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Finite projective schemes in linearly general position

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Abstract

We give a local structure theory for finite subschemes of projective space in linearly general position, and use it to generalize a lemma of Castelnuovo about subschemes of rational normal curves.

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Introduction

If X is a reduced irreducible variety of codimension c in $\mathbb{P}^r$ over an algebraically closed field F of characteristic 0, then a general plane of dimension c meets X in a set of reduced points in **linearly general position**; that is, no k+2 of them are contained in a k-plane for k<c. For this reason, reduced sets of points in linearly general position play a significant role in many arguments of algebraic geometry, perhaps most notably those of Castelnuovo theory, which gives a bound on the genus of a variety in terms of its degree -- see for example Harris [1982] and [1981].

In certain applications, however, it is desirable to extend the theory to more general subschemes of projective space. For example, given any reduced curve of degree d and genus g in $\mathbb{P}^r$, and a rank 1 subbundle of its normal bundle having degree h, one can form a 1-dimensional subscheme of $\mathbb{P}^r$, a "ribbon" in the sense of Bayer and Eisenbud [****], having degree 2d and (arithmetic) genus 2g-1+h, by doubling the curve along the normal directions specified by the line bundle, as in Ferrand [1975]. A bound on the genus of this subscheme in terms of its degree is then a bound on the degree of a subbundle of the normal bundle of the original curve, something of considerable geometric interest. This application is worked out in Eisenbud-Harris [****Excess intersection...], using the results developed below.

This example led us to the (in any case rather natural) project of studying finite **subschemes** of projective space in linearly general position. The definition is pretty clear: A finite subscheme Γ of $\mathbb{P}^r$ (over some algebraically closed field) is in **linearly general position** if for every proper linear subspace $\Lambda \subset \mathbb{P}^r$ we have

$$\text{degree } \Lambda \cap \Gamma \le 1 + \dim \Lambda.$$

Here of course the intersection $\Lambda \cap \Gamma$ is taken in the sense of schemes; that is, if I is the ideal of Γ and J the ideal of Λ, then I+J defines $\Lambda \cap \Gamma$.

This definition is "good" in the sense that it covers some interesting examples: for instance, at least in characteristic 0, we shall see in section 3 below that the general hyperplane section of any (one-dimensional) ribbon is in linearly general position. Also, some of the standard deductions made in the theory remain true: as we shall also show in section 3, a finite subscheme of degree d in linearly general position in $\mathbb{P}^r$ imposes at least

$$\min(d, rk+1)$$

conditions on forms of degree k, just as if it consisted of reduced points.

The algebraic interpretation of this condition is also interesting. If we let W be an r+1-dimensional vectorspace over F, and write $\mathbb{P}^r = \mathbb{P}(W)$, then a finite subscheme of $\mathbb{P}^r$ corresponds, in a way which is made precise in section 1, to a finite dimensional F-algebra $A = \mathcal{O}_\Gamma(\Gamma)$ and a map from W to A whose image includes the identity element. It turns out that the subscheme is in linearly general position iff for every ideal I of A, the composite map $W \to A \to A/I$ is either a monomorphism or an epimorphism.

Following the idea of the application mentioned above, we will be interested here in whether the lemma of Castelnuovo, which says that a (reduced) set of r+3 points in linearly general position in $\mathbb{P}^r$ must lie on a rational normal curve, remains valid in the context of schemes. Of course for this to be true in the strongest sense it is necessary that each component of Γ look like a point on an unramified smooth curve.

Elementary examples suggest that this necessary condition might not be fulfilled. Any finite subscheme of a projective space, re-embedded by a sufficiently high Veronese map, will be in linearly general position. As another example, the first infinitesimal neighborhood of a point in $\mathbb{P}^r$ is a scheme of degree r+1 in linearly general position, and certainly lies on no smooth curves at all if $r \geq 2$. Similarly, the complete intersection of two plane conics whose intersection is supported at one point is a subscheme of degree r+2 in linearly general position in $\mathbb{P}^r$ for r = 2, and we shall see similar examples for all r later on.

On a more positive note, it is clear that a subscheme Γ in $\mathbb{P}^r$ which has at least r irreducible components cannot contain a "planar triple point" (and it follows that such a scheme must be curvilinear): for if it did, then a hyperplane containing the planar triple point would meet Γ in at least 2 points there, and could be chosen to contain at least r-1 further reduced points.

In general, we shall show that as soon as the degree of Γ is at least r+3, such examples are impossible, and the scheme-theoretic version of Castelnuovo's lemma does hold. The following is the main result of this paper:

Theorem 1: Suppose Γ is a finite subscheme of $\mathbb{P}^r$ in linearly general position over an algebraically closed field.

a) If degree $\Gamma \geq r+3$, then Γ lies on a smooth curve which is unramified at each point in the support of Γ.

b) If degree $\Gamma = r+3$, then Γ lies on a unique (smooth) rational normal curve of degree r.

In terms of the corresponding algebra $A = \mathcal{O}_\Gamma(\Gamma)$, the conclusion of part a) of the theorem implies that if $\dim_F A \geq r+3$, then A is a direct product of rings of the form $F[[t]]/t^n$, for various n, where F is the ground field, and the map $W \to A$ associated to the given projective embedding induces an epimorphism
$W \to A \to F[[t]]/(t^{r+1},t^n)$.

To prove part a) of Theorem 1 we shall give, in section 1, a local structure theory for components of finite subschemes in linearly general position. Having established the validity of part a, we can prove part b by a degeneration argument (taking care of course that the desired degeneration of the rational normal curve in question is still smooth.) This part of the argument is given in section 2.

Theorem 1 has an amusing consequence in the non-classical case. While only r+2 distinct points in linear general position can be freely moved around by automorphisms of $\mathbb{P}^r$, it is possible to move a subscheme of degree r+3 freely as long as its support is small:

Corollary 2: Any two subschemes Γ and Γ' of degree r+3 in linearly general position in $\mathbb{P}^r$ are conjugate by an automorphism of $\mathbb{P}^r$ provided that they are the same as cycles and their supports contain at most 3 points.

Proof: Any two rational normal curves of degree r in $\mathbb{P}^r$ are conjugate; any 3 points of $\mathbb{P}^1$ are conjugate to any other 3; and any automorphism of the curve extends to an automorphism of $\mathbb{P}^r$. //

The example with a planar triple point given above shows that if we add a general point to a subscheme of $\mathbb{P}^r$ in linearly general position the result may fail to be in linearly general position. However, Theorem 1 implies that things are better for subschemes of length $\geq r+3$. It is easy to see that a scheme Γ of the type described in Theorem 1a is in linearly general position iff the osculating spaces to Γ, are transverse. Thus we derive:

Corollary 3: If Γ is a finite subscheme of degree $\geq r+3$ in linearly general position in $\mathbb{P}^r$, and q is a general point, then $\Gamma \cup \{q\}$ is in linearly general position. //

It makes sense to ask how to characterize larger subschemes of rational normal curves, and indeed the first assertion of this sort plays an important role in Castelnuovo theory. Any subscheme of a rational normal curve lies on all the quadrics containing the rational normal curve; Thus for Γ to lie on a rational normal curve, it is necessary that Γ imposes only 2r+1 conditions on quadrics. If Γ is reduced and the degree of Γ is large enough, then Castelnuovo proved that this condition suffices (see for example Griffiths-Harris pp. 531-532 [1978]. We prove that it suffices even without the hypothesis of reducedness:

Theorem 4: Suppose that Γ is a finite subscheme of $\mathbb{P}^r$ in linearly general position. If degree $\Gamma \geq 2r+3$ but Γ imposes only 2r+1 conditions on quadrics, then there is a unique rational normal curve containing Γ.

We shall give a proof of Theorem 4 (which seems to be new even in the classical case where Γ is reduced) in a separate paper, as an application of a simple "excess intersection bound" [****]. The essential idea is that under the Veronese map of $\mathbb{P}^r$ the subscheme Γ will be taken to a subscheme of degree 2r+3 in linearly general position of a plane of dimension 2r, so that we can use Theorem 1 to construct a rational normal curve. The problem then becomes one of proving that this rational normal curve is contained in the Veronese image of $\mathbb{P}^n$, which meets it in the "large" scheme Γ, and this part is handled by the excess intersection result.

Many interesting questions about finite subschemes remain. We single out 3 in particular that bear on nonreduced schemes:

It would be interesting to know about the Γ with degree between r+3 and 2r+3. The only result for this range that we know in the classical case is Mark Green's "Strong Castelnuovo Lemma" [1984, Thm. 3.c.6]; does some analog hold for schemes?

Another interesting project in this direction is to give conditions, as in Eisenbud-Harris [1987 Prop. 3.19], for a subscheme to lie on a rational normal scroll of dimension d.

For certain applications, it would be interesting to know in general what conditions on a 1-dimensional scheme ensure that its general hyperplane section is in linearly general position?

In a different direction, it remains to investigate what happens over a field which is not algebraically closed, and, in the case of Theorem 3.1 and the application to normal bundles, what happens over fields of characteristic p. (This has been done in the meantime by E. Ballico [Preprint, 1991]).

1. Subschemes in linearly general position

We first give an algebraic description of the problem. Let F be a field, and let W be an r+1-dimensional vector space over F. We claim that the following are equivalent data:

i) A finite subscheme $\Gamma \subset \mathbb{P}(W)$, together with a hyperplane of $\mathbb{P}(W)$ which does not meet Γ;

ii) A finite dimensional F-algebra A, with a map of vector spaces $\pi: W \to A$ whose image generates A as an F-algebra, together with a 1-dimensional subspace W_0 of W whose image in A contains 1 (This second datum is important only in the "degenerate" case, when π is not a monomorphism).

To go from i) to ii) we take A to be $\mathcal{O}_\Gamma(\Gamma)$, the affine coordinate ring of Γ. If $x = 0$ is an equation of the given hyperplane, then we write U for the open set $\{x \neq 0\} \cong \mathbb{A}^r$ in $\mathbb{P}(W)$, we take the map the map π to be the "dehomogenization" map

$$W \xrightarrow{1/x} \mathcal{O}_{\mathbb{P}^r}(U) \to \mathcal{O}_\Gamma(\Gamma) = A.$$

The element x goes to 1 under this map, so we may take $W_0 = \langle x \rangle$.

Conversely, to go from ii) to i), let K be the kernel of the map induced by π from the polynomial ring F[W] to A. If we take I to be the ideal generated by the intersection of K with the set of homogeneous forms, then $\Gamma \cong \operatorname{Proj} F[W]/I \subset \operatorname{Proj} F[W] = \mathbb{P}(W)$ is the corresponding subscheme, and it does not meet the hyperplane $\{x=0\}$ for any nonzero element $x \in W_0$. The algebraic fact necessary to show that these two processes are inverse is that if I is a homogeneous ideal of F[W], and x_0 is a nonzerodivisor mod I, then every homogeneous element in (I, x_0-1) is in I.

If Γ corresponds to $W \to A$ as above then an ideal $I \subset A$ corresponds to a subscheme Γ' of Γ, and the kernel of the composite map

$$W \to A \to A/I$$

is the ideal defining the plane spanned by Γ'.

Proposition 1.1: If $\Gamma \subset \mathbb{P}(W) = \mathbb{P}^r$ is a finite subscheme corresponding to $\pi: W \to A$ as above, then the following three conditions are equivalent:

a) Γ is in linearly general position in $\mathbb{P}(W)$.

b) For every ideal $I \subset A$ with $\dim A/I \leq r+1$, the composite map $W \to A \to A/I$ is an epimorphism.

c) For every ideal $I \subset A$, the composite map $W \to A \to A/I$ is of maximal rank (that is, it is either an epimorphism or a monomorphism.)

Proof: If Γ corresponds to $W \to A$ as above then an ideal $I \subset A$ corresponds to a subscheme Γ' of Γ whose degree is the dimension of A/I, and the dimension of the image of the composite map π' in the diagram

$$\begin{tikzcd} W \arrow[r, "\pi"'] \arrow[rr, bend left, "\pi'"] & A \arrow[r] & A/I \end{tikzcd}$$

is one more than the dimension of the plane spanned by Γ'. It follows at once that b) implies a).

Since c) obviously implies b), it remains to show that a) implies c), so we suppose that Γ is in linearly general position. Note that if the composite map $\pi': W \to A/I$ is not a monomorphism, then Γ' lies in a proper subspace of $\mathbb{P}^r$ having dimension $r - \dim \ker \pi'$. Thus

$$\begin{aligned} \deg \Gamma' &= \dim A/I \\ &\leq r+1 - \dim \ker \pi' \\ &= \dim W/\ker \pi' \\ &= \dim \operatorname{image} \pi' \\ &\leq \dim A/I, \end{aligned}$$

so π' is surjective and we are done. //

We shall say that Γ is a **curvilinear point** if

$A \cong F[[x]]/(x^d)$ for some m,

and that Γ is an **unramified curvilinear point** if in addition composite map

$W \to A \to A/(x^{r+1})$

is surjective.

It follows easily from the Chinese Remainder Theorem and Bertini's Theorem that a finite subscheme Γ of $\mathbb{P}^r$ consists of unramified curvilinear points iff it lies on a smooth curve which is unramified at every point in the support of Γ. Indeed, if every component of Γ is an unramified curvilinear point then the general complete intersection of r-1 hypersurfaces of sufficiently high degree containing Γ is a smooth curve, unramified at the points in the support of Γ; and the other implication is trivial.

With these notions we may restate Theorem 1a in the following more local way:

Theorem 1.2: If degree $\Gamma \geq r+3$, then all the components of Γ are unramified curvilinear points.

As a first step toward a proof, we give a complete analysis of the case where Γ consists of one multiple point:

Theorem 1.3: If $\Gamma \subset \mathbb{P}(W)$ has just one component, then Γ is in linearly general position iff it is one of the following types:

I. Γ is an unramified curvilinear point.

II. Γ is not curvilinear, and $W \to A$ is a surjection.

III. Γ is not curvilinear but A is Gorenstein and $W \to A/(\text{socle } A)$ is an isomorphism.

In particular, we see that any curvilinear component of Γ must be unramified.

We shall need the following ideas:

Definition: If A is a local Artinian algebra with maximal ideal $\mathfrak{M}$, then $\operatorname{socle}_i A := (0 : \mathfrak{M}^i)$, the annihilator of the i^{th} power of the maximal ideal in A. We write socle A for $\operatorname{socle}_1 A$; note that $\operatorname{socle}_0 A = 0$.

To identify the algebras in case I of the Theorem we use the following:

Lemma 1.4: If A is a local Artinian $F=A/\mathfrak{M}$ algebra, then A is curvilinear (that is, $A \cong F[[x]]/(x^m)$ for some m) iff
$\dim \operatorname{socle}_2 A \le 2$.

Proof of Lemma 1.4: If A is curvilinear the result is obvious.

Suppose now that $\dim \operatorname{socle}_2 A \le 2$. Since $\mathfrak{M}$ is nilpotent, the condition $\operatorname{socle}_i A = \operatorname{socle}_{i+1} A$ implies that $\operatorname{socle}_i A = A$. Thus if $\dim \operatorname{socle}_2 A < 2$ then $A = F$. If $\dim \operatorname{socle}_2 A = 2$ then the same argument shows that dim socle A = 1, so A is Gorenstein. Thus if we choose an F-linear retraction φ: A → socle A, the composite map

$$A \otimes A \xrightarrow{\text{multiplication}} A \xrightarrow{\varphi} \text{socle } A,$$

is a perfect pairing. This pairing induces isomorphisms

$$(\operatorname{socle}_i A)/(\operatorname{socle}_{i-1} A) \cong \operatorname{Hom}_F (\mathfrak{M}^{i-1}/\mathfrak{M}^i, F),$$

so that in particular

$$\dim \mathfrak{M}^{i-1}/\mathfrak{M}^i = \dim (\operatorname{socle}_i A)/(\operatorname{socle}_{i-1} A).$$

Thus $\dim \operatorname{socle}_2 A = 2$ implies

$$\dim \mathfrak{M}/\mathfrak{M}^2 = \dim (\operatorname{socle}_2 A)/(\operatorname{socle} A) = 2-1 = 1.$$

By Nakayama's lemma, $\mathfrak{M}$ is generated by one element. Since A is Artinian, it follows that A is a homomorphic image of the power series ring F[[x]], so A is curvilinear. //

Proof of Theorem 1.3: We leave to the reader the easy verification that all three types listed in Theorem 1.3 are in linearly general position. Thus we suppose that Γ is in linearly general position, and we must show that it is of one of the three types.

Suppose first that Γ is non-degenerate; that is, W → A is an injection, and identify W with its image in A. The degenerate case will follow easily from this one.

Let $\mathfrak{M}$ be the maximal ideal of A and let p be the largest integer such that $W \cap \mathrm{soc}_p A = 0$. Our first goal is to show that the map $W \to A/\mathrm{soc}_p A$ is an isomorphism, and that for every nonzero $w \in W$, the ideal $\mathfrak{M}w \subset A$ contains $\mathrm{soc}_p A$.

To see that $W \to A/\mathrm{soc}_p A$ is an isomorphism, choose $y \in (\mathrm{socle}_{p+1}A) \cap W$. Writing $\langle y \rangle$ for the 1-dimensional subspace generated by y, we see by Propostion 1.1 that $W/\langle y \rangle$ surjects onto $A/(y)$, which surjects in turn to $A/((y)+\mathrm{socle}_p A)$. Since $y \in \mathrm{socle}_{p+1}A$, we have

$$\dim W - 1 = \dim A/(y) = \dim A/(\mathrm{socle}_p A) - 1,$$

and we see that

$$\dim A/\mathrm{socle}_p A = \dim W,$$

showing that that $W \to A/\mathrm{soc}_p A$ is an isomorphism. This implies that $(y) \supset \mathrm{socle}_p A$, and since $y \notin \mathrm{socle}_p A$ we get

$$\mathfrak{M}y = \mathrm{socle}_p A.$$

We shall next show that for any nonzero element $w \in W$, the ideal $\mathfrak{M}w$ contains $\mathrm{socle}_p A$. Choose an element $a \in A$ so that

$$aw \in \mathrm{socle}_{p+1}A - \mathrm{socle}_p A.$$

By the result of the preceding paragraph, we may write

$$aw = y + z,$$

with $y \in W$ and $z \in \mathrm{socle}_p A$. It follows that $y \in (\mathrm{socle}_{p+1}A) \cap W$; since now

$$\begin{aligned}\mathrm{socle}_p A &= \mathfrak{M}y \\ &\subset \mathfrak{M}aw + \mathfrak{M}z \\ &\subset \mathfrak{M}aw + \mathfrak{M}\,\mathrm{socle}_p A \\ &\subset \mathfrak{M}w + \mathfrak{M}\,\mathrm{socle}_p A,\end{aligned}$$

we see from Nakayama's Lemma that $\mathfrak{M}w$ contains $\mathrm{socle}_p A$ as required.

After these preliminaries we are ready to tackle the statement of Theorem 1.3. Still supposing that Γ is nondegenerate, suppose that $W \to A$ is not onto. It follows that $p>0m$ so $\mathrm{socle}_p A \neq 0$.

First suppose that $p = 1$, so that $\mathrm{socle}_p A = \mathrm{socle}\ A$. We claim that socle A must be 1-dimensional (so that A satisfies condition I or III of the Theorem.) To see this choose a F-linear retraction $\alpha: A \to \mathrm{socle}\ A$, and let $V = W \cap \mathfrak{M}$. Consider the pairing given by the composition

$$V \otimes (\mathfrak{M}/\mathrm{socle}\ A) \xrightarrow{\text{multiplication}} A \xrightarrow{\alpha} \mathrm{socle}\ A,$$

which is well-defined because $V\,(\mathrm{socle}\ A) = 0$. For every $0 \neq v \in V$, the induced map

$$(\mathfrak{M}/\mathrm{socle}\ A) \cong v \otimes (\mathfrak{M}/\mathrm{socle}\ A) \to \mathrm{socle}\ A$$

is onto because $\mathfrak{M}v$ contains socle A. Thus we may consider V as a linear space of linear transformations

$$V \subset \mathrm{Hom}\,(\mathfrak{M}/\mathrm{socle}\ A,\ \mathrm{socle}\ A),$$

and we see that

$$\begin{aligned} \dim V &\leq \dim\,(\mathfrak{M}/\mathrm{socle}\ A) - \dim\,(\mathrm{socle}\ A) + 1 \\ &= \dim V - \dim\,(\mathrm{socle}\ A) + 1, \end{aligned}$$

the right hand side being the codimension of the degeneracy locus of linear transformations from $(\mathfrak{M}/\mathrm{socle}\ A)$ to socle A. Thus $\dim\,(\mathrm{socle}\ A) = 1$ as required.

If on the contrary $p \geq 2$, we shall show that Γ is curvilinear by using the criterion of Lemma 1.4. We may assume that W is at least 2-dimensional (else we are working in $\mathbb{P}^0$!) so there is a nonzero element $w \in \mathfrak{M} \cap W$. Since $\mathfrak{M}w \supset \mathrm{socle}_p A$, we get $\mathrm{socle}_p A \subset \mathfrak{M}^2$. Consequently, to show that Γ is curvilinear, that is, $\mathfrak{M}/\mathfrak{M}^2$ is 1-dimensional, it is enough to check this modulo any ideal contained in

$\mathrm{socle}_p A$. In particular, we may factor out $\mathrm{socle}_{p-2}A$ and we may thus assume that p = 2. If we further factor out socle A = socle_{p-1} A, then A must become Gorenstein by the previous argument; that is, $\mathrm{socle}_p A/\mathrm{socle}\, A$ is 1-dimensional. Thus to apply Lemma 1.4 it will be enough to show that socle A is 1-dimensional.

Choose an F-linear retraction β: A $\to$ socle_2 A, and consider the pairing given by the composite map

$$V \otimes (\mathfrak{M}/\mathrm{socle}\ A) \xrightarrow{\text{multiplication}} A \xrightarrow{\beta} \mathrm{socle}_2\ A.$$

As before, every nonzero element of V induces an epimorphism from $\mathfrak{M}$/socle A to socle_2 A, so that

$$\begin{aligned} \dim V &\le \dim \mathfrak{M}/\mathrm{socle}\ A - \dim \mathrm{socle}_2\ A + 1 \\ &= (\dim V + \dim \mathrm{socle}_2\ A - \dim \mathrm{socle}\ A) - \dim \mathrm{socle}_2\ A + 1 \\ &= \dim V - \dim \mathrm{socle}\ A + 1\ . \end{aligned}$$

It follows that

$$\dim \mathrm{socle}\ A = 1,$$

as required. This completes our analysis in the case where Γ is nondegenerate.

If Γ is degenerate, let $P \cong \mathbb{P}^m$ be the smallest plane containing Γ. In P, Γ is nondegenerate and in linearly general position, so that we can apply the nondegenerate case. Possibilities I and II of this nondegenerate case in $\mathbb{P}^m$ correspond to possibilities I and II in $\mathbb{P}^r$; we must show that case III cannot occur. But this is obvious, since an algebra A as in case III would have degree m+2, and thus would not be in linearly general position in $\mathbb{P}^r$. //

We now analyze subschemes Γ in linearly general position which may have many components. Since each component must be in linearly general position, it suffices to say how components, each in linearly general position, can be put together so as to maintain linearly general position.

We will use the classification of Theorem 1.3, and speak of components of types I, II, and III. The simplest but dullest way to put these together is what we shall call the **direct sum**:

Definition: A finite subscheme $\Delta \subset \mathbb{P}^r$ is the direct sum of subschemes Δ_i if $\Delta = \cup\Delta_i$ and the Δ_i are linearly disjoint in the sense that for each i the linear spans of Δ_i and of $\cup_{j\neq i} \Delta_j$ are disjoint.

Proposition 1.5: If a finite scheme $\Delta\subset\mathbb{P}^r$ is the direct sum of subschemes Δ_i , then Δ is in linearly general position in $\mathbb{P}^r$ iff each of the Δ_i is in linearly general position in $\mathbb{P}^r$. //

Direct sums are not very interesting because if Γ (which we have assumed in linearly general position) is nontrivially a direct sum, then degree $\Gamma \leq r+1$. (Reason: each summand is degenerate, thus of degree at most one more than the dimension of its span.) Nevertheless, if Γ contains a noncurvilinear component, then Γ is almost a direct sum:

Theorem 1.6: If Γ contains a noncurvilinear component $\Gamma_1 \subsetneq \Gamma$, then Γ_1 is of type II. If Γ_1 is not Gorenstein, then it is a direct summand of Γ. If Γ_1 is Gorenstein, then its unique colength 1 subscheme Γ_1' is a direct summand of the corresponding colength 1 subscheme Γ' of Γ.

Example: The following example shows that a Gorenstein noncurvilinear point of type II can fail to be a direct summand:

Let Γ_1 be (abstractly) the planar 4-tuple point

$$\Gamma_1 = \text{Spec } F[x,y]/(x^2,y^2),$$

and let W be the four-dimensional vector space with basis x_0, x_1, x_2, x_3. Embed Γ_1 in $\mathbb{P}^3 = \mathbb{P}(W)$ by the map

$$W \to A$$

sending

$$\begin{aligned} x_0 &\mapsto 1 \\ x_1 &\mapsto x \\ x_2 &\mapsto y \\ x_3 &\mapsto xy. \end{aligned}$$

This makes Γ_1 a nondegenerate Gorenstein point of type II. Let $\Gamma = \Gamma_1 \cup \{q\}$ where q is the reduced point $x_0=x_3$, $x_1=x_2=0$. It is clear that a plane through q cuts Γ_1 in a scheme of degree at most 2, and a line through q cuts Γ_1 in a scheme of degree at most 1, so Γ is in linearly general position. Since Γ_1 is nondegenerate, it cannot be a direct summand of Γ.

Proof of Theorem 1.6: Since we have assumed that Γ_1 is not curvilinear, it must be of type II or III. Supposing that Γ_1 is of type III we will derive a contradiction. Writing A_1 for the local ring of Γ_1, A_1 is Gorenstein of degree r+2, and the natural map $W \to A_1$ induces an isomorphism

$$W \cong A_1/\text{socle } A_1.$$

Let q be a (reduced) point contained in Γ but not in Γ_1. By Lemma 1.4 $\text{socle}_2\, A/\text{socle}_1\, A$ has dimension at least 2, and thus contains a nonzero linear form $x \in W$ vanishing at q. The ideal generated by x in A_1 consists of $\langle x\rangle$ and the socle of A_1 and thus has length 2. It follows that the hyperplane defined by x cuts Γ_1 in a scheme of degree r, and cuts Γ in a scheme of degree r+1, contradicting the hypothesis of linearly general position. Thus Γ_1 is not of type III, and must be of type II.

Let Γ_2 be the union of the components of Γ other than Γ_1. If Γ_1 is Gorenstein, it contains a unique colength 1 subscheme Γ_1'. By Lemma 1.4, Γ_1' is not Gorenstein. Let $\Gamma' = \Gamma_1' \cup \Gamma_2$. The schemes Γ_1' and Γ' fulfill the same hypotheses as Γ_1 and Γ, so we may, changing notation, assume from the outset that Γ_1 is not Gorenstein, and we must show that Γ is the direct sum of Γ_1 and Γ_2; that is, writing Λ_i for the linear span of Γ_i we must show that $\Lambda_1 \cap \Lambda_2 = \emptyset$.

Suppose on the contrary $\Lambda_1 \cap \Lambda_2 \neq \emptyset$. Since Γ_2 is in linearly general position we may, for some number m, find a subscheme Γ_2' of degree m+1 spanning an m-plane Λ_2' which meets Λ_1 in a single point q. It now suffices to find a hyperplane H in Λ_1 containing q and meeting Γ_1 in a subscheme of colength 1; since the degree of Γ_1 is $1+\dim \Lambda_1$ the join of H with Λ_2' will be a plane of dimension

$$\dim H + m = \dim \Lambda_1 - 1 + m$$

meeting Γ in a subscheme of degree at least

$$\dim \Lambda_1 + 1 + m,$$

contradicting linearly general position, and establishing the Theorem.

Let $\mathbb{P}^r = \mathbb{P}(W)$, and write W_1 for the image of W in the local ring A_1 of Γ_1, so that $\Lambda_1 = \mathbb{P}(W_1)$. Let $V \subset W_1$ be the hyperplane of linear forms vanishing at q. We must show that for some element $v \in V$ we have length $A_1/(v)$ = length $A_1 - 1$; that is, we must show that some element of V lies in the socle of A_1. Since A_1 is of type II in the classification of Theorem 1, W_1 is all of A_1 and V is of codimension 1 in A_1. Since A_1 is not Gorenstein, its socle is of length ≥ 2, and meets V as required. //

With all this in hand, the proof of Theorem 1.2 is easy:

Proof of Theorem 1.2: If Γ consists only of curvilinear components then we are done by the remark following the statement of Theorem 1.3. If on the other hand Γ had a noncurvilinear component, then Γ would contain a colength 1 subscheme Γ' which is nontrivially a direct sum by Theorem 1.6. By the remark preceding the statement of Theorem 1.6, degree $\Gamma' \leq r+1$, so we would have degree $\Gamma \leq r+2$.//

2. Finite subschemes of rational normal curves

In this section we shall prove Theorem 1b.

In general, one may ask when a finite subscheme Γ of $\mathbb{P}^r$ is contained in a rational normal curve of degree r (that is, a copy of $\mathbb{P}^1$ embedded in $\mathbb{P}^r$ by the complete linear series of degree r). Of course a subscheme of a rational normal curve must be curvilinear, and it is not hard to see that it must be in linearly general position as well. Any curvilinear subscheme in linearly general position is contained in another such subscheme of degree $\geq r+3$, so it suffices to treat the case of a curvilinear scheme in linearly general position of degree $r+3$. By the results of the last chapter, it suffices to assume simply that Γ **is in linearly general position.**

Theorem 2.1: If Γ is a finite subscheme of $\mathbb{P}^r$ in linearly general postion, and if degree Γ = r+3 then there is a unique rational normal curve containing Γ.

Proof: The uniqueness statement follows from the fact that the union of two distinct rational normal curves meeting in a subscheme of degree $\geq r+3$ would be a curve of degree 2r and genus $\geq r+2$, with general hyperplane section consisting of 2r (reduced) points in linearly general position imposing $\leq 2(r-1)$ conditions on quadrics in $\mathbb{P}^{r-1}$; this is easily seen to be impossible. Alternately, the result we want is the special case of Corollary ** of Eisenbud-Harris [****Excess intersection] in which X and C are both rational normal curves.

To prove existence we may, by Theorem 1a, work by degeneration: Regard $\Gamma = \Gamma_0$ as a limit of schemes $\{\Gamma_t | t \in D\}$ all containing a common point p, and such that the Γ_t are reduced for $t \neq 0$. By the classical version of Theorem 1b (see for example Griffiths-Harris [1978 p. 530] or Eisenbud-Harris [1987]) there is a unique rational normal curve C_t containing Γ_t for $t \neq 0$, and we may take its limit C_0 in the Hilbert scheme of curves of degree r and genus 0 in $\mathbb{P}^r$ as t goes to 0. We will prove that C_0 is again a smooth rational normal curve by following a "good" representation of the equations of C_t as $t \to 0$.

To this end, suppose that the ideal of p is generated by the linear forms $x_1, \dots, x_r$. Consider the vector space U of r-tuples of linear forms modulo the r-tuple $x_1, \dots, x_r$. For each point $y \in \mathbb{P}(U^*)$, choose a representative r-tuple $y_1, \dots, y_r$ of linear forms, and let C_y be the subscheme of $\mathbb{P}^r$ defined by the 2×2 minors of the matrix

$$M = \begin{pmatrix} x_1 \dots x_r \\ y_1 \dots y_r \end{pmatrix}.$$

Note that C_y really depends only on y, and not on the choice of the r-tuple $y_1, \dots, y_r$. Further, every rational normal curve containing p can be written as C_y for a uniquely determined y. Thus C_t is of the form C_{y_t} for some path $t \mapsto y_t$ in $\mathbb{P}(U^*)$. Let y_0 be the limit of the y_t as $t \mapsto 0$. The limiting scheme C_0 must be contained in the scheme C_{y_0}. We shall show thatC_{y_0} is in fact a smooth rational normal curve. Note that C_{y_0} contains Γ, so this concludes the argument.

We shall show that if any C_y contains a finite subscheme Γ of $\mathbb{P}^r$ in linearly general position and of degree $\geq r+3$, then M is 1-generic in the sense of Eisenbud [1986]; that is, that every linear combination of the rows of M consists of linearly independent linear forms. It follows easily from this that C_y is a rational normal curve (see for example Harris [1982 p. 104-5] or Eisenbud [1986]. Suppose on the contrary that M were not 1-generic -- that is, that some linear combination of the rows consisted of linearly dependent linear forms. We may without loss of generality assume that this linear combination is $y_1, \dots, y_r$ itself. Making a linear transformation of the columns, we may suppose that $y_1, \dots, y_k$ are linearly independent, while $y_{k+1}, \dots, y_r$ are zero. But then the ideal of 2×2 minors of **M** contains the ideal

$$(y_1,\dots,y_k)(x_{k+1},\dots,x_r).$$

If Γ were a set of reduced points, the impossibility of this would be immediate. In our case the result is slightly more complicated. The following Lemma completes the proof:

Lemma 2.4: Let Γ be a curvilinear subscheme of $\mathbb{P}^r$ in linearly general position. If Γ is contained in the scheme defined by an ideal of the form I_1I_2 where I_i is generated by a_i independent linear forms, and $a_1+a_2 = r$, then degree $\Gamma \le r+2$.

Proof: We first remark that if A is any Artinian ring $I_1 \subset A$ is an ideal, and $I_2 = (a)$ is a principal ideal, then length $A/I_1I_2 \le$ length A/I_1 + length A/I_2 simply because $I_2/I_1I_2 = (a)/aI_1$ is a homomorphic image of A/I_1. If now A is the coordinate ring of a finite curvilinear scheme, then A is a principal ideal ring, so this formula holds for any two ideals.

Returning to the case at hand, note that the subscheme of Γ defined by I_j has degree $\le r - a_j + 1$ because Γ is in linearly general position. Because Γ is curvilinear we may apply the above remark and conclude that the subscheme defined by I_1I_2 has degree $\le (r - a_1 + 1) + (r - a_2 + 1) = r+2$ as required.//

Returning to the proof of the existence statement in Theorem 2.1, let $X \subset D \times \mathbb{P}(A^*)$ be the subvariety

$$X = \{(t,y) \mid \Gamma_t \subset C_y \}.$$

Since X projects onto D-{0}, and $X \to D$ is proper, we see that there is a $y \in \mathbb{P}(U^*)$ such that C_y contains $\Gamma_0 = \Gamma$, and we are done.//

3. Hyperplane sections of ribbons, and the number of conditions imposed

A **ribbon**, according to Bayer and Eisenbud [****] is a scheme C such that the ideal sheaf $\mathcal{J}$ of C_{red} in C has square 0 and the conormal sheaf $\mathcal{J} = \mathcal{J}/\mathcal{J}^2$ is a line bundle on C. Here we shall be concerned exclusively with the case where C_{red} is an irreducible curve. We shall prove:

Theorem 3.1: If $C \subset \mathbb{P}^r$ is a ribbon over an algebraically closed field of characteristic 0, such that C_{red} is not contained in a hyperplane of $\mathbb{P}^r$, then the general hyperplane section of C is a subscheme of $\mathbb{P}^{r-1}$ in linearly general position.

To discuss the hyperplane sections conveniently we introduce some terminology: a subscheme of degree 2 of C supported at a point will be called a **doublet** of C, and the associated reduced point will be called a **singleton**. Thus any subscheme of the general hyperplane section will be a union of doublets and singletons.

Proof: Say a minimal linearly dependent subset Γ of a general hyperplane section $H \cap C$ consists of m doublets and n singletons. Assume that $H \cap C$ is not in linearly general position in H, so that $2m+n \leq r$.

The monodromy action on the points of a general hyperplane section of a reduced irreducible curve is the full symmetric group (see for example Griffiths-Harris [1978] p. 249.) Thus if there is one singleton in Γ, then every reduced point in the hyperplane section is dependent on the remaining m doublets and n-1 singletons, and thus every point in the general hyperplane section of C_{red} lies in a codimension 2 plane; this contradicts the fact that the points of a hyperplane section of C_{red} are in linearly general position. Thus we may assume $n = 0$, $2m \leq r$.

Using the monodromy argument again, we see that the 2m-3-plane Λ spanned by m-1 of the doublets of Γ meets the line spanned by a general doublet of C (at a point not on C_{red}).

Consider for each point $p \in C_{red}$ the 2-plane T_p which is the tangent plane to C at p. If, for a general point p, this plane met Λ in at most a point, then the general doublet in C supported at p would violate the condition above. It follows that Λ meets T_p in at least a line.

This situation leads to a contradiction: if each T_p meets Λ in a line, then the tangent line to C_{red} at each point meets Λ, and projection from Λ is a nowhere smooth map contradicting generic smoothness (or, over $\mathbb{C}$, Sard's Theorem.) //

The next result addresses a well known property of reduced sets of points in linearly general position:

Theorem 3.2: A subscheme Γ of degree d in linearly general position in $\mathbb{P}^r$ imposes at least

$$\min(d, rk+1)$$

conditions on forms of degree k.

Proof: We shall do induction on k, the case k = 1 being equivalent to the definition of linearly general position.

Suppose that $\Gamma \subset \mathbb{P}^r$ corresponds to $W \to A$ in the sense described at the beginning of section 1. If we choose an ideal of colength r in A, then it meets W in a 1-dimensional space, spanned, say, by $x \in W$. Since Γ is in linearly general position, we actually have length $A/(x) = r$, and the natural map $W/\langle x\rangle \to A/(x)$ is an isomorphism. Let Γ' be the subscheme of Γ defined by the ideal I which is the kernel of multiplication by x on A, so that we have an exact sequence

$$0 \to I \to A \xrightarrow{x} A \to A/(x) \to 0,$$

and let Γ'' be the subscheme defined by (x). It follows that

$$\text{degree } \Gamma' = \text{degree } \Gamma - r,$$
$$\text{degree } \Gamma'' = r.$$

Writing $S_k W$ for the k^{th} symmetric power of W, consider the diagram

$$\begin{array}{ccccc}
 & 0 & & 0 & & 0 \\
 & \downarrow & & \downarrow & & \downarrow \\
 & S_{k-1}W & \twoheadrightarrow & N' & \hookrightarrow & A/I \\
 & \downarrow & & \downarrow & & \downarrow \\
 & S_k W & \twoheadrightarrow & N & \hookrightarrow & A \\
 & \downarrow & & \downarrow & & \downarrow \\
 & S_k(W/\langle x\rangle) & \twoheadrightarrow & N'' & \hookrightarrow & A/(x) \\
 & \downarrow & & \downarrow & & \downarrow \\
 & 0 & & 0 & & 0
\end{array}$$

where the two outer columns are exact, and the middle column represents the sequence of images of the left hand column in the right hand column under the maps induced by multiplication in A/I, A and A/(x). The middle column is thus a complex, exact at N' and at N'', so we have

$$\dim N \geq \dim N' + \dim N''.$$

By induction, Γ' imposes at least

$\min(d-r, r(k-1)+1)$

conditions on forms of degree k-1, while Γ'' imposes exactly r conditions on forms of degree 1 (and thus r conditions on forms of every degree.) But the number of conditions imposed by Γ is just the dimension of N, while the number imposed by Γ' and Γ'' are the dimensions of N' and N'' respectively. Thus we see from the inequality above that the number of conditions imposed by Γ is at least

$$\min(d-r, r(k-1)+1) + r = \min(d, rk+1),$$

as required. //

References

Dave Bayer and David Eisenbud: Ribbons. In preparation. ****

David Eisenbud: Linear sections of determinantal varieties. Am. J. Math. (1986) ****

David Eisenbud and Joe Harris: On Varieties of minimal degree (a centennial account). Proc. Symp. Pure Math. 46, ed. Spencer Bloch. Amer. Math. Soc. 1987)

David Eisenbud and Joe Harris: An excess intersection bound and applications to Castelnuovo and Clifford theory. In preparation. ****

Joe Harris: Curves in Projective Space. (Chapter III by Eisenbud and Harris) University of Montreal Press. 1982.

Joe Harris: A bound on the geometric genus of projective varieties. Ann. Scuola Norm. Sup. Pisa 8 (1981) 35-68.

Daniel Ferrand: Courbes gauches et fibres de rang 2. C. R. Acad. Sci. Paris ser. A, t. 281 (1975) 345-347.

Phillip Griffiths and Joe Harris: Principles of Algebraic Geometry. John Wiley and Sons, 1978.

Mark Green: Koszul cohomology and the geometry of projective varieties. J. Diff. Geom. 19 (1984) 125-171.