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The dimension of the Chow variety of curves

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1. Introduction, statement of results and conjectures

The purpose of this paper is to determine the dimension of the Chow variety $\mathscr{C}_{d,r}$ parametrizing curves of degree d in $\mathbb{P}^r$. Of course, this variety has many components; by its dimension we mean as usual the maximum of the dimension of its components. The answer (stated precisely in Theorem 3 below) is not unexpected: except for a few exceptional cases of low degree, the component of the Chow variety having maximal dimension is the component whose general member is a plane curve, so that

$$\begin{aligned}\dim \mathscr{C}_{d,r} &= \dim \mathbb{G}(2, r) + \dim \mathbb{P}(H^0(\mathbb{P}^2, \mathscr{O}(d)) \\ &= 3(r-2) + d(d+3)/2.\end{aligned}$$

This result is, however, a by-product of a much more interesting investigation, and one that is not yet complete. This concerns the question: "What is the largest dimension of a component of $\mathscr{C}_{d,r}$ whose general member is an irreducible, nondegenerate curve in $\mathbb{P}^r$, and which components achieve that dimension?". Here is what we have proved:

- in $\mathbb{P}^3$, the maximum dimension of a component of the Chow variety $\mathscr{C}_{d,3}$ whose general member is irreducible and nondegenerate is achieved by the component parametrizing curves on a quadric surface of balanced bidegree (that is, either complete intersections with the quadric or residual to single lines in complete intersections). For $d \geqslant 8$, this is the unique component of that dimension.
- in $\mathbb{P}^r$ with $r \geqslant 4$, for sufficiently large d the maximum dimension of a component of the Chow variety $\mathscr{C}_{d,r}$ whose general member is irreducible and nondegenerate is achieved by a component parametrizing curves lying on a rational normal scroll and having certain classes in the Picard group of the scroll. By contrast, for low values of d the maximum dimension is attained by the component parametrizing rational curves. We conjecture that one of these two components always achieves the maximal dimension,

and that these are the only components to do so. We can at present prove this for small values of d and for large values of d; there is an intermediate range where the conjecture is not yet proved.

To make these statements more precise, let us introduce some notation and terminology. To begin with, we define the *restricted Chow variety* $\tilde{\mathscr{C}}_{d,r}$ to be the union of those irreducible components of the Chow variety $\mathscr{C}_{d,r}$ whose general point corresponds to an irreducible, nondegenerate curve in $\mathbb{P}^r$. As a first approximation to the dimension of a component of $\tilde{\mathscr{C}}_{d,r}$, we define the *Hilbert number* $h(d, g, r)$ to be

$$h(d, g, r) = (r + 1)d - (r - 3)(g - 1).$$

This number arises in several ways. To begin with, it is the Euler characteristic of the normal bundle N_C of a smooth curve C of degree d and genus g in $\mathbb{P}^r$. Since the space $H^0(N_C)$ of global sections of N_C is the Zariski tangent space to the Hilbert scheme at the point $[C]$, the Hilbert number is an a priori estimate on the dimension of a component of the Hilbert scheme through C. The Hilbert number may also be realized as the sum of the *Brill-Noether number* $\rho = g - (r + 1)(g - d + r)$, the dimension $3g - 3$ of the moduli space of abstract curves of genus g, and the dimension $(r + 1)^2 - 1$ of the group PGL_{r+1} of automorphisms of $\mathbb{P}^r$; this is another reason why $h(d, g, r)$ is a naive estimate for the dimension of the Hilbert scheme.

In fact, from the latter point of view we can see that *every component Σ of the restricted Chow variety $\tilde{\mathscr{C}}_{d,r}$ whose general member has geometric genus g has dimension at least $h(d, g, r)$*. To see this, suppose that $[C_0]$ is a general point of the component Σ of $\tilde{\mathscr{C}}_{d,r}$; let g be the geometric genus of C_0. The standard determinantal representation of the variety of linear systems on curves of genus g shows that every component of the variety $\mathscr{G}(d, g, r)$ parametrizing pairs $(C, \mathscr{D})$ with C a smooth curve of genus g and $\mathscr{D}$ a linear series on C has dimension at least $3g - 3 + \rho$. Since Σ is birational to a PGL_{r+1}-bundle over a component of $\mathscr{G}(d, g, r)$, the basic dimension estimate follows. (Brill-Noether theory also tells us that for any d, r and g such that $\rho \geqslant 0$ and $r \geqslant 3$ there exists a unique component of $\tilde{\mathscr{C}}_{d,r}$ dominating the moduli space $\mathscr{M}_g$, and that this component has dimension exactly $h(d, g, r)$.) We can also see the inequality $\dim(\Sigma) \geqslant h(d, g, r)$ from the normal bundle point of view, at least in case the general point $[C] \in \Sigma$ corresponds to a smooth curve: Jonathan Wahl has pointed out to us that a modification of Mori's argument in §1 of [M] shows that the dimension of $\tilde{\mathscr{C}}$ at such a point (C) is at least the difference $h^0(N_C) - h^1(N_C) = h(d, g, r)$. (Indeed, the same argument can be made when C is singular, but we have to be careful what we mean by the normal bundle N_C; and the bound we get is in terms of the arithmetic genus g' of C and so is weaker.)

By way of terminology, we will call an irreducible component Σ of the Chow variety *general* if its dimension is exactly $h(d, g, r)$, where g is the genus of a general member of Σ; we will call it *exceptional* if $\dim(\Sigma)$ is strictly greater than $h(d, g, r)$.

The next objects we want to introduce are the curves that, as we will show, move with the greatest degree of freedom among irreducible, nondegenerate curve of degree d in $\mathbb{P}^r$ when d is large. To do this, let $X \subset \mathbb{P}^r$ be a rational normal scroll. The Picard group of X is then generated by the class H of a hyperplane section, together with the class F of a line of the ruling of X. In these terms, we have the formula for the canonical bundle of X:

$$K_X = -2 \cdot H + (r-3) \cdot F.$$

We can use this to calculate the dimension of the linear system $|C|$ associated to an irreducible nondegenerate curve $C \sim \alpha H + \varepsilon F$ on X. It's not hard to check that the line bundle $\mathcal{O}_X(C)$ has no higher cohomology, and so by Riemann-Roch

$$\begin{aligned}\dim|C| &= \frac{(\alpha H + \varepsilon F) \cdot ((\alpha + 2)H + (\varepsilon - r + 3)F)}{2} \\ &= \frac{\alpha(\alpha+1)(r-1) + 2(\alpha+1)(\varepsilon+1) - 2}{2}\end{aligned}$$

Noting that the degree d of a curve $C \sim \alpha H + \varepsilon F$ on X is $\alpha(r-1) + \varepsilon$, we may write

$$\alpha = \frac{d - \varepsilon}{r - 1}$$

and substituting we find that

$$\dim|C| = \frac{d^2 + (r+1)d + (\varepsilon+2)(r-\varepsilon-1)}{2(r-1)} - 1$$

This may be maximized for given d by taking ε between -1 and $r-2$; this motivates the

DEFINITION. We say that a curve $C \subset \mathbb{P}^r$ is a *Chow curve* if it lies on a rational normal scroll X and has class

$$C \sim \alpha \cdot H + \varepsilon \cdot F \in \operatorname{Pic}(X)$$

with $-1 \leqslant \varepsilon \leqslant r-2$.

Note that if d is not congruent to $-1 \bmod r-1$, ε is determined by d and so we see that the family of Chow curves is irreducible. If $d \equiv -1 \pmod{r-1}$, on the other other hand, there will in general be two components of this family (if $r=3$ there will only be one because of the ambiguity in the choice of ruling).

It should be observed that Chow curves are not in general Castelnuovo curves – that is, curves of maximal genus for their degree – even though the description is similar and indeed the two classes overlap substantially. Indeed, a Castelnuovo curve may be characterized as a smooth curve having class $C \sim \alpha H + \nu F$ on a rational normal scroll, with $-(r-2) \leqslant \nu \leqslant 1$. Thus, a Chow curve is Castelnuovo if and only if $\varepsilon = -1, 0$ or 1; but in the remaining cases it still has very close to the maximal genus for an irreducible, nondegenerate curve of degree d in $\mathbb{P}^r$: the genus of the Chow curve is just $\varepsilon - 1$ less than the genus $\pi(d, r)$ of a Castelnuovo curve of the same degree.

In particular, it follows from the result of [EH] for large d that any curve of that genus and degree must lie on a scroll, so that Chow curves will form an open subset of an irreducible component of the Chow variety (or of two components if $d \equiv -1 \bmod (r-1)$).

If $d > 3r/2 - 1$, a general Chow curve C will lie on a unique scroll X (to see this, note that except in the cases $d = 2r-3$ or $2r-2$ and $\varepsilon = -1$ or 0, X may be characterized as the intersection of the quadrics containing C; in the two exceptional cases as the union of lines joining points conjugate under the hyperelliptic involution on C). We may thus calculate the dimension of the family of Chow curves of degree d: it is the dimension (calculated above) of the linear system $|\mathcal{O}_X(C)|$ associated to a Chow curve C on a scroll X, plus the dimension $r^2 + 2r - 6$ of the family of rational normal scrolls in $\mathbb{P}^r$. We will call this number

$$\delta(d, r) = \frac{d^2 + (r+1)d + (\varepsilon+2)(r-\varepsilon-1)}{2(r-1)} + r^2 + 2r - 7$$

the *Chow number*. Again, bear in mind that this is known to be the dimension of a component of the Chow variety containing Chow curves only for d large; for small d it may not be true that a deformation of a general Chow curve is a Chow curve (for example, if $d = 3r-4$ and $\varepsilon = r-2$ the Hilbert number exceeds the Chow number by $2r-6$, so that the Chow curves cannot be dense in a component of the Chow variety $\mathscr{C}_{d,r}$ for $r \geqslant 4$). At present when $r \geqslant 4$ we do not know for which values of d (and ε, where there is ambiguity) this is the case. This does not affect our results, since in the case $r \geqslant 4$ they apply only for large d; but in order to settle the remaining unknown cases this will have to be answered.

With all this said, we can now describe our results.

1. The situation in $\mathbb{P}^3$

First, observe that the Hilbert number $h(d, g, 3) = 4d$ is independent of g, i.e., all general components of the Chow variety of curves of degree d have the same dimension. The Chow number $\delta(d, 3)$ is less than or equal to $4d$ for $d \leqslant 7$, strictly greater when $d \geqslant 8$. This suggests the

THEOREM 1. *For $d \leqslant 7$ every component of the restricted Chow variety $\tilde{\mathscr{C}}_{d,3}$ has dimension $h(d, g, 3) = 4d$. For $d \geqslant 8$, the dimension of $\tilde{\mathscr{C}}_{d,3}$ is $\delta(d, 3)$, and the unique component of this dimension is the component containing Chow curves.*

Comparing the dimension $\max(4d, \delta(d, 3))$ of $\tilde{\mathscr{C}}_{d,3}$ to that of the family of plane curves of degree d in $\mathbb{P}^3$, we have the

COROLLARY. *For $d > 1$, the dimension of the Chow variety $\mathscr{C}_{d,3}$ is $3 + d(d + 3)/2$; and for $d \geqslant 4$ the unique component of this dimension is the component whose general member is a plane curve of degree d.*

2. The situation in $\mathbb{P}^r$, $r \geqslant 4$

The picture here is somewhat more complicated, and our results only partial. To begin, we look at the restricted Chow variety $\tilde{\mathscr{C}}_{d,r}$. We note in this case that the Hilbert number $h(d, g, r)$ is a decreasing function of g; among general components of $\tilde{\mathscr{C}}_{d,r}$ the one of maximal dimension is the one parametrizing rational curves, which has dimension

$$h(d, 0, r) = (r + 1)(d + 1) - 4.$$

For low values of d – specifically, for $d \leqslant 2r$ – the normal bundle of any irreducible, nondegenerate curve $C \subset \mathbb{P}^r$ is non-special; thus every component of $\tilde{\mathscr{C}}_{d,r}$ is general and the dimension of $\tilde{\mathscr{C}}_{d,r}$ is $h(d, 0, r)$. On the other hand, for large d the Chow number $\delta(d, r)$ is much larger, and this we claim is the dimension of $\tilde{\mathscr{C}}_{d,r}$. Precisely, we can prove

THEOREM 2. *Let $r \geqslant 4$. For $d \leqslant 2r$, the dimension of every component of the restricted Chow variety $\tilde{\mathscr{C}}_{d,r}$ whose general member has geometric genus g is $h(d, g, r) \leqslant h(d, 0, r) = (r + 1)(d - r) - 4$, and the unique component of dimension $h(d, 0, r)$ is the one parametrizing rational curves. For*

$$d \geqslant 4r^2 - 4r + 3 \tag{**}$$

the dimension of $\tilde{\mathscr{C}}_{d,r}$ is $\delta(d, r)$, and the only components of this dimension are the ones containing Chow curves.

On the basis solely of "experimental evidence", it seems likely that this behavior holds as well in the intermediate range of degrees not covered in the statement of Theorem 2. Specifically, we make the

CONJECTURE. For any d, the dimension of the restricted Chow variety is

$$\dim(\tilde{\mathscr{C}}_{d,r}) = \max(h(d, 0, r), \delta(d, r)).$$

Moreover, (assuming $r \geqslant 4$) any component of maximal dimension in $\tilde{\mathscr{C}}_{d,r}$ parametrizes either rational curves or Chow curves.

The situation with regard to the unrestricted Chow variety $\mathscr{C}_{d,r}$ is likewise slightly more complicated for $r \geqslant 4$ than in the case $r = 3$. In both cases the largest dimension of a component whose general member is irreducible is that of the component parametrizing plane curves; but in $\mathbb{P}^r$ for low values of the degree d components of $\mathscr{C}_{d,r}$ whose general member is reducible may have larger dimension than this. To state this precisely, we have

THEOREM 3. *The dimension of the Chow variety is given by*

$$\dim(\mathscr{C}_{d,r}) = \max(2d(r-1), 3(r-2) + d(d+3)/2),$$

and the general member of a component of this dimension is either a union of d lines or a plane curve.

Conjectures

The discussion so far is concerned only with Chow varieties parametrizing curves. It is natural to ask what may be true for Chow varieties parametrizing higher-dimensional subvarieties of projective space. As in the curve case, we can break up the problem by introducing the restricted Chow variety: if $\mathscr{C}_{d,m,r}$ is the Chow variety of varieties of degree d and dimension m in $\mathbb{P}^r$, we let $\tilde{\mathscr{C}}_{d,m,r}$ be the union of the irreducible components of $\mathscr{C}_{d,m,r}$ whose general member is irreducible and nondegenerate. There are then statements clearly analogous to what is stated above and proved below for the Chow varieties of curves of large degree. Thus by analogy we may make the

GUESS. Given r and m, for all sufficiently large d

(i) The components of maximal dimension of the restricted Chow variety $\tilde{\mathscr{C}}_{d,m,r}$ parametrize divisors on rational normal scrolls $X \subset \mathbb{P}^r$ of dimension $m + 1$; and
(ii) The component of maximal dimension of the Chow variety $\mathscr{C}_{d,m,r}$ parametrizes hypersurfaces in linear spaces $\mathbb{P}^{m+1} \subset \mathbb{P}^r$.

What is likely to go on in low-degree cases is less clear. One obvious candidate for a component of maximal dimension in the restricted Chow variety would be the component parametrizing projections of rational normal scrolls (the ones parametrizing projections of Veronese varieties are slightly smaller). These do in fact seem to be the largest in many cases, but not always – for example, in the first nontrivial case, surfaces of degree r in $\mathbb{P}^r$, projections of rational normal scrolls move in a larger-dimensional family than do del Pezzo surfaces for any $r \geqslant 5$, but for $r = 4$ the del Pezzos do better. You might counter this by saying that in case $r = 4$ the degree 4 is large enough that the conjecture above takes over (del Pezzos in $\mathbb{P}^4$ are divisors on rational normal threefold scrolls, which in this case are just quadrics of rank 4 or less); but the situation remains unclear. Similarly, for the unrestricted Chow variety, that the component whose general member is a union of m-planes is largest seems a natural guess.

A note about the techniques used in this paper. They are very simple, even though the arithmetic gets complicated at points. Basically, we know the dimension of the family of Chow curves, and we know that they are the largest-dimensional family of curves of given degree lying on scrolls. To prove our results, then, it suffices to show that any curve moving in a family of dimension greater than $\delta(d, r)$ must lie on a scroll. We do this by looking at the normal bundle N of C: for $p_1, \ldots, p_{r-2} \in C$ general points we exhibit an exact sequence

$$0 \to \bigoplus_{i=1}^{r-2} \mathcal{O}_C(1)(p_i) \to N \to M \to 0,$$

where $M \cong K_C(3)(-\Sigma p_i)$; and we use this sequence to bound the dimension $h^0(N)$ of the space of global sections of N in terms of the genus of C. (See also the discussion of Lazarsfeld's Lemma in [Ein] for an application of a similar exact sequence for the restricted tangent bundle of a projective curve.) To finish, we invoke a result of Halphen and its generalization by Eisenbud-Harris to the effect that a curve in $\mathbb{P}^3$ (respectively, $\mathbb{P}^r$) of sufficiently high genus relative to its degree must lie on a quadric surface (resp., a scroll).

Finally, one note about the Chow variety and the Hilbert scheme. We have chosen to state our results in terms of the Chow variety, even though most of our techniques relate to the geometry of the Hilbert scheme. In the present circumstances: the irreducible components of the Chow variety correspond to a subset of the irreducible components of the Hilbert scheme. There are two differences, however. The first is simply a matter of convention: the Chow variety $\mathscr{C}_{d,r}$ parametrizes all curves of degree d, while the Hilbert scheme is broken up into subschemes according to the genus of the curves parametrized. Since we are concerned with the maximal dimension of a component of the family of curves of degree d irrespective of genus, it seemed natural to express our results in terms of the Chow variety.

The second reason for working with Chow rather than Hilbert is that the Hilbert scheme will have additional components whose general members are not of pure dimension (e.g., consist of the union of a reduced curve and a zero-dimensional subscheme). For all we know these components may have larger dimension than those corresponding to components of the Chow variety.

2. Irreducible, nondegenerate curves in $\mathbb{P}^3$

In this section we will deal with the case of irreducible, non-degenerate space curves; our goal will be to prove Theorem 1 above. The first part of the statement of Theorem 1 – that for $d \leqslant 7$ every component of the restricted Chow variety $\tilde{\mathscr{C}}_{d,3}$ is general – is relatively easy. We recall first Clifford's theorem (that if $C \subset \mathbb{P}^r$ is a nondegenerate curve of degree $d < 2r$ then $\mathcal{O}_C(1)$ is nonspecial) and the fact that the normal bundle N of such a curve is a quotient of a direct sum of copies of $\mathcal{O}_C(1)$ (so that if $\mathcal{O}_C(1)$ is nonspecial then N is). It follows in case $r = 3$ that except for the case $d = 7$, $g \geqslant 5$ the normal bundle N of the curve $C \subset \mathbb{P}^3$ corresponding to a general point in any component of $\tilde{\mathscr{C}}_{d,3}$ is nonspecial, and the remaining cases $d = 7$ and $g = 5$ or 6 can be checked directly (these are curves on quadrics). We are thus in the following situation:

Let Σ be any irreducible component of the restricted Chow variety $\tilde{\mathscr{C}}_{d,3}$ with $d \geqslant 8$, and let $C_0 \subset \mathbb{P}^3$ be the curve corresponding to a general point of Σ; let g be the geometric genus of C_0. Our object is to show that *if the dimension of Σ is at least $\delta(d, 3)$, then C_0 is a Chow curve.* Note that since we have already checked that the component of $\tilde{\mathscr{C}}_{d,3}$ parametrizing Chow curves has maximal dimension among components whose general member lies on a quadric, it is sufficient to show that C_0 lies on a quadric.

To do this, let $\pi: C \to C_0 \subset \mathbb{P}^3$ be the normalization of C_0, let g be the genus of C, and let $N = N_{C/\mathbb{P}^3}$ the normal sheaf of the map $C \to \mathbb{P}^3$ – that is, the quotient of the pullback $\pi^* T_{\mathbb{P}^3}$ by the image of T_C under the differential map $d\pi: T_C \to \pi^* T_{\mathbb{P}^3}$. Now, since $[C]$ is a general point of Σ, every deformation of C_0 is equisingular; this means in particular that every deformation of C_0 comes from a deformation of the map $\pi: C \to \mathbb{P}^3$, and hence that *the dimension of the Chow variety at $[C_0]$ is at most the dimension of the space of global sections of N*:

$$\dim(\Sigma) \leqslant h^0(C\ N).$$

We may thus assume that $h^0(N) \geqslant \delta(d, 3)$.

The argument is now very straightforward, subject to two simplifying hypotheses: that C_0 *is immersed* (so that the normal sheaf N is a vector bundle) and that *the linear series $|\mathcal{O}_{\mathbb{P}^3}(1)||_C$* is complete – that is, the map $C \to \mathbb{P}^3$ does not factor through a nondegenerate map $C \to \mathbb{P}^r$ for $r > 3$. We will give the argument first in case these two hypothesis are satisfied, and then consider the cases in which they are not.

In this case, we claim that for a general point $p \in C$ there exists an exact sequence for the normal bundle N:

$$0 \to L \to N \to M \to 0, \tag{1}$$

where $L \cong \mathcal{O}_C(1)(p)$ and correspondingly $M \cong K_C(3)(-p)$. To exhibit such a sequence is simple: just choose $H \subset \mathbb{P}^3$ a general plane, choose coordinates Z on $\mathbb{P}^3$ with H given by $Z_0 = 0$ and the image $\bar{p} = \pi(p) = [1, 0, 0, 0]$, and consider the vector field

$$v = Z_0 \frac{\partial}{\partial Z_0}.$$

This is the vector field on $\mathbb{P}^3$ associated to the one-parameter subgroup

$$[Z_0, \ldots, Z_3] \mapsto [tZ_0, Z_1, Z_2, Z_3]$$

flowing from H to $\bar{p}$. As a vector field on $\mathbb{P}^3$, it is zero exactly along H and at $\bar{p}$; its restriction to C thus gives rise to a section of the normal bundle N vanishing on the divisor $H \cdot C + p$, plus at any points $q \in C$ whose tangent line passes through $\bar{p}$. If p is chosen generically, there will be no such points ([K]), so that the subbundle $L \subset N$ spanned by this section will be isomorphic to $\mathcal{O}_C(1)(p)$, as desired.

We can use the exact sequence (1) to estimate $h^0(N)$. To begin with, note that $M \cong K_C(3)(-p)$ is nonspecial. We may thus assume the line bundle $O_C(1)$ is special (otherwise we would have $h^1(N) = 0$ and $\dim(\Sigma) = 4d$). Since p is general it follows that

$$h^0(\mathcal{O}_C(1)(p)) = h^0(\mathcal{O}_C(1)) = 4.$$

Next, we have

$$\begin{aligned} h^0(N) &\leqslant h^0(\mathcal{O}_C(1)(p)) + h^0(K_C(3)(-p)) \\ &\leqslant 4 + (3d + 2g - 3) - g + 1 \\ &= g + 3d + 2. \end{aligned}$$

Using our hypothesis that this is greater than or equal to the Chow number

$$\delta(d, 3) = \begin{cases} \dfrac{d^2}{4} + d + 9, & \text{if } d \text{ is even; and} \\ \dfrac{d^2}{4} + d + 8\tfrac{3}{4}, & \text{if } d \text{ is odd.} \end{cases}$$

we arrive at the inequality

$$\frac{d^2}{4} + d + 8\tfrac{3}{4} \leqslant g + 3d + 2$$

so that

$$g \geqslant \frac{d^2}{4} - 2d + 6\tfrac{3}{4} \tag{2}$$

At this point we may use a classical result of Halphen [Hal] (see also [GP] and [Har] for modern treatments), which says that if a curve $C_0 \subset \mathbb{P}^3$ does not lie on a quadric, then its geometric genus g satisfies

$$g \leqslant \frac{d^2}{6} - \frac{d}{2} + 1 \tag{3}$$

with equality holding (in the sense that g is equal to the integer part of the right hand side) only if C_0 is a complete intersection with a cubic surface, or residual to a line or a conic in a complete intersection with a cubic surface. Comparing the two inequalities (2) and (3), we see that we have a contradiction whenever $d \geqslant 13$. In the remaining cases $8 \leqslant d \leqslant 12$, we can list the allowable values of g:

degree d	$\dfrac{d^2}{4} - 2d + 6\frac{3}{4}$	$\dfrac{d^2}{6} - \dfrac{d}{2} + 1$	allowable g
8	$6\frac{3}{4}$	$7\frac{2}{3}$	7 only
9	9	10	9 or 10
10	$11\frac{3}{4}$	$12\frac{2}{3}$	12 only
11	15	$15\frac{2}{3}$	15 only
12	$18\frac{3}{4}$	19	19 only
13	23	$22\frac{2}{3}$	none

At this point, we can dismiss all but one of the possibilities by using the strong form of Halphen's result to deduce that, except in the case $d = g = 9$, C_0 must be residual to a plane curve in a complete intersection with a cubic surface S. The dimension of the family of such curves may be readily computed and seen to be strictly less than $\delta(d, 3)$ in each of the above cases; alternately, since the cubic surface S must be smooth for a general such C_0, the exact sequence

$$0 \to N_{C_0/S} \to N_{C_0/\mathbb{P}^3} \to N_{S/\mathbb{P}^3} \to 0$$

associated to the inclusion of C_0 in S reads

$$0 \to K_{C_0}(1) \to N \to \mathcal{O}_{C_0}(3) \to 0.$$

We thus have

$$h^1(N) = h^1(\mathcal{O}_{C_0}(3)) \leqslant 1$$

since $3d \geqslant 2g - 2$ in all the above cases. The remaining case $d = g = 9$ can be handled similarly: such a curve C_0 must, by the Riemann-Roch formula applied to the line bundles $\mathcal{O}_C(3)$ and $\mathcal{O}_C(4)$, lie on a cubic surface S and on a quartic surface T not containing S; the residual intersection of S and T will be a twisted cubic curve. Again, the dimension of the family can be worked out directly, or we can use the fact that S is smooth for a general such C_0 to obtain an exact sequence as above and conclude that in fact Σ has dimension $4d$.

We have thus established our result in case C_0 is immersed and $h^0(C, \mathcal{O}(1)) = 4$. We may also deal with the case of C_0 immersed and $h^0(C, \mathcal{O}(1)) > 4$ in pretty much the same fashion. In this case, C_0 is the regular projection of a nondegenerate, linearly normal curve $\tilde{C}_0 \subset \mathbb{P}^r$, $r \geqslant 4$ (here "regular" means with center of projection disjoint from the curve $\tilde{C}_0$). This means that the first inequality (2) above is weakened by $r - 3$: we have $h^0(\mathcal{O}_C(1)(p)) = g - d + r - 1$ rather than $g - d + 2$, so that instead of (2) we have

$$g \geqslant \frac{d^2}{4} - 2d + 9\tfrac{3}{4} - r$$

On the other hand, since C_0 is birational to an irreducible, nondegenerate curve of degree d in $\mathbb{P}^r$, we may apply Castelnuovo's bound on the genus to conclude that

$$g \leqslant \pi(d, r) = \frac{d^2 - (r + 1)d - (\varepsilon - 2)(r + \varepsilon - 1)}{2(r - 1)}$$

where $\varepsilon \equiv d \,(\mathrm{mod}\, r - 1)$ and $-r + 2 \leqslant \varepsilon \leqslant 1$. This formula is equivalent to the one given on page 87 of [EH] though the ε here is not equal to the ε there; it is derived by writing the class of a curve C of degree d on a 2-dimensional scroll as $C \sim ((d - \varepsilon)/(r - 1))H + \varepsilon F$, where F is the ruling, and maximizing the formula for the genus of this class, given d. For example, if $r = 4$, the two inequalities imply

$$g \geqslant \frac{d^2}{4} - 2d + 5\tfrac{3}{4}$$

and

$$g \leqslant \frac{d^2 - 5d + 4}{8}$$

which is a contradiction for any d. As r increases by 1, moreover, the first of these two inequalities is weakened by 1; since $\pi(d, r + 1) \leqslant \pi(d, r) - 1$ for all r and $d \geqslant r$, the contradiction holds for all $r \geqslant 4$ as well.

Finally, if $C \to \mathbb{P}^3$ is not an immersion the same argument applies once we make one additional observation. In this case the normal sheaf N of the map π has a torsion subsheaf N_{tors} supported at the points of C lying over the cusps of C_0. We still have, as before, an exact sequence

$$0 \to \mathcal{O}_C(1)(p) \to N \to M \to 0,$$

But now we cannot simply use the fact that $c_1(M) > 2g - 2$ to deduce that $h^0(M) \leqslant c_1(M) - g + 1$: M will have torsion, and the quotient M/M_{tors} may be special, so that

$$\begin{aligned} h^0(M) &= h^0(M_{\text{tors}}) + h^0(M/M_{\text{tors}}) \\ &> c_1(M) - g + 1. \end{aligned}$$

The key observation that saves us here is one made by Arbarello and Cornalba [AC]: that the *torsion sections of the normal bundle N do not give rise to equisingular first-order deformations of the map π*. Since C_0 is presumed to be general in a component Σ of the Hilbert scheme, all first-order deformations of it are equisingular. It follows then that the space of sections of N coming from the tangent space to the reduced Hilbert scheme does not intersect the subspace $H^0(N_{\text{tors}})$, so that

$$\begin{aligned} \dim(\Sigma) &\leqslant h^0(N/N_{\text{tors}}) \\ &\leqslant h^0(\mathcal{O}_C(1)(p)) + h^0(M/M_{\text{tors}}). \end{aligned}$$

But since $\deg(M/M_{\text{tors}}) < 3d + 2g - 3$, we have $h^0(M/M_{\text{tors}}) < 3d + g - 2$, and the argument now proceeds exactly as before.

3. Irreducible, nondegenerate curves in $\mathbb{P}^r$, $r \geqslant 4$

In this section we will give a proof of Theorem 2. In fact, the proof follows the same lines as that of the preceding section; the main differences are that the formulas are more complicated and less effective. In particular, the range of degrees not covered by the general argument expands: for example, while in $\mathbb{P}^3$ the only degrees that had to be checked on an ad hoc basis were the special cases $8 \leqslant d \leqslant 12$, in case $r = 4$ the general argument fails to cover the cases $10 \leqslant d \leqslant 32$. We could perhaps check these individually, but haven't done so.

As in the previous case, we first dispense with the first half of the statement by observing that in case $d \leqslant 2r$ the normal bundle to an irreducible, nondegenerate curve $C \subset \mathbb{P}^r$ is nonspecial, so that every component of the Chow variety of such curves has dimension equal to the Hilbert number. To prove the second half, let Σ be any component of the Hilbert scheme of curves of degree d in $\mathbb{P}^r$ of dimension greater than or equal to the Chow number $\delta(d, r)$; let $C_0 \in \Sigma$ be a general point of this component and let $\pi: C \to C_0 \subset \mathbb{P}^r$ be the normalization map. We want to show that C_0 is a Chow curve; since we have already checked that among curves on a rational normal scroll the Chow curves move in the largest dimensional family, it will suffice to prove that C_0 lies on such a scroll.

As in the previous argument, we will do this first under the hypothesis that C_0 is immersed and that $h^0(\mathcal{O}_C(1)) = r + 1$. We proceed by exhibiting an exact sequence for the normal bundle of the map $C \to \mathbb{P}^r$ coming from simple vector fields in $\mathbb{P}^r$. In this case, we choose general points $p_1, \ldots, p_{r-2} \in C$, let $\bar{p}_1, \ldots, \bar{p}_{r-2} \in C_0$ be their images, and choose general hyperplanes $H_1, \ldots, H_{r-2} \subset \mathbb{P}^r$ as well. Let v_i be the vector field on $\mathbb{P}^r$ flowing from $\bar{p}_i$ to H_i; as before, v_i gives us a sub-line bundle of the normal bundle N isomorphic to $\mathcal{O}_C(p_i)$. Moreover, those line subbundles will fail to be linearly independent only at points $q \in C$ whose tangent lines meet the codimension 3 subspace of $\mathbb{P}^r$ spanned by $\bar{p}_1, \ldots, \bar{p}_{r-2}$; and if the p_i are chosen generically the results of [K] say as before no such points q will exist. We thus have a sequence

$$0 \to \bigoplus_{i=1}^{r-2} \mathcal{O}_C(1)(p_i) \to N \to M \to 0,$$

where $M \cong K_C(3)(-\Sigma p_i)$.

We can use this exact sequence to estimate $h^0(N)$. To begin with, we may assume the line bundle $\mathcal{O}_C(1)$ is special (otherwise we would have $h^1(N) = 0$ and $\dim(\Sigma) = h(d, g, r) \leqslant h(d, g, 0)$, which in the range of degrees considered is strictly less than δ). Since the p_i are general we have

$$h^0(\oplus \mathcal{O}_C(1)(p_i)) = (r - 2) \cdot h^0(\mathcal{O}_C(1)) = (r - 2)(r + 1).$$

Next, $M \cong K_C(3)(-\Sigma p_i)$ is nonspecial, so we have

$$\begin{aligned} h^0(N) &\leqslant h^0(\oplus \mathcal{O}_C(1)(p_i)) + h^0(K_C(3)(-\Sigma p_i)) \\ &\leqslant (r-2)(r+1) + (3d + 2g - r) - g + 1 \\ &= g + 3d + r^2 - 2r - 1. \end{aligned}$$

Since $h^0(N) \geqslant \dim(\Sigma)$, which we may assume is greater than the Chow number

$$\delta(d, r) = \frac{d^2 + (r+1)d + (\varepsilon+2)(r-\varepsilon-1)}{2(r-1)} + r^2 + 2r - 7$$

we have

$$\begin{aligned} g &\geqslant \frac{d^2 + (r+1)d + (\varepsilon+2)(r-\varepsilon-1)}{2(r-1)} - 3d + 4r - 6 \qquad (4) \\ &\geqslant \frac{d^2 + (r+1)d + r}{2(r-1)} - 3d + 4r - 6. \end{aligned}$$

On the other hand, Theorem (3.15) of [EH], analogous to the result of Halphen quoted above in relation to curves in $\mathbb{P}^3$, says that an irreducible, nondegenerate curve of genus

$$g > \pi_1(d, r) = \frac{d^2 + (r+1)d + r}{2r}$$

in $\mathbb{P}^r$ must lie on a rational normal scroll. Comparing these two expressions for the genus, we have succeeded in showing that C_0 lies on a scroll whenever

$$\frac{d^2 + (r+1)d + r}{2(r-1)} - 3d + 4r - 6 > \frac{d^2 + (r+1)d + r}{2r}.$$

This will hold if

$$d^2 - 2r(2r-3)\cdot d + (8r^3 - 21r^2 + 14r) \geqslant 0.$$

Regarding the left hand side of this inequality as a quadratic polynomial in d, the larger root γ of the two roots is

$$\gamma = r(2r-3) + \sqrt{r^2(2r-3)^2 - (8r^3 - 21r^2 + 14r)}$$

and applying the inequality

$$\sqrt{a^2 - t} \leqslant a - \frac{t}{2a}$$

we see that

$$\begin{aligned}\gamma &\leqslant 2r(2r-3) - \frac{8r^2 - 21r + 14}{2(2r-3)}\\ &< 2r(2r-3) - \frac{4r-5}{2}\\ &< (2r-1)(2r-3).\end{aligned}$$

We have thus proved that C_0 lies on a scroll whenever $d \geqslant (2r-1)(2r-3)$, and we deduce the statement of Theorem 2 (still subject to the hypothesis that the map $C \to C_0 \subset \mathbb{P}^r$ is linearly normal and an immersion).

The remaining cases can be handled just as in the case $r = 3$. The possibility that the map $C \to \mathbb{P}^r$ is not an immersion is dealt with exactly as before. The case in which the map $C \to \mathbb{P}^r$ is not linearly normal, on the other hand, requires one further calculation. To carry this out, suppose that in fact

$$h^0(C, \mathcal{O}_C(1)) = s + 1.$$

The basic inequality (4) on the genus is then weakened by $(r-2)$ times the difference $s - r$, so that we have

$$g \geqslant \frac{d^2 + (r+1)d + r}{2(r-1)} - 3d + 4r - 6 - (r-2)(s-r).$$

On the other hand, we can apply to C Castelnuovo's inequality for the genus of an irreducible, nondegenerate curve of degree d in $\mathbb{P}^s$, which is to say

$$\begin{aligned}g \leqslant \pi(d, s) &= \frac{d^2 - (s+1)d - (\varepsilon - 2)(s + \varepsilon - 1)}{2(s-1)}\\ &\leqslant \frac{d^2 - (s+1)d + (s+1)^2/4}{2(s-1)}\end{aligned}$$

Comparing the two inequalities, multiplying through by $2(r-1)(s-1)$ and collecting terms of like degree in s, we have

$$a\cdot s^2 - b\cdot s + c \geqslant 0$$

where

$$a = 2(r-1)(r-2) + (r-1)/4,$$

$$b = d^2 + (r+1)d + r + 2(r-1)(-3d + 4r - 6 + r(r-2) + r - 2) \\ -(r-1)d + (r-1)/2$$

Now, this polynomial in s does assume negative values. Just as in the case $r = 3$, we can check that it is negative when $s = r + 1$ by comparing it to the inequalities used under the assumption that $s = r$. We may conclude that s must be larger than the average of the roots, i.e.,

$$s \geqslant b/2a.$$

To estimate this, note that

$$a \leqslant 2(r-1)^2$$

and

$$b \geqslant d^2 - 5(r-1)d$$

so that

$$\begin{aligned} s &\geqslant \frac{d^2 - 5(r-1)d}{4(r-1)^2} \\ &\geqslant d\cdot\left(\frac{d}{4(r-1)^2} - \frac{5}{4(r-1)}\right) \\ &\geqslant d - 2 \end{aligned}$$

since d is assumed greater than or equal to $4r^2 - 8r + 3$. But then C must be of genus $g \leqslant 2$; and in particular the dimension of the component Σ of the Chow variety will be the Hilbert number $h(d, g, r) \leqslant h(d, 0, r) < \delta(d, r)$.

4. The unrestricted Chow variety

We consider now the unrestricted Chow variety; we will give a proof of Theorem 3. We start by considering components of the Chow variety whose general member is irreducible.

LEMMA. *Let Σ be a component of the Chow variety $\mathscr{C}_{d,r}$, $(d, r) \neq (3, 3)$, whose general member is irreducible and not planar. Then either*

(i) The dimension of Σ *is strictly less than the dimension* $2d(r-1)$ *of the component of* $\mathscr{C}_{d,r}$ *parametrizing d-tuples of lines; or*

(ii) *The dimension of* Σ *is strictly less than the dimension* $3(r-2) + d(d+3)/2$ *of the component of* $\mathscr{C}_{d,r}$ *parametrizing plane curves of degree d.*

Proof. In case $r = 3$ this follows from Theorem 1, so we may assume $r \geqslant 4$. To begin with, if the general member C of Σ spans a $\mathbb{P}^k$ with $k \geqslant 3$, then the arguments of section 1 above show that the dimension of the corresponding component of the Chow variety $\mathscr{C}_{d,k}$ is at most $d^2/4 + d + 9$. Since the Grassmannian of k-planes in $\mathbb{P}^r$ has dimension at most $(r+1)^2/4$, we have the crude estimate

$$\dim \Sigma \leqslant \frac{d^2 + 4d + 36 + (r+1)^2}{4}. \tag{6}$$

If conclusion (i) of the statement of the Lemma is not satisfied, this must be greater than $2d(r-1)$; thus

$$d^2 + 4d + 36 + (r+1)^2 \geqslant 8d(r-1)$$

or in other words,

$$d^2 - (8r - 12)\cdot d + ((r+1)^2 + 36) \geqslant 0.$$

The roots of this polynomial in d are

$$4r - 6 \pm \sqrt{15r^2 - 50r + 1};$$

we will consider separately the case in which d is less than the smaller of the two roots, and the case in which d exceeds the larger root. The latter case is easier: we have

$$\begin{aligned} d &\geqslant 4r - 6 + \sqrt{15r^2 - 50r + 1} \\ &\geqslant 5r - 5. \end{aligned} \tag{7}$$

For this case, compare the maximal dimension of Σ as described in (6) above with the dimension of the family of plane curves of degree d. If the former is larger, we have

$$\frac{d^2 + 4d + 36 + (r+1)^2}{4} \geqslant 3r - 6 + \frac{d(d+3)}{2}.$$

This in turn says that

$$d^2 + 2d - (r^2 - 10r + 61) \leqslant 0$$

so that

$$\begin{aligned} d &\leqslant -1 + \sqrt{r^2 - 10r + 62} \\ &\leqslant r + 2. \end{aligned} \tag{8}$$

Since the ranges of d allowed by (7) and (8) have no overlap, the lemma follows.

In case the degree d is less than the smaller root of the polynomial in d above, we have

$$\begin{aligned} d &\leqslant 4r - 6 - \sqrt{15r^2 - 50r + 1} \\ &< r \end{aligned} \tag{9}$$

since $15r^2 - 50r + 1 > (3r - 6)^2$ for $r \geqslant 4$. We now split the argument into two cases, depending on whether the dimension k of the span of the image of C in $\mathbb{P}^r$ is greater than or less than $d/2$. The point is that the basic estimate (1) uses the bound on the dimension of the family of curves in $\mathbb{P}^3$, rather than $\mathbb{P}^k$, while at the same time using an estimate on the dimension of the Grassmannian $\mathbb{G}(k, r)$ that is maximized for $k \sim r/2$. Once we decide whether k is indeed small or large, we can do better in one part or the other of the inequality.

In case $d < 2k$, we know that the line bundle $\mathcal{O}_C(1)$ is nonspecial, so that we can replace the crude estimate on the dimension of the family of such curves in $\mathbb{P}^k$ by the Hilbert number $h(d, g, k) \leqslant h(d, 0, k)$. We have then

$$\begin{aligned} \dim(\Sigma) &\leqslant \dim \mathbb{G}(k, r) + h(d, 0, k) \\ &\leqslant (k + 1)(r - k) + (k + 1)(d + 1) - 4 \\ &< (k + 1)(d + r - k + 1). \end{aligned}$$

Given that $d < r$ and $k \leqslant d$ (since C spans a $\mathbb{P}^k$), the maximum value of this function of k is achieved when $k = d$; we thus have

$$2d(r - 1) \leqslant \dim(\Sigma) < (d + 1)(r + 1).$$

This implies that

$$d < \frac{r + 1}{r - 3}$$

and all cases allowed by this the statement can be checked directly.

This leaves the possibility $d \geqslant 2k$. In this case, as we said, we have no better control over the dimension of the Chow variety of irreducible, nondegenerate curves in $\mathbb{P}^k$ than the estimate for $\mathbb{P}^3$, but we can bound the dimension of the Grassmannian: since $k \leqslant d/2$,

$$\dim(\mathbb{G}(k, r)) \leqslant \left(\frac{d}{2} + 1\right)\left(r - \frac{d}{2}\right)$$

we have

$$\begin{aligned}\dim(\Sigma) &\leqslant \frac{d^2 + 4d + 36}{4} + \left(\frac{d}{2} + 1\right)\left(r - \frac{d}{2}\right) \\ &= \frac{dr + d + 2r + 18}{2}\end{aligned}$$

Combining this with $\dim(\Sigma) \geqslant 2d(r - 1)$, we have

$$3dr - 5d - 2r - 18 \leqslant 0;$$

but since $d \leqslant r$ and $r \geqslant 4$,

$$\begin{aligned}3dr - 5d - 2r - 18 &\geqslant 3dr - 5d - 2r - 18 + 5(d - r) - 5(r - 4) \\ &= (3d - 12)r + 2\end{aligned}$$

so that we must have $d \leqslant 3$; again, these cases are easy enough to check individually. This completes the proof of the lemma.

From the lemma, we see that if Σ is a component of the Chow variety $\mathscr{C}_{d,r}$ of maximal dimension and $C \in \Sigma$ a general point, then every irreducible component of C must be either a line or a plane curve. Moreover, every plane curve component C_i of C must have degree at least $4r - 7$, since otherwise we could replace it by d lines and arrive at a family of curves of strictly greater dimension. On the other hand, if C contains two plane curve components of degrees e, $f \geqslant 4r - 7$, we can replace their union by a plane curve of degree $e + f$ to obtain a family of strictly larger dimension. Finally, if C contains both a plane curve of degree $e \geqslant 4r - 7$ and a line, we can amalgamate them into a plane curve of degree $e + 1$ to obtain once more a larger component of $\mathscr{C}_{d,r}$. Thus either C is an irreducible plane curve, or a union of lines.

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