Ideals with a regular sequence as syzygy
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In the following sketch we give an alternate approach to Lemma **, reducing it to results of Huneke and Ulrich [H-U] and Kustin [K] (results similar to those of Kustin were also obtained by M. Stillman.) This approach is more general than Beauville's both in that it gives a characteristic free result, not tied to the case of jacobian ideals, and in that it applies to local Noetherian rings as well as graded rings. For the sake of brevity, we mainly give the local statements. The graded case may be deduced by a localization argument, and in any case the local proofs apply equally in the graded situation. In [H-U] the authors work over a ring containing a field; but the results are general, and are done explicitly without this hypothesis in [K].

We will work throughout in the following setting, which generalizes the situation of the partial derivatives of a form in a polynomial ring whose degree is divisible by the characteristic of the ground field: we will assume that (R,m) is a local Noetherian ring, that $x1,...,xn$ is a regular sequence in R and that $f1,...,fn$ are elements of R satisfying the relation

(*) $x1f1 + ... + xnfn = 0$.

We further set

$I = (f1,...,fn)$.

Recall that if $J \subset R$ is an ideal, then we say that R/J is perfect if the projective dimension d of R/J is equal to the grade of J, and of Cohen-Macaulay type s if in addition $Ext^d(R/J,S)$ is minimally generated by s elements. (Thus if R is regular, R/J is perfect iff it is Cohen-Macaulay, and the type is the dimension of the socle of R/J modulo a maxima regular sequence.)

Theorem 1. If

$$\mathrm{grade}(I) > n-2,$$

then

(i) if n is ~~even,~~ odd R/I is perfect of Cohen-Macaulay type 2.

(ii) if n is ~~odd,~~ even there exists an element $f \notin I$ such that

$$I : (x_1,\ldots,x_n) = (I,f),$$

and $R/(I,f)$ is perfect of Cohen-Macaulay type 1, and grade n-1.

Proof. The relation * shows that the vector (f_i) is a linear combination of the syzygies of the x_i. Since the x_i are a regular sequence, their syzygies are given by the first map of the Koszul complex

$$k: \wedge^2 R^n \to \wedge^1 R^n,$$

so (f_i) is the composition of the dual of k with a map a from $\wedge^2 R^{n*}$ to R. Such a map corresponds to a skew-symmetric map $A: R^n \to R^{n*}$, and a direct computation shows that $(f_i) = A(x_i)$.

Consider first the generic case: let Y be a generic n by n skew-symmetric matrix, let X be a generic n by 1 matrix with entries X_i, and let J be the ideal of entries of the column vector YX in the polynomial ring over the integers in the variables which are entries of these matrices. First assume that n is ~~even~~ odd. In this case, Prop. 5.8 of [H-U] shows that J is a perfect prime ideal of grade n-1. Since by hypothesis the grade of I is n-1, it follows by specialization that if n is even, R/I is perfect. From the proof and statement of Prop. 5.9 of [H-U] (see also [K]) it also follows that the C-M type of J is 2; again by specialization the same is true of I, which completes the proof of i).

Now assume that n is ~~odd,~~ even and let K be the ideal generated by J together with the Pfaffian H of Y. From Prop. 5.9 and Lemma 5.12 of [H-U] (or [K]), it follows that K is a perfect prime ideal defining a Gorenstein ring, and the grade of K is n-1. Since there is a matrix Y' such that Y'Y is the identity times H, we see that HX_i is in J for $i = 1,\ldots,n$, though of course H is not in J. Let f be the Pfaffian of A. As the grade of (I,f) is the same as the grade of K, it follows that $R/(I,f)$ is a perfect Gorenstein ideal, and f is not in I. Furthermore

$$I \subset (I,f) \subset I:(x_1,\ldots,x_n).$$

Since (I,f) has no embedded components and $\mathrm{grade}(x_1,\ldots,x_n) = n$, we must have

$$(I,f) = I:(x_1,\ldots,x_n),$$

which finishes the proof of Theorem 1.//

Remark: In the graded case, when n is even, we may deduce the degree of the socle generator f immediately from this construction: For example, if the degrees of the x_i are 1 and the degrees of the f_i are e, then the degrees of the entries of A are $e-1$, so the degree of the Pfaffian f is $n(e-1)/2$.

Corollary 2. In addition to the hypothesis of Theorem 1, suppose that R is regular. If $g \in R$ is such that $\mathrm{ht}(I,g) = n$, then the socle of $R/(I,g)$ is two-dimensional.

Remark: Graded free resolutions for the ideals I and (I,f) as in Theorem 1 can be found in [K] Theorem 6.3. One can easily read off the degrees of the socle elements in Corollary 2 by using this resolution. Alternately, one can use linkage, as was done in [H-U]. In the case of Corollary 3 below, one recovers in this way the degree results of Beauville.

Proof. If n is ~~even~~ odd, the corollary follows at once from Theorem 1 i). If n is ~~odd~~ even, then the element f constructed in Theorem 1 ii) is in the socle of R/I and a fortiori of $R/(I,g)$. Furthermore the socle of $R/(I,f,g)$ is one-dimensional since $R/(I,f)$ is Gorenstein. It thus suffices to show that f is not contained in (I,g). Since (I,f) is perfect and of smaller height than (I,f,g), it follows that g is a nonzerodivisor modulo (I,f). Now if f were in (I,g), then f would already be in I, contradicting Theorem 1 i).//

Specializing, we obtain the result of Beauville:

Corollary 3. Let $R = k[x1,...,xn]$ with k a field of characteristic $p > 0$, and suppose that F is a form of degree d defining an isolated singularity with p dividing d. If $fi = dF/dx$, then the socle of $R/(F,f1,...,fn)$ is two dimensional.

Proof. Since $x1f1 + ... + xnfn = dF = 0$, the fi satisfy (*) as in Corollary 2. Because F defines an isolated singularity $grade(F,f1,...,fn) = n$, so $grade(f1,...,fn) \geq n-1$. We may now apply Corollary 1 with F playing the role of g.//

Remark: The ideals I (for n ~~even~~ odd) and (I,f) (for n ~~odd~~ even) are actually in the linkage class of a complete intersection by [K]. Hence the corresponding ideals (I,F) (n ~~even~~ odd) and (I,F,f) (n ~~odd~~ even) as in Corollary 3 are also in the linkage class of a complete intersection. This fact has strong consequences for the higher cotangent modules for these ideals, and may thus have interesting consequences for their deformation theory.

References:

[H-U] C. Huneke and B. Ulrich, Divisor Class Groups and Deformations, Am. J. Math. 107 (1985) 1265-1303.

[K] A. Kustin, The minimal free resolutions of the Huneke-Ulrich deviation two Gorenstein ideals. J Alg. (100) 1986, 265--304.