August 14, 1991

This version of an article
in the Journal of Algebraic Geometry (Vol. 1, 1991)
contains a number of corrections and revisions that
unfortunately did not make it into the published version

An intersection bound for rank 1 loci, with applications to Castelnuovo and Clifford Theory.

by

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Summary

Let C be a smooth algebraic curve. The main technical result of this paper is a bound on the degree of the rank 1 locus of a map of vector bundles on C, under mild nondegeneracy assumptions. In the simplest special case we are bounding the degree of the greatest common divisor of the 2x2 minors of a matrix of forms over k[s,t], under the assumtion that the matrix is 1-generic, or a little less.

We give applications to bounding the number of intersections of curves with various other varieties, to the study of determinantal loci, to "Castelnuovo Theory" (finding a curve through certain collections of points, and the consequences for the possible degree and genus of curves in space), to normal bundles of curves, and to "Clifford Theory" (studying products of linear series.)

Contents

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* The authors are grateful for the partial support of the NSF during the preparation of this work.

Introduction

Subvarieties of a projective space defined by the vanishing of minors of a matrix, or more generally subvarieties defined as the degeneracy loci of maps

$$\varphi: \mathcal{V} \rightarrow \mathcal{W}$$

between vector bundles on (say) a smooth variety P are ubiquitous in algebraic geometry. In the case when such a rank k locus Y has the "expected" codimension

$$(\text{rank } \mathcal{V} - k)(\text{rank } \mathcal{W} - k)$$

there are familiar methods, such as Porteous' formula that give information about Y in terms of $\mathcal{V}$ and $\mathcal{W}$. Similar results are known in many related cases, for example where the map satisfies some symmetry hypotheses. (See for example Fulton [1984], and for the local theory the recent exposition of Bruns and Vetter [1988].)

There are many natural problems in which degeneracy loci with "unexpected" codimension appear. Thus it seems an important to identify natural conditions on a map φ under which some information about a degeneracy locus can be deduced even when the codimension is not the expected one. Such a case, where P is a projective space and the vector bundles are of the form

$$\mathcal{V} = \mathcal{O}^v$$
$$\mathcal{W} = \mathcal{O}(1)^w$$

(that is, φ "is" a matrix of linear forms) and φ is **1-generic** in the sense of Eisenbud [1988], is the object of study of the papers of Eisenbud [1988], Eisenbud, Koh and Stillman [1988], Koh and Stillman [1989]. Here to say that φ is 1-generic means that no entry of a matrix representing φ can be 0, no matter what bases for the source and target of φ are chosen. This case was picked out as particularly well behaved (in terms of canonical resolution of singularities of the first degeneracy locus) by George Kempf in the unpublished part of his thesis, and also seems to be one of the main objects of study of the somewhat mysterious book of Room [****].

In this paper we focus on the rank 1 loci of 1-generic matrices (or matrices which satisfy a slightly weaker nondegeneracy condition defined in section 1) and how they intersect nondegenerate curves in the projective space. Our main technical results, presented in section 1 below, are of the form:

> If X is a variety defined (scheme-theoretically) in $\mathbb{P}^n$ by the vanishing of the 2×2 minors of a suitably "nondegenerate" matrix f of linear forms, then the number of intersections of X with a nondegenerate smooth curve C of degree d is bounded by some number A depending on d, the complexity of C, and the size of the matrix f.

Here A should be 2d if f is a 2×2 matrix (Bezout's theorem), and is supposed to decrease with the size and "nondegenerateness" of f and the complexity of the curve C, in senses to be made precise in section 1. The curve C is supposed to be nondegenerate in the sense that it lies in no hyperplane.

Pulling back to the curve C we see that what we need are bounds for the degree of the rank 1 locus associated to a suitable map

$$\varphi: \mathcal{E} \to \mathcal{F}$$

of vector bundles on C. We give two results of this sort, neither of which implies the other. Our original result, which is Theorem 1 . 1 below, requires rather special bundles $\mathcal{E}$ and $\mathcal{F}$, such as the ones coming from the original problem, and gives a result with the number A above in the form $2d-\varepsilon$, as one might hope. Rob Lazarsfeld pointed out to us a method of proof that works in a much more general situation, and even deals with the rank k loci for $k > 1$, but gives a considerably more complex formula for A. This version is presented in a form worked out jointly with Lazarsfeld and Lawrence Ein as Theorem 1 . 2.

In the simplest case, where C is $\mathbb{P}^1$, our main result has a very simple algebraic statement, which we now give: Let f be a $w\times v$ matrix of forms of degree d in 2 variables satisfying the condition that, even after an arbitrary scalar row and column operation, no entry of f is 0. Assuming that the generic rank of f is ≥ 2, our main result says that the degree e of the greatest common divisor of the 2×2 minors of f satisfies

$$e \leq 2d - (v-2) - (w-2).$$

This bound is sharp (examples are given in section 1.)

As a first application of these results we give, in section 2, a generalization to schemes of a lemma of Castelnuovo. Our result says that there is a rational normal curve passing through any finite subscheme Γ of degree $\geq 2n+3$ in $\mathbb{P}^n$ which is in linearly general position and imposes at most $2n+1$ conditions on quadrics. The proof given is new (and conceptually simpler than the classical proof) even in the classical case, when the finite subscheme consists of reduced points. It relies on the lemma that there is a unique rational normal curve through any finite subscheme of degree $m+3$ in linearly general position in $\mathbb{P}^m$ (also due to Castelnuovo in the classical case) which we proved in our [1991].

This result, along with other results from our [1991] allow us to imitate the theory of Castelnuovo bounding the genus of a curve in terms of its degree in the case of certain nonreduced curves. Applying the bound in the case of **ribbons** (double structures on smooth curves -- see Bayer and Eisenbud [199?] for information) we deduce new bounds on the degree of the normal bundle of a space curve in section 3..

The significance of studying the rank 1 locus Y of a 1-generic matrix of linear forms on a projective space $\mathbb{P}^n$ comes partly from the fact that, under mild hypotheses, the matrix itself corresponds to a pair of linear series on Y whose "product" is the hyperplane series. This elementary correspondence is explained, for example, in Eisenbud [1988]. To understand it, consider a very ample line bundle $\mathcal{L}$ on a (reduced irreducible) variety Y which happens to have a factorization

$$\mathcal{L} = \mathcal{L}_1 \otimes \mathcal{L}_2$$

in terms of line bundles $\mathcal{L}_i$, and set

$$V = H^0 \mathcal{L}_1$$
$$W = H^0 \mathcal{L}_2$$
$$H = H^0 \mathcal{L}.$$

The "multiplication" map $\mu: V \otimes W \to H$ has adjoint $V \to W \otimes H$, which is of course the same thing as a $v \times w$ map of linear forms over a polynomial ring k[H] or, again equivalently, a map

$$\varphi: \mathcal{V} = V \otimes \mathcal{O}_{\mathbb{P}(H)} \quad \to \quad W \otimes \mathcal{O}_{\mathbb{P}(H)}(1) = \mathcal{W}$$

of vector bundles on the projective space $\mathbb{P}(H)$ in which Y is embedded by $\mathcal{L}$. It is elementary to check that Y, embedded in $\mathbb{P}(H)$ by means of $\mathcal{L}$, lies in the rank 1 locus of φ. Indeed, if $\mathcal{L}_1$ and $\mathcal{L}_2$ are sufficiently ample, it seems that this rank 1 locus is actually equal to Y (precise results of this type for curves are given in Eisenbud-Koh-Stillman [1988].)

Because Y is reduced and irreducible, the pairing μ has the property that if $\sigma \in V$ and $\tau \in W$ are nonzero elements, then their product $\sigma\tau \in H$ is nonzero. We call such a pairing **1-generic** because the property is equivalent to the 1-genericity of the corresponding matrix of linear forms, in the sense defined above.

Given this construction it is not surprising that one group of applications of our results is to the products of linear series. Recall that a linear series L on a scheme X is a pair

$$L = (\mathcal{L}, V),$$

with $\mathcal{L}$ a line bundle on X and V a subspace of $H^0\mathcal{L}$. The dimension of L is by definition one less than the vectorspace dimension of V.

If $L_1 = (\mathcal{L}_1, V_1)$and $L_2 = (\mathcal{L}_2, V_2)$ are linear series, we define the product L_1L_2 to be the series

$$L_1L_2 := (\mathcal{L}_1 \otimes \mathcal{L}_2, \text{ image } (V_1 \otimes V_2 \to H^0(\mathcal{L}_1 \otimes \mathcal{L}_2))).$$

If the dimensions of the L_i are both at least 1, we define the Clifford index of the pair L_1,L_2 to be

$$\mathrm{Cliff}(L_1,L_2) = \dim L_1L_2 - \dim L_1 - \dim L_2.$$

A simple general argument about 1-generic pairings shows that $\mathrm{Cliff}(L_1,L_2) \geq 0$ for any pair of linear series. In Eisenbud [1988] it is shown that when equality is achieved, the linear series must both be pulled back, in a suitable sense, from complete linear series on $\mathbb{P}^1$ -- this generalizes the classical theorem of Clifford. Using the techniques developed here, we are able to classify, in section 4, the pairs of series of Clifford index 1. For example, on a curve, such a pair of series must come from either complete series on a curve of arithmetic genus 1 or from certain pairs of incomplete series on $\mathbb{P}^1$.

Finally, in section 5, we use the previous theory give information about the rank 1 locus Y of a 1-generic $(a+1)\times(b+1)$ matrix of linear forms on $\mathbb{P}^n$. The case where the codimension is ab, the largest possible, is the case of "expected" codimension. In the case when the codimension is a+b-1, the smallest possible value, the matrices and varieties that occur were completely classified by Eisenbud [1988]: only rational normal curves, rational normal scrolls, and the Veronese surface in $\mathbb{P}^5$ can appear as Y. Here we partially deal with the case where the codimension is the next larger value, a+b. We show for example that if Y is a curve then it is the rational normal curve or a curve of arithmetic genus 1 embedded by a complete series. We close with a number of open problems about such loci.

We wish to thank Rob Lazarsfeld and Lawrence Ein, some of whose ideas appear in section 1.

1. Bounds for degeneracy loci on curves

We first discuss the measure of the complexity of a curve C that will appear in our main results. We will always assume that C is a smooth connected projective curve of genus g over an algebraically closed field. The complexity is described by a numerical function $a_C(r)$ related to the Clifford index, and defined for integers $r \geq 0$ as follows:

$$a_C(r) = \min \{ d \mid \text{there is a line bundle } \mathcal{L} \text{ of degree } d \text{ on } C \text{ with } h^0 \mathcal{L} \geq r+1 \}.$$

For example, if $C = \mathbb{P}^1$ then $a_C(r) = r$, while if C is elliptic then $a_C(0) = 0$ while $a_C(r) = r+1$ for all $r > 0$. More generally, $a_C(1)$ is what is usually called the "gonality" of C. It would be quite interesting to know what functions a_C are possible for smooth curves C. We briefly summarize what we know, some of which will be used below:

Brill-Noether theory (see for example Harris [1982] gives us an upper bound, while the Riemann-Roch Theorem and Clifford's Theorem (Hartshorne [1977 p.343]) give us a lower bound for a_C: If C is a curve of genus g, we get

$$\min (2r\ ,\ r + g) \leq a_C(r) \leq r + g - \lfloor g/(r+1) \rfloor.$$

Here the upper equality holds for all r if C is a general curve, while the lower equality holds for some $r \leq g-2$ iff C is a two-sheeted cover of $\mathbb{P}^1$ iff it holds for every r. The lower bound is equal to the upper bound for $r=0$ and for $r \geq g-1$; in particular, we see (what is also clear for more elementary reasons)

$$\begin{aligned} a_C(0) &= 0 \\ a_C(g-1) &= 2g-2 \\ a_C(r) &= r + g \text{ as soon as } r \geq g. \end{aligned}$$

From the fact that vanishing at s general points imposes s linear conditions on sections of a line bundle with r+s sections we derive

(1) $a_C(r+s) \geq a_C(r) + s.$

In particular (1) shows that a_C is strictly increasing.

On the other hand, because any line bundles $\mathcal{L}_i$ with $h^0\mathcal{L}_i \geq 1$ satisfy

$$h^0(\mathcal{L}_1 \otimes \mathcal{L}_2) \geq h^0\mathcal{L}_1 + h^0\mathcal{L}_2 - 1$$

(see Eisenbud [1988] for a study of this old idea) we get

(2) $\quad a_C(r+s) \leq a_C(r) + a_C(s).$

The analogue in higher rank of a line bundle with at least one section is a vector bundle generically generated by its sections, and we can easily give a bound on the degree of such a vector bundle in terms of its global sections: If $\mathcal{F}$ is a bundle on C of rank k, and $\mathcal{F}$ is generically generated by global sections, then we claim that

(3) $\quad \deg \mathcal{F} \geq a_C(h^0(\mathcal{F}) - k)$

(the case rank $\mathcal{F} = 1$ is equivalent to the definition of a_C!) Indeed, since the sub-bundle generated by the global sections of $\mathcal{F}$ has smaller degree and the same rank, we may assume that $\mathcal{F}$ is generated by global sections. Choosing (rank $\mathcal{F}$) - 1 of them generically, we obtain an exact sequence

$$0 \to \mathcal{O}_C{}^{k-1} \to \mathcal{F} \to \mathcal{F}_1 \to 0,$$

where $\mathcal{F}_1$ is a line bundle. Since $h^0\mathcal{F}_1 \geq h^0\mathcal{F} - k + 1$ and $\deg \mathcal{F}_1 = \deg \mathcal{F}$, the desired inequality follows from the definition of a_C.

Next we must describe what we mean by saying that the matrix of linear forms whose minors define X is "suitably nondegenerate." Again, we pull everything back to the curve C, and thus we wish to give the definition in terms of a map of bundles of the form

$$\varphi: V \to W \otimes \mathcal{L}$$

where V and W are vector spaces (regarded as trivial vector bundles on C) and $\mathcal{L}$ is a line bundle on C; that is, φ may be represented as a matrix of sections of $\mathcal{L}$. Let us write v and w for the dimensions of the spaces V and W, respectively.

Since the desired bound is supposed to improve as v and w get larger, we need a hypothesis to ensure that v and w have not been "artificially increased" by adding on some rows or columns of zeros to a legitimate example. Invariantly put, we must assume at the very least that the maps φ and

$$\varphi^*\otimes\mathcal{L}: W^* \to V^*\otimes\mathcal{L}$$

both induce monomorphisms on global sections. This hypothesis is not sufficient to prevent degenerate examples, however; for example, if we tak V = W, and take φ to be the identity matrix multiplied by a section σ of $\mathcal{L}$, then the rank 1 locus is twice the zero locus of σ, independent of v and w.

Fortunately there is a rather natural hypothesis which does suffice: in the cases that arise in the applications, the map φ will be "1-generic" in the sense that for every rank one quotient space W_1 of W the induced map

$$V \to W\otimes\mathcal{L} \to W_1\otimes\mathcal{L} = \mathcal{L}$$

induces a monomorphism on global sections. (This condition is more symmetric than it may seem: it holds for φ iff it holds for $\varphi^*\otimes\mathcal{L}$. See for example Eisenbud [1988] for a study of it and related notions.)

However, we will need a little less than this: For the purposes of this section we will call φ **nondegenerate** if for every 2-dimensional quotient W_2 of W the composite map

$$V \xrightarrow{\varphi} W\otimes\mathcal{L} \to W_2\otimes\mathcal{L} \cong \mathcal{L}\oplus\mathcal{L}$$

induces a monomorphism on global sections, and also

$$\varphi^* \otimes \mathcal{L}: W^* \to V^* \otimes \mathcal{L}$$

induces a map on global sections.

If we choose bases of V and W and thus think of f as a matrix with entries in $H = H^0(\mathcal{L})$, whose columns correspond to the basis vectors of V, then f is nondegenerate iff no row and column transformations can produce either two zero entries in one column of f or one whole row of zeros. Some examples of nondegenerate matrices presented in this way as matrices of linear forms in $H := \langle a, b, \ldots \rangle$ (with, as we will see, the smallest possible spaces H given the dimensions of V and W) are

$$f = \begin{pmatrix} a & 0 & b & 0 & \cdots \\ 0 & a & 0 & b & \cdots \end{pmatrix}$$

and the generic 3×3 skew symmetric matrix

$$f = \begin{pmatrix} 0 & a & -b \\ -a & 0 & c \\ b & -c & 0 \end{pmatrix}.$$

With these definitions we may state the first form of the excess intersection bound:

Theorem 1 . 1: Let $\mathcal{L}$ be a line bundle of degree d on a smooth curve C, and let V and W be vector spaces of dimensions v and w respectively. Let

$$\varphi: \mathcal{O}_C \otimes V \to \mathcal{L} \otimes W$$

be a map of sheaves. If φ is nondegenerate and not everywhere of rank 1, then

$$\deg(\text{rank 1 locus of } \varphi) \leq 2d - a_C(v - 2) - (w - 2).$$

The virtue of Theorem 1 . 1 is that it gives a simple bound of the expected form 2d - ε, and ε does not depend on any knowledge of the degeneracy loci of φ. But it has several peculiarities and disadvantages:

First, it is asymmetric in V and W; the symmetric form with $a_C(w - 2)$ in place of w-2 in the conclusion would be sharper if it were true. The reader might suppose that this is because our hypothesis is asymmetric, but in fact even in examples satisfying the (stronger) symmetric hypothesis that the map φ is 1-generic, Theorem 1 . 1 may be sharp, and the symmetric version false, as we shall see.

Moreover, we have not been able to generalize Theorem 1 . 1 to cases where φ is a map of arbitrary vector bundles, nor to give a version that bounds the degree of the rank k locus for larger k.

These disadvantages may be overcome, at the expense of a more complex bound, by an idea suggested by Rob Lazarsfeld. Here is the result, in a form worked out in discussions with Lazarsfeld and Lawrence Ein:

Theorem 1 . 2: Let $\varphi: \mathcal{E} \to \mathcal{F}$ be a map of vector bundles, and $\mathcal{L}$ a line bundle, on a smooth curve C. Suppose that

1) $\mathcal{E}$ and $\mathcal{F}^*\otimes\mathcal{L}$ are generically generated by global sections.
2) $\varphi: \mathcal{E} \to \mathcal{F}$ and $\varphi^*\otimes\mathcal{L}: \mathcal{F}^*\otimes\mathcal{L} \to \mathcal{E}^*\otimes\mathcal{L}$ induce monomorphisms on global sections.

If the generic rank of φ is r > k, then

deg (rank k locus of φ) ≤

$$A_1(\varphi,\mathcal{E},\mathcal{F},\mathcal{L},k,r) := (1/r-k)\,\{\, dr - a_C(h^0\mathcal{E} - r) - a_C(h^0\mathcal{F}^*\otimes\mathcal{L} - r) + (r-k-1)\ \deg\,(\text{rank } k-1 \text{ locus of } \varphi)\,\}.$$

Thus assuming only that is everywhere of rank ≤ k, we may write

deg (rank k locus of φ)

$$\leq A(\varphi,\mathcal{E},\mathcal{F},\mathcal{L},k) := \max_{k\leq r\leq \min(e,f)} A_1(\varphi,\mathcal{E},\mathcal{F},\mathcal{L},k,r)$$

where e and f are the ranks of $\mathcal{E}$ and $\mathcal{F}$.

If we make some simplifying assumptions and ignore the possible benefits of using the functions $a_C(n)$ in place of n -- that is, ignore the complexity of C -- We can derive a simpler-looking inequality from Theorem 1 . 2:

Corollary 1 . 3: If $\varphi: \mathcal{E} \to \mathcal{F}$ satisfies conditions 1,2 of the Theorem, and if

$$\{ (d+2)k - h^0\mathcal{E} - h^0\mathcal{F}^*\otimes\mathcal{L} - \deg(\text{rank } k-1 \text{ locus of } \varphi) \} \geq 0$$

then

$$\deg(\text{rank } k \text{ locus of } \varphi) \leq (k+1)(d+2) - h^0\mathcal{E} - h^0\mathcal{F}^*\otimes\mathcal{L}.$$

The necessary inequality is satisfied in many cases of interest -- for example if k = 1, the map φ is "1-generic", and the rank 0 locus of φ is empty.

We will not require Theorem 1 . 2 in the sequel, though many of the applications below follow from it as well.

Before turning to the proofs of these results we will give some consequences and examples, including examples showing that neither Theorem 1 . 1 nor Theorem 1 . 2 is consistently sharper than the other. We begin by interpreting Theorem 1 . 1 in the setting of the problem mentioned at the beginning of this section:

Corollary 1 . 4: Let $C \subset \mathbb{P}^n$ be a nondegenerate smooth curve of degree d, and let X be any subscheme which is a component of the rank 1 locus of a 1-generic (or even just nondegenerate) $w \times v$ matrix of linear forms. If $X \cap C$ contains a finite subscheme of degree

$$> 2d - a_C(v-2) - (w-2),$$

then $C \subset X$.

Proof: The matrix of linear forms defining X pulls back to a nondegenerate (even 1-generic) matrix of sections of $\mathcal{L}$ on C because C is nondegenerate in $\mathbb{P}^n$; that is, because no linear form on $\mathbb{P}^n$ vanishes on C.//

Varieties X which are rank 1 loci of 1-generic matrices of linear forms are rather common. Indeed, every variety appears this way for a sufficiently high degree embedding: Take a complete embedding $X \subset \mathbb{P}^m$ such that the ideal of X is generated by quadrics; the double of this embedding is a linear section of the quadratic Veronese embedding of $\mathbb{P}^m$, and as such is cut out by the 2×2 minors of the generic symmetric $m+1 \times m+1$ matrix, restricted to the linear span of X. To see that the restriction is 1-generic, note that the restricted matrix is the matrix corresponding to the multiplication map

$$H^0\,\mathcal{L} \otimes H^0\,\mathcal{L} \to H^0\,\mathcal{L}^2,$$

where $\mathcal{L}$ is the line bundle corresponding to the original embedding. In this setting, 1-genericity corresponds to the statement that no pair of nonzero sections σ,τ of $\mathcal{L}$ can have $\sigma\tau = 0$ in $H^0\,\mathcal{L}^2$. See Eisenbud-Koh-Stillman [1988] for further information. By the main theorem of that paper, every curve of genus g embedded by a complete linear series of degree $\geq 4g+2$ is the rank 1 locus of a 1-generic matrix (in many ways!) Applying Theorem 1.1 to the intersection of such a high degree curve with an arbitrary curve we get:

Corollary 1 . 5: Let $C \subset \mathbb{P}^n$ be a nondegenerate smooth curve of degree d, and let X be a reduced irreducible curve of genus g_X, embedded by a complete series of degree $e \geq 4g_X+2$. If $X \cap C$ contains a finite subscheme of degree

$$> 2d - e + 2\,g_X + 2,$$

or even

$> 2d - a_C(\lceil e/2 \rceil - g_X - 1) - (\lfloor e/2 \rfloor - g_X - 1),$

then C = X.

Proof: We claim that X is the rank 1 locus of a 1-generic $w \times v$ matrix of linear forms with $v = \lceil e/2 \rceil - g_X + 1$ and $w = \lfloor e/2 \rfloor - g_X + 1$; given this, the Corollary follows from the Theorem. Indeed, we may write $\mathcal{O}_X(1) = \mathcal{L}_1 \otimes \mathcal{L}_2$, where $\deg \mathcal{L}_1 = \lceil e/2 \rceil$ and $\deg \mathcal{L}_2 = \lfloor e/2 \rfloor$, (taking distinct bundles if $X \neq \mathbb{P}^1$ but $e = 4g_X+2$ so that $v = w$.) From the theorem of Eisenbud-Koh-Stillman it follows that X is the rank 1 locus of the 1 generic matrix of linear forms coming from the pairing

$$H^0 \mathcal{L}_1 \otimes H^0 \mathcal{L}_2 \rightarrow H^0 \mathcal{O}_X(1).$$

Riemann-Roch identifies the dimensions $h^0 \mathcal{L}_1$ and $h^0 \mathcal{L}_2$ with the given v and w. //

It would be nice to have further theorems about the possible number of intersections of two curves. Gruson, Lazarsfeld, and Peskine [1983] have proved that a reduced irreducible nondegenerate curve of degree d in $\mathbb{P}^r$ is cut out by forms of degree $\leq d - r + 2$, and it follows that a curve of degree d can meet a curve of degree e in at most $e(d - r + 2)$ points. Diaz [1986] has sharpened this result, showing for example that if $6 \leq r+2 \leq d \leq e$, then the number of intersections is at most $e(d - r + 1)$. One could also use Castelnuovo theory to bound the genus of the union of two nondegenerate curves, getting a bound on the number of intersections that would depend on the genera of the curves as well as their degrees. These results are of course much more general than Corollary 1 . 5, but are also much weaker.

Among the familiar varieties which are the rank 1 loci of 1-generic matrices of linear forms are the rational normal scrolls, the Segre varieties, and the quadratic Veronese varieties. In the next section, we will use the result of Corollary 1 . 4 for these last, so we give the statement we need explicitly:

Corollary 1 . 6: Let $m = \binom{n+2}{2}-1$, and let $X \subset \mathbb{P}^m$ be the quadratic Veronese embedding of $\mathbb{P}^n$. If C is a nondegenerate curve of degree d in $\mathbb{P}^m$ which meets X in a subscheme of degree

$$> 2d - 2n + 2,$$

or even

$$> 2d - a_C(n - 1) - (n - 1),$$

then $C \subset X$. //

Examples: These results are sharp in many situations; that is, we do sometimes have

$$\text{degree } C \cap X = 2(\deg C) - a_C(v-2) - (w-2).$$

For examples where $w=2$ we may take two rational normal curves meeting in $n+2$ points in $\mathbb{P}^n$. Another example is given by an elliptic quartic C of type 2,2 on a quadric surface Q in $\mathbb{P}^3$, which meets a twisted cubic X of type 1,2 on Q in a scheme of degree 6.

To get some examples with $v=w=3$, take X to be a Veronese surface in $\mathbb{P}^5$. The intersection of 3 general quadrics containing X is the degenerate K3 surface which is the union of X and another Veronese surface X', meeting along an elliptic normal sextic curve, the image of a plane cubic $E \subset \mathbb{P}^2$ under the Veronese map $\mathbb{P}^2 \to X'$.

Let C be a smooth plane curve of degree d, different from E, and embed C in $X' \subset \mathbb{P}^5$ as a curve of degree 2d via the Veronese map. Of course we have

$$C \cap X = C \cap E$$

as schemes, and this scheme has degree 3d by Bezout's theorem in the plane. On the other hand, $a_C(1)$ is the gonality of C; since C is a smooth plane curve, the maps from C to $\mathbb{P}^1$ of lowest degree are given by projection from a point of $C \subset \mathbb{P}^2$, and thus have degree d-1. Thus

$$2(\deg C) - a_C(1) - 1 = 4d - (d-1) -1 = 3d,$$

and Corollary 1 . 4 is sharp. Note that the tempting symmetric version $2(\deg C) - a_C(1) - a_C(1)$ would fail in this case.

Examples showing that Theorem 1 . 1 is sometimes sharper than Theorem 1 . 2: For simplicity, take $v=w$. We want an example where φ is 1-generic; this means that we are in $\mathbb{P}^n$ with $n \geq 2v-2$; suppose that in fact $n = 2v-2$. For C to be nondegenerate we must embed it by a line bundle with at least $2v-1$ sections; thus d, the degree of C, must be $\geq a_C(2v-2)$, and we suppose that in fact $d = a_C(2v-2)$. We also suppose that g, the genus of C is $\leq v-2$ so that we have $a_C(v-2) = v-2+g$, and of course $a_C(2v-2) = 2v-2+g$. Writing Γ for the rank 1 locus on C, and supposing $\Gamma \neq C$, the bound given by Theorem 1 . 1 is

$$\text{degree } \Gamma \leq 2v+g.$$

On the other hand, the bound given by Theorem 1 . 2 is

$$\max_{2\leq r\leq v}(r/(r-1))(2v-2+g) + (\text{the degree of the rank 0 locus}).$$

$$= 4v-4+2g + (\text{the degree of the rank 0 locus}).$$

which is a good deal worse even if the rank 0 locus is empty.

Example showing that Theorem 1 . 2 is sometimes sharper than Theorem 1 . 1: Let C be an elliptic curve of degree 9 in $\mathbb{P}^5$, and let X be the Veronese surface, which is the rank 1 locus of the generic symmetric 3×3 matrix. In this case Theorem 1 . 1 leads to the bound degree $C.X \leq 15$, while Theorem 1 . 2 gives degree $C.X \leq 14$ (the rank 0 locus of the generic symmetric matrix being empty) or even ≤ 12 if we assume that C is not contained in the secant locus of the Veronese, so that the generic rank is 3.

We turn now to the proofs of Theorems 1 . 1 and 1 . 2.

We will prove Theorem 1 . 1 by induction on w and on the degree of $\mathcal{L}$. We will start the induction with the case $w = 2$. At the end of the proof, we will need in fact a version that covers the case where the target $\mathcal{L}\otimes W$ is replaced by an arbitrary rank 2 bundle:

Lemma 1 . 7: Let $\mathcal{E}$ be a rank 2 vector bundle on a smooth curve C, and let V be a vector space of dimension v. Suppose that

$$\varphi: \mathcal{O}_C \otimes V \to \mathcal{E}$$

is a map such that the induced map

$$f: V \to H^0(\mathcal{E})$$

is a monomorphism. If φ is generically of rank 2, then

$$\deg(\text{rank 1 locus } \varphi) \leq \deg \mathcal{E} - a_C(v - 2).$$

Proof of Lemma 1 . 7: Let $\mathcal{F}$ be the image of φ, and let Γ be the rank 1 locus. Because φ is generically of rank 2, $\mathcal{F}$ has rank 2. As $\mathcal{F}$ is generated by global sections, we may apply property (3) of the function a_C and conclude that

$$\deg \mathcal{F} \geq a_C(v-2).$$

The rank 1 locus Γ of φ is the same as the rank 1 locus of the induced map $\mathcal{F} \to \mathcal{E}$. Thus

$$\begin{aligned} \text{degree } \Gamma &= \deg \mathcal{E} - \deg \mathcal{F} \\ &\leq \deg \mathcal{E} - a_C(v-2), \end{aligned}$$

as required. //

Proof of Theorem 1 . 1: Writing Γ for the rank 1 locus of φ, we must show that

$$\text{degree } \Gamma \leq 2 \deg \mathcal{L} - a_C(v - 2) - (w - 2).$$

Since in any case $0 \leq \text{degree } \Gamma$, we must show that

$$2 \deg \mathcal{L} \geq a_C(v - 2) + (w - 2),$$

and this inequality will suffice for the case in which Γ is empty. To prove it we use a property of nondegenerate maps similar to one proved in the 1-generic case in Eisenbud [1988]: If

$$f: V \to H \otimes W$$

is nondegenerate (in the sense that the correponding map

$$V \otimes \mathcal{O}_{\mathbb{P}(H)} \to V \otimes \mathcal{O}_{\mathbb{P}(H)}(1)$$

is nondegenerate) then $\dim H \geq \lceil v/2 \rceil + w - 2$. To see this note that regarding f as a map $f: V \to \mathrm{Hom}(H^*, W)$, f is nondegenerate iff no element of V maps to a transformation of rank $\leq w-2$. It follows that the dimension of V must be $\leq$ the codimension of the locus of matrices of rank $\leq w-2$; that is,

$$v \leq 2(\dim H - w + 2),$$

so

$$\dim H \geq \lceil v/2 \rceil + w - 2$$

as claimed.

In our case, taking $H = H^0\mathcal{L}$, it follows that

$$h^0\mathcal{L} - 1 \geq \lceil v/2 \rceil + w - 3,$$

so

$$\deg \mathcal{L} \geq a_C(\lceil v/2 \rceil + w - 3).$$

It now suffices to show that

$$2\, a_C(\lceil v/2 \rceil + w - 3) > a_C(v-2) + w - 2.$$

By property (2) of the function a_C we have

$$2\, a_C(\lceil v/2 \rceil + w - 3) \geq a_C(2\lceil v/2 \rceil + 2w - 6)$$
$$\geq a_C(v + 2w - 6)$$
$$\geq a_C(v - 2) + 2(w - 2),$$

the last by property (1) of a_C. Since we have assumed that the generic rank of φ is ≥ 2, we must have $w \geq 2$, so the last expression is $\geq a_C(v - 2) + (w - 2)$, as required. This completes the proof in the case where Γ is empty.

If on the other hand Γ is not empty, we will do induction on w and on the degree of $\mathcal{L}$ (the case of negative degree being vacuous.) The case $w = 2$ is a special case of Lemma 1 . 7, above, so we may assume $w > 2$.

Let $p \in C$ be a point in the support of Γ, so that the rank of φ at p is ≤ 1. Write φ' for the "dual" map

$$\varphi' := \mathcal{L} \otimes \varphi^* : \mathcal{O}_C \otimes W^* \to \mathcal{L} \otimes V^*,$$

and note that the rank 1 locus of φ' is again exactly Γ. Let

$$W^*_p = \{s \in W^* \mid \varphi'(1 \otimes s) \text{ vanishes at } p\}.$$

Because φ has rank ≤ 1 at p the codimension of W^*_p in W^* is at most 1. If $W^*_p = W^*$, then φ' factors through a map

$$\varphi_1: \mathcal{O}_C \otimes W^* \to \mathcal{L}(-p) \otimes V^*,$$

and the rank 1 locus of φ_1 has degree exactly degree $\Gamma - 2$ as one sees from a local computation at p. Further, the map

$$f_1 : V \to \mathcal{L}(-p) \otimes W$$

corresponding to φ_1 is again nondegenerate, so we are done by the induction on deg $\mathcal{L}$. Thus we may suppose that W^*_p has codimension exactly 1 in W^*.

We claim that the rank 1 locus of the map

$$\psi' = \varphi'|_{\mathcal{O}_C \otimes W^*_p} \quad : \quad \mathcal{O}_C \otimes W^*_p \to \mathcal{L} \otimes V^*$$

has degree > degree Γ. To see this, note first of all that the rank 1 locus of ψ' contains that of φ', so it will suffice to show that they are not equal locally at p.

Let t be a local parameter on C near p so that t=0 is the local equation of p. Choose an element of W^* outside W^*_p and complete it with a basis for W^*_p to a basis for W^*. With respect to the corresonding basis of $\mathcal{O}_C \otimes W^*$ and any basis for $\mathcal{O}_C \otimes V^*$ the map φ' will be represented by a matrix

$$f = (x_{ij}) \qquad i=1,\dots,w, \ \ j=1,\dots,v$$

over the power series ring k[[t]] such that for $j=2,\dots,w$ we may write $x_{ij} = ty_{ij}$ with $y_{ij} \in k[[t]]$. Suppose that the degree of the component of Γ at p is m; that is, suppose that the 2×2 minors of φ' generate the ideal (t^m). We must show that the 2×2 minors taken from the last v-1 columns of φ' are contained in (t^{m+1}). If m = 1 this is obvious because the entries in the last v-1 columns are all divisible by t. Thus we may suppose that $m \geq 2$.

By our assumption that $W^*_p \neq W^*$, the first column of φ' cannot be contained in (t). Rearranging the rows if necessary, we may suppose that x_{11} is a unit of k[[t]]. If i,j > 1 then

$$x_{11}x_{ij}-x_{1j}x_{i1} \equiv 0 \pmod{t^m},$$

and "dividing by t" we get

$$x_{11}y_{ij} \equiv y_{1j}x_{i1} \pmod{t^{m-1}}.$$

It follows that the 2×2 minors in the matrx

$$(y_{ij}) \qquad i=1,\dots,w, \ \ j=1,\dots,v-1$$

are all in (t^{m-1}). Since the last v-1 columns of the matrix (x_{ij}) form the matrix (ty_{ij}), the 2×2 minors of this matrix are in (t^{m+1}) as required.

If the map ψ' is generically of rank ≥ 2, then we may apply induction (the map ψ: $V \to W_p \otimes \mathcal{L}$ corresponding to ψ' is nondegenerate because any 2-quotient of W_p is automatically a 2-quotient of W.) Thus

$$\begin{aligned} \text{degree } \Gamma &< \text{degree rank 1 locus of } \psi \\ &\leq 2d - a_C(V) - ((w-1) -2) \\ &= 2d - a_C(V) - (w -2) +1, \end{aligned}$$

which is the desired inequality.

If on the contrary the composite map ψ' is everywhere of rank 1, then we write $\mathcal{M}$ for its image, so that the original map φ' factors through the epimorphism

$$\mathcal{O}_C \otimes W^* = \mathcal{O}_C \oplus (\mathcal{O}_C \otimes W^*_p) \twoheadrightarrow \mathcal{O}_C \oplus \mathcal{M}.$$

Note that $\mathcal{M}$ is a line bundle on C having $h^0(\mathcal{M}) \geq w-1$, and thus deg $\mathcal{M} \geq w - 2$.

Returning to the original map φ, we see that the rank 1 locus Γ is also the rank 1 locus of the induced map

$$\mathcal{O}_C \otimes V \to \mathcal{L} \oplus \mathcal{L} \otimes \mathcal{M}^*.$$

This map satisfies the hypothesis of Lemma 1 . 7, and

$$\deg (\mathcal{L} \oplus \mathcal{L} \otimes \mathcal{M}^*) \leq 2 \deg \mathcal{L} - w + 2,$$

so the conclusion of Lemma 1 . 7 completes the proof. //

Proof of Theorem 1 . 2: Let $\mathcal{E}'$ be the image of $\mathcal{E}$ in $\mathcal{F}$. Since φ is a monomorphism on global sections, $h^0\mathcal{E}' \geq h^0\mathcal{E}$. Since $\mathcal{E}$ is generically generated by global sections, $\mathcal{E}'$ is too. Applying property 3) of the function a_C we get

$$\deg \mathcal{E}' \geq a_C(h^0\mathcal{E} - r).$$

Let $\mathcal{F}'$ be the saturation of $\mathcal{E}'$ in $\mathcal{F}$, so that $\mathcal{F}'$ is a sub-bundle of $\mathcal{F}$ of rank r. Dualizing and twisting by $\mathcal{L}$ we get an epimorphism

$$\mathcal{F}^*\otimes\mathcal{L} \to \mathcal{F}'^*\otimes\mathcal{L},$$

whence by the same argument as for $\deg \mathcal{E}'$,

$$\deg (\mathcal{F}'^* \otimes \mathcal{L}) = rd + \deg \mathcal{F}'^* \geq a_C(h^0\mathcal{F}^*\otimes\mathcal{L} - r),$$

or equivalently

$$\deg \mathcal{F}' \leq rd - a_C(h^0\mathcal{F}^*\otimes\mathcal{L} - r).$$

Since $\mathcal{E} \to \mathcal{E}'$ and $\mathcal{F}' \to \mathcal{F}$ are maps of bundles, the rank k locus of φ is, for any k, the same as the rank k locus of the induced map

$$\varphi': \mathcal{E}' \to \mathcal{F}'.$$

We claim that for such a map φ', with generic rank r, between bundles of rank r,

$$\deg (\text{rank } k \text{ locus of } \varphi') \leq$$
$$(1/r-k) \{ \text{length coker } \varphi' + (r-k-1)\deg (\text{rank } k-1 \text{ locus of } \varphi') \}.$$

Since the length of the cokernel of φ' is

$$\deg \mathcal{F}' - \deg \mathcal{E}' \geq rd - a_C(h^0\mathcal{E} - r) - a_C(h^0\mathcal{F}^*\otimes\mathcal{L} - r),$$

this will suffice for the proof.

To prove the claim we may work locally, and assume that in terms of a local parameter t at a point of C, φ' is given by a diagonal matrix with entries

$$t^{a_1}, \dots , t^{a_r}$$

having

$a_1 \leq \ldots \leq a_r$.

In these terms, the local contributions at t=0 are

length coker $\varphi' = a_1 + \ldots + a_r$
degree rank k locus $= a_1 + \ldots + a_{k+1}$,
degree rank k-1 locus $= a_1 + \ldots + a_k$,

so the local form of the desired inequality is

$$a_1 + \ldots + a_{k+1}$$
$$\leq (1/r-k) \{ a_1 + \ldots + a_r + (r-k-1)(a_1 + \ldots + a_k) \}$$
$$= a_1 + \ldots + a_k + (1/r-k)(a_{k+1} + \ldots + a_r),$$

which holds because of the inequalities

$a_{k+1} \leq \ldots \leq a_r$. //

Proof of Corollary 1 . 3: Using $a_C(n) \geq n$ we derive

deg (rank k locus of φ)

$$\leq d+2 + \deg(\text{rank } k-1 \text{ locus of } \varphi)$$
$$+ (1/r-k) \{ (d+2)k - h^0\mathcal{E} - h^0\mathcal{F}^*\otimes\mathcal{L}$$
$$- \deg(\text{rank } k-1 \text{ locus of } \varphi) \}$$

and thus, independently of the generic rank r,

deg (rank k locus of φ)

$$\leq d+2 + \deg(\text{rank } k-1 \text{ locus of } \varphi)$$
$$+\max_{\{r \mid v \geq r > k\}} (1/r-k) \{ (d+2)k - h^0\mathcal{E} - h^0\mathcal{F}^*\otimes\mathcal{L}$$
$$- \deg(\text{rank } k-1 \text{ locus of } \varphi) \}.$$

The maximum is taken on either at r = k+1 or at r = min (rank $\mathcal{E}$, rank $\mathcal{F}$), respectively, depending on the sign of

$$\{ (d+2)k - h^0\mathcal{E} - h^0\mathcal{F}^*\otimes\mathcal{L} - \deg(\text{rank } k-1 \text{ locus of } \varphi) \},$$

which we have assumed positive. //

2. Finite subschemes of rational normal curves

We will be interested in the conditions under which a finite subscheme Γ is of $\mathbb{P}^n$ is contained in a rational normal curve of degree n in $\mathbb{P}^n$. Of course a subscheme of a rational normal curve must be curvilinear, and it is not hard to see that it must be in linearly general position as well; that is, for $k<n$, no k-plane in $\mathbb{P}^n$ can contain a subscheme of Γ having degree $\geq k+1$.

The following are two of the central results of Castelnuovo theory, generalized to finite schemes. The first was proved in Eisenbud-Harris [199? Corollary 2 and Thm 2.1], where it was also shown that the hypothesis of curvilinearity is superfluous if the degree of Γ is $n+3$:

Theorem 2 . 1: Let Γ be a curvilinear subscheme of $\mathbb{P}^n$ in linearly general position. If degree $\Gamma \leq n+3$ then there is a unique rational normal curve containing Γ.

Any subscheme of a rational normal curve lies on all the quadrics containing the rational normal curve; Thus for Γ to lie on a rational normal curve, it is necessary that Γ imposes only $2n+1$ conditions on quadrics. The second result is that if the degree of Γ is large enough, the converse holds:

Theorem 2 . 2: Let Γ be a curvilinear subscheme of $\mathbb{P}^n$ in linearly general position. If degree $\Gamma \geq 2n+3$ but Γ imposes only $2n+1$ conditions on quadrics, then there is a unique rational normal curve containing Γ.

The proof that we will give for Theorem 2 . 2 seems to be new even in the classical case where Γ is reduced. Its essential idea is that under the quadratic Veronese map $\nu: \mathbb{P}^n \to \mathbb{P}^{\binom{n}{2}-1}$ the subscheme Γ will be taken to a subscheme of degree $2n+3$ in linearly general position in a plane of dimension $2n$, so that we can use Theorem 2 . 1 to construct a rational normal curve. The problem then becomes one of proving that this rational normal curve is contained in the Veronese image of $\mathbb{P}^n$, which it meets in the "large" scheme Γ.

This central point of the proof is handled by Corollary 1 . 6 from the last section.

What about the case of where Γ has degree between n+3 and 2n+3? The only result for this range that we know in the classical case is Mark Green's "Strong Castelnuovo Lemma" [1984 Cor. 3.c.6]; it would be interesting to prove its analogue for schemes.

Proof of Theorem 2 . 2: We claim first that, writing $N = \binom{n}{2}-1$ and

$$\nu: \mathbb{P}^n \rightarrow \mathbb{P}^N$$

for the Veronese embedding of $\mathbb{P}^n$ by quadrics, the scheme $\nu(\Gamma)$ is in linearly general position in its span M, which is a 2n-dimensional plane in $\mathbb{P}^N$. The fact that the span is a 2n-plane is an immediate translation of the statement that Γ imposes only 2n+1 conditions on quadrics. Given this, the statement about linearly general position follows at once from Theorem 3.2 of our [1991].

By Theorem 2 . 1, there is a rational normal curve C in M containing Γ. It suffices to show that C is contained in $\nu(\mathbb{P}^n)$; the preimage of C in $\mathbb{P}^n$ is then the desired rational normal curve containing Γ. For this we may use Corollary 1 . 6, with C the rational normal curve of degree d = 2n. //

3. The normal bundles of curves and Castelnuovo theory for ribbons

We can use the results of the last section and the results of our [199?] to get some information about sub-line-bundles of the normal bundles of smooth curves in $C \subset \mathbb{P}^3$. The technique we will exhibit works for curves in any projective space, and also for curves somewhat more general than smooth curves, but we leave the generalizations to the reader.

Our idea is based on the fact that if $\mathcal{L}$ is a line subbundle of the normal bundle of C then there is a corresonding "ribbon" -- a purely 1-dimensional scheme $C' \subset \mathbb{P}^3$ of degree 2d containing C, such that the normal bundle of C in C' is $\mathcal{L}$. We have an exact sequence

$$0 \to \mathcal{L}^{-1} \to \mathcal{O}_{C'} \to \mathcal{O}_C \to 0,$$

from which we can compute the Euler characteristic of $\mathcal{O}_{C'}$, and thus the arithmetic genus

$$g(C') = 2g(C) - 1 + d.$$

Thus a bound on the genus of C' gives rise to a bound on d. Now the classical methods of Castelnuovo give a (sharp) bound on the genus of a curve in terms of the degree of the curve. Of course they were originally used only for smooth curves, but the results of the last section together with Theorems 3.1 and 3.2 of our [199?] are exactly the results necessary to imitate the proofs given for example in Harris [1982] and thus to extend Castelnuovo theory to the case of ribbons. One obtains:

Propostion 3 . 1: Let C be a reduced irreducible nondegenerate curve of degree d and genus g in $\mathbb{P}^3$. If $\mathcal{L}$ is a line subbundle of the normal bundle of C, then

$$\deg \mathcal{L} \leq d^2-2d+2-2g,$$

and equality holds iff C is a complete intersection with a quadric surface or is residual to a line in a complete intersection with a singular quadric, and the bundle $\mathcal{L}$ is the normal bundle of C in the quadric.//

Remarks: 1) In $\mathbb{P}^n$ a similar argument leads to a result of the form deg $\mathcal{L} \le 2d^2/(n-1) - 2g + O(d) \ldots$)

2) One should compare the bound in the Proposition with the result obtained from regularity theory: By Gruson-Lazarsfeld-Peskine [1983] we know that $I_C(d-1)$ is generated by global sections. Since $I_C(d-1)$ maps onto $\mathcal{L}^*(d-1)$, we see that the degree of this last is nonnegative; that is, deg $\mathcal{L} \le d(d-1)$. The result in Propostion 3 . 1 is sharp when C itself has maximal genus for its degree, but probably not otherwise. An exact answer is known when $g(C) \le 1$ (see Eisenbud and Van de Ven [1981] and Ghione and Sacchiero [1980] in the case of genus 0; Hulek and Sacchiero [1983] in the case of genus 1. Fong [1991] gives some information on the general case.) It would be nice to have more results of this type.

We sketch an application:

Corollary 3 . 2: Let C be a smooth nondegenerate curve of degree d in $\mathbb{P}^3$ with ideal sheaf $\mathcal{I}$ and normal sheaf N.

a) If $m > d - 2$, then

$$H^1 (\mathcal{I}/\mathcal{I}^2(m)) = 0.$$

Further, if $H^1\ \mathcal{I}/\mathcal{I}^2(d-2) \neq 0$, then C lies on a quadric.

b) If $m > d - 2 - (2g+2)/d$ then

$$H^1 (N(m)) = 0.$$

Remark: One can of course give an analogous result in $\mathbb{P}^n$, and a somewhat sharper analysis of the boundary cases above. We let these things wait for an application or an interested reader.

Proof: If $H^1\ \mathcal{I}/\mathcal{I}^2(m) \neq 0$, then $H^0\ N \otimes \omega_C(-m) \neq 0$, so N has a sub-bundle of degree $\ge 2g-2+md$. If follows from the previous Corollary that $m \leq d-2$.

Similarly, if $H^1\ N(m) \neq 0$ then $H^0\ N^* \otimes \omega_C(-m) \neq 0$, so N^* has a subbundle of degree $\geq md - 2g + 2$. It follows that N has a quotient of degree $\leq 2g - 2 - md$. Since the degree of N is $2g - 1 + n$, we see that N has a rank $n-2$ subbundle of degree $\geq (2g-1+n)-(2g-2-md) = n+1+md$. If $n=3$ this is a line bundle, and it follows from the previous Corollary that $4+md \leq d^2-2d+2-2g$, or $m \leq d - 2 - (2+2g)/d$, as claimed. //

4. Pairs of bundles with Clifford index 1:

In this section we will discuss results on the Clifford index of a pair of linear series L_1, L_2 defined on a variety (the definition is given in the introduction, above.) It is clear that $\mathrm{Cliff}(L_1,L_2)$ remains the same if we subtract base divisors from L_1 and L_2, so we will generally assume in talking about the Cliffford index that L_1 and L_2 have no base locus in codimension 1.

If $\varphi: Y \to X$ is a morphism, and (L_1,L_2) is a pair of linear series on X, then $(\varphi^*L_1,\varphi^*L_2)$ is a pair of linear series on Y with the same Clifford index as long as φ^* is a monomorphism on the sections of L_1L_2; in this case we will say that $(\varphi^*L_1,\varphi^*L_2)$ is a faithful pullback of (L_1,L_2). More generally, if φ is only a rational map, we will say that (M_1,M_2) is a faithful pullback of (L_1,L_2) by φ as long as (M_1,M_2) and $(\varphi^*L_1,\varphi^*L_2)$ agree wherever φ is defined.

If X is an irreducible curve, and $L_2=K_X \otimes L_1^{-1}$, then $\mathrm{Cliff}(L_1,L_2) \leq$ the usual Clifford index of L_1, with equality iff $V_1 \otimes V_2 \to H^0(K_X)$ is onto. In general, if X is reduced and irreducible, then the pairing $V_1 \otimes V_2 \to H^0(\mathcal{L}_1 \otimes \mathcal{L}_2)$ is 1-generic, and as Kempf pointed out in his thesis, (see also Eisenbud [1988]) it follows that

$$\mathrm{Cliff}(L_1,L_2) \geq 0.$$

Eisenbud [1988, Cor. 5.2] obtained a generalization of the part of the Clifford theorem which lists the cases of equality when X is a smooth curve and $L_2=K_X \otimes L_1^{-1}$ (the classical result says that in this case, if the L_i are complete of dimensions ≥ 1, and L_1L_2 is complete, then X must be hyperelliptic and the L_i are pulled back from complete series on $\mathbb{P}^1$ via the hyperelliptic involution.) We give the statement for the reader's convenience:

Theorem 4 . 1: If $L_1 = (\mathcal{L}_1, V_1)$ and $L_2 = (\mathcal{L}_2, V_2)$ are linear series of positive dimension on a reduced, irreducible projective variety X, then

$$\mathrm{Cliff}(L_1,L_2) = 0,$$

iff (L_1,L_2) is faithfully pulled back from a pair of complete series on $\mathbb{P}^1$.

Remark: If, on a smooth curve X, we assume that $L_2=K_X\otimes L_1^{-1}$ and that the L_i are complete of dimensions ≥ 1 with $\text{Cliff}(L_1,L_2) = 0$, but do not assume that L_1L_2 is complete, we get some further cases besides the hyperelliptic one. For example, if we take a trigonal curve of genus 4 with only one g_3^1 , then the canonical image of the curve is the complete intersection of a quadric cone and a cubic surface in $\mathbb{P}^3$ and the $g^1{}_3$ is residual to itself in the canonical series. Taking L_1 and L_2 to be the $g^1{}_3$, we see that the product L_1L_2 is incomplete -- it is the projection of the canonical curve from the vertex of the quadric cone -- and (L_1,L_2) has Clifford index 0.

Theorem 1 . 1 leads to an extension of this to pairs of linear series of Clifford index 1:

Theorem 4 . 2: If $L_1 = (\mathcal{L}_1, V_1)$ and $L_2 = (\mathcal{L}_2, V_2)$ are of positive dimension on a reduced, irreducible variety X with

$$\text{Cliff}(L_1,L_2) = 1,$$

then (L_1,L_2) is faithfully pulled back from one of the following:

i) The pair of complete series ($|\mathcal{O}_{\mathbb{P}^2}(1)|$, $|\mathcal{O}_{\mathbb{P}^2}(1)|$) on $\mathbb{P}^2$.

ii) the complete series corresponding to the ruling and a section on a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$.

iii) complete series on a curve of arithmetic genus 1 (an elliptic curve or a rational curve with one simple node or one cusp.)

iv) a pair of series on $\mathbb{P}^1$, one complete and one of codimension 1 in the complete series.

Remarks: 1) When X is a smooth curve, $L_2=K_X\otimes L_1^{-1}$, and L_1L_2 is complete, only the first two cases can occur, and they do occur respectively for plane quintics, and for trigonal non-hyperelliptic curves. The third and fourth possibilities cannot occur here because the canonical image of a curve is never an elliptic curve, or $\mathbb{P}^1$, except in the hyperelliptic case.

2) The cases overlap somewhat, since for example a pair of complete series on an elliptic curve (Clifford index one), one of which is a pencil, may always be pulled back from a scroll, as in ii). Also, if X is smooth, then the cases in iii) where the curve is singular may be dropped, since in these cases the series is pulled back from series on the normalization of the curve.

Proof: It is easy to check that the given series, and thus their faithful pullbacks, really do have Clifford index 1.

Now suppose that (L_1,L_2) as in the theorem has Clifford index 1. Let $L = (\mathcal{L},V)$ be the product of L_1 and L_2. We may replace X by its image under L in $\mathbb{P}(V)$, which is a nondegenerate variety of positive dimension because the linear series L has positive dimension, and thus assume that L is very ample.

Write $a=\dim L_1$, $b=\dim L_2$. The pairing $V_1 \otimes V_2 \to V$ gives rise to an $(a+1)\times(b+1)$ 1-generic matrix M of size $(a+1)\times(b+1)$ whose entries are linear forms on $\mathbb{P}(V)$ and whose rank 1 locus contains X, as explained in Eisenbud [1988]. Since the Clifford index is 1 we have $\mathbb{P}(V)=\mathbb{P}^{a+b+1}$.

Let X' be a component of the rank 1 locus of M which contains X. If the dimension of X' is ≥ 2, then the codimension of the rank 1 locus of M is at most a+b-1, so we are in a position to apply the classification theorem of Eisenbud [1988], and we deduce that we are in case i) or ii), with the desired linear series.

If on the other hand the dimension of X' is 1, then X = X'. By Corollary 5 . 2 we see that $\deg X \leq a+b+2$. Since X is nondegenerate, its degree must be at least a+b+1. If the degree is exactly a+b+1 then X is the rational normal curve, and we are in case iv).

Thus we may suppose that X is a curve of degree a+b+2. In this case X is either a curve of arithmetic genus 1 embedded by a complete series, and we are in case iii), or $\mathbb{P}^1$ embedded by a series of codimension 1 in the complete series. But the second case does not in fact occur, since if X is a smooth rational curve of degree a+b+2 then Theorem 4 . 3, below, shows that X is actually a 2-dimensional rational normal scroll (and one of the L_i is the complete series of degree 1 on $\mathbb{P}^1$.)//

To complete the proof of Theorem 4 . 2, and for general interest, we will now analyze certain pairs of series on $\mathbb{P}^1$. A series

$L = (\mathcal{L}, V)$

on $\mathbb{P}^1$ of degree d is nothing but a subspace V of the vector space S_d of forms of degree d in two variables, say s,t. By the codimension of L we will mean the codimension of V in S_d.

We will make use of the Theorem of Hilbert-Burch (in exactly the form given by Hilbert in [1890]!): If L as above is base-point free, that is, there is no linear form that divides every element of V, and if L has dimension e, then any set of e+1 generators of V consists of the $e \times e$ minors of the $(e+1) \times e$ matrix of forms which is the matrix of relations on those generators, thought of as generators of an ideal in k[s,t]. Each column of the matrix of relations consists of elements of the same degree. If the elements of the i^{th} column have degree d_i, then the sum of the d_i is d, and the d_i (up to permutation) are an invariant of the linear series, not just of the generators chosen.

The invariants d_i above are orthogonal to most of the usual geometric invariants of the map from $\mathbb{P}^1$ to $\mathbb{P}^e$ defined by the series. For example, every base-point free series of codimension 1 will have as d_i the sequence 1,1,...,1,2, since this is the only sequence of d-1 positive integers that sum to d. However, the d_i distinguish two types of series of codimension 2: those with sequence 1,1,...,1,2,2 and those with sequence 1,1,...,1,3; we will call these types I and II, respectively.

Theorem 4 . 3: Let $L = (\mathcal{O}_{\mathbb{P}^1}(d), V)$ be a base point free linear series on $\mathbb{P}^1$ of dimension d-1 (codimension 1 in the complete series.) L may be expressed as the product of a pair of series (L_1,L_2) having Clifford index 1 iff L satisfies one of the following:

i) L is base point free but not very ample. In this case L and therefore the L_i are pulled back from complete series on a rational curve C with one cusp or one node via the normalization map $\mathbb{P}^1 \to C$, so that the L_i must have the form:

$$L_1 = (\mathcal{O}_{\mathbb{P}^1}(a), \langle f_1, g\cdot S_{a-2}\rangle)$$
$$L_2 = (\mathcal{O}_{\mathbb{P}^1}(b), \langle f_2, g\cdot S_{b-2}\rangle),$$

with

$$a,b \geq 2,\ L_1 \neq L_2 \text{ if } a=b=2,$$

where g is a quadratic polynomial vanishing on the preimage of the singular locus of C, and the f_i are polynomials relatively prime to g. Conversely, any series of this type provides an example.

ii) L is very ample, and is the product of the complete series L_1 of degree 1 with a codimension 2 series L_2 of type II. Equivalently, L embeds $\mathbb{P}^1$ as a subvariety of a 2-dimensional rational normal scroll S, and the L_i are pulled back from the complete series on S consisting of the ruling and its complement in the hyperplane series on the scroll.

Proof: i) First suppose that $L = L_1L_2$ is not very ample. Let C be the image of $\mathbb{P}^1$ by L. As in Theorem 4 . 2 we see that L, L_1, L_2 are pulled back from linear series L', L'_1, L'_2 on C. Since L is not very ample, there is a divisor D of degree 2 imposing only one condition on the sections V of L, and C is the curve of arithmetic genus 1 obtained from $\mathbb{P}^1$ by collapsing D. By Riemann-Roch, any complete series of positive degree on C has dimension one less than its degree; since the Clifford index of (L'_1,L'_2) is 1, we see that both the L'_i are complete. It follows that L and the L_i have the given form, and, since each L'_i is at least 1-dimensional, that $a,b \geq 2$. One checks by direct computation (for example use Harshorne [1977 exc. IV, 4.1]) that if $a = b = 2$ and $L'_1 = L'_2$, then $L'_1L'_2$ has dimension only 2, so (L'_1,L'_2) has Clifford index 0 rather than 1.

Conversely, we must show that if L'_1,L'_2 are base point free complete series on C of dimension ≥ 1, and with $L'_1 \neq L'_2$ if both have dimension 1, then their product L' is again a complete series. If $L'_1 \neq L'_2$ then either $H^1(\mathcal{L}_1\mathcal{L}_2{}^{-1})$ or $H^1(\mathcal{L}_1{}^{-1}\mathcal{L}_2)$ is zero, and the desired result follows at once from the base point free pencil trick (see St. Donat [1973]). If $L'_1 = L'_2$ it follows from a knowledge of the ring $\oplus H^0(\mathcal{L}_1{}^{\otimes\nu})$; again, see for example Hartshorne [1977, exc. IV 4.1].

ii) Next suppose that $L = L_1L_2$ is very ample. Because Cliff $(L_1,L_2) = 1$, the sum of the codimensions of L_1 and L_2 is 2. By Lemma 4 . 4, ii) below, one of them, say L_2, must be of codimension ≥2, and the other must be complete. From the number of linear and quadratic syzygies among a set of generators of V_2 we can find the dimensions of $\langle s,t\rangle V_2$, and of $\langle s^2, st, t^2\rangle V_2$, and we see that the first is complete unless L_2 is of type II, while the second is always complete. This shows that L_1 must be the complete series of degree 1, and L_2 must have type II, and as in ii).

It follows at once from the argument above that in any embedding of $\mathbb{P}^1$ as a curve C of degree d on a 2-dimensional rational normal scroll in $\mathbb{P}^{d-1}$, the restriction to C of the complement of the ruling in the hyperplane series on the scroll is a series of codimension 2 and type II.

It remains to show that conversely given any series L_2 of codimension 2 and type II, the product of this series with the complete series L_1 of degree 1 is very ample (every smooth rational curve of degree d in $\mathbb{P}^{d-1}$ lies on a 2-dimensional scroll whose ruling cuts out the complete series of degree 1 on the curve, so the rest is automatic.) That is, we must show that any divisor $D = p+q$ of degree 2 on $\mathbb{P}^1$ imposes 2 conditions on L (we allow the case $p = q$). Let λ be the linear form vanishing at p. Since L_2 has type II it must be base point free. But the subseries L' of L vanishing at p contains λV_2, which has only a simple base point at p, and no further base points, so we are done.//

To complete the proof, we still must prove:

Lemma 4 . 4: Let $V \subset S_d$ be a series of codimension 1, corresponding to projection of the rational normal curve in $\mathbb{P}(S_d)$ from a point p. If, for some m and n with m+n=d, V contains the product of series $V_1 \subset S_m$ and $V_2 \subset S_n$ having codimensions c_1 and c_2 respectively, and if

$$c_1+c_2 \leq \min(m, n),$$

then p lies in a (c_1+c_2)-secant (c_1+c_2-1)-plane of the rational normal curve; that is, V contains a subspace of the form $gS_{d-(c_1+c_2)}$ with degree $g = c_1+c_2$. In particular:

i) if V contains the product of a complete series of positive degree and a codimension 1 series, then V has a base-point; and

ii) if V has no base-points and contains the product of two codimension 1 series of degrees ≥ 2, then V is not very ample.

Proof: We may regard p as an element of $(S_d)^*$ whose kernel is V. Suppose $W_1W_2 \subset V$, with W_1, W_2 of degree m, n respectively.

For each element a of W_1, the elements b of W_2 satisfy the linear condition p(ab)=0. This may be interpreted as the condition (a(p))(b)=0, where a(p) indicates the action of S_m on $(S_d)^*$ induced from the Hopf algebra structure of the symmetric algebra (in characteristic 0, where the dual of the symmetric algebra on a vectorspace U is isomorphic to the symmetric algebra on U^*, this action is obtained by identifying $(S_d)^*$ with the degree d polynomials in "dual" variables, and identifying S_m with the polynomial differential operators of order m.) Since W_2 has codimension c_2, the linear functionals a(p) on S_n coming from various $a \in W_1$ can together have rank at most c_2, and since W_1 has codimension c_1 in S_m, the functionals on S_n of the form a(p) for all $a \in S_m$ can together have rank at most c_1+c_2.

Now write

$$p = \Sigma x_i \sigma_i,$$

where $\{\sigma_i\}$ is the dual basis to $\{s^i t^{d-i}\}$. The elements $\{(s^j t^{m-j})(\sigma_i)\}$ are the dual basis to $\{s^{i-j} t^{n-i+j}\}$, so that

$$((s^j t^{m-j})(p))(s^k t^{n-k}) = x_{j+k}.$$

Thus the condition that the functionals $((s^j t^{m-j})(p))$ have together rank at most c_1+c_2 is the condition that the $(m+1)\times(n+1)$ Catalecticant matrix

$$\begin{pmatrix} x_0 & x_1 & \cdots & x_n \\ x_1 & x_2 & \cdots & x_{n+1} \\ . & . & \cdots & . \\ . & . & \cdots & . \\ x_m & x_{m+1} & \cdots & x_d \end{pmatrix}$$

whose (i,j) entry is x_{i+j-2} has rank $\le c_1+c_2$. But if $c_1+c_2 \le \min(m,n)$ then this condition is equivalent to the condition that p lie on a (c_1+c_2)-secant (c_1+c_2-1)-plane to the rational normal curve. (See our [1988] for a proof based on Lemma 2.3 of Gruson-Peskine [1982].)//

5. The rank 1 locus of a 1-generic matrix on projective space

In this section we use the excess intersection bounds of section 1 to get some information on the determinantal loci of 1-generic matrices:

Proposition 5 . 1 : Let L be an $(a+1)\times(b+1)$ 1-generic matrix of linear forms on $\mathbb{P}^{a+b}$. If the 2×2 minors of L vanish on a finite subscheme Γ of degree $\geq a+b+3$ in linearly general position, then the ideal they generate is the homogeneous ideal of the unique rational normal curve passing through Γ; in particular, the matrix is equivalent to a catalecticant matrix.

Proof: By Eisenbud-Harris [199? Theorem 2.1], there is a unique rational normal curve through Γ. By the classification theorem of Eisenbud [1988, Cor. 5.2], it is enough to show that the minors vanish identically on this curve. The conclusion now follows from Corollary 1 . 4.//

Remarks: If the minors vanish on a set of dimension ≥ 1, then they have minimal codimension, and thus by the classification theorem L is equivalent to a generic catalecticant (≡ Hankel ≡ persymmetric) matrix, and its 2×2 minors generate the homogeneous ideal of a rational normal curve.

The hypothesis of 1-genericity is necessary (though curiously not if $a=b=1$). For instance if $a=1$, $b=2$, a matrix of the form

$$\begin{pmatrix} x & 0 & 0 \\ y & z & 0 \end{pmatrix}$$

on $\mathbb{P}^3$ can easily vanish on 6 points in linearly general position, and will not vanish on any rational normal curve passing through the points.

On the other hand, it is not so clear whether the hypothesis that the points be in linearly general position is required, except to produce the rational normal curve:

Problem: Can the hypothesis of linearly general position be dropped? That is, if L is an $(a+1)\times(b+1)$ 1-generic matrix of linear forms in $\mathbb{P}^{a+b}$ whose 2×2 minors vanish on a 0-dimensional subscheme of degree $\geq a+b+3$, do they vanish on a subscheme of dimension ≥ 1?

If $a=1$ this is trivial: the minors automatically generate the ideal of a rational normal curve. If $a=b=2$, then and the rank 1 locus of L is 0-dimensional, then it has the same codimension as the rank 1 locus of the generic 3×3 matrix, and thus the same degree, which is $a+b+2$, so the answer is yes in this case as well.

Examples: The zero locus of the 2×2 minors of a matrix as in the Theorem can be $a+b+1$ <u>or</u> $a+b+2$ reduced points in general position. Of course if a or b = 1, then the locus is always a curve; while if $a=b=2$ and the locus consists of points then it is a subscheme of degree exactly $7=a+b+3$, since the codimension, 4, is the generic one in this case. However we can manufacture examples with $a=2$, $b=3$ as follows:

Let L' be a generic 3×5 catalecticant matrix, thought of as a matrix of linear forms on $\mathbb{P}^6$. Let L'_1 and L'_2 be the 4×4 1-generic matrices obtained from L' by deleting the third and fourth column, respectively. Direct computation (using for example the program Macaulay of Bayer and Stillman [1982-1990]) shows that the ideal of 2×2 minors of L'_1 is the same as the ideal of 2×2 minors of L', that is, the ideal of the rational normal sextic curve, while the ideal of 2×2 minors of L'_2 is the ideal of the union of the rational normal sextic and one of its tangent lines, a curve of degree 7 and arithmetic genus 1 (the flat limit of the union of the rational normal curve and one of its secant lines, as the secant line degenerates to a tangent line.) In both cases, the locus is arithmetically Cohen-Macaulay, with ideals generated by the minors.

Now let L_1 and L_2 be the matrices obtained from L'_1 and L'_2 by reducing modulo a general linear form. These matrices will still be 1-generic, since we still have enough variables, and by Bertini's Theorem and the general position lemma, their ideals of 2×2 minors define reduced sets of 6 and 7 points respectively, in linearly general position. The minors actually generate the ideals of the points, as one sees from the fact that the curves associated to the minors of the L'_i were arithmetically Cohen-Macaulay.

One could vary the example of type L_2 by taking a general hyperplane section of a smooth elliptic normal curve in $\mathbb{P}^6$; by Eisenbud-Koh-Stillman [1988] such a curve has homogeneous ideal generated by the 2×2 minors of a matrix associated to any decomposition of the embedding line bundle of degree 7 into the product of line bundles of degrees 3 and 4, and we correspondingly obtain matrices 3×4 1-generic matrices on $\mathbb{P}^5$ whose 2×2 minors generate the ideal of the general hyperplane section of the elliptic normal curve.

Qestion: What other finite subschemes can be the rank 1 loci of 1-generic matrices of this size?

Corollary 5 . 2: If L is an $(a+1)\times(b+1)$ 1-generic matrix of linear forms on $\mathbb{P}^r$ whose rank 1 locus R has a nondegenerate reduced component X of codimension a+b, then the degree of X is $\leq a+b+2$.

Proof: A generic a+b-plane section will satisfy the conditions of Proposition 5 . 1.//

It seems likely that much more is true. For the moment, we can complete the analysis only when the dimension of X is 1:

Theorem 5 . 3: If L is an $(a+1)\times(b+1)$ 1-generic matrix of linear forms on $\mathbb{P}^r$ whose rank 1 locus R has a nondegenerate reduced 1-dimensional component of codimension a+b , then either:

i) R is a rational normal curve, and L is a submatrix of a catalecticant matrix,

or

ii) R is a curve of arithmetic genus 1 embedded by a complete linear series.

In the second case, the 2×2 minors of L generate the homogeneous ideal of R, which is arithmetically Cohen-Macaulay.

Proof: Let X be a nondegenerate reduced component of R having dimension 1 and codimension a+b. Of course we have $r = a+b+1$, so the pair of linear series on X cut out by the planes defined by the rows (respectively columns) of L has Clifford index ≤ 1.

If the Clifford index were 0, then the entries of L would not span the space of all linear forms, so R would be a cone. Thus any 1-dimensional component of R would be a line, which contradicts the nondegeneracy of X. It follows that the Clifford index is 1, so we can apply Theorem 4 . 2 to X and the two linear series corresponding to the rows and columns of L . From the list of possibilities, we see that X is either the rational normal curve or a curve of arithmetic genus 1, and that in the genus 1 case the linear series on X corresponding to L are both complete. In this second case, the main theorem of Eisenbud-Koh-Stillman [1988] shows that R = X, and that the 2×2 minors of L generate the homogeneous ideal of X. //

Problems: 1) If L is an $(a+1)\times(b+1)$ 1-generic matrix of linear forms on $\mathbb{P}^r$ whose rank 1 locus R has a nondegenerate reduced component X of codimension a+b, is R then homogeneously (a cone over) either a rational normal curve, and L is a submatrix of a catalecticant matrix, or an arithmetically Gorenstein scheme of degree a+b+2, (and thus Δ-genus 1)? That is, if R is a curve, is it a rational normal curve or of arithmetic genus 1, if it is a surface is it del Pezzo, ... ?.

2) Which schemes of this type actually arise?

Remark and further conjectures in the case of $\mathbb{P}^1$: It seems plausible that in case i) as well, the 2×2 minors of L generate the homogeneous ideal of R.

In general, if we take a submatrix of a catalecticant matrix, leaving out one generalized row or column -- that is, if we take as pair of series on $\mathbb{P}^1$ the complete series and a codimension 1 series -- then we get a matrix whose 2×2 minors may, as the examples after

Proposition 5 . 1 show, generate either the ideal of the rational normal curve or the ideal of a curve of arithmetic genus 1, the union of a rational normal curve and a secant. It seems that the first case should occur if the codimension 1 subseries is nonsingular, the second if it is singular. (Of course the series must each have dimension >1 for this to make sense; if one has dimension 1, we get a 2-dimensional scroll.) This suggests that the theorem of Eisenbud-Koh-Stillman [1988] should be true for some incomplete series, in particular for a pair of series on $\mathbb{P}^1$ as long as one is complete, the other is very ample of codimension 1, and neither is a pencil. It seems likely that this could be generalized ... .

References

A. Lascoux. Syzygies des variétés déterminantales. Adv. Math. 30 (1978) 202-237.

D. Bayer and M. Stillman. Macaulay, a computer algebra program, for many machines including the Macintosh, IBM-PC, Sun, Vax, and others. Available free from the authors (or ftp 128.103.1.107, login ftp, password any, cd Macaulay) (1982-1990).

D. Bayer and D. Eisenbud: Ribbons. In preparation ****

W. Bruns and U. Vetter: Determinantal rings. Springer Lect. Notes in Math. 1327 (1988).

S. Diaz: Space Curves that intersect often. Pacific J. Math. 123 (1986) 263-267.

D. Eisenbud: Linear Sections of Determinantal varieties. Am. J. Math. 110 (1988) 541-575.

D. Eisenbud and J. Harris: Finite projective schemes in linearly general position. Preprint (199?)

D. Eisenbud, J. Koh, and M. Stillman: Determinantal equations for curves of high degree. Am. J. Math. 110 (1988) 513-539.

D. Eisenbud and A. Van de Ven: On the normal bundles of smooth rational space curves. Math. Ann. 256 (1981) 453-463.

D. Eisenbud and A. Van de Ven: On the variety of smooth rational space curves with given degree and normal bundle. Invent. Math. 67 (1982) 89-100.

L.-Y. Fong: Thesis, Brandeis (1991).

W. Fulton: Intersection Theory. Springer-Verlag, New York (1984).

F. Ghione and G. Sacchiero: Normal bundles of rational curves. Manuscripta Math. 33 (1980) 111-128.

M. Green: Koszul cohomology and the geometry of projective varieties. (with an appendix by M. Green and R. Lazarsfeld.) J. Diff. Geom. 19 (1984) 125-171.

L. Gruson, R. Lazarsfeld, and C. Peskine: On a theorem of Castelnuovo and the equations defining space curves. Invent. Math. 72 (1983) 491-506.

Joe Harris: Curves in Projective Space. (Chapter III by Eisenbud and Harris) University of Montreal Press. 1982.

R. Hartshorne, Algebraic Geometry. Springer-Verlag, New York (1977).

K. Hulek and G. Sacchiero: On the normal bundle of smooth elliptic space curves. Archiv. der Math. 40 (1983) 61-68.

D. Hilbert: Über die Theorie der algebraischen Formen. Math. Ann. 36 (1890) 473-534. (For the result cited, see pp. 239 ff. of Bd. II in the Ges. Abh., Springer-Verlag Berlin 1933 and 1970.)

J. Koh and M. Stillman: Linear syzygies and line bundles on an algebraic curve. J. Alg. 125 (1989) 120-132.

T. G. Room: Determinantal Varieties. ****

B. St.-Donat: On Petri's analysis of the linear system of quadrics through a canonical curve. Math. Ann. 206 (1973), 157-175.