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Graph Curves*

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With an appendix by Sung Won Park

We study a family of stable curves defined combinatorially from a trivalent graph. Most of our results are related to the conjecture of Green which relates the Clifford index of a smooth curve, an important intrinsic invariant measuring the "specialness" of the geometry of the curve, to the "resolution Clifford index," a projective invariant defined from the canonical embedding. Thus we study the canonical linear series and its powers and identify them in terms of combinatorial data on the graph; we given combinatorial criteria for the canonical series to be base point free or very ample; we prove the analogue of Noether's theorem on the projective normality of smooth canonical curves; we define a combinatorial invariant of a graph which we conjecture to be equal to the resolution Clifford index of the associated graph curve, at least for "most" graphs; and we prove our conjecture for planar graphs and for graphs of Clifford index 0. Along the way we prove a result of some independent interest on the canonical sheaves of (not necessarily arithmetically Cohen–Macaulay) face varieties. The Appendix establishes a formula connecting the combinatorics of a trivalent graph G and the minimal degree of an admissible covering of a curve of arithmetic genus 0 by the corresponding graph curve. © 1991 Academic Press, Inc.

* We dedicate this paper to the memory of our student and friend Sung Won Park, who died in the first flush of his mathematical career in January 1990.

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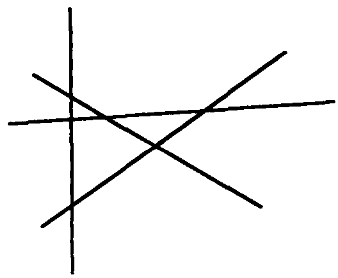
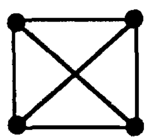
INTRODUCTION

A *graph curve* C is a connected, projective algebraic curve which is a union of copies of the projective line, each meeting exactly three others, transversely at distinct points; for example,



is the only graph curve that can be embedded in the projective plane. Since the automorphism group of $\mathbb{P}^1$ is 3-transitive, the points of attachment do not really matter, and the isomorphism class of a graph curve is specified in terms of the purely combinatorial data of how the lines meet. It is convenient to express this in terms of the *dual graph* G which has a vertex for each component of C and an edge for each node of C, so that for example the dual graph of the curve above is the complete graph on four vertices:



Of course the dual graph of a graph curve is trivalent—that is, every vertex lies on exactly three edges. We will write $C = C(G)$ for the graph curve corresponding to the trivalent graph G. Every invariant of C is, of course, a combinatorial invariant of the graph G—for example, the genus of C is one more than half the number of vertices of the graph, and the space of global sections of the canonical sheaf of C is the first cohomology group of the graph. Indeed, the set of images of nodes of the graph curve, under the map induced by the canonical series on the curve, has been studied by graph theorists as the "canonical realization of the bond matroid of the graph" (Tutte, [1, Theorem 7.47, p. 359]).

Graph curves are interesting because they are the only stable curves with smooth components that admit no degenerations; they are in some sense ultimate degenerations of smooth curves. There has been intense development over the last 15 years in our knowledge of the properties of "general" smooth curves of genus g, and the main tools of that development have been ever more refined uses of degenerations of smooth curves to singular curves, mostly to stable singular curves; in other words uses of the Deligne–Mumford compactification of the moduli space of smooth curves. Roughly speaking, it has seemed that the more "special" a curve could be chosen and studied, the more different loci in the moduli space it would avoid, and thus the more it would behave like a "general" curve. Nice properties of special curves translate into theorems about properties of the general curve, because such properties are usually invariant under deformation. In particular, one can hope that by studying graph curves one could approach some of the more resistant problems about smooth curves. In the paper below we will follow this program as it related to the Clifford index of smooth curves.

Recall that the Clifford index of a smooth curve of genus g (which we take $\geqslant 3$ to avoid trivialities) may be defined as the minimum, over all line bundles $\mathscr{L}$ on the curve satisfying $h^0\mathscr{L}$, $h^1\mathscr{L} \geqslant 2$, of the quantity

$$\begin{aligned}\operatorname{Cliff} \mathscr{L} &= g + 1 - h^0\mathscr{L} - h^1\mathscr{L}\\ &= \deg \mathscr{L} - 2h^0\mathscr{L} + 2,\end{aligned}$$

(the last equality being Riemann–Roch). Clifford's theorem says in this language that if C is a smooth curve then $\operatorname{Cliff} C \geqslant 0$, with equality iff C is hyperelliptic; it is known from Brill–Noether theory that

$$\operatorname{Cliff} C \leqslant [(g-1)/2]$$

for every curve, with equality holding for the general curve of each genus, and from a result of Ballico [3] that every value between 0 and $(g-1)/2$ is actually taken on.

The most naive approach to defining the Clifford index for a singular (perhaps stable curve) would be to follow the usual definition for smooth curves and define it as the least value of the expression

$$g + 1 - h^0\mathscr{L} - h^1\mathscr{L},$$

taken over all the line bundles $\mathscr{L}$ on C with both $h^0\mathscr{L}$ and $h^1\mathscr{L} \geqslant 2$.

However, experience has shown that as soon as the curve becomes reducible, this straightforward extension is rarely the right one. There are several other possibilities:

Perhaps most interesting from the point of view of the study of smooth curves nearby C would be to take the Clifford index of a stable curve C to be the minimum of the Clifford indices of curves in arbitrary small (Zariski) neighborhoods of C in the compactified moduli of curves of genus—we will call this the *limit Clifford index of* C. One might hope that the limit Clifford index would be measured by the Clifford indices of some appropriately defined "limit linear series" on C, where the Clifford index of a series is its degree minus twice its (projective) dimension, as usual. In the case of pencils one has an adequate notion of limit series coming from the theory of admissible covers; these admissible covers of genus 0 curves by graph curves are the subject of the appendix by Park. One might call the Clifford index defined in this way, in terms of admissible covers, the *admissible Clifford index* of C. Since admissible covers are always limits of covers of $\mathbb{P}^1$ by nearby smooth curves, one obtains a direct link to the limit Clifford index. However, it seems likely that there is a notion of limit linear series of arbitrary dimension like that for curves of compact type (see Eisenbud and Harris [8] and Ran [16]) and it is not so clear that a Clifford index computed in terms of these would be equal to the limit Clifford index as defined above; the variety of curves of some low Clifford index might well have components contained entirely in the boundary of moduli.

Another approach to defining the Clifford index of a graph curve C would be to invert Green's conjecture [9] for smooth curves and define the *resolution Clifford index* of C to be the largest integer c such that $\mathrm{Tor}^S_{g-2-c}(\mathbb{C}, R)_{g-1-c} \neq 0$, where R is the homogeneous coordinate ring of C, and S is the homogeneous coordinate ring of the projective space in which C is canonically embedded.

It is known (Green-Lazarsfeld [10]) that for smooth curves there is at least an inequality between the resolution Clifford index and the ordinary Clifford index, defined as above, though the proof of this inequality does not immediately extend to graph curves. However, in this section we will compute the naive Clifford index of certain line bundles on graph curves and prove for them an inequality of the above type. This will suggest yet another definition of the Clifford index—perhaps one should call it the "combinatorial Clifford index" of a graph curve $C(G)$, namely $f-2$, where f is the "homoliferous disconnection number of G," that is, the smallest number of edges which disconnect G into two connected pieces, each with nontrivial homology. Park has shown that with the exception of a small number of graphs, this combinatorial Clifford index really is linked with the limit Clifford index defined in terms of admissible coverings.

As was pointed out to us by Joe Harris, a curve with given resolution Clifford index, defined as above, need not be the limit of smooth curves with such a low Clifford index. For example, if we take a graph G' of high Clifford index, and replace one vertex v with a triangle to obtain G, then the curve $C(G)$, which has Clifford index 1, is not the limit of similar smooth curves (Reason: the quadrics containing the canonical image of $C(G)$ cut out the union of $C(G'-v)$ and the plane containing the triangle, whereas the quadrics containing the limit of smooth curves (assuming that the limit is arithmetically Cohen–Macaulay, and thus lies on the correct number of quadrics) will contain a nondegenerate surface. Harris, Miranda, and Ciliberto (unpublished) have also checked the nonexistence of a degree 3 admissible cover of a genus 0 curve by such a curve.

All of these invariants, for graph curves, must of course be combinatorial invariants of the graph. In this paper we present some evidence for the

Combinatorial Clifford Index Conjecture. The combinatorial Clifford index Cliff G of a trivalent 3-connected graph G is equal to the resolution Clifford index of the corresponding graph curve $C(G)$ as long as Cliff $G < [(g-1)/2]$.

By the theory of extremal graphs, the last inequality is satisfied for all large graphs—see Section 5. We prove the conjecture if either of the two Clifford indices are 0, and we prove it for planar graphs. We had originally hoped that a proof of this conjecture would lead to a proof of Green's conjecture for general curves, but this is not the case because the combinatorial Clifford index of a graph curve of genus g is never more than about $2\log_2(g)$ (see Section 5).

We would like to mention three other projects using graph curves that have recently been undertaken:

Park, in an appendix to this paper, shows how to approach the admissible covers by graph curves of curves of genus 0 and has related this to the combinatorial Clifford index.

Ciliberto, Harris, and Miranda [7] have used graph curves (including some of the foundational material developed below) to prove a conjecture of Jonathan Wahl, concerning the natural map

$$\Lambda^2 H^0(\omega_C) \to H^0(\omega_C^{\otimes 3})$$

on a smooth curve C, called the Wahl map. Wahl had proved that if this map is surjective, then C cannot be embedded in a smooth $K3$ surface, and he conjectured that the map is surjective if the genus of C is 10 or $\geqslant 12$. This conjecture could not be attacked directly using degenerations to curves of compact type because the Wahl map is never surjective for these curves; but Ciliberto, Harris, and Miranda proved the conjecture by showing that it is surjective for certain graph curves (corresponding to certain generalizations of the Petersen graph) of the appropriate genera.

In another direction, Kra [14] has observed that the smoothing parameters of the $3g-3$ nodes of a graph curve of genus g can be chosen rather naturally to give a kind of canonical system of coordinates on the moduli space in the neighborhood of a graph curve.

In the body of the paper we often deal with somewhat more general curves than the above, corresponding to allowing the dual graphs to have multiple or returning edges and to be other than trivalent and connected, though we always take each component of the curve to be of geometric genus 0. Of course such curves may not be determined entirely in terms of the combinatorial data of the dual graph, but some interesting properties of them depend only on the dual graph anyway.

We now describe the contents of the paper in more detail. The first four sections are foundational in nature, although they only develop the part of the foundations related to the canonical embedding. In Section 1 we give expressions for the canonical series on a graph curve and its powers in terms of the 0-chains, 1-chains, and 1-cycles of the graph.

In Section 2 we study the ampleness of the canonical series. We prove an ampleness result for graphs other than trivalent ones; our main result is that the canonical series on the graph curve of a trivalent graph is very ample iff the graph is 3-connected. The main idea is to show that the restriction of the canonical series to certain curves made from subgraphs is a complete series.

In Section 3 we explain how to compute the canonical embeddings of graph curves explicitly, by computing the ideals of the curves. We made extensive computations using this method on the computer algebra system Macaulay of Bayer and Stillman [4], and the results of these computations lead to the conjecture above and to a number of theorems proved in the text.

In Section 4 we show that if the graph curve of a trivalent graph is embedded by its canonical series, then the homogeneous coordinate ring is arithmetically Cohen–Macaulay, and thus Gorenstein.

In Section 5 we begin the main work of the paper, on Clifford indices. Starting from a connected subgraph V of G we construct some line bundles on a graph curve whose Clifford indices depend only on the combinatorics of V in G and whose Clifford indices bound the resolution Clifford index of $C(G)$. We show that the maximum of the Clifford indices of such line bundles is the combinatorial Clifford index defined above.

In Section 6 we study face varieties coming from compact simplicial manifolds. We show in particular that the canonical class of such a variety is trivial iff the manifold is orientable (else its square is trivial), generalizing work of Hochster in the case of homology spheres. In case the manifold has dimension 2, the face variety is a singular algebraic surface, and we show how such surfaces fit into the classification of algebraic surfaces (for example, the real projective plane gives rise in this way to Enriques surfaces!) The motivation for this is that the graph curve of the dual graph of the simplicial surface is embedded along with the surface, and the embedding is by a subseries of the canonical series if the surface is orientable. In this way we recover some classical results about embeddability and unique embeddability of graphs due to Whitney [19] and Steinitz–Rademacher [18].

In Section 6 we use the material on face varieties to reduce the combinatorial Clifford index conjecture for planar graphs to a question about the resolutions of face varieties, which can be answered using the methods of Reisner [17], proving the conjecture for these graphs. Unfortunately, planar graphs all have Clifford index $\leqslant 3$, so we still have no graphs of large Clifford index for which we know the conjecture to be true. In the last section we offer of problems, not all terribly deep, from this newly forming area.

We began working on this material following suggestions from Joe Harris which go back to ideas of his and Ciro Ciliberto's. We also thank Rob Lazarsfeld, Dave Morrison, and Henry Pinkham for some helpful discussions of the algebraic geometry, and Joe Buhler, Sung Won Park, and Richard Stanley for enlightening us on various aspects of the combinatorics.

0. Definitions and Notation

For any projective curve C we define $g(C)$ to be $1-\chi(\mathcal{O}_C)$; if C is reduced, for example if C is a graph curve, then this becomes

$$g(C) = h^0\omega_C - (\text{the number of connected components of } C) + 1.$$

A *graph* is allowed to have multiple and returning edges.

The number of edges incident to a vertex v is its *valency*, $\operatorname{val}(v)$. G is *trivalent* if every vertex has exactly three incident edges. It is *at most trivalent* if every vertex has $\leqslant 3$ incident edges.

If V is a set of vertices of G, then $G-V$ is the graph obtained by removing the vertices V and all the edges incident to them. We will identify V with the full subfgraph on V, consisting of the vertices in V and all the edges incident only to vertices in V. If E is a set of edges of G, then $G-E$ is the graph obtained by removing all the edges E but leaving all the vertices.

G is k-connected (more properly "k-edge-connected") if the removal of $<k$ edges does not increase the number of connected components of G (in

particular, G need not be connected to begin with.) If G contains a vertex of valency r which is not itself a connected component, then clearly G is not $(r+1)$-connected. A trivalent graph with double or returning edges is never 3-connected.

G is k-vertex-connected if the removal of $<k$ vertices does not disconnect G.

The following are defined for any graph G:

$dG =$ number of vertices in G.

$eG =$ number of edges in G. If E is a set of edges of a graph G, we will similarly write eE for the number of edges in E.

$nG =$ number of connected components of G.

Coch $G =$ group of 1-cocycles of G (with some chosen orientation).

$gG = \dim H_1(G, \mathbb{C}) - nG + 1$, called the genus of G. By the *Euler formula* this may be written as $gG = eG - dG + 1$.

$C(G) =$ curve with rational components having dual graph G. C contains a $\mathbb{P}^1$ for each vertex of G, a node for each edge of G.

Throughout the following, G will be a graph, $C = C(G)$ will be the corresponding graph curve, and g, d, e, and n will denote gG, dG, eG, and nG, respectively.

We fix once and for all an orientation of G. In particular, this gives us a canonical pair of dual bases for the spaces of 1-chains and 1-cochains of G, in terms of which these dual spaces may be identified and thought of as inner-product spaces. In this way, for example, the 1-cohomology of G with coefficients in a field, say, is identified with the space of 1-cochains orthogonal to the 1-coboundaries, and the homology and cohomology of G are identified.

1. The Sections of Powers of the Canonical Bundle

In this section we will identify the canonical series of a graph curve and its tensor powers (the identifications will depend on the choice of orientation). We begin by identifying the space of sections of the canonical sheaf of C with the 1-cohomology of G.

Proposition 1.1. *There is a natural isomorphism $H^0\omega_{C(G)} \cong H^1(G, \mathbb{C})$; in particular, $g(C) = g(G)$.*

Proof. The restriction of ω_C to a component of C is the canonical series twisted by the set of points that are nodes of C. A section of this restriction is identified by its residues at the nodes. If two components A and B of C

meet in a node p, then any section of ω_C, restricted to A and B, must have residues at p that sum to 0. We may thus make each section ω of ω_C correspond to the 1-cochain on G whose value at the oriented edge E going from the vertex v corresponding to A to the vertex w corresponding to B is the residue of ω at the point p of B.

Since $A \cong \mathbb{P}^1$, there is a differential having poles at a given collection of points, and having specified residues there, if and only if the sum of these residues is 0; thus a 1-cochain on G corresponds to a differential form iff it is orthogonal to the 1-coboundaries—that is, iff it is an element of $H^1(G, \mathbb{C})$. ∎

There is a natural bilinear map from pairs of 1-cocycles to 1-cochains of G taking cocycles φ, ψ to their "pointwise product" $\varphi\psi(e) = \varphi(e)\,\psi(e)$ for each edge e. Since the dimension of the space of 1-cochains is $3g-3$, the same as the dimension of the space of sections of the square of the canonical bundle, one may hope that the 1-cochains may be identified with the sections of $H^0(\omega_C^2)$ in such a way that the bilinear map becomes the multiplication

$$H^0(\omega_C) \otimes H^0(\omega_C) \to H^0(\omega_C^2).$$

We show that this is so at least if the graph is trivalent:

PROPOSITION 1.2. (i) *There is a natural map φ from $H^0(\omega_C^2)$ to* Coch G *given as follows: If $\alpha \in H^0(\omega_C^2)$ and x is an edge of G incident to v, corresponding to a node p of C, then in terms of a local parameter t on $C(v)$ at p and an expression $\alpha|_{C(v)} = \beta(t)(dt)^2$ we have $\varphi(\alpha)(x) = (t^2\beta)(0)$.*

(ii) *φ takes the multiplication map $H^0(\omega_C) \otimes H^0(\omega_C) \to H^0(\omega_C^2)$ to the pointwise product $H^1(G, \mathbb{C}) \otimes H^1(G, \mathbb{C}) \to$* Coch^1 G.

(iii) *If G is trivalent, then φ is an isomorphism.*

Proof. (i), (ii). One checks by a direct local calculation that this definition is independent of the choice of parameter t and of the choice of component containing p. Part ii is also immediate from the local computation.

(iii) Since both $H^0(\omega_C^2)$ and Coch G are vectorspaces of dimension $3g-3$, it suffices to show that the map is a monomorphism. Since $C(v)$ contains val v nodes, where val v is the valency of the vertex v in G, we must show that a section of $(\omega_{\mathbb{P}^1}(\mathrm{val}\, v))^2$, vanishing at the nodes, vanishes identically; that is, a meromorphic quadratic differential on $\mathbb{P}^1$, regular at all but val v points and without double poles, is identically 0. But such a form is a section of $(\omega_{\mathbb{P}^1})^2\,(\mathrm{val}\, v) = \mathcal{O}_{\mathbb{P}^1}(\mathrm{val}\, v - 4)$, so the assertion is clear for val $v \leqslant 3$. ∎

We may treat the other powers of ω similarly: On each component $C(v)$ of C. Choose k distinct points, $p_{v,1}, ..., p_{v,k}$ other than the nodes of C, and local parameters $t_{v,i}$ at $p_{v,i}$. We may then map $H^0(\omega_C^{2+k})$ to $\mathrm{Coch}^1 G \oplus (\mathrm{Coch}^0 G)^k$: Let $\alpha \in H^0(\omega_C^{2+k})$. We will send α to a sum of cochains $\varphi(\alpha)$ defined as follows: First, for each directed edge x from u to v, corresponding to a node p on C we may write α in terms of any local prameter t at p, as $\alpha|_{C_v} = \beta(t)(dt)^{2+k}$, and we let the 1-cochain component of $\varphi(\alpha)$ have value $\varphi(x) = t^{2+k}\beta(t)$. Next, for each vertex v, i, we write $\alpha = \beta_{v,i}(t_{v,i})\, dt_{v,i}$ near $p_{v,i}$ and let the ith 0-cochain component of $\varphi(\alpha)$ have value $\beta_{v,i}(t_{v,i})$ at v.

PROPOSITION 1.3. *If G is trivalent then φ is an isomorphism for all k.*

Proof. We have $\dim \mathrm{Coch}^1 G \oplus (\mathrm{Coch}^0 G)^k = (k+1)(g-1) = \dim H^0(\omega_C^{2+k})$ by Riemann–Roch, so it is enough to show that φ is a monomorphism. But if the 1-cochain component of $\varphi(\alpha)$ is 0, then $\alpha|_{C(v)}$ is a section of $\omega_C^{2+k}|_{\mathbb{P}^1}(-\mathrm{val}\, v) = \mathcal{O}_{\mathbb{P}^1}(k+2-\mathrm{val}\, v)$, and if the 0-cochain components vanish as well, then it is a section of $\mathcal{O}_{\mathbb{P}^1}(-1)$, and thus identically 0 as required. ∎

2. When Is the Canonical Series Very Ample?

We can look for base points by examining the restrictions of the canonical series to individual components of C; and we can decide whether the canonical series is very ample by looking at its restriction to pairs of components. For this reason the following result is central to our discussion:

PROPOSITION 2.1. (i) *The curve $C(V)$ associated to a full subgraph V of G imposes*

$$h^1 V - h^0 V + (\textit{the number of edges between } V \textit{ and } G - V) \\ - (n(G-V) - n(G))$$

conditions on the canonical series of C.

(ii) *If no connected component of G is contained in V then the restriction of the canonical series on C to $C(V)$ is non-special. If in addition $nG = n(G-V)$ then the restriction is a complete series.*

Remark. If G is trivalent, it is not hard to show that the number in part (i) takes the amusing form $dV - gV + 1 - (n(G-V) - n(G))$.

Proof. (i) A section of ω_C vanishing on $C(V)$ corresponds to a cocycle whose value is 0 on all the edges incident to vertices in V—that is, a cocycle of $G - V$. Thus the number of conditions imposed is simply

$$\begin{aligned} &h^1G - h^1(G-V) \\ &\quad = h^1V + (\text{the number of edges between } V \text{ and } G-V) \\ &\qquad - nV + n(G-V) - nG \\ &\quad = h^1V - h^0V + (\text{the number of edges between } V \text{ and } G-V) \\ &\qquad - (n(G-V) - n(G)), \end{aligned}$$

giving the first formula, and

$$\begin{aligned} &h^1V - h^0V + (\text{the number of edges between } V \text{ and } G-V) \\ &\qquad = eV - dV + (\text{the number of edges between } V \text{ and } G-V) \\ &\quad = eG - e(G-V) - dV, \end{aligned}$$

as required for the second.

(ii) The restriction of the canonical line bundle to $C(V)$ is the canonical line bundle on $C(V)$ twisted by the points which are nodes on C corresponding to the edges between V and $G-V$. If V contains no connected component of G, then there is at least one such node on every connected component of $C(V)$, so h^1 of this bundle vanishes; that is, the bundle is non-special. Thus by Riemann–Roch and Proposition 1.1, the number of sections of the bundle is

$$\begin{aligned} &gV - 1 + (\text{the number of edges between } V \text{ and } G-V) \\ &\qquad = h^1(V) - h^0(V) + (\text{the number of edges between } V \text{ and } G-V). \end{aligned}$$

Since the dimension of the image series is the number of conditions imposed by V, we are done. ▮

We can easily deduce a sufficient condition for the canonical series to be very ample. Since a vertex of G of valence $\leqslant 1$ would correspond to a whole component of C in the base locus of ω_C, we assume that there are none:

COROLLARY 2.2. (i) *If G has no vertices of valence $\leqslant 1$ and is 2-vertex-connected then ω_C has no base points.*

(ii) *If G has no vertices of valence $\leqslant 2$, is 3-vertex connected, and has no multiple or returning edges, then ω_C is very ample.*

Remark. The connectivity hypotheses in 2.2 are probably not necessary in general (but see below for the case of trivalent graphs.) The hypothesis that there are no multiple edges is certainly not necessary; it would be quite interesting to give a sharper criterion.

Proof. We may reduce to the case where G is connected.

(i) If G has only one component the result is obvious. Else we apply Proposition 2.1(ii) to each singleton $V = \{v\}$, and conclude that the restricted series is complete. Since the valence of v is $\geqslant 2$, the restricted series has no base points.

(ii) The hypotheses imply that G has at least four vertices. First apply Proposition 2.1(ii) to each singleton $V = \{v\}$; since $\mathrm{val}\, v \geqslant 3$, we see that the restricted canonical series is very ample on $C(V)$. Next we apply Proposition 2.1(ii) to each pair of vertices $V = \{v, w\}$. If V is disconnected, the result is now obvious. If, on the other hand, v and w are joined by an edge, then the restriction of the canonical series to $C(V)$ is of codimension 1 in the direct sum of the restrictions to $C(\{v\})$ and $C(\{w\})$, corresponding to a linear series that embeds these two lines into linear subspaces that meet only in the common point that the lines share as components of $C(G)$. This concludes the proof. ∎

The situation for sets of edges is even simpler:

PROPOSITION 2.3. *If E is a set of edges of G, then the number of conditions imposed by the corresponding set of nodes on the canonical series is*

$$eE + nG - n(G - E).$$

In particular, the conditions imposed by the nodes corresponding to edges of E are independent iff removing E does not disconnect any connected component of G.

Proof. The equations defining cocycles on G which vanish on E are the same as those defining cocycles on $G - E$, and

$$h^1G - h^1(G - E) = eE + nG - n(G - E)). \quad ∎$$

Using this we can extend Proposition 2.1 to a necessary and sufficient condition for the completeness of the restriction of the canonical series to a single component:

PROPOSITION 2.4. *The restriction of the canonical series on C to a component corresponding to a vertex v is complete iff either v is a connected component of G; or* $\mathrm{val}(v) \leqslant 1$; *or* $nG = n(G - v)$, *that is, iff v is not disconnecting.*

Proof. The restriction of ω_C to the component corresponding to v is $\mathcal{O}_{\mathbb{P}^1}(\mathrm{val}(v) - 2)$. Thus if $\mathrm{val}(v) \leqslant 1$ there is nothing to prove. Similarly, if v is a connected component of G, the result is obvious. Otherwise, the series is complete iff the set of values taken by sections of ω_C at the nodes corresponding to edges incident to v (with respect to some local trivialization there) form a vectorspace of dimension $\mathrm{val}(v) - 1$. By Proposition 2.3, this is so iff $n(G-E) = nG + 1$. Because v is not a connected component of G, $G - E$ is the disjoint union of $G - v$ and the isolated vertex v, this is equivalent to the desired statement. ▌

Using these tools we can give necessary and sufficient conditions for ω_C to be base point free or very ample in the case of greatest interest to us, namely the case where G is trivalent.

PROPOSITION 2.5. *If G is trivalent then*

(i) *ω_C is base point free iff G is 2-connected.*

(ii) *ω_C is very ample iff G is 3-connected.*

Remark. The proof will show that if there is a base point, then one of the nodes is a base point.

Proof. (i) By Proposition 2.3, a node imposes no conditions on sections of ω_C, and so it is a base point iff the corresponding edge is disconnecting. Now since G is trivalent, and thus the restriction of ω_C to each component has degree 1, some point of a component corresponding to a vertex $v \in G$ is a base point iff the restriction of $H^0\omega_C$ to that component is incomplete. By Proposition 2.4, this happens iff v is disconnecting.

It remains to show that G has a disconnecting vertex iff G has a disconnecting edge. If G has a disconnecting edge then each of the vertices on it is disconnecting. Conversely suppose that v is a disconnecting vertex, so that edges incident to v meet components V_1, V_2 of $G - v$ that are in the same connected component of G. Since the valence of v is 3 (at least), one of the V_i shares only one edge with v. This edge is disconnecting.

(ii) First suppose that ω_C is very ample. If E is a pair of disconnecting edges, then by Proposition 2.1, the corresponding nodes together impose only one condition on the sections of ω_C, so ω_C is not very ample.

We may complete the argument by using Corollary 2.2(ii), thanks to the following combinatorial lemma:

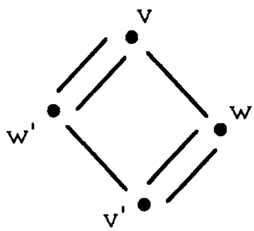
LEMMA 2.6. *If G is trivalent then G is 3-connected iff G has no multiple or returning edges and is 3-vertex connected.*

Proof of Lemma. We will prove that if G is trivalent and 3-connected, then the second condition holds. We will not need the converse (which was pointed out to us by S.-W. Park), and we leave its proof to the reader.

We may assume that G is connected. It is obvious that G can have no multiple or returning edges.

Let V be a set of two vertices of G, and suppose that $G - V$ has at least two connected components. Adding some edges to $G - V$ if necessary to reduce the number of connected components to 2, and contracting each one to a single vertex, we may assume that G is a 3-connected graph with four vertices, two in V of valence exactly 3 and two others w, w', and that $G - V$ is disconnected. We will show that this is impossible.

Clearly w and w' are not joined by an edge. If at most two of the six edges emanating from v, v' went to w then G could be disconnected by removing these $\leqslant 2$ edges, contradicting 3-connectivity, and similarly for w'. Thus exactly three of the edges from v, v' go to each of w and w', and up to exchanging v and v', G has the form



This graph is visibly not 3-connected, again a contradiction. ∎

It is an easy exercise—which we leave to the interested reader—to extend Proposition 2.5 to the case of at most trivalent graphs. Perhaps the most interesting point is that if ω_C is very ample, Then under this hypothesis G must actually be trivalent.

3. Equations of the Canonical Image

In this section we will assume that G is trivalent and 2-connected, with no multiple or returning edges so that, by Proposition 2.5(i), the canonical mapping of C is defined, and by Proposition 2.1(ii), it carries each component of C to a line. We will show how to compute the homogeneous ideal of the canonical image of C.

We first establish some notation to be used in the rest of the paper: We will write S for the homogeneous coordinate ring of the canonical space; by Proposition 1.1,

$S = \mathrm{Symm}(H^0\omega_C) = \mathrm{Symm}(H^1(G, \mathbb{C}))$. S is naturally a subring of the polynomial ring

$T = \mathrm{Symm}(\mathrm{Coch}\ G)$.

We set

$\mathscr{R} = \bigoplus_n H^0(\omega_C^{\otimes n})$, the canonical ring of C, and we write

$R =$ the image of the natural map $S \to \mathscr{R}$, for the homogeneous coordinate ring of the canonical image of C. We let

$I = \ker S \to R = \ker S \to \mathscr{R}$ be the canonical ideal of C.

We have:

PROPOSITION 3.1. *I is the intersection of S and the ideal of T generated by all monomials of the forms*

$$xy, \qquad \textit{where } x, y \textit{ are dual to disjoint edges of } G$$

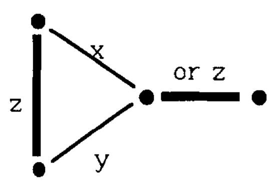
and

$$xyz, \qquad \textit{where } x, y, z \textit{ are dual to the edges of a triangle of } G.$$

Proof. Since the canonical image of C is a union of lines corresponding to the vertices v of G, its ideal is the intersection of the ideals $I(v)$ of the lines. Now an element of $H^0(C, \omega_C)$ that has, as a meromorphic differential form on the component of C corresponding to v, no poles, is identically 0 on that component, so $I(v)$ is generated by the set of cocycles whose supports do not contain edges incident to v. It follows that $I(v)$ may be written as $I(v) = S \cap J(v)$, where $J(v) \subset T$ is the ideal generated by all duals of edges not incident to v, and $I = \bigcap_v I(v) = S \cap (\bigcap_v J(v))$.

It now suffices to show that $J = \bigcap_v J(v)$ is the ideal generated by monomials of the form given in the proposition. Since each ideal $J(v)$ has as generators a set of dual edges, J will have a basis consisting of those monomials m such that for each vertex v, there is a factor of m dual to an edge not incident to v. It is thus clear that all the monomials in the statement of the Proposition are in J, and also that every monomial of degree 2 in J is the product of the duals of two disjoint edges.

Let $m = xyz \cdots$ be a monomial in J, and suppose that no two factors of m are dual to a pair of disjoint edges. The edges dual to x and y thus must meet in some vertex, say v; and z and all further factors, if any, then must either meet v or form a triangle with x and y:



Since not all factors can be dual to edges incident to v, we see that m is divisible by a product of three factors dual to the edges of a triangle, as required. ∎

4. Noether's Theorem for Graph Curves

We continue to use the notation introduced at the beginning of Section 3.

Noether's theorem states that for any smooth curve for which the canonical map is an embedding, the canonical image is projectively normal. Here is the analogue for graph curves corresponding to trivalent graphs:

Proposition 4.1. *If G is trivalent and 3-connected then the natural map $R \to \mathscr{R}$ is an isomorphism; that is, the canonical image of C is arithmetically Cohen–Macaulay.*

Proof. R is generated in degree 1, and $R_1 \cong \mathscr{R}_1$ by definition, so we must show that the multiplication map

$$\mathscr{R}_1 \otimes \mathscr{R}_i \to \mathscr{R}_{i+1}$$

is an epimorphism for all i.

For $i \geqslant 3$ this follows from a well-known general argument, the "base-point free pencil trick": Let $V \subset H^0(\omega_C)$ be a two-dimensional subvectorspace of sections such that the elements of V have no common zeros in C (a base-point-free pencil). There is an exact sequence of sheaves,

$$0 \to \Lambda^2 V \otimes \omega_C^{\otimes i-1} \to V \otimes \omega_C^{\otimes i} \to \omega_C^{\otimes i+1} \to 0,$$

which shows that even $V \otimes H^0(\omega_C^{\otimes i}) \to H^0(\omega_C^{\otimes i+1})$ is onto as soon as $H^1(\omega_C^{\otimes i-1}) = 0$, that is, as soon as $i-1 \geqslant 2$ (see, for example, Arbarello *et al.* [2, p. 151, "Castelnuovo's lemma"]).

It remains to prove the surjectivity for $i=1$ and 2, which we do by using the combinatorial description of these multiplication maps given in Propositions 1.2 and 1.3.

The combinatorial information that we need is given to us by a special case of the "edge form" of Menger's theorem (see Bollobás [6, Chap. III, Theorem 5, part ii], for example): If u and v are distinct vertices of a graph Γ, and if there is no edge whose removal disconnects Γ, then there are two edge-disjoint paths connecting u to v in Γ.

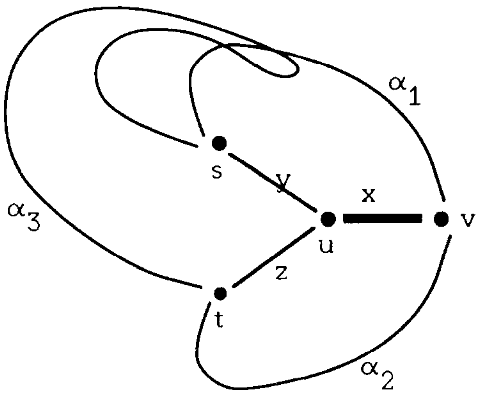
We first prove that μ: $R_1 \otimes R_1 \to \mathscr{R}_2$ is onto. By Proposition 2.2 and the first remark following it, it suffices to show that given any edge x of G, connecting vertices v and w, say, there are two cycles in G whose only common edge is x; the restriction of the product of the corresponding sections α_1 and α_2 to the various components of C will have double poles only at the point of C corresponding to x, and thus $\varphi(\alpha_1 \alpha_2)$ will be the cochain dual to the edge x. But after removing x from G we get a graph Γ which is not disconnected by the removal of any edge; thus we may apply Menger's theorem to conclude that there are two edge-disjoint paths from v to w in Γ. Together with x these form the desired cycles.

Next we prove that the multiplication map μ_3: $R_1 \otimes R_1 \otimes R_1 \to \mathscr{R}_3$ is onto. First, if we follow

$$R_1 \otimes R_1 \otimes R_1 \xrightarrow{\mu_3} \mathscr{R}_3 = H^0(\omega_C^3) \xrightarrow{\varphi} \mathrm{Coch}^1 G \oplus \mathrm{Coch}^0 G$$

by the projection onto the 1-cochain component, then it is onto; for if we choose α_1, α_2 as above, then $\alpha_1^2\alpha_2$ restricted to the various components of C will have triple poles only at the point of C corresponding to the edge x.

It now suffices to produce, for any vertex $u \in G$, a product $\alpha_1\alpha_2\alpha_3$ of sections $\alpha_i \in H^0(\omega_C)$ whose restriction to the various components of C has no triple poles whatever and is identically 0 except on the component corresponding to v, where it is nonzero. Let x be one of the edges incident to u, say from u to v, and let α_1, α_2 be chosen as above to correspond to cycles having only the edge x in common. Let s, t be the other two vertices of G which are neighbors of u. By the argument used in Proposition 1.5, $G - \{u, v\}$ is connected, so we may choose a path in $G - \{u, v\}$ from s to t and take α_3 to be the section of ω_C corresponding to the cycle made from this path together with the edges y from s to u and z from u to t:



Since no edge appears in all three cycles, it is clear that the restriction of $\alpha_1\alpha_2\alpha_3$ to a component of C has no triple poles. Further, as one easily checks, no vertex of G except u appears in all three of the cycles, so on every component of C except $C(u)$ the product is a section of $\omega_{\mathbb{P}^1}^3(4) = \mathcal{O}_{\mathbb{P}^1}(-2)$, and so it is identically 0. Finally, on $C(u)$, $\alpha_1\alpha_2\alpha_3$ is the product of three nonzero sections of $\omega_C|_{C(u)} = \omega_{\mathbb{P}^1}(3)$, and it is thus not identically 0. As it is a differential without triple poles, it is actually a section of $\omega_{\mathbb{P}^1}^3(6) = \mathcal{O}_{\mathbb{P}^1}$, and as such it is a nonzero constant. Thus $\varphi(\alpha_1\alpha_2\alpha_3)$ is the 0-cocycle dual to u. ∎

5. Cohomology and Clifford Indices of Line Bundles Determined by Subgraphs

To simplify the notation, *we will assume throughout this section that G is connected.* The reader will have no difficulty if he wants to extend the results to the more general case.

What interesting special line bundles does a graph curve C have? The most obvious ones are those obtained by taking the line bundle associated with a divisor having one point on each of several components of C. Since we are looking for examples which are as special as possible, it is natural to look for examples of this type for which the points are actually contained in a hyperplane in the span of $C(V)$—that is, they form a hyperplane section of $C(V)$. In this section we will analyze such examples from the point of view of the naive Clifford index (see the Introduction for definitions), and we compare the result with the resolution Clifford index of the graph curve.

First a remark that holds for any reduced curve:

Lemma 5.1. *Let $C \subset \mathbb{P}^r$ be a reduced nondegenerate curve and let D be a hyperplane section which meets every component of C transversely. If n of the connected components of C span $\mathbb{P}^r$, then the codimension of the linear span of D in $\mathbb{P}^r$ is at most n. Equality holds if the embedding of C is complete, or more generally, if the connected components of C are embedded in linearly disjoint subspaces.*

Remark. In general, the dimension of the span of D will depend on the hyperplane chosen, as is the case for three skew lines in $\mathbb{P}^3$, for example. It would be nice to have a formula for the minimum possible dimension.

Proof. We may assume that C has only n components C_i. The result follows easily from the fact that the codimension of the linear span of D is the dimension of the intersection of

$$\mathbb{C}^n = H^0\mathcal{O}_C = \sum_i H^0\mathcal{O}_{C_i} \subset \sum_i H^0\mathcal{O}_{C_i}(1) = H^0\mathcal{O}_C(1)$$

and

$$H^0\mathcal{O}_{\mathbb{P}^r}(1) \subset H^0\mathcal{O}_C(1). \quad \blacksquare$$

Now let $V \subset G$ be a set of vertices, regarded as a full subgraph. Let D be a proper hyperplane section of $C(V)$ which does not contain any nodes of C, and set $\mathcal{L} = \mathcal{O}_C(D)$. We will write $\mathcal{L}(V)$ for any line bundle $\mathcal{L}$ made from V in this way.

PROPOSITION 5.2. *If G is trivalent, then*

$$h^0\mathscr{L}(V) \leqslant h^1V + n(G-V)$$

and

$$h^1\mathscr{L}(V) \leqslant h^1(G-V) + nV,$$

so that

$$\operatorname{Cliff}\mathscr{L}(V) \geqslant g+1-h^1V-h^1(G-V)-nV-n(G-V).$$

All three inequalities become equalities if either V or $G-V$ is connected. In particular, if V and $G-V$ are connected, and $f = 3dV - eV$ is the number of edges in G joining V to $G-V$, then

$$\operatorname{Cliff}\mathscr{L}(V) = f-2.$$

Proof. The given formula for f follows at once from the trivalency of G. For simplicity, let $\mathscr{L} = \mathscr{L}(V)$. From Proposition 2.1 and Lemma 5.1 we see that

$$h^1\mathscr{L} \leqslant g-(dV-h^1V-(n(G-V)-1)),$$

with equality if $C(V)$ is either connected—that is, V is connected—or $C(V)$ is embedded by a complete series—that is, (by the second statement of Proposition 2.1) $G-V$ is connected. Elementary manipulations using the Euler formula and the formula for f given above show that the right-hand side is equal to $h^1(G-V)+nV$. The Riemann–Roch formula on C, applied to the inequality as written above, now yields $h^0\mathscr{L} \leqslant h^1V+n(G-V)$, and we get

$$\begin{aligned}\operatorname{Cliff}\mathscr{L} &= g+1-h^0\mathscr{L}-h^1\mathscr{L}\\ &\geqslant g+1-h^1V-h^1(G-V)-nV-n(G-V)\end{aligned}$$

as desired. Further applications of the Euler formula and the formula for f yield

$$g+1-h^1V-h^1(G-V)-nV-n(G-V) = f+2(1-nV-n(G-V)),$$

which becomes the desired formula in case V and $G-V$ are connected, so that $nV = n(G-V) = 1$. ∎

Remark. The symmetry in the formulas for h^0 and h^1 in the proposition is not surprising; if we take a general hyperplane section of C we see that the corresponding $\mathscr{L}(V)$ and $\mathscr{L}(G-V)$ satisfy $\mathscr{L}(G-V) = \omega_C \otimes \mathscr{L}(V)^{-1}$.

Recall that a line bundle $\mathscr{L}$ *contributes to the Clifford index of C* if both $h^0\mathscr{L}$ and $h^1\mathscr{L}$ are $\geqslant 2$. We would like to say that V contributes to the Clifford index of C if $\mathscr{L}(V)$ does; however, the criterion above only allows us to decide this from the combinatorics when V or $G-V$ is connected, so we will only say that *V contributes to the Clifford index if V or $G-V$ is connected and $\mathscr{L}(V)$ contributes to the Clifford index of C.*

COROLLARY 5.3. *If G is trivalent, and V is a subgraph such that either V or $G-V$ is connected, then V contributes to the Clifford index of C iff*

$$h^1V+n(G-V)\geqslant 2 \qquad \textit{and} \qquad h^1(G-V)+nV\geqslant 2. \quad \blacksquare$$

For V and $G-V$ connected we have

$$\operatorname{Cliff}\mathscr{L}(V)=g-1-h^1V-h^1(G-V).$$

The next result shows that this number is related to the resolution Clifford index of C:

PROPOSITION 5.4. *If G has a subgraph V with both h^1V and $h^1(G-V)$ nonzero then setting $a=h'V$, $b=h'(G-V)$, and $t=a+b$, we have*

$$\dim_{\mathbb{C}} \operatorname{Tor}^S_{k-1}(\mathbb{C},R)_k\geqslant\binom{t}{k}-\binom{a}{k}-\binom{b}{k},$$

so that the resolution Clifford index of C is $\leqslant g-1-t$, which is Cliff $\mathscr{L}(V)$ *if V and $G-V$ are connected.*

Proof. Writing $A=H^1(V,\mathbb{C})$ and $B=H^1(G-V,\mathbb{C})$, we see that $AB\subset I$. Thus the linear part of the resolution of AB is a direct summand of the linear part of the resolution of I. On the other hand, this resolution is easily computed: since $A\cap B=0\subset S_1$ we have $(A)\cap(B)=(AB)$, so the resolution of $(A)\cap(B)$ may be computed as the cokernel of the natural map from the direct sum of the Koszul complexes resolving (A) and (B) to the Koszul complex resolving $(A+B)$. This gives the last assertion. The other assertions follow directly from the definitions. $\blacksquare$

The next result shows the Clifford index is minimized when both V and $G-V$ are connected.

PROPOSITION 5.5. *Let G be a connected, 3-connected graph, and let V be a subgraph. Suppose that $G-V$ is connected, but V is not. If V_0 is a connected component of V, then $G-V_0$ is connected and*

$$\operatorname{Cliff}\mathscr{L}(V_0)<\operatorname{Cliff}\mathscr{L}(V).$$

Proof. Since G is connected, $G-V$ must be connected by an edge of G to every connected component of V, whence the connectedness of $G-V_0$. By Proposition 1.8 it is enough to show that

$$h^1V_0+h^1(G-V_0)+2>h^1V+h^1(G-V)+1+nV,$$

or in other terms,

$$\begin{aligned}h^1(G-V_0)&>h^1(G-V)+(h^1V-h^1V_0)-1+nV\\&=h^1(G-V)+h^1(V-V_0)+n(V-V_0).\end{aligned}$$

Let f be the number of edges of G joining $G-V$ and $V-V_0$. Since $G-V_0$ is connected, G must contain at least one edge joining each component of $V-V_0$ to $G-V$, and thus we have

$$h^1(G-V_0)=h^1(G-V)+h^1(V-V_0)+f-n(V-V_0),$$

so the desired inequality is equivalent to $f>2n(V-V_0)$. However, since G is 3-connected we must in fact have $f\geqslant 3n(V-V_0)$, and we are done. ∎

It thus makes sense to define

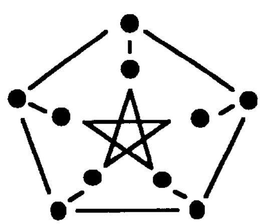
$$\text{Cliff}\, G=\min\{\text{Cliff}\,\mathscr{L}(V)\mid V \text{ and } G-V \text{ are connected}\}.$$

From Propositions 5.2 and 5.5 we have immediately:

COROLLARY 5.6. Cliff $G=f-2$, *where f is the smallest number of edges of G which disconnect G into two connected pieces, both with homology* (*the "homoliferous disconnection number"*), Cliff G *is* $\geqslant$ *the resolution Clifford index of $C=C(G)$.*

COROLLARY 5.7. *If G is trivalent, connected, and* 2*-connected, then the canonical series on C is very ample iff* Cliff $G>0$ *iff the resolution Clifford index of C is* >0.

Is the resolution Clifford index of C equal to Cliff G? Not in general. The graph $K(3,3)$ has associated graph curve of genus 4, and resolution Clifford index 1, *although there are no triangles*, and no way to disconnect the graph into homoliferous pieces with three edges. The "Peterson graph"



is an interesting example here. The resolution Clifford index of the corresponding curve of genus 6 is $2 = [(g-1)/2]$; but one sees by inspection that the graph contains no cycle of length < 5, and thus no connected subgraph contributing to the Clifford index of C and having Clifford index < 3. Thus a line bundle computing the Clifford index must *not* be of the form $\mathscr{L}(V)$ with V connected. However, in cases where the Cliff G is small compared to g there is some experimental evidence, obtained with the program Macaulay of Bayer and Stillman [4], that the resolution Clifford index is measured by Cliff G. To be definite, we make:

Conjecture 5.8. If $\operatorname{Cliff} G < [(g-1)/2]$, then $\operatorname{Cliff} G$ is equal to the resolution Clifford index of $C(G)$.

We will prove this conjecture in the case of planar graphs dual to triangulations of the sphere in Section 7. The condition on Cliff G is actually not very restrictive; in particular it is satisfied for all G of genus $\geqslant 11$, as the following results show:

Recall that the *girth* of a graph G having nontrivial homology is the length of the shortest cycle in G. We need the following standard combinatorial information (see, for example, Bollobás [6, Theorem IV.1]):

LEMMA 5.9. *If G is a connected trivalent graph of genus g and girth γ, then*

$$g \geqslant 2^{\gamma/2} \qquad \text{if } \gamma \text{ is even,}$$

$$g \geqslant \tfrac{3}{2} 2^{(\gamma-1)/2} > 2^{\gamma/2} \qquad \text{if } \gamma \text{ is odd.}$$

(This estimate is sharp, for example, for $g = 4$ and the complete bipartite graph $K(3, 3)$; for $g = 6$ and the Petersen graph; for $g = 8$ and the Heawood graph; and for $g = 16$; see Behzad *et al.* [5, pp. 63, 64] for pictures.)

PROPOSITION 5.10. *If $g \geqslant 5$, and G is trivalent of genus $g \geqslant 5$, then the homoliferous disconnection number of G is $\leqslant$ girth G, the length of the shortest cycle in G. In particular, if $g \geqslant 11$ then* $\operatorname{Cliff} G < (g-1)/2$.

Proof Sketch. Let γ be the girth of G, and let f be the homoliferous disconnection number. If V is a cycle of length γ, then the number of edges joining V to $G - V$ is again γ, so it suffices by Proposition 5.5 to check that $G - V$ has homology. Euler's formula gives

$$h^0(G-V) - h^1(G-V) = \gamma - g + 1,$$

so we are done if $g > \gamma + 1$. The formulas in Bollobás [6, Theorem IV.1] yield this conclusion at once when $g > 6$. In the cases $g = 5$, $g = 6$ we get from these formulas $\gamma \geqslant g - 1$, and it is then easy to examine all possible cases. ∎

6. Face Varieties and Their Hyperplane Sections

In the next section we will analyze the Clifford index and the resolution Clifford index of a planar trivalent graph dual to a triangulation of a 2-sphere. The essential reason that this is possible is that then the associated graph curve is a hyperplane section of the "face variety" associated to the triangulation. In this section we provide some motivating background for this construction, and prove what seems to be a new theorem about face varieties.

In order to obtain a canonical curve as the hyperplane section of a surface, the canonical bundle of the surface must be trivial (or at least meet the hyperplane section in a trivial class). Thus we are led to ask about combinatorially defined varieties with trivial canonical bundle.

Recall that if M is a simplicial complex, the corresponding *face variety* X (in the sense of Stanley, Hochster, Reisner; see, for example, Hochster [12]) is the variety defined, in a projective space over a field k, with coordinate functions corresponding to the vertices of M, by monomials corresponding to the non-simplices of M. Its dimension (as a projective variety) is the same as the dimension of M. From work of Reisner and Hochster and Roberts (see, for example, Hochster and Roberts [13, p. 171]) it follows that if M is a manifold, then

$$H^i(X, \mathcal{O}_X) = \tilde{H}^i(M, k) \text{ the reduced simplicial cohomology of } M, \quad \text{if } i > 0,$$

while

$$H^i(X, \mathcal{O}_X(n)) = 0 \qquad \text{if } 0 < i < d \text{ and } n \neq 0.$$

Further, if $\mathcal{I}_X$ is the ideal sheaf of X, then

$$H^1(X, \mathcal{I}_X) = \tilde{H}^0(M, k), \qquad H^1(X, \mathcal{I}_X(n)) = 0 \qquad \text{for all } n \neq 0.$$

Thus in particular, X is projectively Cohen–Macaulay iff M has no reduced homology below dimension d, and X is projectively Gorenstein iff M is a homology sphere. But if M is a manifold, then Reisner also proves that X is locally Gorenstein; so it is quite interesting to ask about its canonical sheaf even if X is not projectively Cohen–Macaulay. We have:

Theorem 6.1. *If M is a simplicial compact manifold of dimension d over $\mathbb{R}$, and X is the corresponding face variety, then the square of the canonical bundle of X is trivial; the canonical bundle itself is trivial iff M is orientable.*

Joe Harris suggested to us that the second part of the above theorem might be true, and he also contributed some ideas to the proof. In the Cohen–Macaulay case, the result is also a consequence of an unpublished theorem of Hochster.

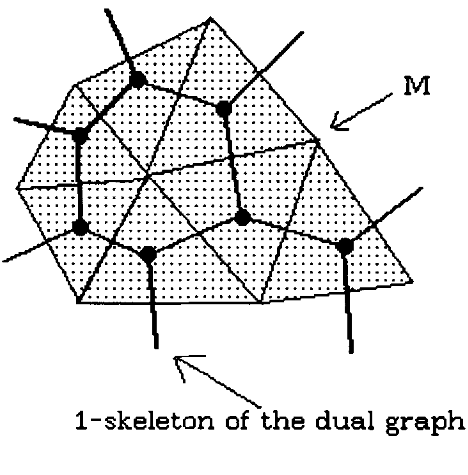
Proof Sketch. Without loss of generality M is connected. Reduce by étale double cover to statements about the orientable case. If M is orientable, then via Reisner's description of the free resolution of the homogeneous coordinate ring S_X of X we get a non-zero element of $\mathrm{Ext}^{\mathrm{codim}\, X}(S_X, S)$ in degree $r+1 = \{$the number of vertices of $M\}$, which gives a nonzero section of ω_X. But since the restriction of ω_X to each component of X is trivial (each component meets the rest along $d+1$ hyperplanes), this section is nowhere 0, and thus trivializes ω_X.

Conversely, if M is non-orientable, Reisner's description implies that $\mathrm{Ext}^{\mathrm{codim}\, X}(S_X, S) = 0$ in degree the number of vertices of M. Since M is connected, S_X has depth $\geqslant 2$, so $\mathrm{Ext}^{\mathrm{codim}\, X}(S_X, S)(-r-1) = \Gamma_* \omega_X$, so ω_X has no global sections, and we are done. ∎

Consider the case of dimension 2: here X as in the theorem will be an algebraic surface with Kodaira dimension $\kappa = 0$, and it is amusing to see how these fit into the classification of surfaces. Recall that the irregularity q and the geometric genus p_g of a surface X are defined by $q = h^1(\mathcal{O}_X)$, $p_g = h^0(\omega_X)$. We obtain:

Topology of the compact surface M	Invariants of X	Type of X
Sphere	$\omega_X = \mathcal{O}_X$; $q = 0$ $p_g = 1$	$K3$
Projective plane	$\omega_X \neq \mathcal{O}_X$, $\omega_X^{\otimes 2} = \mathcal{O}_X$; $q = p_g = 0$	Enriques
Torus	$\omega_X = \mathcal{O}_X$; $q = 2$, $p_g = 1$	Abelian
Other; $\chi(M) < 0$	$\kappa = 0$, $\chi(\mathcal{O}_X) < 0$	Non-smoothable

Recall that if M is a compact simplicial 2-manifold, then the dual graph of M is defined by taking a vertex for each 2-simplex of M and an edge for each edge of M:



The connection with graph curves is given by:

COROLLARY 6.2. *Let M be a simplicial 2-manifold, and let X be the corresponding surface. A general hyperplane section C of X is a graph curve with dual graph equal to the* 1*-skeleton of the dual subdivision of M. The curve C is a canonically embedded graph curve iff M is orientable, and C is embedded by the complete canonical series iff M is a sphere.*

Proof. Suppose first that M is non-orientable. We must show that $\mathcal{O}_C(1) \neq \omega_C$, which is $\omega_X(1) \otimes \mathcal{O}_C$ by the adjunction formula; that is, we must prove that $\omega_X \otimes \mathcal{O}_C \neq \mathcal{O}_C$. But from the exact sequence

$$0 = H^0(\omega_X) \to H^0(\omega_X \otimes \mathcal{O}_C) \to H^1(\omega_X(-1)) = H^1(\mathcal{O}_C(1)) = 0,$$

we see that $\omega_X \otimes \mathcal{O}_C$ has no global sections.

Now suppose that M is orientable. By Reisner's results, X is embedded by a complete series, so C is embedded by a complete series iff $H^1(\mathcal{I}_{C/X}(1)) = 0$; but $\mathcal{I}_{C/X} = \mathcal{O}_X(-1)$, so the condition is equivalent to $H^1(\mathcal{O}_X) = 0$, which is the condition that M is a sphere by the result of Hochster and Roberts cited above. ∎

7. PROOF OF THE MAIN CONJECTURE FOR PLANAR GRAPHS

We will make use of:

THEOREM 7.1 (Steinitz-Rademacher [18]—see also Harary [11, p. 106]). *A graph without multiple edges is the* 1*-skeleton of a convex* 3*-dimensional polyhedron iff it is planar and* 3*-connected.*

Since the dual of the 1-skeleton of a convex polyhedron is the 1-skeleton of the dual convex polyhedron, we immediately obtain the following characterization (which was pointed out to us by Richard Stanley):

COROLLARY 7.2. *A planar graph without multiple edges is the dual of a triangulation of the* 2*-sphere iff it is* 3*-connected.*

Actually we have already proved the necessity of 3-connectedness, even in a more general context: If a graph G is dual to a triangulation of any orientable 2-manifold then of course G is trivalent, and by Corollary 6.2 its canonical map is an embedding, so by Proposition 2.5, G is 3-connected. (Since the canonical embedding is the only embedding of a graph curve of a 3-connected trivalent graph in $g-1$ space in which the components go to lines, we also see that the planar embedding of a 3-connected planar trivalent graph is unique, which is a special case of a theorem of Whitney [19]—see Harary [11, p. 105].)

In this section we will prove Conjecture 5.8 for planar graphs:

PROPOSITION 7.3. *If $C = C(G)$ is the graph curve associated to a trivalent planar graph G then the resolution Clifford index of C is equal to* Cliff G.

In fact, the proof will yield an expression for the graded Betti numbers of C.

In evaluating the weight of evidence for the combinatorial Clifford index conjecture provided by Proposition 7.3, one must bear in mind the fact that the Clifford index of a trivalent connected planar graph is always $\leqslant 3$. (*Proof.* If the graph has genus g, then it divides the sphere into $g+1$ faces. Each of the $3g-3$ edges borders two of these faces, so the average number of sides of a face is $(6g-6)/(g+1) < 6$; thus there must be a face with five or fewer edges, and the edges meeting its perimeter but not bordering it form a homoliferous disconnection (if g is not too small) of $\leqslant 5$ edges, whence the bound. ∎)

Proof. We know from Proposition 5.6 that Cliff G is $\geqslant$ the resolution Clifford index of C, so it suffices to prove the opposite inequality.

By the results above, G is the dual graph to a simplicial sphere M. Letting X be the face variety associated to M, Corollary 6.2 tells us that the general hyperplane section of X is $C(G)$ in its canonical embedding. Since X is arithmetically Cohen–Macaulay by Reisner's theorem (Reisner [17]), the graded Betti numbers for the resolution of the homogeneous coordinate ring S_X of X are the same as those for the homogeneous coordinate ring of C. Letting S be the polynomial ring on the vertices of M (the homogeneous coordinate ring of the projective space in which X is embedded) and setting $c = \operatorname{Cliff} G$, it thus suffices to show that

$$\operatorname{Tor}^S_{g-1-c}(\mathbb{C}, S_X)_{g-c} = 0.$$

To this end we apply Reisner's description: Since S_X is graded by the semigroup of monomials m in the vertices of M, we may decompose $\operatorname{Tor}^S_*(\mathbb{C}, S_X)$ by monomials. Writing $\operatorname{Tor}^S_i(\mathbb{C}, S_X)_m$ for the mth graded piece of $\operatorname{Tor}^S_i(\mathbb{C}, S_X)$, Reisner shows that if m is divisible by a square then $\operatorname{Tor}^S_i(\mathbb{C}, S_X)_m = 0$, while if m is square-free then

$$\operatorname{Tor}_i(\mathbb{C}, S_X)_m = \tilde{H}_{|m|-i-1}(\operatorname{supp} m, \mathbb{C}),$$

where $\tilde{H}_*(\operatorname{supp} m, \mathbb{C})$ is reduced simplicial cohomology.

In particular, when $|m| = i+1$, we have

$$\operatorname{Tor}_i(\mathbb{C}, S_X)_m = \tilde{H}_0(\operatorname{supp} m, \mathbb{C}),$$

whose rank is one less than the number of components of supp m. It follows that $\mathrm{Tor}^S_{g-1-c}(\mathbb{C}, S_X)_{g-c} \neq 0$ iff we can find some disconnected full subcomplex m of M involving $g-c$ distinct vertices.

We now suppose that M has a disconnected full subcomplex m involving $g-c$ distinct vertices. We will show there is a connected subgraph V such that each V and $G-V$ has homology and such that $h^1V + h^1(G-V) \geqslant g-c$. Using Propositions 5.2 and 5.5, gives Cliff $G \leqslant c-1$, a contradiction which shows that m cannot exist and thus yields the desired conclusion.

Let m' be a connected component of m and let m'' be the complement of m' in m. Of course the vertices of M correspond to faces of the subdivision of the sphere corresponding to the graph G. We take V to be the subgraph consisting of all vertices and edges around the boundaries of faces in m'. V is obviously connected.

Since V is inscribed on the sphere, the faces of V generate the homology of V and satisfy in this homology the single linear relation that the sum of all the faces is 0. Thus the vertices of m' correspond to linearly independent 1-cycles in H^1V, and the vertices in m'' correspond to linearly independent 1-cycles in $H^1(G-V)$, a total of $g-c$ cycles, as claimed. This concludes the proof. ▮

The proof above is based on a correspondence between disconnected subcomplexes of M and homoliferous disconnections of G which may also be described as follows:

Given f edges which disconnect G, color each face of the sphere blue whose boundary contains one or more of these edges, and color the remaining faces green. If V and $G-V$ are each connected, then on the sphere they bound two green "continents" surrounded by an "ocean" of blue, and the green faces give a basis for the homology of the full subgraphs V and $G-V$. We compute at once by Euler's formula that $h^1V + h^1(G-V) = g+1-f$, so this is the number of green faces. If the f edges disconnect G into more than two connected subgraphs, then there are more than $g+1-f$ green faces. If at least two of these subgraphs are homoliferous, then the green faces form at least two continents.

One sees from this picture that the minimum value of f for a homoliferous disconnection of G is achieved by choosing a band of f blue faces connected in a cycle on the sphere, which leave two connected sets of green faces: Any set of blue faces corresponding to a homoliferous disconnection contains such a cycle of faces, and if there are blue faces not included in this cycle, we can lower f by recoloring them green. The f edges which remain bounded on both sides by blue faces give the new, smaller homoliferous disconnection. It follows that for the minimum value of f, there are exactly two green continents.

The f edges of a minimal homoliferous disconnection of G thus give rise to a set of $g+1-f$ vertices of M, whose full subcomplex in M has two connected components. Conversely, $g+1-f$ vertices of M whose full subcomplex in M is disconnected give rise to f or fewer edges of G that homoliferously disconnect G.

Remark. Any disjoint pair of faces yields a quadratic element of the ideal of C, and by taking all such pairs, we get the right number of quadratic generators. This corresponds to the fact that we have written the canonical ideal of C as the hyperplane section of the variety X.

8. Further Remarks and Problems

1. *Uniqueness of the canonical embedding.*

THEOREM 8.1. *If G is a 3-connected trivalent graph, then there is, up to invertible linear transformations, only one embedding of the bond matroid of G in a space of dimension g.*

Proof. The proof of the fact that the canonical series on $C(G)$ is very ample given above shows that any embedding of G extends to an embedding of the graph curve. Such an embedding corresponds to a linear series on the graph curve restricting to $\mathcal{O}(1)$ on each line. But there is only one such linear series of (vectorspace) dimension g and degree $2g-2$ on a curve: it must be the canonical series, by Riemann and Roch. ∎

PROBLEM. Is it possible to drop the hypothesis of 3-connectedness? Is there such a result for other graphs?

2. *Graph Varieties.* It seems quite interesting to look for a good higher-dimensional analogue of graph curves—graph varieties, say. Perhaps a graph variety should be defined as the union of planes such that the restriction of canonical sheaf to each component is trivial—that is, with $d+1$ d-planes meeting a given d-plane. (Some of these, canonically embedded, appear as generic hyperplane sections of face varieties associated to homology spheres, as in Sections 6 and 7, above.) When is the canonical map an embedding? We seem to have $H^0(\omega)=H^d\Delta$, where Δ is the simplicial complex made from the incidence complex (dual complex) of V; the double point locus of V, inside each d-plane, seems to be a graph variety; perhaps there should be some restriction placed on it, such as that it should be the "standard graph variety" with incidence graph as the complete graph on $d+1$ vertices, where d is the dimension.

3. When does the canonical image of C (say for G connected, 3-connected, and trivalent) lie on some kind of degenerate $K3$ surface? *Guess*: when G is the 1-skeleton of a cell-decomposition of the 2-sphere. (Stable models of $K3$'s consisting of planes correspond to *triangulations* of the sphere; but other things can occur as limits in projective space.)

4. From Proposition 1.2 we see directly why the product of two edge-disjoint cycles belongs to I. But also, we can "see" some quadrics of rank 4: If a pair of cycles a, b has exactly one edge x in common, and if two further cycles c, d also have exactly the edge x in common, then there will be a rank 4 quadric in I of the form ab-(constant) cd. *Problem*. When do such rank 4 quadrics generate I?

5. Does the isomorphism of Propositions 1.2, 1.3 hold for arbitrary graph curves?

6. In the proof of Corollary 2.2 we showed that if G is trivalent and 3-connected, and if V has at most two vertices, then the restriction of the canonical series on C to $C(V)$ is complete. Can this be extended to some nice class of subgraphs?

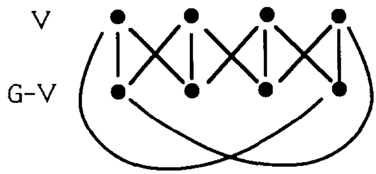
7. The connectivity hypotheses in Corollary 2.2 are probably not necessary in general. The hypothesis that there are no multiple edges is certainly not necessary; it would be interesting to give a sharper criterion.

8. From the theorem of Section 4 we see that each trivalent 3-connected graph leads in a simple way to a Gorenstein ring. What are the properties of these rings? For example, are they in the liaison classes of complete intersections?

9. When is $\mathscr{L}(V)$ (from Section 5) independent of the hyperplane chosen? It would seem not in general, since if, for example, V consists of 3 nonadjacent vertices such that $C(V)$ only spans a 3-space, then some hyperplane sections will have different dimensional spans than others, and thus $h^1\mathscr{L}$ will vary! But this dimension trouble does not occur if the canonical series is restricted to a complete series on $C(V)$, or if V is connected.

10. Recall that *$\mathscr{L}$ contributes to the Clifford index of C* if both $h^0\mathscr{L}$ and $h^1\mathscr{L}$ are $\geqslant 2$. We would like to say that V contributes to the Clifford index of C if $\mathscr{L}(V)$ does; however, Corollary 5.3 only allows us to decide whether $\mathscr{L}(V)$ contributes from the combinatorics when V or $G-V$ is connected, so we have restricted ourselves to that case.

PROBLEM. Decide what the possible values of Cliff $\mathscr{L}(V)$ are when neither V nor $G-V$ is connected. For example, what are the possible Clifford indices for the line bundles of the form $\mathscr{L}(V)$ for V as in the following example:



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APPENDIX: HOMOLIFEROUS CONNECTIVITY OF GRAPH CURVES

Sung Won Park[1]

1. *Basic Definitions*

In the preceding paper, D. Bayer and D. Eisenbud introduce the *combinatorial Clifford index* for trivalent graphs and prove that this is the same as the *resolution Clifford index* for plane trivalent graphs. On the other hand, for the Clifford index Cliff(C) and the gonality $\mu = \mu(C)$ of a smooth curve C,

$$\mathrm{Cliff}(C) \leqslant \mu - 2.$$

We will show that with a few exceptions, the above is true for graph curves if we use the combinatorial Clifford index Cliff(C) and the gonality $\mu = \mu(C)$, which is defined by the minimum of degrees of admissible covers. We will freely use definitions and notations of Bayer and Eisenbud.

Recall that an admissible cover is a map between curves such that the image of a node is a node and the inverse image of a node is a union of nodes. It is more convenient to work with graphs than curves. On the other hand, to get results on curves, we need to define a good class of maps between graphs which correspond to admissible covers of curves. It turns out to be good enough to get a class of maps between the graphs which is slightly larger than the class of maps induced from admissible covers of curves. We will define such a class of maps below and call the maps in it semi-admissible. And then we will define the equivalence of graphs which corresponds to the successive blowings up and down of graph curves.

DEFINITION 1.1. We will keep the following notations:

$h_0(G) = \dim H_0(G, \mathbb{C}) =$ the number of connected component of G

$h_1(G) = \dim H_1(G, \mathbb{C})$.

Then the genus g of a graph G is:

$$g(G) = h_1(G) - h_0(G) + 1.$$

DEFINITION 1.2. For each vertex v of a graph G, the *star* of v is the set of edges incident to v:

$$\mathrm{St}(v) = \{e \in E(G) \colon e \text{ is incident to } v\}.$$

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Definition 1.3. A *map* φ: $G \to H$ between two graphs G and H consists of a vertex map φ: $V(G) \to V(H)$ between the sets of vertices and an edge map φ: $E(G) \to E(H)$ between the sets of edges preserving the incidence of vertices and edges; that is, if a vertex v and an edge e of G are incident, then $\varphi(v)$ and $\varphi(e)$ are incident in H. A map φ between two graphs is called surjective if both the vertex and the edge maps are surjective.

A surjective map φ: $G \to T$ from a graph G to a tree T is called a *semi-admissible cover* of T by G if its restriction for each open star φ: $\mathrm{St}(v) \to \mathrm{St}(\varphi(v))$ is surjective for all $v \in V(G)$. We will define the *degree* of a semi-admissible cover φ by:

$$\deg(\varphi) = \max\{|\varphi^{-1}(e)| : e \in E(T)\}.$$

Proposition 1.4. *If f: $C \to D$ is an admissible cover of degree n of a graph curve then the induced map φ: $G \to T$ on the dual graphs is a semi-admissible cover of degree at most n.*

Proof. For each $v \in V(G)$, define $\varphi(v)$ as the vertex of T which corresponds to $f(C_v)$, where C_v is the component of C which corresponds to v, and similarly for edges. This defines a surjective map φ between graphs in the sense of Definition 1.3. (Since f maps a node to a node, φ maps an edge to an edge.) To show that φ is semi-admissible, it is enough to look at each open star $\mathrm{St}(v)$ of G. Fix a vertex v of G and the corresponding component C_v of C. Let $w = \varphi(v)$ and $D_w = f(C_v)$. Every edge e of $\mathrm{St}(w)$ corresponds to a node q of D_w. Because f is admissible, there is a node p of C_v with $f(p) = q$. If a is the edge which corresponds to the node p, then $a \in \mathrm{St}(v)$ and $\varphi(a) = e$. Q.E.D.

Figure shows some examples of semi-admissible covers.

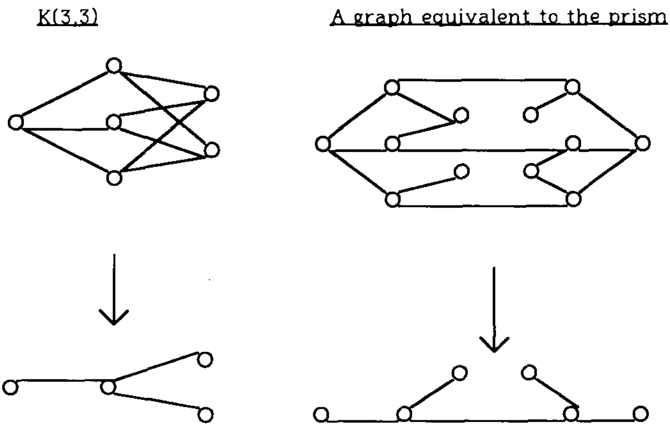

Fig. 1. Semi-admissible covers.

DEFINITION 1.5. Let T be a tree. For any edge e of T with the vertices v and w, $T-e$ has exactly two components. Let $T[v, e]$ be the unique component of $T-e$ which contains v. If the edge e is obvious, then we will use the notation $T[v]$ instead of $T[v, e]$. Let φ: $G \to T$ be a semi-admissible cover of a tree T by a graph G. Let us define $G[v, e; \varphi] = \varphi^{-1}(T[v, e])$. If φ is fixed then we drop it, and similarly for e. When both of e and φ are fixed, then we will simply write $G[v]$.

DEFINITION 1.6. Let φ: $G \to T$ be a semi-admissible cover of a tree T by a graph G. A vertex v of T is *critical* if $h_1(G[v, e]) \neq 0$ but $h_1(G[w, e]) = 0$ for each vertex w incident to v and the edge e between v and w.

DEFINITION 1.7. For each graph G, we define the underlying space $|G|$ of G as for a simplicial complex (see [4]). Following [2], two graphs are *homeomorphic* if also their underlying spaces are. A graph G is a *retract* of a graph G' if $|G|$ is a deformation retract of $|G'|$. Two graphs G and H are *equivalent* if they have a common retract; that is, there is a graph F such that $|F|$ is a deformation retract of $|G|$ and $|H|$, or equivalently, G and H have subgraphs G^+ and H^+, respectively, such that G^+ and H^+ are retracts of G and H, respectively, and G^+ and H^+ are homeomorphic.

DEFINITION 1.8. For any set A of vertices of a graph G of valency 1, define a retraction operator r_A by $r_A(G) = G - A$. If A is *the* set of vertices of valency 1, then we will write $r(G)$ for $r_A(G)$.

PROPOSITION 1.9. *A connected subgraph G^+ of a connected graph G is a retract of G if and only if $G^+ \supset r^n(G)$ for some n.*

Proof. If $G^+ \supset r^n(G)$, then define $A[i] \subset V(D)$, $1 \leqslant i \leqslant n$, inductively, by $A[i] = \{v \in V(r_{A[i-1]} \cdots r_{A[1]}(G))\colon \operatorname{val}(v) = 1\} - V(G^+)$. Since G and G^+ are connected, $G^+ = r_{A[n]} \cdots r_{A[1]}(G)$. Thus G^+ is a retract of G.

Conversely, if G^+ is a retract of G, then G^+ can be obtained from G by removing vertices of valency 1 successively. If n is larger than the number of vertices in $G - G^+$, then $G^+ \supset r^n(G)$. Q.E.D.

LEMMA 1.10. *A retract G' of a connected trivalent graph G is a subdivision of G.*

Proof. Let G^+ be the subgraph of G which is an image of G', that is, $|G^+| = |G'|$. By Proposition 1.9, $G^+ \supset r^n(G)$ for some n. Since G is trivalent, $r^n(G) = G$, hence $G^+ = G$. Because G' is homeomorphic to a trivalent graph G, it is a subdivision of G. Q.E.D.

PROPOSITION 1.11. *A graph H is equivalent to a trivalent graph G if and only if $r^n(H)$ is a subdivision of G for some n.*

Proof. If $r^n(H)$ is a subdivision of G, then G is equivalent to H. Conversely, if H is equivalent to G, then it has a subgraph H' which is a retract of G and H. By Lemma 1.10 and Proposition 1.9, H' is a subdivision of G and $H' \supset r^n(H)$. Since H' is homeomorphic to G, it has no vertex of valency 1 and $H' = r^n(H') \subset r^n(H)$. Thus, $r^n(H) = H'$ is a subdivision of G. Q.E.D.

DEFINITION 1.12. As in [2], a *$u-v$ walk* of a graph G is an alternating sequence v_0, e_1, v_1, ..., e_k, v_k of vertices and edges of G such that $u = v_0$, $v = v_k$ and for each i, v_{i-1} and v_i are the vertices incident to e_i. Also, we will follow the definitions of a *trail* and a *path* in G. A trail (resp. a path) is a walk with all edges (resp. vertices) in it distinct. The number k of occurrences of edges is the *length* of the above walk. If $k \neq 0$ and $v_0 = v_k$, then the trail is a *circuit*. If in addition, all v_i, $i \neq 0$, are distinct, then the circuit is called a *cycle*. A *k-cycle* is a cycle of length k. The *distance* $\operatorname{dist}(v, w)$ of two vertices v and w of a graph G is the length of the shortest path between v and w. The distance $\operatorname{dist}(A, B)$ between two disjoint subsets A and B of $V(G)$ is defined by: $\operatorname{dist}(A, B) = \min\{\operatorname{dist}(v, w) \colon v \in A,\ w \in B\}$. For a subgraph H of G, the diameter of H is defined by:

$$\operatorname{diam}(H) = \max\{\operatorname{dist}(v, w) \colon v, w \in V(H)\}.$$

DEFINITION 1.13. If $h_1(G) \neq 0$, then the *girth* $\gamma(G)$ of a graph G is defined by the length of the shortest cycle in G.

LEMMA 1.14. *A trivalent graph has a minimal girth among equivalent graphs; that is, if a trivalent graph G is equivalent to a graph G', then $\gamma(G) \geqslant \gamma(G')$.*

Proof. By Proposition 1.11, there is an integer n such that $r^n(G')$ is a subdivision of G. Thus we have $\gamma(G) \leqslant \gamma(r^n(G')) \leqslant \gamma(G')$. Q.E.D.

See Fig. 2.

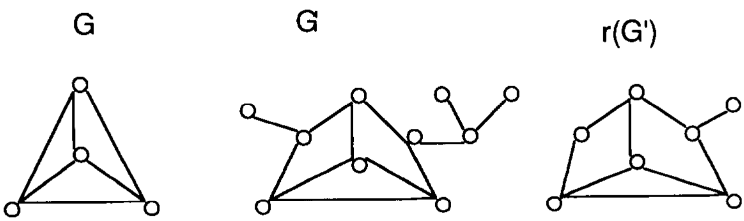

FIG. 2. Retractions.

2. *The Main Theorem*

In this section, we will show that $\mathrm{Cliff}(C) \leqslant \mu(C)-2$ for all graph curves C with four exceptions. First, we will fix some notations. Through this section, we will fix a graph curve C and its dual graph G which we will assume to be connected. Note that G is trivalent from the definition of a graph curve. Because of the duality, we will deal with graphs. We will use the following notations and definitions:

DEFINITION (Notation) 2.1. We will fix the genus g of the graph G. The *gonality* $\mu = \mu(G)$ is the gonality of the curve $C = C(G)$. Following [1], the *homoliferous connectivity* $\eta = \eta(G)$ of the graph G is the smallest number of edges in G disconnecting it into two parts with non-trivial first homology groups. We will also use Corollary 5.6 of [1] for the definition of the *combinatorial clifford index* $\mathrm{Cliff}(G)$ of the graph G, that is $\mathrm{Cliff}(G) = \eta - 2$. We will define a number $\nu = \nu(G)$ by the minimum of degrees of all semi-admissible covers of trees by graphs equivalent to G:

$$\nu(G) = \min\{\deg(\varphi)\colon \varphi \text{ is a semi-admissible cover of a tree by a graph equivalent to } G\}.$$

We will show that $\eta \leqslant \nu$ unless G is one of the following graphs: $B(2)$, $K(4)$, $K(3, 3)$, Peterson graph, and Heawood graph (see Fig. 3). Because $\nu \leqslant \mu$ from Proposition 1.5, this proves $\mathrm{Cliff}(C) \leqslant \mu(C)-2$.

We need to know when the homoliferous connectivity of a trivalent graph G is defined, that is, when there is a set of edges disconnecting the graph into two parts with nontrivial homology. In the following lemmas, we will find all trivalent graphs of genus $g \leqslant 4$ for which the homoliferous connectivity is not defined. For genus $g \geqslant 5$, see [1, Proposition 5.10].

LEMMA 2.2. *A trivalent 3-edge connected graph G of genus $g \leqslant 4$ is one of* $B(2)$, $K(4)$, $K(3, 3)$, *or the* prism.

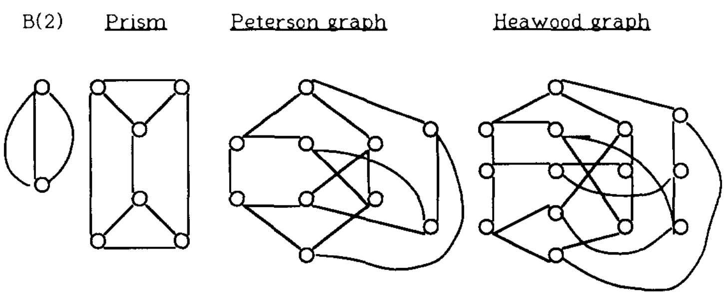

FIGURE 3

Proof. Because G is 3-connected, $g \geqslant 2$. If $g=2$, then $d(G)=2$ and $e(G)=3$, since G is trivalent. Thus G is $B(2)$ which is the graph with two vertices connected by three edges. In fact, this is the only trivalent, connected, and 3-connected graph with multiple edges. (If there is a triple edge, then $G=B(2)$. Any other graph with a double edge cannot be 3-connected.) If $g=3$ then $d(G)=4$ and $e(G)=6$. Since G is 3-connected, there is no double or returning edge. So each vertex of G is incident to three other vertices; that is, any two vertices are connected by a unique edge. By definition, G is the complete graph $K(4)$ on four vertices.

Assume $g=4$. Then $d(G)=6$ and $e(G)=9$. There are two cases: G contains a triangle or not. If G contains a triangle A, then there are only three edges in G between A and $G-A$. Thus $G-A$ is a graph on three vertices with three simple edges, which is a triangle. So G is a prism if it has a triangle. If G has no triangle, then split $V(G)$ into two parts as follows. Choose a vertex v_1 of G. There are three vertices, say, w_1, w_2, and w_3 incident to v_1. Because G has no triangle, no two of w_i's are incident. Let v_2, v_3 be the other two vertices. Using w_j, $1 \leqslant j \leqslant 3$, instead of v_1, we can see each w_j is incident to v_i, $1 \leqslant i \leqslant 3$, and no two of v_i's are incident. That is, G is the complete bipartite graph $K(3,3)$ generated by $V=\{v_i \mid 1 \leqslant i \leqslant 3\}$ and $W=\{w_j \mid 1 \leqslant j \leqslant 3\}$. Q.E.D.

LEMMA 2.3. *If a trivalent graph G of genus $g \leqslant 4$ is not 3-edge connected, then the girth γ of G is less than or equal to 2.*

Proof. Take a disconnecting set E of edges of G with $e(E)=k \leqslant 2$. If A and B are the connected components of $G-E$, then we have $3d(A)=2e(A)+k$ and $g(A) \geqslant 1$ since G is trivalent. Combining with Euler's formula, we get $d(A)=2g(A)-2+k$. By the same way, we have $d(B)=2g(B)-2+k$ and $g(B) \geqslant 1$. We may assume $1 \leqslant g(A) \leqslant g(B)$. Because $g(A)+g(B)+k-1=g \leqslant 4$, either $g(A)=1$ or $g(A)=2$ and $k=1$. If $g(A)=1$, then $\gamma \leqslant \gamma(A) \leqslant d(A)=k \leqslant 2$. If $g(A)=2$, $k=1$, then $d(A)=3$ and $e(A)=4$. Since $k=1$, A has a unique vertex v of valency 2. Since $d(A-v)=e(A-v)=2$ and $g(A-v)=g(A)-1=1$, $A-v$ is a cycle of length 2 in G; that is, $\gamma \leqslant 2$. Q.E.D.

To proceed, we need the following estimation from [3].

LEMMA 2.4 [3, Theorem IV.1]. *If G is a connected, trivalent graph of genus g and girth γ, then*

$$g \geqslant 2^{\gamma/2} \qquad \textit{if } \gamma \textit{ is even}$$

$$g \geqslant (3/2)\, 2^{(\gamma-1)/2} > 2^{\gamma/2} \qquad \textit{if } \gamma \textit{ is odd.}$$

Using the above lemma, we make the following estimations.

LEMMA 2.5. *The homoliferous connectivity η of a 3-valent graph G is well defined unless G is one of $B(2)$, $K(4)$, or $K(3, 3)$. In addition, $\eta(G) \leqslant \gamma(G) \leqslant g(G) - 1$ whenever $\eta(G)$ is defined.* (*See also* [1, *Proposition* 5.10].)

Proof. Choose a cycle Γ of length γ in G. Since the set E of edges between Γ and $G - \Gamma$ has $e(E) \leqslant \gamma$, $h_1(G - \Gamma) \geqslant g(G - \Gamma) = g - e(E) \geqslant g - \gamma$.

If $\gamma \leqslant g - 1$, then $\eta(G) \leqslant e(E) \leqslant \gamma$ and $\eta(G)$ is well defined. Otherwise, Lemma 2.4 gives $g = \gamma \in \{2, 3, 4\}$. By Lemma 2.2 and Lemma 2.3 G is one of $B(2)$, $K(4)$, or $K(3, 3)$. Q.E.D.

LEMMA 2.6. *Let G be a connected trivalent graph of genus g and girth γ, then $\gamma \leqslant [(g + 3)/2]$ unless $G = K(3, 3)$, Peterson graph, or Heawood graph, for which $\gamma = (g/2) + 2$.*

Proof. If $\gamma > [(g + 3)/2]$, then by Lemma 2.4, $(g, \gamma) = (4, 4)$, $(6, 5)$, $(8, 6)$. If $(g, \gamma) = (4, 4)$, then $G = K(3, 3)$ by Lemmas 2.2 and 2.3 above. If $(g, \gamma) = (6, 5)$ or $(8, 6)$, then G is the 5-cage or the 6-cage, which are Peterson graph and Heawood graph, respectively. (See [2, pp. 63, 64].)

LEMMA 2.7. *Let G' be an at most trivalent graph of genus g, homoliferous connectivity η. Let φ: $G' \to T$ be a semi-admissible cover of degree m of a tree T by G'. If $m < \eta$, then there is a unique* critical *vertex v of T which is trivalent.*

Proof. If e is an edge of T and v, w are vertices incident to e, then either $h_1(G'[v, e]) = 0$ or $h_1(G'[w, e]) = 0$, since $|\varphi^{-1}(e)| \leqslant m < \eta$. Because $\deg(\varphi) = m \leqslant \eta - 1 \leqslant g - 2$, by Euler's formula,

$$\begin{aligned} &h_1(G'[v, e]) + h_1(G'[w, e]) - h_0(G'[v, e]) - h_0(G'[w, e]) + 1 \\ &= g - |\varphi^{-1}(e)| \; h_1(G'[v, e]) + h_1(G'[w, e]) \\ &\geqslant g - m + 1 \geqslant 3. \end{aligned}$$

Hence, exactly one of $h_1(G'[v, e])$ and $h_1(G'[w, e])$ is nontrivial. We will define a partial order by $v > w$ if $h_1(G'[v, e]) \neq 0$.

Since T is a tree, we can extend this as a *partial order* on $V(T)$. Since we can compare any two incident vertices and T is connected, the above partial order is *directed*. Since $V(T)$ is a finite set, we have the maximum v of $V(T)$ under the above order. By definition, the maximum v is *the critical vertex*.

We want to show $\text{val}(v) = 3$. If $\text{val}(v) = 1$, then $G'[v, e] = \varphi^{-1}(v)$. Thus, $H_1(G'[v, e]) = 0$, a contradiction. Assume that $\text{val}(v) = 2$. Let a, b be the edges incident to v and let u, w be the vertices incident to a and b, respectively. Let $A = \varphi^{-1}(a)$, $B = \varphi^{-1}(b)$, and $V = \varphi^{-1}(v)$. From Euler's formula,

$$d(G') - e(G') = 1 - g$$
$$d(G'[u]) - e(G'[u]) = h_0(G'[u])$$
$$d(G'[w]) - e(G'[w]) = h_0(G'[w]).$$

On the other hand, we have the following relations:

$$d(G') = d(G'[u]) + d(G'[w]) + d(V)$$
$$e(G') = e(G'[u]) + e(G'[w]) + |A| + |B|$$
$$|A| + |B| \leqslant 3d(V).$$

Combining the above relations, we obtain

$$1 - g = h_0(G'[u]) + h_0(G'[w]) + d(V) - |A| - |B| \geqslant 2 - (2/3)(|A| + |B|).$$

Note that $g = g(G')$ and $\eta = \eta(G')$ are invariant under equivalences but the girth of G' is not. To get an invariant under equivalence, we need the minimum of the girths of graphs equivalent to G', which is, by Lemma 1.9, the girth of the trivalent graph G which is equivalent to G'.

If $\gamma = \gamma(G)$ is the gonality of the trivalent graph G equivalent to G', then we can use all the above inequalities between η, γ, and g for trivalent graphs. From Lemmas 2.5 and 2.6, $\eta \leqslant \gamma \leqslant (g/2) + 2$. Thus, $m \leqslant \eta - 1 \leqslant (g/2) + 1$. Combining two inequalities, we have $3 - 3g \geqslant 6 - 4m \geqslant 2 - 2g$, which is a contradiction. Q.E.D.

THEOREM 2.8. *If G is a connected and trivalent graph, then $\eta(G) \leqslant \nu(G)$, unless G is one of $B(2)$, $K(4)$, $K(3, 3)$, Petersen graph, or Heawood graph.*

Proof. By Lemma 2.3, we may assume G is 3-edge connected. By Lemmas 2.2 and 2.5 we may assume $g \geqslant 5$ and $\eta = \eta(G)$ is well defined. We want to show that if $\nu = \nu(G) < \eta = \eta(G)$, then G is either Peterson graph or Heawood graph.

Take a semi-admissible cover $\varphi: G' \to T$ of degree ν of a tree T by a graph G' equivalent to G. By Lemma 2.7 T has a unique trivalent *critical* vertex v. If w_i, e_i are the vertices and edges incident to v and $T_i = T[w_i]$, $G'_i = G'[w_i]$, and $E_i = \varphi^{-1}(e_i)$, $1 \leqslant i \leqslant 3$, then

$$h_1(G'_i) = 0 \qquad \text{for all} \quad i = 1, 2, 3.$$

Since G and G' are equivalent and G is trivalent, by Lemma 1.9, G is a subdivision of a retract G^+ of G' which is a subgraph of G'. Let $G_i^+ = G'_i \cap G^+$ for $1 \leqslant i \leqslant 3$. Define $G_i = G_i^+ \cap G$ by the subgraph of G whose underlying topological space is $|G_i^+|$. By the same way, if we define $V' = \varphi^{-1}(v)$, $V^+ = V' \cap G^+$, $V = V^+ \cap G$, and $k = d(V^+) - d(V)$, then k is the number of edges in G between G_1, G_2, and G_3. The number of edges of G between V and $\bigcup_{1 \leqslant i \leqslant 3} G_i$ is

$$3d(V) = e(G) - \sum e(G_i) - k.$$

By Euler's formula, we obtain

$$e(G) = d(G) + g - 1$$

$$e(G_i) = d(G_i) - h_0(G_i) + h_1(G_i) \qquad \text{for all} \quad i = 1, 2, 3.$$

Since G_i is equivalent to G_i' for any $i = 1$, 2, and 3,

$$h_1(G_i) = h_1(G_i') = 0 \qquad \text{for all} \quad i = 1, 2, 3.$$

Hence,

$$e(G) - \sum e(G_i) = d(G) - \sum d(G_i) + \sum h_0(G_i) + g - 1$$

$$= d(V) + \sum h_0(G_i) + g - 1.$$

By using $d(V^+) = d(V) + k$, we obtain

$$2d(V^+) = g - 1 + k + \sum h_0(G_i).$$

Since $d(VB^+) \leqslant d(V') \leqslant v \leqslant \eta - 1 \leqslant (g/2) + 1$, we have $k + \sum h_0(G_i) \leqslant 3$. Thus, $k = 0$ and $h_0(G_i) = 1$ for all i, which imply $d(V) = v$, $g = 2v - 2$. Since G is trivalent and $\operatorname{dist}(G_i, G_j) = 2$ for $i \neq j$,

$$d(G_i) = v - 2 \qquad \text{for all} \quad i$$

$$v + 1 \leqslant \eta \leqslant \gamma \leqslant \operatorname{diam}(G_i) + 3 = v + 1 \qquad \text{for all} \quad i.$$

Thus each G_i is a $(v-2)$-pointed line segment.

Let $V = \{v_i\colon 1 \leqslant i \leqslant v\}$, $G_1 = \{a_j\colon 1 \leqslant j \leqslant v - 2\}$, $G_2 = \{b_j\}$, *and* $G_3 = \{c_j\}$. We may assume that the indices are ordered along the line segments. We may assume that v_1 and v_2 (v_{v-1} and v_v, respectively) are incident to a_1 (a_{v-2}, respectively). If b_i and b_j are incident to v_1 and v_2, respectively, for $i \leqslant j$, then we have

$$v + 1 \leqslant \operatorname{dist}(b_i, b_j) + 4 \leqslant \operatorname{diam}(G_2) + 4 \leqslant v + 1$$

$$j - i = \operatorname{dist}(b_i, b_j) = v - 3$$

$$i = 1 \qquad \text{and} \qquad j = v - 2.$$

This implies b_1, b_{v-2} are incident to v_1 and v_2, respectively. Similarly, b_1, b_{v-2} are incident to v_{v-1} and v_v, respectively (after interchanging v_{v-1} and

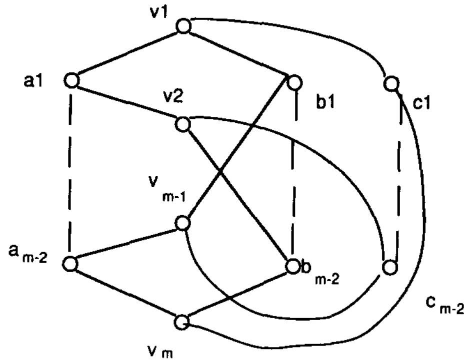

FIGURE 4

v_ν if necessary). For the same reason, $\{c_1, c_{\nu-2}\}$ is the set of vertices of G_3, each of which is incident to one of the pairs of vertices $\{v_1, v_2\}$ and $\{v_{\nu-1}, v_\nu\}$. We may assume that c_1 is incident to v_1. It follows that $c_{\nu-2}$ is incident to v_2. If in addition, c_1 is incident to $v_{\nu-1}$, then G contains a cycle $[v_1, b_1, v_{\nu-1}, c_1, v_1]$ of length 4; that is, $\eta \leqslant \gamma \leqslant 4$. Since $g = 2\nu - 2$ and $g \geqslant 5$, we obtain $\nu \geqslant 4 \geqslant \eta$, which is a contradiction. Thus, c_1 is incident to v_ν and $c_{\nu-2}$ is incident to $v_{\nu-1}$. Because of the cycle $(a_1, v_1, b_1, v_{\nu-1}, c_{\nu-2}, v_2, a_1)$, we get $\gamma \leqslant 6$. Since $4 \leqslant \nu < \gamma \leqslant 6$, we obtain $\nu = 4$ or 5. Thus the only possibilities are Peterson graph and Heawood graph (see Fig. 4). Q.E.D.

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