the order of $H_R(t)\, t^i$ is $d+i$. Then replace $t$ by $-t$ and solve for $P_M^R(t)$.)

The following consequence of Theorem 1, which was conjectured by Kempf, was the starting point of our investigation. It extends a result of Eisenbud and Goto [6] from the regular case.

COROLLARY 2. *In the situation above, if* $r \geqslant \operatorname{reg}_R M$, *then the truncation* $M_{\geqslant r} := \bigoplus_{j\geqslant r} M_j$ *has a linear resolution.*

*Proof.* Given Theorem 1 and the fact that $\operatorname{reg}_R k=0$, the proof used by Eisenbud and Goto [6, Prop. 1.1] can be applied, replacing the Koszul complex wherever it appears by a minimal free resolution of $k$ over $R$. ∎

The proof we give for the first statement of the theorem works just as well whenever $\operatorname{reg}_R k<\infty$, but this probably occurs only for Koszul algebras:

*Conjecture.* Koszul algebras are the only rings for which $k$ has finite regularity.

The conjecture is obvious for complete intersections. In general, if $k$ has finite regularity, then the off-diagonal generators in the Tate resolution must *all* occur in odd homological degree. It follows that they yield central elements in the homotopy Lie algebra $\pi^*R$. Jacobsson has conjectured that the center of $\pi^*R$ is concentrated in degrees 1 and 2, and this would suffice to prove our conjecture. Jacobsson's conjecture—and thus the conjecture above—was proved by Avramov [1, Sect. 4] in the case where the embedding codepth $\dim R_1 - \operatorname{depth} R$ is $\leqslant 3$, and in some other cases as well.

Before proving Theorem 1, we record a few easy remarks about regularity:

LEMMA 3. (a) *If* $0 \to M' \to M \to M'' \to 0$ *is a short exact sequence of R-modules, then*

$$\operatorname{reg}_R M \leqslant \max(\operatorname{reg}_R M', \operatorname{reg}_R M''),$$

*with equality unless*

$$1 + \operatorname{reg}_R M'' = \operatorname{reg}_R M'.$$

(b) *If* $x \in R_1$ *is a nonzerodivisor on M then*

$$\operatorname{reg}_R M = \operatorname{reg}_R M/xM.$$

(c) *If Q is a polynomial ring, and M is a Q-module of finite length, then*

$$\operatorname{reg}_Q M = \max\{r \mid M_r \neq 0\},$$

*and in fact if* $\dim Q_1 = n$ *then*

$$\operatorname{Tor}_n^Q(M, k)_{n + \operatorname{reg}_Q M} \neq 0.$$

*Proof.* Parts (a) and (b) follow directly from the long exact sequence in Tor applied to the sequences

$$0 \to M' \to M \to M'' \to 0$$

and

$$0 \to M(-1) \xrightarrow{x} M \to M/xM \to 0,$$

respectively.

To prove (c), write $K$ for the Koszul complex on a basis of $Q_1$, so that $\operatorname{Tor}^Q(M, k)$ is the homology of $M \otimes K$, and let $t = \max\{r \mid M_r \neq 0\}$. Since

$$(M \otimes K_i)_j = (M^{\binom{n}{i}})_{j-i},$$

we see that $\operatorname{Tor}_i^Q(M, k)_j = 0$ for $j > i + t$, and $\operatorname{Tor}_n^Q(M, k)_{n+t} = M_t \neq 0$, proving both claims. ∎

*Proof of Theorem* 1. First consider the case where $M$ is a module of finite length. Applying part (c) of the lemma we see that

$$\operatorname{reg}_Q M = \max\{r \mid M_r \neq 0\},$$

and using induction on the length of $M$, part (a) of the lemma, and the assumption $\operatorname{reg}_R k = 0$, we see at once that $\operatorname{reg}_R M \leqslant \operatorname{reg}_Q M$, as required.

Next we argue by Noetherian induction, and assume that the result holds for any proper homomorphic image of $M$.

If the homogeneous maximal ideal $R_+$ is not an associated prime of $M$, then supposing as we may that $k$ is infinite there will exist an element $x \in R_1$ which is a nonzerodivisor on $M$, and the result follows at once by applying part (b) of the lemma to $M$, both as an $R$-module and as a $Q$-module.

If $M$ is not of finite length, but $R_+$ is associated to $M$, then let $M' \subset M$ be the largest submodule of finite length contained in $M$, and set $M'' = M/M'$. Note that $M' \neq 0$ and

$$\operatorname{reg}_Q M' \leqslant \operatorname{reg}_Q M;$$

indeed, since $R_+$ is not associated to $M''$, we have

$$\operatorname{Tor}_n^Q(M'', k) = (0 : Q_+)_{M''}(-n) = 0,$$

so $\operatorname{Tor}_n^Q(M, k) = \operatorname{Tor}_n^Q(M', k)$, and the inequality follows from the last statement of part (c) of the lemma.

If $\operatorname{reg}_R M'' \leqslant \operatorname{reg}_Q M'$ then by part (a) of the lemma, the finite length case treated above, and the preceding inequality, we obtain

$$\begin{aligned} \operatorname{reg}_R M &\leqslant \max(\operatorname{reg}_R M', \operatorname{reg}_Q M') \\ &= \operatorname{reg}_Q M' \\ &\leqslant \operatorname{reg}_Q M, \end{aligned}$$

so we may assume that $\operatorname{reg}_Q M' < \operatorname{reg}_R M''$. Using our induction hypothesis and the finite length case above, we may expand this to the sequence of inequalities

$$\operatorname{reg}_R M' \leqslant \operatorname{reg}_Q M' < \operatorname{reg}_R M'' \leqslant \operatorname{reg}_Q M''.$$

Now using part (a) of the lemma, both for $R$ and for $Q$, we obtain

$$\begin{aligned} \operatorname{reg}_R M &= \operatorname{reg}_R M'' \\ \operatorname{reg}_Q M &= \operatorname{reg}_Q M''. \end{aligned}$$

Since $\operatorname{reg}_R M'' \leqslant \operatorname{reg}_Q M''$ by the induction, we are done. ∎

## References

1. L. L. Avramov, Homological asymptotics of modules over local rings, *in* "Commutative Algebra" (M. Hochster, C. Huneke, and J. D. Sally, Eds.), pp. 33–62, Springer-Verlag, New York, 1989.
2. J. Backelin, On the rates of growth of the homologies of Veronese subrings, *in* "Algebra, Algebraic Topology, and Their Interactions" (J.-E. Roos, Ed.), pp. 79–101, Lecture Notes in Mathematics, Vol. 1183, Springer-Verlag, New York, 1986.
3. J. Backelin and R. Fröberg, Koszul algebras, Veronese embeddings, and rings with linear resolutions, *Rev. Roumaine Math. Pures Appl.* **30** (1985), 85–97.
4. S. Barcanescu and N. Manolache, Betti numbers of Segre–Veronese singularities, *Rev. Roumaine Math. Pures Appl.* **26** (1981), 549–565.
5. D. C. Butler, Normal generation of vector bundles over a curve, preprint.
6. D. Eisenbud and S. Goto, Linear free resolutions and minimal multiplicity, *J. Algebra* **88** (1984), 89–133.
7. M. Finkelberg and A. Vishik, Canonical ring of a general projective curve is Koszul, to appear.
8. R. Fröberg, Determination of a class of Poincaré series, *Math. Scand.* **37** (1975), 29–39.
9. G. Kempf, Some wonderful rings in algebraic geometry, *J. Algebra* **134** (1990), 222–224.
10. G. Kempf, Projective coordinate rings of abelian varieties, *in* "Algebraic Analysis, Geometry, and Number Theory" (J.-I. Igusa, Ed.), pp. 225–234, The Johns Hopkins Press, Baltimore, MD, 1989.
11. Yu Manin, "Quantum Groups and Noncommutative Geometry," Centre de Recherches Mathématiques, Université de Montréal, Montréal, 1988.
12. D. Mumford, "Lectures on Curves on an Algebraic Surface," Ann. of Math. Studies, Vol. 59, Princeton Univ. Press, Princeton, NJ, 1966.
13. S. B. Priddy, Koszul resolutions, *Trans. Amer. Math. Soc.* **152** (1970), 39–60.