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Some Linear Syzygy Conjectures

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In this paper we introduce four conjectures refining a very useful result of Mark Green's on the linear part of the free resolution of a graded module over a polynomial ring. We prove a number of special cases of the conjectures, including all cases pertaining to modules over a polynomial ring in at most four variables. In a later paper we will give a complete classification of the nets of skew-symmetric transformations on a 5-dimensional vector space over an algebraically closed field, from which we deduce that our conjectures are true in five variables as well.

Contents.

INTRODUCTION

If $X \subset \mathbb{P}^r$ is a projective variety, then the Hilbert polynomial of X is a fundamental invariant. A considerable refinement of this invariant is given by the free resolution of the homogeneous ideal of X; as Hilbert pointed out already 100 years ago, the Hilbert function of the homogeneous coordinate ring of X, and thus in particular the Hilbert polynomial, can easily be computed from the free resolution. But although the geometric significance of the Hilbert polynomial has been widely studied, the finer

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invariants of the free resolution have remained quite mysterious until very recently, when a series of investigations and conjectures of Mark Green, Rob Lazarsfeld, and others has brought fresh attention to the area (see, for example [12, 13, 19, 5, 17, 1, 16, 8]).

Most of this recent work concerns the "linear part of the resolution," or can be interpreted in terms of this linear part. To be explicit, consider a graded module $\mathscr{M}$ generated by elements of degree 0 over the homogeneous polynomial ring $S=F[x_0, ..., x_r]$; in the geometric applications, $\mathscr{M}$ is usually either the ideal sheaf of a projective scheme, or the module of sections of twists of the canonical bundle of a projective scheme, suitably twisted, in either case, to bring the generators into degree 0. We can write the (unique) minimal graded free resolution of $\mathscr{M}$ in the form

$$\mathscr{F}\colon \cdots \to F_i \to F_{i-1} \to \cdots \to F_1 \to F_0 \to \mathscr{M} \to 0,$$

where

$$F_i = \sum_j S(-a_{ij}) \qquad \text{with} \quad a_{ij} \geqslant i$$

because we have assumed that $\mathscr{M}$ is generated in degree 0. The *linear part of $\mathscr{F}$* is the subcomplex

$$\mathscr{F}_{\text{lin}}\colon \cdots \to F_i' \to F_{i-1}' \to \cdots \to F_1' \to F_0' \to \mathscr{M} \to 0,$$

where

$$F_i' = \sum_j S(-a_{ij}) \qquad \text{with} \quad a_{ij} = i.$$

It is called the linear part because the maps between the F_i' are given by matrices of linear forms.

In many ways the linear part $\mathscr{F}_{\text{lin}}$ seems simpler than the rest of the resolution. For example, $\mathscr{F}_{\text{lin}}$ depends only on the parts of $\mathscr{M}$ of degrees 0 and 1, which we will call M and M_1 throughout this paper, and the part of the module structure expressed by the map $W \otimes M \to M_1$, where W is the vector space of linear forms in S. There is even a sort of "closed form" construction of $\mathscr{F}_{\text{lin}}$ in terms of this data; see [15; 4, p. 108, Theorem] for treatments of this elementary fact. The ranks of the free modules F_i' given by this construction is

$$\operatorname{rank} F_k' = \dim \ker \left(\bigwedge^k W \otimes M \to \bigwedge^{k-1} W \otimes M_1 \right),$$

where the map may be represented, for example, as a graded piece of the differential of the tensor product of $\mathscr{M}$ and the Koszul complex of S. (The homology of this complex is $\mathrm{Tor}^S_*(F, \mathcal{M})$, so the equality is immediate.) In Green's terms [11] this is the dimension of the Koszul homology group $\mathcal{K}_{p,0}(\mathcal{M}, W)$.

Perhaps the simplest invariant attached to $\mathcal{F}_{\mathrm{lin}}$ is what we will call the "linear projective dimension" of $\mathcal{M}$: it is the largest k for which F'_k is nonzero. A good deal of the recent geometric work mentioned above rests on a simple "Vanishing Theorem" of Mark Green which gives in effect an upper bound for the linear projective dimension of $\mathcal{M}$ in certain cases. The purpose of this paper is to introduce a number of conjectures strengthening and generalizing this vanishing theorem, and to prove them in some special cases.

As Mark Green has pointed out, even the weakest of our conjectures (the linear syzygy conjecture, 2.1 below) implies a conjecture of Green and Lazarsfeld [13] on the geometric significance of the linear part of the resolution of a finite set of points; see [5] for an exposition of this matter.

To explain our results and conjectures, we begin by establishing notation that we will use throughout this paper:

We work over an arbitrary ground field F.

W will denote a vector space of dimension w, thought of as the space of linear forms in the polynomial ring $S = F[W]$.

$\mathcal{M}$ will denote a graded S-module

$$\mathcal{M} = M \oplus M_1 \oplus \ldots$$

which we may take to be generated by $M = M_0$. We write m for the dimension of M.

We will often assume that $\mathcal{M}$ has a *linear k-syzygy*: that is, if $\mathcal{F}_{\mathrm{lin}}$ as above is the linear part of the minimal free graded resolution for $\mathcal{M}$, then $F'_k \neq 0$, or equivalently $\mathrm{Tor}_k^{F[W]}(F, \mathcal{M})_k$, the degree k part of $\mathrm{Tor}_k^{F[W]}(F, \mathcal{M})$, is nonzero.

With this notation, Green's Vanishing Theorem says that if $\mathcal{M}$ has a linear k-syzygy, and if $\mathcal{M}$ satisfies a certain "global" hypothesis, then $\dim M = m > k$. (See Theorem 1.1 for a precise statement.) Since the hypothesis that $\mathcal{M}$ has a linear k-syzygy, and also the conclusion of this theorem, only depend on the data involved in the multiplication map $W \otimes M \to M_1$, it is natural to hope for a version of the theorem in which the global hypothesis is eliminated in favor of a "local" hypothesis on this map alone. For example, Green's global hypothesis implies the local hypothesis that no element of W annihilates any element of M. After a study of examples, we became convinced that this should suffice, whence we have the following primitive version of one of our conjectures:

Conjecture. If $\mathscr{M}$ has a linear k-syzygy and no element of W annihilates any element of M, then $m > k$.

The versions of this conjecture stated in the body of this paper improve the one given above by giving *quantitative* consequences of the existence of a linear k-syzygy, both in cases where $m > k$ and in cases where $m \leqslant k$. For example, consider the variety

$$R_1 = \{(x \otimes m) \in W \otimes M \mid xm = 0\}$$

consisting of pairs of linear forms and elements of M that they annihilate. The conjecture above can be reformulated as follows:

Conjecture. If $\mathscr{M}$ has a linear k-syzygy and $m \leqslant k$, then $R_1 \neq 0$.

But looking at examples, one sees that the following quantitative version is justified:

Conjecture. If $\mathscr{M}$ has a linear k-syzygy and $m \leqslant k$, then $\dim R_1 \geqslant k$.

To understand this estimate in at least one case, consider the situation where $\mathscr{M} \cong S/I$, that is, $m = 1$. The linear part of the free resolution of $\mathscr{M}$ is then a Koszul complex on whatever linear forms happen to lie in I, and thus $\mathscr{M}$ has a linear k-syzygy iff at least k independent linear forms lie in I iff $\dim R_1 \geqslant k$. This shows that the conjecture is correct and sharp in this case. The reader may convince himself that all the conjectures in this paper are satisfied in a similarly trivial way in the case $m = 1$.

This conjecture may be modified to apply to modules with any values of k and m (2.1 below), but all the cases may be deduced from the case where $m = k$, which we think of as the *critical case*. In this critical case, various strengthenings are possible. One of the most interesting is perhaps the "epimorphism conjecture." To state it we will restrict ourselves to the "minimal case," and assume that for any $M' \subsetneq M$, the submodule

$$\mathscr{M}' = M' \oplus M_1 \oplus \cdots \subset \mathscr{M}$$

has no linear k-syzygy. We have:

Conjecture. If $m = k$ and $\mathscr{M}$ is a minimal module with linear k-syzygy in the sense above, then every element of M is annihilated by some element of W; that is the projection map $R_1 \to M$ is an epimorphism.

We can actually prove a statement closely related to this last conjecture, and, aside from the examples and special cases that we know, this provides the main evidence for believing the conjectures. To explain this statement, we must introduce another idea.

A linear k-syzygy of $\mathscr{M}$ corresponds to an element e of the kernel of the

Koszul map $\kappa\colon \bigwedge^k W \otimes M \to \bigwedge^{k-1} W \otimes M_1$ induced by the multiplication map $W \otimes M \to M_1$. Let R be the kernel of the multiplication map, so that R is the set of linear first syzygies (and R_1, defined above, is the set of rank 1 tensors in R.) The map κ factors as

$$\bigwedge^{k} W \otimes M \xrightarrow{\Delta \otimes M} \bigwedge^{k-1} W \otimes W \otimes M \longrightarrow \bigwedge^{k-1} W \otimes M_1,$$

where Δ is the diagonal in the exterior algebra. It is easy to check that with these notations $\Delta \otimes M(e) \in \bigwedge^{k-1} W \otimes R$, and taking adjoints we get from e a map $\bigwedge^{k-1} W^* \to R$. Instead of asking about the locus $R_1 \subset R$, we may now ask about its preimage in $\bigwedge^{k-1} W^*$. In fact, we can do even better.

The elements of $\bigwedge^{k-1} W^*$ which are "most likely" to map to elements of R_1 seem to be the "pure $(k-1)$-vectors," that is, the products $a_1 \wedge \cdots \wedge a_{k-1}$. These can be thought of as elements of the affine cone Γ over the Grassmannian G of $(k-1)$-quotients of W, which is naturally embedded in the projective space of lines in $\bigwedge^{k-1} W^*$. Let

$$\Gamma_1 = \{\gamma \in \Gamma \mid \gamma \text{ goes to an element of } R_1 \text{ in } R\}.$$

Our strongest conjectures concern the image of Γ_1 in R_1. What we can actually prove, except in special cases, concerns Γ_1 itself.

The plan of the paper is as follows. Section 1 contains an algebraic version of Green's proof of his original vanishing theorem. In Section 2, which is the heart of this paper we discuss various conjectures strenghthening it.

Section 3 contains some elementary remarks on linear syzygies in general, which serve, for example, to reduce the Linear Syzygy conjecture to the case $m = k$. Section 4 contains our main theorem on the set Γ_1 introduced above.

The remaining sections treat various special cases. The first two are rather trivial, and we treat them very briefly: In Section 5 we do the "monomial" case where (in the notation above) the image of

$$e\colon M^* \to \bigwedge^{k} W$$

is generated by elements of the form $x_{i_1} \wedge \cdots \wedge x_{i_k}$ for some basis $x_1, ..., x_w$ of W. In Section 6 we do the cases $k = w$ and $k = w - 1$.

In Section 7 we classify the "basic" examples of 2-generator modules with linear second syzygies ($k = 2$). Surprisingly, there is a unique example over each polynomial ring in w variables with $w = 2$ or w odd, none when $w > 2$ is even. The reduced support of the example in $w = 2d + 1$ variables is the affine cone over a rational normal curve of degree d; we conjecture that the examples are associated with some geometrically familiar sheaves on the rational normal curves.

With all these reductions and special cases done, the simplest remaining open case is that of 3-generator modules with a linear 3rd syzygy over a polynomial ring $F[W]$ in five variables (that is, W is 5-dimensional). Such things correspond, by the method of analysis described in Section 2 below, to 3-dimensional families of forms in $\bigwedge^3 W \cong \bigwedge^2 W^*$. Thus we may regard them as coming from nets of skew-forms in five variables.

It turns out that the classification of such nets is finite: In [6] we give the classification and deduce that our conjectures hold in five variables.

Much, of course, remains to be done. Aside from proving or disproving the conjectures we have left, the following seems to us some of the more central problems:

1. Find stronger conditions that hold in the case $m > k$. In particular, it would seem natural to study a linear k-syzygy of a module $\mathcal{M}$ by looking at it as a linear $(k-1)$-syzygy of the first syzygy module of $\mathcal{M}$, but this syzygy module usually has many generators. Find some conditions strong enough to make an induction.

2. The relations between linear syzygies: How many linear $(k-1)$-syzygies does one obtain from a given linear k-syzygy? Which linear k-syzygies imply the existence of linear $k+1$ syzygies? What conditions on a module $\mathcal{M}$ are implied by the existence of two (or more) linear k-syzygies?

3. What subvarieties of $\mathbb{P}M \times \mathbb{P}W$ can be the rank 1 locus, or the image of Γ_1, for a module with a linear k-syzygy?

4. Study the derivative to prove that the image of Γ_0 is small and the image of Γ_1 is large; that is, analyze the map $\Gamma \to R$ in the neighborhood of a point of Γ_0. For example, if one could show that the differential to this map could not map the tangent space of Γ_0 onto R, one could use the ideas of Section 4 to prove at least part of the conjectures.

5. Could something similar to the epimorphism conjecture hold—for codimension t subspaces—in case $\dim M = k + t$? Does this correctly predict the numbers observed in the special cases? What should be the generalization for $m < k$?

6. Specialization theorems for linear syzygies: Under what conditions does a module with a linear k-syzygy specialize to one with a linear k-syzygy? For example, it has been shown that the ideal of 2×2 minors of a 1-generic $p \times q$ matrix (regarded as a module generated in degree 0 by twisting by 2) always has a linear $r + s - 4$ syzygy (Eisenbud, unpublished), and this has been extended by Koh and Stillman [17] to certain ideals of 4×4 pfaffians.

7. It seems reasonable to hope that the result on the monomial case in Section 5 could be strenghthened to include some of the stronger versions

of the conjecture, (or perhaps used to find counter-examples!) At any rate, it should be possible to give a combinatorial analysis of the situation in this case.

This work was begun while Mike Stillman and the second author held NSF postdoctoral fellowships at Brandeis University, and early parts of it owe much to the collaboration with Stillman; in particular, the idea of doing the monomial case (Section 5, below) is his, and we are grateful to him for his permission to include it here, and also for many other useful discussions of the material. In addition all our conjectures were derived from examples made using the program Macaulay of [2].

1. Green's Vanishing Theorem Revisited

Here is an algebraic version of the Vanishing Theorem of Mark Green [11, Theorem 3.a.1]. Though the statement is considerably more general than Green's, the proof is simply a translation of his proof into algebraic language.

Vanishing Theorem 1.1. *If $\mathcal{M}$ is a graded module over the polynomial ring $S := F[W]$ and $\mathcal{M}$ is torsion free as a module over $R := F[W]/P$ for some (absolutely irreducible) homogeneous prime ideal P not containing a linear form, then the length of the linear part of the free resolution of $\mathcal{M}$ is $<$ the vector space dimension of M; that is,*

$$\operatorname{Tor}_p^S(F, \mathcal{M})_p = 0 \qquad \text{for} \quad p \geqslant \dim_F M.$$

Proof. Note that the projective dimension of $\mathcal{M}$ is $\leqslant \dim W - 1$, since $\mathcal{M}$ has depth at least 1. Thus if $\dim_F M \geqslant \dim W$ there is nothing to prove, and we may assume that $\dim_F M < \dim W$. But if $\mathcal{M}_j \neq 0$ for some $j < 0$ we would have $\dim M \geqslant \dim W$; thus it follows from our assumption that $\mathcal{M}_j = 0$ for all $j < 0$.

Since P is absolutely irreducible, we may assume that F is algebraically closed.

The proof consists of a Koszul cohomology computation and a general position result. First the Koszul computation:

Since $\mathcal{M}_{<0} = 0$, the module $\operatorname{Tor}_p^S(F, \mathcal{M})_p$ is the kernel of the map

$$\bigwedge^p W \otimes M \to \bigwedge^{p-1} W \otimes M_1$$

sending $t = \sum_I x_I \otimes e_I$ to $\sum_{j \in I} \pm x_{I-j} \otimes x_j e_I$. Here I runs over the set of p-tuples of the elements of a fixed basis x of W. If we let J range over the $(p-1)$-tupels, we may rewrite this as

$$\sum_J x_J \otimes \sum_{j \notin J} \pm x_j e_{J+j}.$$

Suppose now that t is in the kernel of this map. The elements x_{I-j} are independent, so for each $(p-1)$-tuple of J of basis elements, $\sum_{j \notin J} x_j e_{J+j} = 0$. Collecting all the equations that bear on a fixed p-tuple I, we see that e_I is knocked by each x_j for $j \in I$ into the submodule of $\mathcal{M}$ generated by the $x_k \mathcal{M}$ for $k \notin I$; that is, $e_I \in \operatorname{socle} \mathcal{M}/(\{x_k\}_{k \notin I} \mathcal{M}$. We wish to show that for a general choice of coordinate system, this represents p independent conditions on e_I (one for each element of I). This is an algebraic translation of the fact that for a global section of a vector bundle to vanish at p general points is at least p linear conditions:

LEMMA 1.2. *Let $\mathcal{M}$ be as above. If a basis $x_0, ..., x_w$ of W is chosen generally, then for all subsets $x_{i_1}, ..., x_{i_r}$ of W, the set*

$$M \cap \operatorname{socle} \mathcal{M}/(x_{i_1}, ..., x_{i_r}) \mathcal{M} = \{m \in M \mid Wm \subset (x_{i_1}, ..., x_{i_r}) M\}$$

has codimension $\geqslant w+1-r$ in M or is 0; in particular it is 0 if $\dim M \leqslant w+1-r$.

Proof. Given any $e \in M$, there is an open set of closed points p in $\operatorname{Proj} R$ such that, writing $\mathcal{P}$ for the prime ideal associated to p, e becomes a free generator of the free $R_{\mathcal{P}}$-module $\mathcal{M}_{\mathcal{P}}$, and there is a homomorphism $\varphi: \mathcal{M}_p \to \kappa(p)$ sending $e/1$ to something nonzero (here $\mathcal{M}_p$ denotes $\mathcal{M}/\mathcal{P}\mathcal{M}$, the "fiber" of $\mathcal{M}$ at p).

It follows that there is an open set of $w+1$-tuples of such points and maps $(p_0, \varphi_0), ..., (p_w, \varphi_w)$ such that the kernel of the corresponding homomorphism

$$M \to \bigoplus \kappa(p_i)$$

has codimension $w+1$ or is 0. Because $\operatorname{Proj} R$ is irreducible and non-degenerate in $\operatorname{Proj} S$ and F is algebraically closed, a general such $w+1$-tuple corresponds to a general choice of coordinate system $x_0, ..., x_w$ in which the p_i are the coordinate points

$$(1, 0, ..., 0) \cdots (0, ..., 0, 1).$$

We claim that such a coordinate system satisfies the condition of the lemma.

To check this, simply note that if

$$e \in \operatorname{socle} \mathcal{M}/(x_{i_1}, ..., x_{i_r}) \mathcal{M}$$

then certainly

$$e \in \operatorname{socle}(\mathcal{M}_{p_k} := \mathcal{M}/(x_0, ..., (x_k \text{ left out}), ..., x_w) \mathcal{M})$$

for $k \notin i_1, \ldots, i_r$, so $e \to 0$ in $\mathcal{M}_{p_k}$ for $w+1-r$ values of k. The desired conclusion follows. ∎

Problem. We have used irreducibility and torsion-freeness to say that a general choice of points satisfies the condition that $M \to \bigoplus \kappa(p_i)$ is of maximal rank, and spans the space. Can this somehow be done without algebraic closure?

Remark. The lemma implies a restriction on which 1-generic pairings $W \otimes M \to M_1$ come from a nondegenerate module in the sense above.

2. Conjectures on Linear Syzygies

Notation. If $R \subset M \otimes W$ is a subspace and $j \geqslant 0$ is an integer, then we write R_j for the subvariety of all tensors of rank $\leqslant j$ in R. (Despite our other conventions, this is NOT the jth graded piece of R!) Equivalently, regarding R as a subspace of $\operatorname{Hom}(M^*, W)$, we take R_j to be the set of maps in R of rank $\leqslant j$.

Linear Syzygy Conjecture 2.1. If $\mathcal{M} = M \oplus M_1$ has linear kth syzygies over $F[W]$ ($\mathrm{lpd} \geqslant k$), then

$$R := \ker M \otimes W \to M_1$$

satisfies:

$$\text{if } \dim M \leqslant k \text{ then } \dim R_1 \geqslant k$$

$$\text{if } \dim M \geqslant k \text{ then } \dim R_{m-k+1} \geqslant k,$$

where $m = \dim M$.

Remarks. (1) (Green) This conjecture (in the case $m = k$) implies the Green–Lazarsfeld conjecture [13] on resolutions of small numbers of points in projective space. See, for example, the discussion in [5].

(2) The conjecture can be reduced to the case $\dim M = k$—see the reduction following Proposition 3.1.

(3) From examples, the conjecture seems fairly sharp in the case $\dim M \leqslant k$, but can probably be improved in the case $\dim M > k$.

Method of Analysis and Stronger Conjectures

By Koszul homology analysis, a linear kth syzygy corresponds to an element of $\Lambda^k W \otimes M$ that goes to zero under the composition

$$\Lambda^k W \otimes M \to \Lambda^{k-1} W \otimes W \otimes M \to \Lambda^{k-1} W \otimes M_1,$$

that is, setting $N = M^*$, a map $N \to \bigwedge^k W$ such that for all $\alpha \in \Lambda^{k-1}W^*$ the composition

$$N \longrightarrow \Lambda^k W \xrightarrow{\tilde{\alpha}} W$$

lies in $R \subset \mathrm{Hom}(N, W)$, where we have written $\tilde{\alpha}$ for the map corresponding to α. Thus we may without loss of generality suppose that R is the set of all maps obtained in this way, and the conjecture, in the case $\dim N \leqslant k$, say, amounts to proving that for every map (equivalently, inclusion) $N \subset \Lambda^k W$, the image R of

$$\Lambda^{k-1} W^* \to \mathrm{Hom}(N, W)$$

has rank 1 locus of dimension $\geqslant k$.

Strengthenings of the conjecture, and a heuristic reason to believe in it, come by restricting our attention to the locus Γ of pure vectors in $\Lambda^{k-1}W^*$, the cone over the grassmannian $G = \mathrm{Gr}(W \to k-1)$. Let N_G and W_G be the trivial bundles obtained by pulling back N and W to G, and let

$$0 \longrightarrow S \longrightarrow W_G \xrightarrow{q} Q \longrightarrow 0$$

be the tautological exact sequence on G. Composing the diagonal map

$$\Delta_{W_G} \colon \Lambda^k W_G \to \Lambda^{k-1} W_G \otimes W_G$$

with $\bigwedge^{k-1} q \otimes W_G$ we derive a map of vector bundles

$$\rho \colon \Lambda^k W_G \to \Lambda^{k-1} Q \otimes W_G$$

on G whose fiber $\rho|_\alpha$ over the point of G corresponding to an element $\alpha \in \Lambda^{k-1} W^*$ is, up to a scalar, the map $\tilde{\alpha}$ (note that $\tilde{\alpha}$ does not define a map of vector bundles $\Lambda^k W_G \to W_G$ because when α is multiplied by a factor r, the map $\tilde{\alpha}$ is multiplied by r as well; the $\otimes\, \Lambda^{k-1} Q$ serves to "cancel out" this scalar.) Restricting ρ to $N_G \subset \Lambda^k W_G$, we obtain

$$\rho_N \colon N_G \to \Lambda^{k-1} Q \otimes W_G.$$

The image of Γ in $\mathrm{Hom}(N, W)$ is just the set of maps obtained fiber by fiber from this map of vector bundles.

We can make a stronger conjecture by requiring the inequalities of the linear syzygy conjecture to hold for the subvarieties of the R_s consisting of points in the image of Γ; that is, we may look in Γ at the locus Γ_s where the map $\rho_N \colon N_G \to \Lambda^{k-1} Q \otimes W_G$ has rank $\leqslant s$, and require that the image of Γ_s in R_s have at least the given dimension:

Strong Linear Syzygy Conjecture 2.2. The image of $\Gamma_s \to R$ has the dimension predicted for R_s in the Linear Syzygy Conjecture; for example, if $m \leqslant k$, then the dimension of the image of Γ_1 is at least k.

Two kinds of degenerate examples of modules with linear kth syzygies limit the possible strengthenings of the conjectures:

(1) If M contains an element annihilated by a k-dimensional space of linear forms, then $\mathcal{M}$ (or even the submodule generated by that element) obviously has a linear k-syzygy.

(2) If $W = W' \oplus \langle x \rangle$, $\mathcal{M}$ has a linear $(k-1)$-syzygy over W', and $\mathcal{M}$ is annihilated by x, then $\mathcal{M}$ has a linear k-syzygy. Of course if $m = k$, then M may have no rank 1 relations as a module over W', so the rank 1 locus of the relations on $\mathcal{M}$ will be exactly the space $\langle x \rangle \otimes M$.

Remark. In (2), if $\langle x \rangle \otimes M$ does not consist of images of pure vectors, then of course the strong conjecture is false!

Of course the simplest way for the strong conjecture to hold would be for the following still stronger condition to be true. For simplicity we deal only with $m \leqslant k$:

Generic Injectivity Conjecture (*Also Suggested by Mark Green*) 2.3. If $m \leqslant k$ and $\mathcal{M}$ does not have a linear kth syzygy over a polynomial ring on any $W' \subsetneqq W$, then the map $\Gamma_1 \to R$ is generically injective. In particular

$$\dim R_1 \geqslant k + (k-m)(w-k).$$

Note that if $m = k$ this reduces to the old conjecture. Note that the restriction that $\mathcal{M}$ does not have a linear kth syzygy over a polynomial ring on any $W' \subsetneqq W$ is necessary: given an $\mathcal{M}$ with $m < k$, we can increase w arbitrarily by tensoring everything with a polynomial ring in some new variables (that is, add elements to W, but no new relations at all). Thus generic injectivity cannot hold in general.

Note that one should almost never hope for injectivity, as opposed to generic injectivity, again because Γ_0 is mapped to the 0 element of $\operatorname{Hom}(N, W)$. For example, when

$$m \leqslant \binom{w}{k} - (w - k + 1),$$

as is usually the case, we may guarantee $\Gamma_0 \neq 0$ by simply choosing an M in the kernel of the map induced by some pure element of $\Lambda^{k-1} W^*$.

Perhaps more manageable strengthenings of the Strong form are the following, which give "reasons" for the image of Γ_1 in R_1 to be large:

Image Conjecture 2.4. If $\mathcal{M}$ is not annihilated by any linear form, and $m \leqslant k$, then the closure of the set of points in W which are the images of rank 1 transformations from Γ_1 form a subvariety of dimension $\geqslant k$ in W.

Another version is the following, which is heuristically supported in the case $w > k > 2$, or $k = 2$, w odd by Theorem 4.2. (In the cases $k = w$, and $k = 2$, w even, it holds vacuously, as every $\mathcal{M}$ with $\dim M = k$ and a linear kth syzygy has a proper submodule with a linear kth syzygy over a smaller W: In case $k = w$ this is elementary, as such an $\mathcal{M}$ must contain the residue class field in degree 0. In case $k = 2$ it is proved below.)

Epimorphism Conjecture 2.5. In the situation of the Linear Syzygy Conjecture, suppose that $\dim M = k$ and that no submodule of M supports a linear k-syzygy. Then the maps induced by nonzero elements of Γ_1 have as kernels every codimension 1 subspace of N—that is, every element of M is annihilated by some linear form.

A Wrong Idea Exposed. This conjecture says that if no subspace of M supports a linear k-syzygy, and $m = k$, then every codimension 1 subspace of N is *closed* in the sense that there is an element of Γ_1 vanishing on it but not on all of N. A natural hope would be that it is actually closed in all of $\Lambda^k W$—one might even hope that every $k-1$ space might be closed in this sense. *This is quite false.* In fact already in the case $k = 2$, the closure of an element of $\Lambda^2 W$, thought of as a skew-symmetric map from W^* to W, is the set of transformations with kernel containing the kernel of the given one. Thus *no* 1-dimensional set is closed in $\Lambda^2 W$! What saves the conjecture in that case is that under the hypothesis that no element of M is annihilated by an element of W, N cannot contain any transformation of rank $< w-1$, and this implies that it cannot contain any two transformations with the same (or even with an inclusion of) kernels, so 1-dimensional subspaces of N are closed in N.

3. General Remarks on Linear Syzygies

We begin by showing that Conjecture 2.1 reduces to the case $m = k$. The reduction rests on two simple constructions of new modules with linear syzygies from old ones.

Proposition 3.1. *Suppose that $\mathcal{M}$ has a linear k-syzygy.*

(a) *For any subspace $W' \subset W$ of codimension c, the $F[W']$ module $M \oplus M_1$ obtained by restricting scalars has a linear $(k-c)$-syzygy.*

(b) *For a generic subspace $M' \subsetneq M$, the module*

$$\mathcal{N} = \mathcal{M}/(F[W]\,M') = (M/M') \oplus (M_1/WM') \oplus \cdots$$

has a linear k-syzygy.

Proof. (a) We may inductively assume that $c = 1$, and write

$$W = W' \oplus \langle x \rangle.$$

Let

$$e = e_1 + e_2 \in \bigwedge^k W \otimes M = \left(x \wedge \bigwedge^{k-1} W' \otimes M\right) \oplus \left(\bigwedge^k W' \otimes M\right)$$

be the Koszul cycle corresponding to the linear kth syzygy, where e_1 and e_2 are the components corresponding to the given direct sum decomposition. Let

$$\kappa: \begin{tikzcd} \left(x \wedge \bigwedge^{k-1} W' \otimes M\right) \arrow[r, "\kappa_1"] \arrow[dr, "\kappa_{12}"'] & \left(x \wedge \bigwedge^{k-2} W' \otimes M\right) \\ \oplus & \oplus \\ \left(\bigwedge^{k} W' \otimes M\right) \arrow[r, "\kappa_2"'] & \left(\bigwedge^{k-1} W' \otimes M_1\right) \end{tikzcd}$$

be the corresponding decomposition of the koszul map κ. Note that κ_2 is the Koszul map for W', and $\kappa_1 = x \wedge \kappa_2$. From $\kappa(e) = 0$ we derive $\kappa_1(e_1) = 0$ and $\kappa_2(e_2) = -\kappa_{12}(e_1)$. Thus if $e_1 \neq 0$ then M has a linear $(k-1)$-syzygy over W', whereas if $e_1 = 0$ then $\kappa_2(e_2) = 0$ and M has a linear k-syzygy over W'.

(b) Again suppose that $e \in \bigwedge^k W \otimes M$ corresponds to a linear k-syzygy. The commutativity of the diagram

$$\begin{tikzcd} \bigwedge^k W \otimes M \arrow[r] \arrow[d] & \bigwedge^{k-1} W \otimes M_1 \arrow[d] \\ \bigwedge^k W \otimes M/M' \arrow[r] & \bigwedge^{k-1} W \otimes M_1/WM' \end{tikzcd}$$

shows that if the image of e in $\bigwedge^k W \otimes M/M'$ is nonzero, then $\mathcal{N}$ has a linear k-syzygy. The image is certainly nonzero for generic M', so we are done. ∎

Remark. The conclusion of (a) can be rephrased as saying, with obvious notation, that if W' is a subspace of codimension

$$c \leqslant \text{lin proj dim}_W\, \mathcal{M}$$

then

$$\text{lin proj dim}_W \mathcal{M} - c \leqslant \text{lin proj dim}_{W'} \mathcal{M}.$$

It seems plausible that we will have equality for generic choices of W'. If this is so, it would give an interesting condition on the Koszul kth homology for the $(k+1)$st homology to be 0.

Reduction of Conjecture 2.1 *to the Critical Case.* First suppose that $\mathcal{M}$ has a linear k-syzygy and $m > k$. In this case, by Proposition 3.1(b), if we choose a generic subspace $M' \subset M$ of dimension $m-k$, then $\mathcal{M}/F[W]\, M'$ has a k-linear syzygy. If Conjecture 2.1 holds in the critical case, then there is a k-dimensional family of linear relations of rank 1 on this module. However, each such relation gives rise to a relation of rank at most $m-k+1$ on $\mathcal{M}$, so we are done.

Next suppose that $\dim M < k$. Using Proposition 3.1(a) in the case when W' has codimension 1 in W, and writing $\mathbb{P}R_1$ for the projective variety associated to R_1 we see by induction that since $\mathcal{M}$ has a linear $(k-1)$ syzygy over $F[W']$, the spaces

$$(\mathbb{P}W' \times \mathbb{P}M) \cap \mathbb{P}R_1 \subset \mathbb{P}(W \otimes M)$$

all have dimension at least $k-2$. Since the intersection of all the $\mathbb{P}W' \times \mathbb{P}M$ is empty, it follows that $\mathbb{P}R_1$ has dimension $\geqslant k-1$, and thus that R_1 has dimension $\geqslant k$ as required. ∎

Remark. Since the choice of M' in the first part of the reduction is arbitrary, it should be possible to obtain more from this. It seems likely that the conjecture is far from sharp in the case $m > k$.

Similarly, the image conjecture can be reduced to the critical case:

PROPOSITION 3.2. *If the image conjecture is true in all cases of k-linear syzygies of modules with $m = k$ generators, then it holds in general.*

Proof. We may suppose by induction that the image conjecture holds in all cases where $k - m < d$, and we must prove it under the assumption that $k - m = d$. If $\mathcal{M}$ has a linear kth syzygy then of course it also has a linear $(k-1)$st syzygy—in fact many, in general. Among these are the ones corresponding to composite maps $N \to \Lambda^k W \to \Lambda^{k-1} W$, where the second factor is obtained by operating with an element y of W^*. By Lemma 3.5, below, we may assume that no such y annihilates N. Thus $y(N)$, and with it N, has a $(k-1)$-dimensional family of rank 1 images in W under elements of Γ_1. All of these images are contained in the hyperplane $\ker y \colon W \to F$.

This shows that every hyperplane contains a $(k-1)$-dimensional family of rank 1 images, so there must at least a k-dimensional family of them to start with. ∎

Now let $\mathcal{M}$ be a module with a linear k-syzygy corresponding to

$$e \in \bigwedge^{k} W \otimes M = \operatorname{Hom}\left(N, \bigwedge^{k} W\right).$$

We will suppose that $\mathcal{M}$ is minimal in the sense that e represents an inclusion $N \subset \bigwedge^k W$, and we will identify N with this subspace of $\bigwedge^k W$.

The following is a very weak version of our conjectures:

PROPOSITION 3.3. *For any $N \subset \bigwedge^k W$ there is a subspace V of dimension k in W^* such that the image of $\Lambda^{k-1} V \subset \Gamma$ in* $\operatorname{Hom}(N, W)$ *has dimension k.*

Proof. Write $n \in N$ as a sum of pure vectors, say

$$n = x_1 \wedge \cdots \wedge x_k + \cdots,$$

with respect to some basis of W, with dual basis $y_1, ..., y_w$. By our hypothesis, we may assume that $V = \langle y_1, ..., y_k \rangle$. We can act with the k elements

$$y_1 \wedge \cdots (\wedge y_i \text{ omitted}) \wedge \cdots \wedge y_k$$

to obtain elements of the form

$$x_i + (\text{linear combination of } x_{k+1}, ..., x_w).$$

Since these span a k-space, the proposition is proven. ∎

Notation. Define L by

$$0 \to N \to \Lambda^k W \to L^* \to 0.$$

LEMMA 3.4. *If $L \subset \Lambda^k V$ has codimension $\leqslant k$, then $\wedge : L \otimes V \to \Lambda^{k+1} V$ is onto.*

Proof. $\Lambda^{k+1} V$ is spanned by subspaces $\Lambda^{k+1} V'$, where V' ranges over the $k+1$-dimensional subspaces of V. Further, L meets each such in codimension $\leqslant k$, and in particular nontrivially. Thus it suffices to prove the lemma when $\dim V = k+1$. In this case, indeed, it suffices for L to be nonzero, as required. ∎

Define K by

$$0 \to K \to \Lambda^{k-1} W^* \to R \to 0.$$

We have

$$\begin{array}{ccccccccc} 0 \longrightarrow & \operatorname{Hom}(W^*, L) & \longrightarrow & \operatorname{Hom}(W^*, \Lambda^k W^*) & \longrightarrow & \operatorname{Hom}(W^*, M) & \longrightarrow 0 \\ & \uparrow\cup & & \uparrow\cup & & \uparrow\cup & \\ 0 \longrightarrow & K & \longrightarrow & \Lambda^{k-1} W^* & \longrightarrow & R & \longrightarrow 0 \end{array}$$

Note that we thus have $L \supset K \wedge W^*$.

LEMMA 3.5. *If L contains a subspace of the form $y \wedge \Lambda^{k-1} W^*$ then $\mathscr{M}$ has lpd over $W' = \ker y$; in fact the relations R only involve variables from W'. A fortiori, the same is true if K contains a subspace of the form $y \wedge \Lambda^{k-2} W^*$.*

Proof. In this case M^* is contained in $\Lambda^k W' \subset \Lambda^k W$, so the images of M under the maps of R are contained in W'. ▮

4. A Schubert Calculus Proof of a Weak Version of the Conjectures

We may weaken the versions of the various linear syzygy conjectures involving the images of the Γ_s by requiring the dimension inequalities to hold for the Γ_s themselves rather than for their images, *and we will prove this weakened version, in a version corresponding to the epimorphism conjecture, below.*

From this point of view it is perhaps more natural to look at the corresponding loci in G rather than in Γ, and we will accordingly define G_s to be the set of points in G where the rank of the map ρ_N is at most s. Of course G_s is simply the projective variety corresponding to the affine cone Γ_s, so $\dim \Gamma_s = \dim G_s + 1$.

The key advantage of this viewpoint is that the image of ρ is contained in the subbundle

$$\Lambda^{k-1} Q \otimes S \subset \Lambda^{k-1} Q \otimes W_G.$$

To see this, we need only show that the composition of ρ with the natural map $\Lambda^{k-1} Q \otimes W \to \Lambda^{k-1} Q \otimes Q$ is zero. This however follows from the commutativity of the following diagram, whose top row is ρ:

$$\begin{array}{ccccc} \Delta_{W_G}\colon \Lambda^k W_G & \longrightarrow & \Lambda^{k-1} W_G \otimes W_G & \longrightarrow & \Lambda^{k-1} Q \otimes W_G \\ \downarrow & & & & \downarrow \\ \Delta_Q\colon \Lambda^k Q & \longrightarrow & & & \Lambda^{k-1} Q \otimes Q \\ \| & & & & \\ 0 & & & & \end{array}$$

We can give an upper bound for the codimension of any component of G_s from the usual formulas for the codimensions of determinantal loci. If we treat ρ as a map to $\Lambda^{k-1}Q \otimes W_G$ we do not obtain an interesting result; but if we treat ρ as a map to $\Lambda^{k-1}Q \otimes S$, then because $\Lambda^{k-1}Q \otimes S$, has rank only $w-k+1$, the result is much sharper: if $s \leqslant \min(m, w-k+1)$, the codimension of any component is at most $(m-s)(w-k+1-s)$, so we obtain the

Basic Dimension Estimates 4.1. If $s \leqslant \min(m, w-k+1)$, and G_s is nonempty, then

$$\begin{aligned}\dim G_s &\geqslant (k-1)(w-k+1)-(m-s)(w-k+1-s)\\ &= (k-m+s-1)(w-k+1)+(m-s)s,\end{aligned}$$

and

$$\dim \Gamma_s = \dim G_s + 1.$$

We will prove this nonemptyness for most values of k and w in Corollary 4.3. Thus when $m=k$, for example, we obtain

$$\dim \Gamma_s \geqslant (s-1)(w-k+1)+(m-s)\,s+1,$$

and when $s=1$, m arbitrary, we obtain

$$\dim \Gamma_1 \geqslant k+(k-m)(w-k)$$

when these are nonempty. In particular, we see that if $m \leqslant k$, then $\dim \Gamma_1 \geqslant k$, which is to the promised weakening of the strong Linear Syzygy Conjecture.

It remains to show the nonemptyness of G_s in some cases. Actually, we will prove a still stronger result, corresponding to the "epimorphism conjecture" below: Given any $P \subset N$ with $\dim P \leqslant k-1$, there is a nonzero element of Γ which annihilates P.

We are grateful to Joe Harris and Jerzy Weyman for their help with the proof of the following result:

THEOREM 4.2. *Let $P \subset \Lambda^k W$ be a subspace, and let $\Gamma(P)$ be the set of pure vectors $\alpha \in \Lambda^{k-1} W^*$ such that $\tilde{\alpha}(P)=0$. Suppose that $k<w$ and that either $k>2$ or $k=2$ and w is odd. If* $\dim P \leqslant k-1$, *then $\Gamma(P)$ is nonempty and of dimension* $\geqslant 1+(k-1-\dim P)(w-k+1)$.

COROLLARY 4.3. *G_s is nonempty if $m-s+1 \leqslant k < w$ and either $k>2$ or $k=2$ and w is odd.*

Proof. Set $d = \dim P$. We will prove instead the corresponding statement for the subvariety of the Grassmannian G

$$G(P) := \{x \in G \mid \rho|_x(P) = 0\}$$

which corresponds to the affine cone $\Gamma(P)$; that is, we will prove that $\dim G(P) \geqslant (k-1-\dim P)(w-k+1)$.

Picking a basis for P, we see that $G(P)$ is the intersection of the zero loci of d sections of the bundle $\Lambda^{k-1}Q \otimes S$, a bundle of rank

$$r := w - k + 1.$$

The rth Chern class $c = c_r(\Lambda^{k-1}Q \otimes S)$ of this bundle in the Chow ring of the Grassmannian is supported on the zero locus of any section. Thus if $c \neq 0$, the components of these zero loci have codimension at most r. Since the dth power c^d of c in the Chow ring is supported on the intersection of the zero loci of the d sections, it suffices to show that $c^d \neq 0$; then

$$\begin{aligned} \dim G(P) &\geqslant \dim G - dr \\ &= (k-1)(w-k+1) - d(w-k+1) \end{aligned}$$

as required. Of course it is enough to show that $c^d \neq 0$ for the largest allowable value, and thus we may take $d = k-1$.

We will use some facts about the Chow ring of G which may be found, for example, in [9, p. 271]. First there is a basis of the Chow ring of G represented by the *Schubert cycles*, which are named by symbols of the form $\{\lambda_1, ..., \lambda_r\}$, with

$$k-1 \geqslant \lambda_1 \geqslant \cdots \geqslant \lambda_r \geqslant 0.$$

The number $\sum_i \lambda_i$ is the codimension of the cycle representing the class $\{\lambda_1, ..., \lambda_r\}$. The class of a point is thus $\{k-1, ..., k-1\}$. This basis has the very important *positivity* property: the product of two Schubert cycles is a linear combination of Schubert cycles with positive integral coefficients (determined by the "Littlewood–Richardson rule," [9, p. 264–265]).

The Chern classes of the universal subbundle S on G are given by

$$c_i(S) = (-1)^i \{1^i\},$$

where we have written $\{1^i\}$ for the string of i ones followed by $w-k+1-i$ zeros, $\{1^i\} = \{1, 1, ..., 1, 0, ..., 0\}$, and the notation is that of [9, p. 271]. (This fact is obtained by putting together [9, p. 270 Example 14.6.5] and [9, p. 54, Remark (a)].) Writing

$$\{1\} = c_1(Q) = c_1\left(\bigwedge^{k-1} Q\right)$$

for the first Chern class of Q, we obtain (from [9, p. 55, formula on the top of page]).

$$c = c_r(\Lambda^{k-1}Q \otimes S) = \sum_{i=0}^{r} (-1)^i \{1\}^{r-1} \{1^i\}.$$

First we treat the case $k=2$, where it is easy to be explicit. Here G is simply projective space, $\{1\}$ is the class of $\mathcal{O}(1)$, and $\{1^i\} = \{1\}^i$. Hence, $\{1^i\}\{1^j\} = \{1^{i+j}\}$, and

$$\begin{aligned} c &= \sum_{i=0}^{r} (-1)^i \{1^r\} \\ &= \begin{cases} 0 & \text{if } \ w = r+1 \text{ is even} \\ \{1^r\} & \text{if } \ w = r+1 \text{ is odd,} \end{cases} \end{aligned}$$

as claimed.

Now suppose that $k>2$. We group the terms in the formula for c into pairs:

$$c = \sum_{i \geqslant 0} P_i$$

with

$$P_i = \{1\}^{r-2i} \{1^{2i}\} - \{1\}^{r-2i-1} \{1^{2i+1}\},$$

where the last term vanishes is $2i+1>r$.

Examining this formula, one sees trivially that $P_0 = 0$. On the other hand, if $i>0$ then

$$\{1\}\{1^{2i}\} = \{1^{2i+1}\} + \{2, 1^{2i-1}\}$$

by Pieri's formula (see [9, p. 271]), and thus

$$P_i = \begin{cases} \{1\}^{r-2i-1}\{2, 1^{2i-1}\} & \text{if } \ 0 < 2i < r \\ \{1^r\} & \text{if } \ 2i = r, \\ 0 & \text{if } \ 2i > r. \end{cases}$$

In particular, these P_i are all positive linear combinations of the Schubert cycles. We will show that $P_1^{k-1} \neq 0$; by the fundamental positivity property of the Schubert cycles, this will show that $c^{k-1} \neq 0$.

Since we have assumed that $k<w$, we have $r = w-k+1 \geqslant 2$. If $r=2$, then only $P_1 = \{1^2\}$ occurs, and we have

$$c^{k-1} = P_1^{k-1} = \{1^2\}^{k-1} = \{k-1, k-1\} \neq 0$$

by (for example) the dual of Pieri's formula.

Thus we may assume that $r \geqslant 3$. Using Pieri's formula again we have

$$\begin{aligned} P_1 &= \{1\}^{r-3}\{2, 1^1\} \\ &= \{2, 1^{r-2}\} + (\text{a positive linear combination of} \\ &\qquad\qquad \text{other Schubert cycles}), \end{aligned}$$

so that it suffices to show

$$\{2, 1^{r-2}\}^{k-1} \neq 0.$$

By the Littlewood–Richardson rule, $\{a^r\}\{b^r\} = \{(a+b)^r\}$ for any integers a, b. Thus it is enough to show that

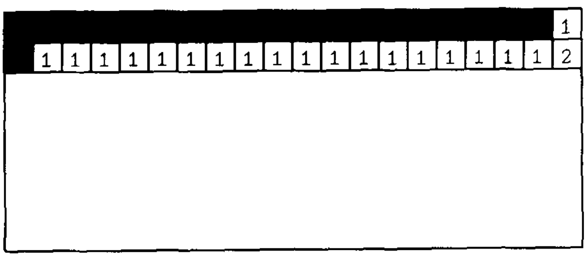
$$\{2, 1^{r-2}\}^2 = \{2^r\} + \{3, 2^{r-2}, 1\}$$

and that

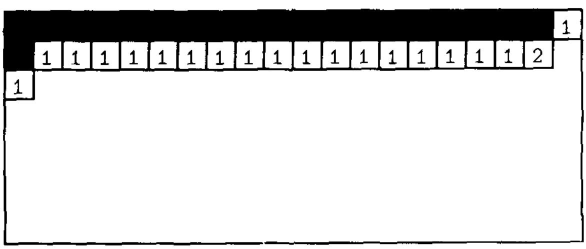
$$\begin{aligned} \{2, 1^{r-2}\}^3 &= \{3^r\} + (\text{a positive linear combination of} \\ &\qquad\qquad \text{other Schubert cycles}). \end{aligned}$$

The first of these formulas follows at once from the Littlewood–Richardson rule; the "strict $r-1$, 1-expansions of $\{2, 1^{r-2}\}$" that are required [9, pp. 264–265] are represented by the following diagrams:

$\{2^r\}$ =



$\{3,2^r,1\}$ =

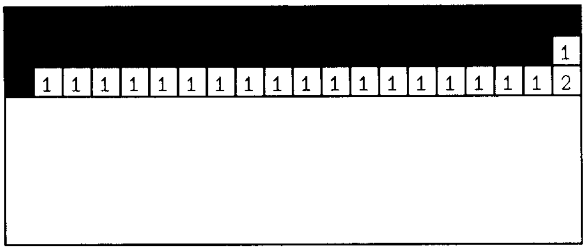


To check the second of the required formulas, we use the first, and thus see that it suffices to show

$$\{3,2^{r-2},1\}\{2,1^{r-2}\}=\{3^r\}+(\text{a positive linear combination of other Schubert cycles}).$$

The "strict $r-1$, 1-expansion of $\{3,2^{r-2},1\}$" that we require is given in the following diagram:

$\{3^r\}$ =



This completes the proof. ∎

5. The Monomial Case

In this section we will deal by combinatorial methods with the case when $N\subset\bigwedge^k W$ is generated by "monomials" $x_{i_1}\wedge\cdots\wedge x_{i_k}$ with respect to some basis $x_1,\ldots,x_w$ of W. The results in this section are due to Mike Stillman. We prove only a weak version of our conjectures:

Theorem 5.1 (Stillman). *Suppose that M has a linear k-syzygy represented by the koszul homology class $e\in\Lambda^k W\otimes M$, and that the corresponding map*

$$e\colon M^*\to\bigwedge^k W$$

is a monomorphism with image generated by "monomials" $x_{i_1}\wedge\cdots\wedge x_{i_k}$ with respect to some basis $x_1,\ldots,x_w$ of W. If the dimension of M is $\leqslant k$, then M is spanned by elements which are annihilated by linear forms, and these relations come from elements of Γ.

Proof. Let $y_1,\ldots,y_w\in W^*$ be a dual basis to $x_1,\ldots,x_w\in W$. For any subset $I=\{i_1,\ldots,i_k\}$ we write

$$x_I=x_{i_1}\wedge\cdots\wedge x_{i_k}\in\bigwedge^k W,$$

and similarly for y_J.

By hypothesis, N admits a basis consisting of elements $x_{I_1}, ..., x_{I_m}$, with $m \leqslant k$. We will show that for each i, there is a $(k-1)$-subset J_i such that

$$y_{J_i}(x_{I_i}) \neq 0 \qquad \text{while} \qquad y_{J_j}(x_{I_i}) = 0 \qquad \text{for} \quad j \neq i.$$

The y_{J_i} are pure vectors, and thus in Γ; from the above formula it follows that if $a_1, ..., a_m$ are a basis for M dual to the basis $x_{I_1}, ..., x_{I_m}$ of N, then y_{J_i} maps to a relation on $\mathscr{M}$ of the form $xa_i = 0$ for a suitable linear form $x \in W$, as desired.

To this end recall that the map $\bigwedge^k W \to W$ induced by an element $b_J \in \bigwedge^{k-1} W^*$ takes a_I to 0 if $J \not\subset I$, and to x_{I-J} if $J \subset I$. Thus it suffices to show that for each i there is a $k-1$ element subset of I_i which is not contained in any other I_j. Of course I_i and I_j can have at most one $k-1$ element subset in common, and there are k distinct $k-1$ element subsets of I_i, but only $k-1$ other I_j, so the result is immediate. ∎

6. The cases $k = w$ and $k = w - 1$

These two cases are quite simple compared to the general ones, and nearly everything can be worked out. Here is a sample:

Case $k = w$. $\Lambda^w W$ is 1-dimensional, so a linear wth syzygy is determined by the 1-dimensional quotient $M^* \to \Lambda^w W$, that is, by a 1-dimensional subspace M' of M, which must be annihilated by all of W. Thus $R_1 \supset M' \otimes W$, and has dimension $\geqslant w$, as required. In this case $G_1 = G$ is projective $w-1$-space, Γ_1 is mapped injectively, and Γ_0 is 0.

Note that this is a case where the expected dimension of Γ_0 is $w - m > 0$, even though Γ_0 is 0 (that is, G_0 is empty.)

Case $k = w - 1$. Using the duality in the exterior algebra we identify $\Lambda^k W$ with W^*, and $\Lambda^{k-1} W^*$ with $\Lambda^2 W$. The maps $W^* \to W$ induced by the elements of $\Lambda^2 W$ are precisely the skew-symmetric maps, and those induced by Γ are the skew-symmetric maps of rank 2. We must study the compositions of these with a map $M^* = N \to \Lambda^{w-1} W = W^*$ corresponding to a linear $(w-1)$st syzygy.

First note that if $m > k$, then in any case we may take $m = w = k + 1$, since the syzygy in question must be induced by one involving only that large a subspace of M (the dual of the image of N in W^*). Taking $m = k + 1$, we obtain $N = W^*$, so there is essentially only one example of this type. Of course there are no skew-symmetric transformations of rank 1, so $R_1 = 0$ in this case. On the other hand we have

$$\dim \Gamma_s = \dim \Gamma = 2k - 1 \qquad \text{for all} \quad s \geqslant 2,$$

and $\Gamma=\Gamma_2$ maps isomorphically to R_2, while in general $\dim R_{2t}=\dim R_{2t+1}=\binom{w}{2}-\binom{w-2t}{2}$.

A natural generalization of the epimorphism conjecture is true here: there is in this case a unique element of Γ_2 whose kernel is a given 2-codimensional subspace (that is, a unique rank 2 relation involving a given 2-dimensional subspace of elements of M).

In the case where $m \leqslant k = w-1$, our conjectures predict that Γ_1 and R_1 are of dimension $\geqslant k$. In order for a pure vector $a \wedge b \in \Lambda^2 W$ to induce a map of rank 1 on N, it is necessary for some element of the space $\langle a, b\rangle$ to annihilate N; for if a, b are independent modulo the annihilator of N, then $a \wedge b$ acts as a nonzero element of $\Lambda^2 N^*$, and thus induces a transformation of rank 2 on N. On the other hand, if $a \in N^\perp$, then $a \wedge b$ induces the transformation $n \mapsto b(n)a$, so $a \wedge b$ and $a \wedge b'$ induce the same transformation on N iff $b \equiv b' \bmod N^\perp$.

Thus G_1 may be identified with the schubert cycle

$$\sigma_{m-1,0,0\ldots} \qquad \text{(notation from [14, p. 197])}$$

of all 2-planes in W meeting $N^\perp$ in a space of dimension $\geqslant 1$, and identifying $W/N^\perp$ with N, the image of

$$\Gamma_1 \to R_1 \subset M \otimes W = \mathrm{Hom}(N, W)$$

is the set of all rank 1 transformations in $\mathrm{Hom}(N, N^\perp)$, the affine cone over $\mathbb{P}N^\perp \times \mathbb{P}(N)$. Thus, for any m with $1 \leqslant m \leqslant k$, the image of Γ_1 in R_1 has dimension k. On the other hand,

$$\dim \Gamma_1 = \dim G - (m-1) + 1 = 2(w-2) - m + 2 = 2k - m,$$

the generic value, so the general fiber has dimension $k-m$, and the fiber over 0 has larger dimension (except when $m=k$, $k-1$):

$$\dim \Gamma_0 = 2(w-m-2)+1 = 2k-2m-1 \text{ for } m<k \qquad (\text{but } =0 \text{ for } m=k).$$

In particular, if we take $w-1=k>m>1$ we arrive at a situation where the map $\Gamma_1 \to R_1$ is not generically injective, and $\mathcal{M}$ does not have $\mathrm{lpd}=w-1$ over any proper subspace of W, since no element of M is annihilated by a codimension 1 subspace of W (the images of M under the maps in Γ_1 are all contained in $N^\perp$).

We claim that in fact the image of Γ_1 is all of R_1 in this case. To see this, write $W=N^* \oplus N^\perp$, so that $\Lambda^2 W = \Lambda^2 N^* \oplus N^* \otimes N^\perp \oplus \Lambda^2 N^\perp$. Since elements of $\Lambda^2 N^\perp$ induce the zero map on N, we may ignore them, and it suffices to prove that any vector inducing a transformation of rank 1 must have its component in $\Lambda^2 N^*$ equal to 0. This is, however, clear, since an element of $\Lambda^2 N^*$ induces a rank 2 transformation *with image in* N^*, whereas the images of the transformations in the other components are all in $N^\perp$.

One can write the relations forced by a linear $(w-1)$-syzygy rather concretely. Choosing a basis $a_1, ..., a_m$ of M and a basis $x_1, ..., x_w$ of W such that a dual basis to $a_1, ..., a_m$ in N maps to the first m elements of a dual basis to $x_1, ..., x_w$ in W^*, the implied relations R appear as the 2×2 minors of the matrix

$$\begin{pmatrix} x_1, ..., x_m, x_{m+1}, ..., x_w \\ a_1, ..., a_m, \quad 0, \quad ..., \ 0 \end{pmatrix}.$$

Since this case is so accessible one might hope to complete the linear syzygy conjecture in this case to a necessary and *sufficient* condition for the existence of a linear $(w-1)$-syzygy. Some preliminary experimentation suggests that the variety R_1 must be rather special.

7. The Case $k=2$

In the case $k=2$ the strong form of Linear Syzygy is the same as the weak form. But actually something stronger is true:

Proposition 7.1. *If $\mathcal{M}$ has a linear second syzygy and $m \geqslant 2$, then*

$$\dim R_{m-1} \geqslant m.$$

Remark. If $m=1$, then of course lpd $\mathcal{M} \geqslant 2$ iff $\dim R = R_1 \geqslant 2$.

Proof. If $k=2$ then as soon as $K = \ker W^* \to R \neq 0$ we may apply Lemma 3.5 to see that not all the variables are involved. Thus we may assume $K=0$, so that the natural map $\bigwedge^{k-1} W^* = W^* \to R$ is a monomorphism. With this assumption, we have $\Gamma_{m-1} \subset R_{m-1}$, so it will be (more than enough) to show that $\dim \Gamma_{m-1} \geqslant m$. By the basic dimension estimates, any nonzero component˙ of Γ_{m-1} has dimension $\geqslant (m-2)(w+1)+m$, which is $\geqslant m$ as soon as $m \geqslant 2$. Thus it suffices to show that if $m \geqslant 2$, then Γ_{m-1} is nonzero.

To say that $\Gamma_{m-1} = 0$ means that every nonzero element of W^* maps to an element of rank m in $M \otimes W$; that is, regarding $M \otimes W$ as $\operatorname{Hom}(M^*, W)$, every nonzero element of W^* induces a monomorphism $M^* \to W$. However, the rank $m-1$ locus has codimension only $w-m+1 < w$ in the space of maps, so this is impossible. ∎

We can go further and completely characterize the modules $\mathcal{M}$ with 2 generators having a linear second syzygy. Given such a module over a ring

$F[W']$, for some $W' \subsetneq W$ we may of course tensor it over $F[W']$ with $F[W]$ to obtain such a module over $F[W]$. Also, given such a module, we can factor out further relations of degrees $\geqslant 1$ and again obtain such a module. We will say that $\mathcal{M}$ is a basic module with a linear second syzygy if it cannot be obtained by either of these procedures. The following result classifies the basic modules with linear second syzygy:

THEOREM 7.2. *Suppose $m=2$. If $\mathcal{M}$ is a basic module with a linear second syzygy iff either:*

(1) *$w=2$ and $\mathcal{M}$ has presentation matrix*

$$\begin{pmatrix} x_1 & x_2 \\ 0 & 0 \end{pmatrix}$$

for any basis x_1, x_2 of W, or

(2) *w is odd, say $w=2d+1$, and $\mathcal{M}$ is presented by the matrix*

$$\begin{pmatrix} x_1, x_2, ..., & x_d, & 0, & x_{d+1}, x_{d+2}, ..., & x_{2d} \\ 0, & x_1, ..., x_{d-1}, & x_d, & x_{d+2}, x_{d+3}, ..., & x_{2d+1} \end{pmatrix}$$

for some basis $x_1, ..., x_{2d+1}$ of W.

Proof. Suppose first that $\mathcal{M}$ is a basic module with a linear second syzygy. If $w=2$, then $\mathcal{M}$ has a linear second syzygy iff M contains an element annihilated by all linear forms, that is, iff a presentation matrix for $\mathcal{M}$ contains the matrix given in (1). This proves the theorem in the case $w=2$, so we may assume that $w>2$.

Under these circumstances, no element of M is annihilated by two independent linear forms; else $\mathcal{M}$ comes from a module with a linear second syzygy over $F[x_1, x_2]$ (perhaps by adding further relations) so that since $w>2$, $\mathcal{M}$ is not basic.

Adopting the notation of Lemma 3.5, the fact that $\mathcal{M}$ is basic implies $K=0$, so that $W^* \to R$ is a monomorphism. Since $\mathcal{M}$ is basic, it follows that $W^* \to R$ is an epimorphism. In this case we will show that w is odd and that the presentation matrix for $\mathcal{M}$ is as in (2) above.

No submodule of $\mathcal{M}$ generated by a proper subspace of M can have a linear second syzygy; for such a submodule is cyclic, and if it had a linear second syzygy its generator would be annihilated by two independent linear forms. Thus the map

$$N = M^* \to \bigwedge^2 W$$

is a monomorphism. We may regard $N \subset \Lambda^2 W$ as a space of skew symmetric maps from $R = W^*$ to W. We claim that under the hypothesis of the theorem, every nonzero element of N corresponds to a map whose rank is precisely $w-1$; in particular, this will show that w is odd.

First, suppose that N contains some element n with

$$\operatorname{rank} n < w-1,$$

and choose two independent elements $a, b \in R$ in $\ker n$. The elements a, b are both of rank 1 as maps $N \to W$, since they both kill n, so they correspond to rank 1 relations. Let n, n' be a basis for N and let $m, m' \in M$ be a dual basis, so that $n(m) = n'(m') = 0$ and $n(m') = n'(m) = 1$. The relation corresponding to any $c \in R$ is then

$$n(c) \otimes m' + n'(c) \otimes m,$$

so the two relations corresponding to a and b are of the form

$$n'(a) \otimes m \qquad \text{and} \qquad n'(b) \otimes m.$$

Since a, b are independent rank 1 transformations from N to W with the same kernel, they must have independent images, $n'(a) = x_1$ and $n'(b) = x_2$, say. Thus m is annihilated by the two independent linear forms x_1 and x_2, contradicting our hypothesis. It follows that no element of N has rank $< w-1$.

Next suppose that some element n of N had rank w. It follows that this is so for all n, since the rank of a skew symmetric matrix is always even, and thus we could not have transformations of ranks w and $w-1$. But then no element of R has rank 1, contradicting Proposition 7.1.

Since the rank of every element of N is even and exactly $w-1$, we may write $w = 2d+1$. We can now apply the following classification of pencils of skew-symmetric matrices with constant rank:

LEMMA 7.3. *If $N \subset \bigwedge^2 W$ is a 2-dimensional linear space of matrices of constant rank $w-1 = 2d$, then N can be represented by a skew-symmetric matrix of linear forms of the form*

$$B = \begin{pmatrix} 0 & L_d \\ -L_d^t & 0 \end{pmatrix},$$

where L_d denotes the $d \times (d+1)$ matrix

$$\begin{pmatrix} s & t & 0 & \cdot & \cdot & 0 \\ 0 & s & t & \cdot & \cdot & 0 \\ \vdots & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & \cdot & \cdot & \cdot & \cdot & \vdots \\ 0 & \cdot & \cdot & 0 & s & t \end{pmatrix}.$$

Proof. We will use the results developed by Gantmacher [10, Chap. XII, Sect. 4] (note that our L is the homogenization of Gantmacher's). Gantmacher shows that after changing bases in source and target we may express any 1-dimensional linear space of matrices N, by a matrix of the form

$$A=\begin{pmatrix}
L_{e_1} & 0 & \cdot & \cdot & & & & & & 0\\
0 & L_{e_2} & 0 & \cdot & & & & & & 0\\
\cdot & 0 & \cdot & & & & & & & \cdot\\
\cdot & \cdot & & \cdot & & & & & & \cdot\\
\cdot & \cdot & & & L_{e_p} & 0 & & & & \cdot\\
 & & & & 0 & L^t_{n_1} & 0 & & & \\
 & & & & & 0 & L^t_{n_2} & & & \\
\cdot & & & & & & & \cdot & & \cdot\\
\cdot & & & & & & & \cdot\ 0 & & \cdot\\
0 & & & & & & & \cdot\ 0 & L^t_{n_q} & 0\\
0 & & & & & & & & 0 & M
\end{pmatrix},$$

where M is a square matrix of linear forms in s and t with nonzero determinant. In our situation M has to be 0 because N is of constant rank whereas any generically nonsingular one-parameter family of matrices M drops rank. Since N can be expressed by a skew-symmetric matrix of linear forms, the degrees of the minimal relations among the rows and among the columns of A is the same; thus $p=q$ and (after rearranging) $e_i=n_i$ for all $1 \leqslant i \leqslant p$. Since $2\sum_{1\leqslant i\leqslant p} e_i = \operatorname{rank} A = \operatorname{rank} N = 2v$ and $p+2\sum_{1\leqslant i\leqslant p} e_i =$ size of $A=$ size of $N=w$, we have $p=1$ and $e_1=v$. Using row and column operations, we may transform A to B above. Since N and B are the same up to changes of basis, and both skew-symmetric, they are congruent (that is, differ by a transformation of the form

$$B \mapsto P^t B P,$$

where P is an invertible scalar matrix) by Gantmacher [10, Chap. XII, Theorem 6] and we are done. (Theorem 6 was stated over the field of complex numbers but it stays valid over any field which contains square-roots of its elements.) ∎

It is easy to check that the presentation matrix given in (2) above is precisely the matrix of linear relations corresponding to the subspace $N \subset \bigwedge^2 W$ given in Lemma 7.3. This shows that every basic module with a linear second syzygy, for $w \geqslant 3$, has the given form.

It remains to show that the given examples are basic. Now if the example for a given $w=2d+1$ were not basic, there would be no basic example with w variables, by the above classification. Thus the given example would be a homomorphic image of a module with a linear second syzygy and a smaller presentation matrix. The space N corresponds to the module $\mathcal{M}$ together with a linear second syzygy. It is easy to see that this linear syzygy does not come from a module over a ring with fewer variables; thus if $\mathcal{M}$ were basic, it would have some other linear second syzygy. However one can check directly that there is only one linear relation on the columns of the matrix given in 2): it is the column vector with entries

$$-x_{d+1}, ..., -x_{2d+1}, x_1, ..., x_d.$$

This concludes the proof. ▮

Remark. Actually it is easy to show slightly more: if $w = 2d+1$ and the linear part of the presentation matrix for $\mathcal{M}$ properly contains the one given in (2) above, then some element of $\mathcal{M}$ is annihilated by two independent linear forms. Of course, we may assume that the presentation matrix has exactly one more column. But the only variable not present in the first row of the given matrix is x_{2d+1}, and the only element not present in the second row is x_{v+1}, so the new column may be taken to have entries of the form x_{2d+1}, ax_{v+1}. This new column, together with the last v columns of the original matrix, forms a $2 \times (v+1)$ matrix in $v+1$ variables, and thus is not 1-generic. Since every linear combination of the rows in the matrix made of the first $v+1$ columns of the original matrix obviously has dependent entries, we see that some linear combination of the rows of the augmented matrix has two entries dependent on the rest, showing that some element of M is annihilated by two independent linear forms.

Here are the betti numbers of the module $\mathcal{M}$ for a few values of d, computed with Macaulay:

;	$d=1$								
;	total:	2	3	1					
;									
;	0:	2	3	1					
;	$d=2$								
;	total:	2	5	9	10	5	1		
;									
;	0:	2	5	1	—	—	—		
;	1:	—	—	8	10	5	1		
;	$d=3$								
;	total:	2	7	33	70	73	43	14	2
;									
;	0:	2	7	1	—	—	—	—	—
;	1:	—	—	32	70	73	43	14	2

The annihilator of $\mathscr{M}$ contains the 2×2 minors of the presenting matrix of $\mathscr{M}$, so that the reduced support of $\mathscr{M}$ is the rational normal curve

$$x_1 = \cdots = x_d = 0,$$
$$x_{d+i} = s^{d-i+1}t^{i-1} \qquad \text{for} \quad i = 1, ..., d+1.$$

We do not know a nice geometric interpretation of this module ... .

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