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Irreducibility of some families of linear series with Brill-Noether number. I

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IRREDUCIBILITY OF SOME FAMILIES OF LINEAR SERIES WITH BRILL-NOETHER NUMBER −1

BY DAVID EISENBUD AND JOE HARRIS [1]

TABLE OF CONTENTS

ABSTRACT. — In this paper we study families of linear series of dimension r and degree d on curves of genus g with negative Brill-Noether number

$$\rho=g-(r+1)(g-d+r)$$

and, allowing specified ramification at a point, on smooth pointed curves. In particular, if $\rho=-1$ we prove that there is exactly one irreducible family of curves and linear series whose curves vary in codimension 1 in the family of all smooth curves. Thus there is at most one component of the Hilbert scheme of nondegenerate smooth curves of genus g and degree d in $\mathbb{P}^r$ dominating a codimension 1 subvariety of the moduli space of curves of genus g.

1. Introduction and statement of results

Throughout this paper, curves will be reduced, connected, projective curves over the complex numbers.

We will denote by $\mathscr{H}_{d,g,r}$ the union of the components of the Hilbert scheme whose general members are smooth curves of genus g and degree d nondegenerately embedded in $\mathbb{P}^r$. A basic problem in the study of $\mathscr{H}_{d,g,r}$ is to determine the dimensions of its (usually numerous) components, and to describe the family of abstract smooth curves that appear in a given component. In general, the character of the answer obtained

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seems to be strongly affected by the *Brill-Noether number*

$$\rho := g-(r+1)(g-d+r).$$

Very little of the solution to this problem is known in general, but in the case when $\rho>0$, an interesting partial answer is furnished by results of Kleiman, Laksov, Gieseker, Fulton and Lazarsfeld (*see* [F-L] and the references found there) among others. From the point of view of moduli of curves—that is, considering a curve in $\mathbb{P}^r$ essentially as an abstract curve together with a linear series of degree d and dimension r, or in classical language a g^r_d—they show that in case $\rho>0$ that the scheme G^r_d parametrizing g^r_d's on a general curve C is smooth and irreducible of dimension ρ (*see* [ACGH], vol. 1, for a formal definition of the scheme $G^r_d(C)$). It follows from this, in conjunction with the result that a general such g^r_d really corresponds to an embedding [E-H4], that whenever $\rho>0$ there exists a unique component of $\mathscr{H}_{d,g,r}$ dominating the moduli space $\mathscr{M}_g$ of curves; and that this component will have the "expected" dimension

$$\dim(\mathscr{M}_g)+\rho+\dim(\mathrm{PGL}_{r+1})=3g-3+\rho+(r+1)^2-1$$

([ACGH], § 5 and references therein).

Some of these results were extended in [E-H1] to the case when $\rho=0$ by considering not just the family $G^r_d(C)$ of linear series on a general curve C but the family $\mathscr{G}^r_d$ parametrizing pairs consisting of a curve $C\in\mathscr{M}_g$ (without automorphisms, say) and a linear series on C—that is, the union of all the $G^r_d(C)$'s; *see* [E-H2], [E-H3] and the forthcoming second volume of [ACGH] for a formal definition—and showing that as in the case $\rho>0$ there is a unique component dominating $\mathscr{M}_g$. In the present paper we will prove some analogous results for $\rho<0$, and obtain in particular the analogous irreducibility result for $\rho=-1$:

THEOREM (1.1). — *For any d, g and r let* $\rho=g-(r+1)(g-d+r)$. *Then*

(i) *If* $\rho<-1$, *the image in* $\mathscr{M}_g$ *of any component of the variety* $\mathscr{G}^r_d$ *has codimension at least two; and*

(ii) *If* $\rho=-1$, *there is a unique irreducible component of the variety* $\mathscr{G}^r_d$ *whose image in* $\mathscr{M}_g$ *is of codimension one.*

From the equations defining the variety $\mathscr{G}^r_d$ locally it is apparent that each component of this variety has dimension $\geqq \dim\mathscr{M}_g+\rho$ (and a similar remark holds for $\mathscr{H}_{d,g,r}$). But no such principle is known for the images of these components in $\mathscr{M}_g$.

Note that the fact that we have only defined $\mathscr{G}^r_d$ over the locus in $\mathscr{M}_g$ of curves without automorphisms is harmless, since for $g>3$ the set of curves with automorphisms is of codimension $g-2\geqq 2$ in $\mathscr{M}_g$, while for $g\leqq 3$ the only examples of smooth curves possessing g^r_d's with negative ρ are hyperelliptic curves of genus 3.

We immediately derive:

COROLLARY (1.1 A). — *For any d, g and r such that* $\rho=g-(r+1)(g-d+r)=-1$ *there is at most one component of the variety* $\mathscr{H}_{d,g,r}$ *whose members vary in a subvariety of codimension one in* $\mathscr{M}_g$.

The "at most" in the statement stems from the fact that we do not know whether the general member of the component of $\mathcal{G}^r_d$ whose existence is asserted in Theorem (1.1) is very ample; if it were not, there would be no component of $\mathcal{H}_{d,g,r}$ corresponding to this component of $\mathcal{G}^r_d$. We do believe that indeed this component of $\mathcal{H}_{d,g,r}$ exists when $r \geqq 3$; this statement (for sufficiently large g) is a special case of the general conjecture made in the following section.

We may also derive a related statement for abstract curves:

COROLLARY (1.1 B). — *The locus $\mathcal{M}^r_{g,d} \subset \mathcal{M}_g$ of curves that possess a linear series of degree d and dimension r has a unique irreducible component of codimension one.*

We also extend the principle involved in Theorem (1.1) to pointed curves. Given a curve C with a point $p \in \mathrm{C}$ and a sequence $\underline{\alpha}=(\alpha_0, \ldots, \alpha_r)$ of natural numbers, we can define a variety $\mathrm{G}^r_d(\mathrm{C}, p; \underline{\alpha})$ parametrizing linear series g^r_d on C having ramification sequence $\alpha_i(\mathcal{D}) \geqq \alpha_i$ greater than or equal to $\underline{\alpha}$ at p. For a general pointed curve $(\mathrm{C}, p) \in \mathcal{C}_g$, the variety $\mathrm{G}^r_d(\mathrm{C}, p; \underline{\alpha})$ has dimension equal to the *adjusted Brill-Noether number*

$$\rho' := \rho(g, r, d) - |\alpha| \qquad (\text{here } |\alpha| := \Sigma\, \alpha_i),$$

and is empty if $\rho' < 0$ (*see* [EH-2], section 1).

To express a relative version of this construction, let $\mathcal{X} \to \mathcal{C}$ be any family of smooth curves with section $\sigma: \mathcal{C} \to \mathcal{X}$. Using a Grassmann bundle over the relative Picard variety of the family one can define a variety $\mathcal{G}^r_d(\mathcal{C}/\mathcal{X}, \sigma; \underline{\alpha})$ and a map of $\mathcal{G}^r_d(\mathcal{C}/\mathcal{X}, \sigma; \underline{\alpha})$ to $\mathcal{C}$ whose fiber over a point $\mathrm{C} \in \mathcal{C}$ will be the variety $\mathcal{G}^r_d(\mathrm{C}, \sigma(\mathrm{C}); \underline{\alpha})$ parametrizing g^r_d's on C with ramification at least $\underline{\alpha}$ at $\sigma(\mathrm{C})$. Applying this construction to the universal pointed curve $\mathcal{M}_{g,1} = C_g$ and the universal family with section $\pi: \mathcal{X}_g \to \mathcal{C}_g$ (that is, $\mathcal{X}_g = \mathcal{C}_g \times_{\mathcal{M}_g} \mathcal{C}_g$, with π the projection on the first factor and σ the diagonal map), always keeping away from pointed curves with automorphisms, we arrive at a variety $\mathcal{G}^r_d(\underline{\alpha})$ that is in effect the union of $\mathcal{G}^r_d(\mathrm{C}, p; \underline{\alpha})$ over all pairs (C, p) without automorphisms. Our theorem is then

THEOREM (1.2). — *For any d, g, r and $\underline{\alpha}$, let $\rho' = g-(r+1)(g-d+r)-\Sigma\,\alpha_i$.*

(i) If $\rho' < -1$, then the image in $\mathcal{C}_g$ of any component of $\mathcal{G}^r_d(\underline{\alpha})$ will have codimension at least two; and

(ii) If $\rho' = -1$, there is a unique irreducible component of $\mathcal{G}^r_d(\underline{\alpha})$ whose image in $\mathcal{C}_g$ has codimension one.

One can of course extend the statements of this Theorem to multiply pointed curves; we would conjecture that they remain true, but new ideas seem to be required for a proof.

The proofs of Theorems (1.1) and (1.2) proceed by looking at certain stable curves (resp. pointed curves) lying in the closure in $\overline{\mathcal{M}}_g$ of the image in $\mathcal{M}_g$ of any component of $\mathcal{G}^r_d$ [resp. the closure in $\overline{\mathcal{C}}_g$ of the image in $\mathcal{C}_g$ of any component of $\mathcal{G}^r_d(\underline{\alpha})$]. Specifically, in the case of Theorem (1.1) we first prove that a certain class of curves belongs to the closure of every codimension 1 component of $\mathcal{M}^r_{g,d}$ in the compactified moduli space of stable curves: these are curves consisting of a union of a curve of genus $g-2$ and a smooth curve of genus 2, attached at a Weierstrass point of the curve of genus 2. It turns out that on such a curve there are generally many g^r_d's with $\rho=-1$ (so that $\mathcal{G}^r_d$ and $\mathcal{H}_{d,g,r}$, under the best of circumstances, will many-sheeted covers of $\mathcal{M}^r_{g,d}$), but that the monodromy transformations coming from certain families of reduced curves of compact type permute the sheets transitively. Thus over a suitable small neighborhood of a point in the closure of every component of $\mathcal{M}^r_{g,d}$, the variety $\mathcal{G}^r_d$ is irreducible. At the same time, we will see that such a curve C_0 possesses no g^r_d's with $\rho=-2$, which will suffice to establish part (i) of Theorem (1.1). Theorem (1.2) involves an analysis of the behavior of $\mathcal{G}^r_d(\underline{\alpha})$ near similarly defined points in the compactification $\overline{\mathcal{C}}_g$ of $\mathcal{C}_g$.

As will be clear from the above, there should be a general result on the irreducibility of the variety $\mathcal{G}^r_d$ of all g^r_d's on all smooth curves; we might hope that such a result could be obtained by looking at an extension $\overline{\mathcal{G}}^r_d$ of $\mathcal{G}^r_d$ parametrizing limits of g^r_d's on stable curves. Unfortunately we know how to construct such a variety only locally, in the neighborhood of a given linear series on a curve of compact type. Thus, it is *a priori* possible that additional components of the variety $\mathcal{G}^r_d$ exist that do not contain any curve of compact type in their closure, and so are "undetectable" by these methods. Two possible solutions to this problem exist: one is to extend the theory of limit linear series from curves of compact type to all stable curves, and the other is to prove more refined theorems about the intersection of closed subvarieties of $\overline{\mathcal{M}}_g$ with the boundary. At present, though, neither has been successfully carried out.

2. Counter-examples and a conjecture

If ρ is strictly positive then there are a number of theorems describing the geometry of $\mathcal{G}^r_d$, that taken in sum we will call the *Brill-Noether-Petri Principle*, or BNPP:

(i) the variety $\mathcal{G}^r_d$ contains a unique irreducible component Σ dominating $\mathcal{M}_g$, and this component has dimension $\rho+3g-3$;

(ii) a general fiber of the map from Σ to $\mathcal{M}_g$ is smooth;

(iii) when $r\geqq 3$, a general $g^r_d\in\Sigma$ is very ample (when $r\geqq 1$ it is base point free and when $r\geqq 2$ it is immersive); and

(iv) the multiplication map $\mu_0: H^0(L)\otimes H^0(KL^{-1})\to H^0(K)$ is of maximal rank (that is, injective when $\rho\geqq 0$ and surjective when $\rho\leqq 0$) for $|L|$ a general $g^r_d\in\Sigma$.

Note that these statements taken together imply that there is a corresponding component of the Hilbert scheme $\mathcal{H}_{d,g,r}$ that has dimension $\rho+3g-3+(r+1)^2-1$). (*See* for example [ACGH] or [E-H4] and [E-H5] for recent proofs and the original references for these statements).

This principle fails beyond the case $\rho>0$, in the following sense. There are clear analogues of the statements of the BNPP: we can replace (i) and (ii) above with

(i′) the variety $\mathcal{G}^r_d$ contains a unique irreducible component Σ dominating a subvariety of codimension $-\rho$ or less in $\mathcal{M}_g$, and this component has dimension $\rho+3g-3$; and

(ii′) a general fiber of the map from Σ to its image in $\mathcal{M}_g$ is reduced of dimension 0;

while statements (iii) and (iv) above make sense as they stand. As it turns out, though, most of the statements above are violated for some values of g, r and d (the one exception is the statement about the smoothness of the fibers of the map from $\mathcal{G}^r_d$ to its image in $\mathcal{M}_g$, to which we know of no counterexample). In general, the Hilbert scheme $\mathcal{H}_{d,g,r}$ may be reducible. Examples abound of values of d, g and r for which $\mathcal{H}_{d,g,r}$ has many components, even having different dimensions, some nonreduced, many of dimension strictly greater than predicted, and some consisting entirely of singular curves [H2]. (We will see that even in cases where the Brill Noether number is positive, so BNPP holds, there are sometimes "extra" components of $\mathcal{H}_{d,g,r}$, living over proper subvarieties of the moduli space of curves and having dimension greater than that predicyed by the BNPP. On the other hand recent results of Ein [E1], [E2] show that $\mathcal{H}_{d,g,r}$ *is* irreducible in some ranges of cases if ρ is sufficiently large—for example when $r=3$ and $d\geqq g+3$, that is, $\rho\geqq g$. This suggests that perhaps the Brill-Noether number is not the correct way of distinguishing components of the Hilbert scheme satisfying the BNPP from those violating it.)

In this section we first present some (very isolated) examples of families of curves violating the BNPP, to give some sense of what goes on. We then propose the conjecture that a certain aspect of the examples holds in general. We will focus here primarily on the issue of when the dimension of the Hilbert scheme is that predicted by the BNPP; by way of terminology we will call a component of the Hilbert scheme of dimension strictly larger than that predicted an *exceptional component* of $\mathcal{H}$.

a. EXAMPLES OF EXCEPTIONAL COMPONENTS. — Where should we look for examples of aberrant behavior? To begin with, the BNPP holds in the case $r=1$ [ACGH], § 5 and *loc. cit.* In the case $r=2$, it is easy to characterize counterexamples to BNPP: if we restrict to components of $\mathcal{G}^r_d$ whose general members are birationally very ample (in effect, looking at the Severi variety) we find that the BNPP is known to hold in most respects; specifically, the variety parametrizing curves together with birational embeddings in the plane is irreducible of the correct dimension [H1], § 3 and *loc. cit.* (It is an interesting open problem to deal with the rank of the μ_0 map for the general member of a Severi variety.) Most components of $\mathcal{G}^2_d$ whose general members fail to be birational will violate the dimension statement; for example, an element of the variety $\mathcal{G}^2_4$ consists of a hyperelliptic curve with the double of its g^1_2 so that $\mathcal{G}^2_4$ will have dimension $2g-1$; on the other hand, the predicted dimension is $3g-3+\rho=g+3$.

Complications emerge when we get to $r\geqq 3$. One way of arriving at exceptional components of $\mathcal{H}_{d,g,r}$ is to use multiples of a linear series. For example, when the genus g is sufficiently large we can construct one such component out of trigonal curves: a general trigonal curve C of genus $g\geqq 8$ may be embedded, by the linear series $\mathcal{D}=\mathrm{K}-2.g^1_3$ residual in the canonical series K to twice the pencil of degree 3, as a curve of degree $2g-8$ in $\mathbb{P}^{g-5}$ (that $\mathcal{D}$ is very ample follows from looking at the rational normal scroll $\mathrm{X}\subset\mathbb{P}^{g-1}$ containing the canonical image of C: if F is the class of a ruling of X, the linear series $\mathcal{D}$ is cut out by the linear series $|\mathcal{O}_{\mathrm{X}}(1)(-2\mathrm{F})|$, which embeds X as a rational normal scroll in $\mathbb{P}^{g-5}$). According to the BNPP the codimension in moduli of the locus of curves so expressible (it's not hard to see that if a curve is so expressible it is uniquely so) should be

$$\begin{aligned}-\rho &= -g+(r+1)(g-d+r)\\ &= -g+(g-4)(3)\\ &= 2g-12,\end{aligned}$$

whereas in fact the locus of trigonal curves has codimension $g-4$ in moduli. Thus for $g>8$ the Hilbert scheme $\mathscr{H}_{2g-8,g,g-5}$ violates the BNPP.

Needless to say, we can make up similar examples using multiples other than 2 of linear series other than a g^1_3. To give one example, we can consider a plane curve $C\subset\mathbb{P}^2$ of degree d, re-embedded in projective space by the nth Veronese map for any $n\leqq d-5$. Whether or not such curves are actually dense in a component of the Hilbert scheme is not so clear, but in any event the component of the Hilbert scheme containing them will map to a subvariety of $\mathscr{M}_g$ of dimension at least $(d+1)(d+2)/2-9=g+3d-9$, *i.e.*, of codimension at most $2g-3d+6$. On the other hand, the predicted codimension in $\mathscr{M}_g$ of curves in this Hilbert scheme is

$$\begin{aligned}-\rho &= -g+(r+1)(g-d+r)\\ &= -g+\binom{n+2}{2}\left(g-d+\binom{n+2}{2}-1\right)\end{aligned}$$

which will in general be huge compared to $2g-3d+6$.

We can use another generalization of this construction to find examples of exceptional components of the Hilbert scheme even when the Brill-Noether number is positive: all we have to do is to consider projections of curves described as in the previous example. Specifically, look again at trigonal curves, and for any k consider the series residual to k times the pencil of degree 3; this will generically have degree $2g-2-3k$ and dimension $g-1-2k$ (that it is in general very ample follows from an argument identical to that given above in the case $k=2$). Now look at projections of these curves to a projective space $\mathbb{P}^r$. Choosing a projection means choosing an $(r+1)$-dimensional quotient of a $(g-2k)$-dimensional vector space, and so involves $(r+1)(g-r-2k-1)$ additional parameters; thus the space of such curves has dimension

$$(2g+1)+(r+1)(g-r-2k-1)+(r+1)^2-1=2g+(r+1)(g-2k).$$

On the other hand the BNPP predicts a dimension of

$$\begin{aligned}&3g-3+\rho+(r+1)^2-1\\ &\qquad=3g-3+g-(r+1)(g-(2g-2-3k)+r)+(r+1)^2-1\\ &\qquad=4g-4+(r+1)(g-1-3k).\end{aligned}$$

Thus by choosing k relatively large [*i.e.*, slightly larger than $2g/(r+1)$] we will get a component of the Hilbert scheme violating the BNPP.

Another way of finding components of the Hilbert scheme violating the BNPP is by looking either at complete intersection curves or curves of large degree on a fixed surface in projective space. To carry out some cases of this explicity in $\mathbb{P}^3$, note first that the BNPP predicts a particularly simple answer for the dimension of the Hilbert scheme of curves in $\mathbb{P}^3$: we have

$$3g-3+\rho+(r+1)^2-1=3g-3+g-4.(g-d+3)+15=4d.$$

Now consider first the case of complete intersections of surfaces of degrees d_1 and d_2, with $d_1<d_2$. To specify such a curve we have to specify first the surface of degree d_1, which introduces $\binom{d_1+3}{3}-1$ parameters, and then the surface of degree d_2, modulo those containing the surface of degree d_1, which introduces $\binom{d_1+3}{3}-\binom{d_2-d_1+3}{3}-1$ more; so that the dimension of the component Hilbert scheme parametrizing such curves is $(2d_1^3+3d_2^2d_1-3d_2d_1^2+12d_2d_1+22d_1-6)/6$. (Note that in case $d_1=d_2$ this is off by one, since the first surface is not uniquely determined by the curve; the correct number is the dimension of the Grassmannian of 2-dimensional subspaces of the vector space of polynomials of degree d_1, which is one less than that given by the formula.) On the other hand, the degree of such a curve is of course $d_1.d_2$, so that the BNPP predicts a dimension of $4d_1d_2$; it follows that for all but the pairs of values $(d_1, d_2)=(2, 2)$, $(2, 3)$ and $(3, 3)$, this component of the Hilbert scheme will be exceptional.

To give an example of the second type suggested above—curves of high degree on a fixed surface—consider curves of type (a, b) on a quadric surface in $\mathbb{P}^3$ with $a, b\geqq 4$. To specify such a curve we have to specify first the quadric, which involes 9 parameters; and then a bihomogeneous polynomial of degree (a, b) on $Q\cong\mathbb{P}^1\times\mathbb{P}^1$ (up to scalars), which involves $(a+1)(b+1)-1$ more; the dimension of the component of the Hilbert scheme (such curves do form a dense open subset of a component of the Hilbert scheme when $a, b\geqq 3$) will thus be $(a+1)(b+1)+8$, which will be greater than the number $4(a+b)$ predicted by the BNPP except in the case where a or b is 3.

Note that the "first" case of each of the last two examples coincide: they are both the family of complete intersections of quadrics and quartics in $\mathbb{P}^3$. Here the dimension estimate of the BNPP fails by just one: the actual dimension of the family is 33, as opposed to the predicted dimension of 32.

b. WILD SURMISE. — Given the above, what might still be true? At first glance the profusion of counterexamples to virtually every extension of the BNPP might suggest that "exceptional component" is something of a misnomer, being more aptly used to describe a component of $\mathscr{H}$ actually satisfying the BNPP rather than those violating it. On further examination of these and other examples, however, we may note one common phenomenon: *all the components violating the principle lie over relatively small subvarieties of the moduli space*, specifically in codimension on the order of g or greater. For example, our first two collections of counterexamples to the BNPP all consisted just of trigonal curves and plane curves, which have codimension $g-4$ and $2g-3d+6$ in $\mathscr{M}_g$ (the latter of these numbers is never smaller than $g-3$ as long as

$d \geqq 5$, *i. e.*, as long as the corresponding component of $\mathscr{H}$ is exceptional). Of our last two classes of examples the simplest case, complete intersections of quadrics and quartics, lives over a subvariety of dimension $33-15=18$ in the moduli space, which has dimension $3g-3=24$, so that the codimension is $g-4$; in every other case of either of these two classes of examples the codimension is strictly greater than $g-4$.

All of the above leads us to formulate the questions:

(1) Do there exist subvarieties $\Sigma_g \subset \mathscr{M}_g$ of codimension $\sigma(g) \sim g$ such that the Brill-Noether-Petri principle holds in every possible respect over $\mathscr{M}_g - \Sigma_g$?

(2) Does there exist a function $\sigma(g)$, of the order of g, or perhaps a constant times g, with the properties that for any d, g and r with $\rho \geqq -\sigma(g)$.

(i) the variety of curves of genus g possessing a linear series of degree d and dimension r has a unique irreducible component of codimension less than $\sigma(g)$ in $\mathscr{M}_g$, and this component has codimension $-\rho$ (2);

(ii) the variety $\mathscr{G}^r_d$ has a unique irreducible component mapping onto a subvariety of codimension less than $\sigma(g)$ in $\mathscr{M}_g$, and this component has dimension $\rho+3g-3$;

(iii) the variety $\mathscr{H}_{d,g,r}$ has a unique irreducible component mapping onto a subvariety of codimension less than $\sigma(g)$ in $\mathscr{M}_g$, and this component has dimension $\rho+3g-3+(r+1)^2-1$?

Alternatively, if we were really only concerned with the question of dimension, we could phrase the whole thing this way: Let $\sigma_0(g)$ be the minimum over all d and r and exceptional components Σ of the Hilbert scheme $\mathscr{H}_{d,g,r}$ of the codimension of the image of Σ in $\mathscr{M}_g$. Then we conjecture that

$$\liminf_{g \to \infty} (\sigma_0(g)/g) > 0.$$

Perhaps the idea of this paper—analyzing the intersection of the loci $\mathscr{M}^r_{g,d} \subset \mathscr{M}_g$ with the boundary in $\overline{\mathscr{M}}_{\mathbb{G}}$—could shed some light on cases in higher codimension.

Of course, as we indicated in the previous section, we can also ask analogous questions for pointed curves $(C; p_1, \ldots, p_k)$ and linear series on C with specified ramification at the points p_i. In this setting the counterexamples to the analogous statements having smallest known codimension come from the family of Weierstrass points with first non-gap $g/2$; this will have codimension roughly $g/2$ ([E-H6]). In particular, one could certainly conjecture that the analogue for multiply pointed curves of the theorems actually proved below should still hold; that is, that the locus in the moduli space $\mathscr{M}_{g,k}$ of k-pointed curves $(C; p_1, \ldots, p_k)$ possessing a linear series g^r_d with ramification at least $\underline{\alpha}^i$ at p_i will have codimension at least two when $\rho' = \rho(g, r, d) - \Sigma |\underline{\alpha}^i| < -1$ and will have at most one irreducible component of codimension one when $\rho' = -1$, and so forth.

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(2) The existence of at least one component of codimension exactly $-\rho$ is known for a range of d, g and r; *see* [E-H3].

3. Proof of Theorem (1.1)

In this section and the next we assume familiarity with the basic ideas of [EH1-3].

First, let us fix notation. Except when the contrary is explicitly stated, g, r, and d will denote integers such that $\rho(g, r, d)=g-(r+1)(g-d+r)=-1$. We will denote by $\mathcal{M}^r_{g,d}\subset\mathcal{M}_g$ the locus of curves of genus g possessing a linear system of degree d and dimension r, and by $\overline{\mathcal{M}}^r_{g,d}$ its closure in $\overline{\mathcal{M}}_g$. As in the previous sections, $\mathcal{G}^r_d$ will denote the variety of pairs (C, $\mathcal{D}$) where C is a curve of genus g without automorphisms and $\mathcal{D}$ a linear system of degree d and dimension r.

If $\mathcal{C}\to\mathrm{B}$ is a family of stable curves of compact type, with B local (that is, in the algebraic setting, $\mathrm{B}=\mathrm{Spec}\ \mathrm{R}$ with R local; in an analytic context, B may be taken to be a polydisc), then we further define

$$\tilde{\mathcal{G}}^r_d(\mathcal{C}/\mathrm{B})\to\mathrm{B}$$

to be the associated family of crude limit linear series (basic ideas about limit linear series and curves of compact type may be found in [E-H2] and [E-H3], or in the forthcoming second volume of [ACGH]). Thus a point of $\tilde{\mathcal{G}}^r_d(\mathcal{C}/\mathrm{B})$ is essentially a pair (C, $\mathcal{D}$) where C is a stable curve of compact type, a fiber of $\mathcal{C}\to\mathrm{B}$, and $\mathcal{D}$ is a *limit linear series* of degree d and dimension r on C. If $\mathrm{B}^0\subset\mathrm{B}$ is the germ of the set of points of B corresponding to smooth curves without automorphisms, and $\mathrm{C}^0\to\mathrm{B}^0$ is the restriction of $\mathcal{C}\to\mathrm{B}$, then we define $\overline{\mathcal{G}}^r_d(\mathcal{C}/\mathrm{B})$ to be the closure of $\mathcal{G}^r_d(\mathcal{C}^0/\mathrm{B}^0)$ in $\tilde{\mathcal{G}}^r_d(\mathcal{C}/\mathrm{B})$. Applying this to the Kuranishi space of any stable curve C of compact type, we obtain a variety $\overline{\mathcal{G}}^r_d$ (depending on C) which maps naturally to the intersection of the closure $\overline{\mathcal{M}}^r_{g,d}$ with the open set $\mathcal{U}\subset\overline{\mathcal{M}}_g$ of curves of compact type.

To prove the Theorem, we will first exhibit a locus in $\overline{\mathcal{M}}_g$ that must lie in any codimension 1 component of the closure $\overline{\mathcal{M}}^r_{g,d}$ of $\mathcal{M}^r_{g,d}$, and then show that at most one component of $\overline{\mathcal{G}}^r_d$ passes above this locus.

For the first part, we consider first the following stable curve $\mathrm{C}\in\Delta_2\subset\overline{\mathcal{M}}_g$: let B and C_0 be general curves of genus 2 and $g-2$ respectively. Let $q\in\mathrm{C}_0$ be a general point and $p\in\mathrm{B}$ an arbitrary point, and form a curve C by identifying p and q:

$$\mathrm{C}=\mathrm{C}_0\cup\mathrm{B}/q\sim p.$$

LEMMA (3.1). — *Any component of $\overline{\mathcal{M}}^r_{g,d}$ having codimension one in $\overline{\mathcal{M}}_g$ contains all curves of the form* C *for which p is a Weierstrass point.*

Proof. — Let Ψ be any such component. From [E-H2] (Theorem 1. Note that the proof given there yields the result of Theorem 1 for every component. Note also the misprint "e" for "3" in the formula Theorem 1) we see that the class of the divisor $\overline{\mathcal{M}}^r_{g,d}$ in $\mathrm{Pic}(\overline{\mathcal{M}}_g)$ is given by

$$[\overline{\mathcal{M}}^r_{g,d}]=c.\left((g+3).\lambda-\frac{g+1}{6}.\delta_0-\sum_{i=1}^{[g/2]} i(g-i).\delta_i\right)$$

for some constant c; and çoreover that the class of any codimension one locus in $\bar{\mathcal{M}}_g$ lying in the closure of the locus of smooth curves possessing a linear series with negative Brill Noether number must be a multiple of this. In particular, the class of Ψ is expressible in this way, with $c \neq 0$ since any nonzero effective divisor has nonzero class.

Now let $\Gamma = \{C_p\} \subset \Delta_2$ be the family of curves obtained by identifying a variable point p on the curve B with a fixed general point q on C_0. By the computations given in [H-M p. 81] we see that the degrees of all the divisors λ and δ_i for $i \neq 2$ are zero on Γ, while $\deg_\Gamma(\delta_2) = -2$, so that Ψ will intersect Γ; on the other hand by Lemma 3.2 below Ψ can meet Γ only at the points $C_p \in \Gamma$ corresponding to Weierstrass points p of B. Since the set $W \subset \Delta_2$ of stable curves obtained by attaching a curve of genus $g-2$ to a curve of genus 2 at a Weierstrass point of the latter is an irreducible (because the family of Weierstrass points is irreducible, as one sees in the case of curves of genus 2 by constructing the curves as double covers of $\mathbb{P}^1$) codimension 1 locus of Δ_2, it follows that Ψ must contain the locus of such curves. ■

Remark. — In fact, we do not really have to write down explicitly the class of $\bar{\mathcal{M}}^r_{g,d}$ or determine the degrees of the divisors λ and δ_i on the curve Γ introduced there: by Theorems (2.1), (3.1) and (4.1) of [E-H2], any codimension 1 component of the variety $\bar{\mathcal{M}}^r_{g,d}$ that failed to contain the locus $W \subset \Delta_2$ would have class 0.

LEMMA (3.2). — *Let* C *be as above. If* C *possesses a crude limit* g^r_d $\mathcal{D}$, *then* $\mathcal{D}$ *is actually refined,* p *is a Weierstrass point on* B, *and* $\mathcal{D}$ *consists of the linear system* $|(r+2)p| + (d-r-2)p$ *on* B (*that is, the linear system formed by sums of divisors* $D \in |(r+2)p|$ *and the fixed divisor* $(d-r-2)p$) *and a* g^r_d *on* C_0 *with ramification sequence* $(0, 1, 2, \ldots, 2)$ *at the point* q.

Proof. — The last part of this statement—the ramification sequence of the aspect of $\mathcal{D}$ on C_0—follows from the rest by the definition of limit linear series, given that the ramification of the series $|(r+2)p| + (d-r-2)p$ on B at p is $(d-r-2, \ldots, d-r-2, d-r-1, d-r)$. Now, since (C_0, q) is a general pointed curve of genus $g-2$, by Theorem (1.1) of [E-H2] the adjusted Brill Noether number of any linear series on C_0 with respect to q is nonnegative; by the additivity of adjusted Brill Noether numbers (*see* [E-H3]) it follows that the adjusted Brill Noether number with respect to p of the aspect on B of any limit linear series on C having negative Brill-Noether number is negative. Lemma (3.2) now follows from

LEMMA (3.3). — *For any* r *and* d, *any* g^r_d $\mathcal{D}$ *on a curve* B *of genus* 2, *and any point* $p \in B$, *the adjusted Brill-Noether number of* $\mathcal{D}$ *with respect to* p *is greater than or equal to* -1, *with equality holding if and only if* p *is a Weierstrass point of* B *and* $\mathcal{D} = |(r+2)p| + (d-r-2)p$.

Proof. — For any linear series g^r_d on B with $r \geqq 2$ we have $d \geqq r+2$; for any g^1_d we have $d \geqq 2$ with equality holding if and only if the g^1_2 is the canonical pencil. If the vanishing sequence of the linear series $\mathcal{D}$ of dimension r and degree d on B at p is $a_0, \ldots, a_r$ the linear series $\mathcal{D}(-a_i p)$ has degree $d - a_i$ and dimension $r-i$; applying this we have

$$a_i \leqq d-r+i-2, \qquad \text{for} \quad i \leqq r-2;$$
$$a_{r-1} \leqq d-2$$

and

$$a_r \leqq d.$$

Thus the ramification indices satisfy $\alpha_i \leqq d-r-2$ for $i \leqq r-2$; $\alpha_{r-1} \leqq d-r-1$ and $\alpha_r \leqq d-r$. Adding these up we have that the total ramification of $\mathscr{D}$ at p is at most $(r+1)(d-r-2)+3$, and the adjusted Brill Noether number is thus at least

$$\begin{aligned}\rho &\geqq g-(r+1)(g-d+r)-(r+1)(d-r-2)-3\\ &=-1.\end{aligned}$$

We see at the same time that equality can hold only if $a_r = d$ (so that $\mathscr{D} \subset |dp|$), $\alpha_0 = d-r-2$ (so that $\mathscr{D} = \mathscr{D}' + (d-r-2)p$ for some $\mathscr{D}' \subset |(r+2)p|$, which must then equal $|(r+2)p|$ because they have the same dimension) and $\alpha_{r-1} = d-r-1$ (so that $\mathscr{D}(-a_{r-1}p) = |2p| + (d-r-2)p$ is the g_2^1 and hence p is a Weierstrass point of B). ■

Returning to our assumption that $\rho(g, r, d) = -1$, we consider $\bar{\mathscr{M}}_{g,d}^r$ in a neighborhood of the point C. Let $\mathscr{C} \to \mathscr{B}$ be the Kuranishi space ([ACGH], vol. 2) of the curve C. Let μ be the number of linear series on the curve C_0 having ramification sequence $(0, 1, 2, \ldots, 2)$ at the point p. There are by the above at most μ distinct points of $\bar{\mathscr{G}}_d^r(\mathscr{C}/\mathscr{B})$ over the point of $\mathscr{B}$ corresponding to C, corresponding to those of the μ linear series on C_0 that are aspects of smoothable limit series on C. In fact each of the μ limit series on C is smoothable, as follows from our smoothing results in [E-H3], so there are exactly μ points in this fiber. We will show next that $\bar{\mathscr{G}}_d^r$ is unibranch at each of these points:

LEMMA (3.4). — *Let* C *be as above and let* $(C, \mathscr{D}) \in \bar{\mathscr{G}}_d^r$ *be any limit linear series on* C. *Then* $\bar{\mathscr{G}}_d^r$ *is smooth in a neighborhood of* $(C, \mathscr{D})$.

Proof. — With notation as above, let $\mathscr{G} = \bar{\mathscr{G}}_d^r(\mathscr{C}/\mathscr{B})$; observe that since all limit linear series on C are smoothable, this coincides with the space $\tilde{\mathscr{G}}_d^r(\mathscr{C}/\mathscr{B})$. Also, let $\Delta \subset \mathscr{B}$ be the discriminant divisor, $\mathscr{C}_\Delta \to \Delta$ the restriction of the family $\mathscr{C} \to \mathscr{B}$ to Δ, and $\mathscr{G}_\Delta = \bar{\mathscr{G}}_d^r(\mathscr{C}_\Delta/\mathscr{B}) \to \Delta$ the corresponding family of limit linear series. Our proof that $\mathscr{G}$ is smooth at $(\mathscr{C}, D)$ will proceed in two steps: first, we will argue that $\mathscr{G}_\Delta$ is smooth, using an explicit description of this family; then we will argue that this implies the smoothness of $\mathscr{G}$. Note that this would follow immediately if we knew that $\mathscr{G}_\Delta$ was in fact the (scheme-theoretic) inverse image of Δ in $\mathscr{G}$, *i.e.*, that formation of the family $\bar{\mathscr{G}}_d^r(\mathscr{C}/\mathscr{B})$ communed with base change; but, despite the assertion to this effect in [E-H3], this is proved only on the set level.

For the first part, to see that $\mathscr{G}_\Delta$ is smooth at $(C, \mathscr{D})$, observe first that Δ is isomorphic to the product of the Kuranishi spaces $(\pi_1 : \mathscr{C}_1 \to \mathscr{B}_1, \sigma_1 : \mathscr{B}_1 \to \mathscr{C}_1)$ and $(\pi_2 : \mathscr{C}_2 \to \mathscr{B}_2, \sigma_2 : \mathscr{B}_2 \to \mathscr{C}_2)$ for the pointed curves (B, p) and (C_0, q); and moreover by the definition of $\mathscr{G}_\Delta$ given in [E-H3] $\mathscr{G}_\Delta$ is in a neighborhood of the point $(C_0, \mathscr{D})$ the fiber product of the pullbacks of the schemes

$$\mathscr{G}_1=\mathscr{G}_d^r(\mathscr{C}_1/\mathscr{B}_1;\ \sigma_1,\ (d-r-2,\ \ldots,\ d-r-1,\ d-r))$$

and

$$\mathscr{G}_2=\mathscr{G}_d^r(\mathscr{C}_2/\mathscr{B}_2;\ \sigma_2,\ (0,\ 1,\ 2,\ \ldots,\ 2)).$$

Of course, since the pointed curve (C_0, q) is general the scheme $\mathscr{G}_2$ is etale over $\mathscr{B}_2$ at $(C_0, \mathscr{D}_{C_0})$, and $\mathscr{B}_2$ is smooth because C is 1-dimensional, so we need only verify the smoothness of $\mathscr{G}_1$ at the point $(B, (d-r-2)p+|(r+2)p|)$. But we see from the definitions that

$$\begin{aligned}&\mathscr{G}_d^r(\mathscr{C}_1/\mathscr{B}_1;\ \sigma_1,\ (d-r-2,\ \ldots,\ d-r-1,\ d-r))\\ &\quad=\mathscr{G}_{r+2}^r(\mathscr{C}_1/\mathscr{B}_1;\ \sigma_1,\ (0,\ \ldots,\ 0,\ 1,\ 2))\\ &\quad=\mathscr{G}_3^1(\mathscr{C}_1/\mathscr{B}_1;\ \sigma_1,\ (1,\ 2))\\ &\quad=\mathscr{G}_2^1(\mathscr{C}_1/\mathscr{B}_1;\ \sigma_1,\ (0,\ 1))\end{aligned}$$

by using alternately the observations that for any family $\mathscr{C}\to\mathscr{B}$,

$$\mathscr{G}_d^r(\mathscr{C}/\mathscr{B};\ \sigma,\ (\alpha_0,\ \ldots,\ \alpha_r))=\mathscr{G}_{d-\alpha_0}^r(\mathscr{C}/\mathscr{B};\ \sigma,\ (0,\ \alpha_1-\alpha_0,\ \ldots,\ \alpha_r-\alpha_0));$$

and for $d=2g-1+\delta$, $r=g-1+\delta$ with $\delta\geqq 0$ we must have $\alpha_0=\ldots=\alpha_\delta=0$ (or else $\mathscr{G}=\varnothing$) and

$$\mathscr{G}_d^r(\mathscr{C}/\mathscr{B};\ \sigma,\ (0,\ \ldots,\ 0,\ \alpha_{\delta+1},\ \ldots,\ \alpha_r))=\mathscr{G}_{2g-1}^{g-1}(\mathscr{C}/\mathscr{B};\ \sigma,\ (\alpha_{\delta+1},\ \ldots,\ \alpha_r)).$$

But now $\mathscr{G}_2^1(\mathscr{C}_1/\mathscr{B}_1)$ maps isomorphically to $\mathscr{B}_1$ (the fiber of $\mathscr{G}_2^1(\mathscr{C}_1/\mathscr{B}_1)$ over (B, p), for example, is the scheme $G_2^1(B)$, which is reduced by *e.g.* Theorem (4.2) of [ACGH]). $\mathscr{G}_2^1(\mathscr{C}_1/\mathscr{B}_1;\ \sigma_1,\ (0,\ 1))$ is correspondingly isomorphic to the locus $\mathscr{W}\subset\mathscr{B}_1$ of Weierstrass points, which is an etale cover of the Kuranishi space of B. Thus $\mathscr{G}_\Delta$ is smooth at the point $(C, \mathscr{D})$ as claimed.

This leaves us with the second part, to deduce that $\mathscr{G}$ is smooth. Referring to the construction of $\bar{\mathscr{G}}_d^r(\mathscr{C}/\mathscr{B})$ in [E-H3], recall the definition of the auxilliary scheme $\bar{\mathscr{F}}_d^r(\mathscr{C}/\mathscr{B})$ of framed limit linear series, whose equation are given explicitly on page 358 of [E-H3]. From these equations, we see that the formation of $\bar{\mathscr{F}}_d^r(\mathscr{C}/\mathscr{B})$ does commute with base change, though the same cannot necessarily be said of $\bar{\mathscr{G}}_d^r(\mathscr{C}/\mathscr{B})$, which is simply defined to be the image of $\bar{\mathscr{F}}_d^r(\mathscr{C}/\mathscr{B})$. In the present circumstances, however, we may use this to prove the chain of implications

$$\begin{aligned}&\bar{\mathscr{G}}_d^r(\mathscr{C}_\Delta/\Delta)\ \text{smooth}\\ &\quad\Rightarrow\ \bar{\mathscr{F}}_d^r(\mathscr{C}_\Delta/\Delta)\ \text{smooth}\\ &\quad\Rightarrow\ \bar{\mathscr{F}}_d^r(\mathscr{C}/\mathscr{B})\ \text{smooth}\\ &\quad\Rightarrow\ \bar{\mathscr{G}}_d^r(\mathscr{C}/\mathscr{B}).\end{aligned}$$

The first of these implications is immediate, since $\bar{\mathscr{F}}^r_d(\mathscr{C}_\Delta/\Delta)$ is simply a fiber bundle over $\bar{\mathscr{G}}^r_d(\mathscr{C}_\Delta/\Delta)$. The second follows from the fact that formation of $\mathscr{F}$ does commute with base change, which says that $\bar{\mathscr{F}}^r_d(\mathscr{C}_\Delta/\Delta)$ is a Cartier divisor in $\bar{\mathscr{F}}^r_d(\mathscr{C}/\mathscr{B})$. Finally, the third implication is a consequence of our present circumstances: first, as we have said, $\bar{\mathscr{F}}^r_d(\mathscr{C}_\Delta/\Delta)$ is a fiber bundle over $\bar{\mathscr{G}}^r_d(\mathscr{C}_\Delta/\Delta)$, so that the differential of the map $\pi_\Delta : \bar{\mathscr{F}}^r_d(\mathscr{C}_\Delta/\Delta) \to \bar{\mathscr{G}}^r_d(\mathscr{C}_\Delta/\Delta)$ has the maximal possible rank $\dim(\mathscr{G}_\Delta)$; and since the divisor $\mathscr{F}_\Delta \subset \mathscr{F}$ is reduced, it follows that the differential of $\pi : \bar{\mathscr{F}}^r_d(\mathscr{C}/\mathscr{B}) \to \bar{\mathscr{G}}^r_d(\mathscr{C}/\mathscr{B})$ likewise has the maximal rank $\dim(\mathscr{G})$. For some (indeed, any) point x of $\bar{\mathscr{F}}^r_d(\mathscr{C}/\mathscr{B})$ lying over C_0, then, we can choose a subvariety of $\bar{\mathscr{F}}^r_d(\mathscr{C}/\mathscr{B})$ transverse to the fiber of π at the point x. Since the fiber of π though x is connected, a neighborhood of x in this cross-section will then map isomorphically onto a neighborhood of $\pi(x)$ in $\mathscr{G}$, which is therefore smooth. Indeed, we see as well that the divisor Δ in $\mathscr{G}$ is reduced, and so coincides with $\mathscr{G}_\Delta$, *i. e.*, in this case formation of $\bar{\mathscr{G}}^r_d$ commutes with restriction to Δ. ■

Remark. — In fact, we see from the above that the variety $\mathscr{G}$ is transverse to the curves Γ introduced in the proof of Lemma (3.1). It follows that the intersection number of $\bar{\mathscr{M}}^r_{g,d}$ with Γ is 6μ, where μ is the number of g^r_d's on the curve C_0 with ramification $(0, 1, 2, \ldots, 2)$ at q, and hence that the constant c appearing in the expression above for the class of $\bar{\mathscr{M}}^r_{g,d}$ is $3\mu/2(g-2)$.

Putting together all the above, we see that there are at most μ distinct branches of $\bar{\mathscr{G}}^r_d$ passing over the point C, and the closure of every irreducible component of $\mathscr{G}^r_d$ contains at least one of these branches. Now, let the curve C vary in Δ_2 by fixing the pair (B, p) and letting the pair (C_0, q) vary in the moduli space of pointed curves of genus $g-2$. We claim that the branches of $\bar{\mathscr{G}}^r_d$ over C are in this way permuted transitively, *i. e.*, that the branches of $\bar{\mathscr{G}}^r_d$ passing over C belong to a *unique* irreducible component of $\mathscr{G}^r_d$. Of course, Theorem (1.1) above will follow from this statement, which in turn amounts to the

LEMMA (3.5). — *Let $\mathscr{C} = \mathscr{C}^0_{g-2}$ be the moduli of pointed smooth automorphism-free curves of genus $g-2$, and let $\Psi \to \mathscr{C}$ be the cover whose fiber over $(C_0, q) \in \mathscr{C}$ is the set of linear series g^r_d's on C_0 with ramification sequence $(0, 1, 2, \ldots, 2)$ at q. Then the monodromy on the points of a general fiber of Ψ over $\mathscr{C}$ is transitive, that is Ψ has a unique irreductible component dominating $\mathscr{C}$.*

Proof. — This is really just an extension of the theorem proved in [E-H1] that, for any g, r and d with Brill Noether number $\rho = 0$ the family $\mathscr{G}^r_d$ of g^r_d's on curves of genus g has transitive monodromy over $\mathscr{M}_g$. Specifically, in the paper [E-H1], this statement is reduced to the fact that the simplicial complex of chains of Schubert cycles (relative to a fixed flag) on the Grassmannian $\mathbb{G}(r, g-d+2r)$ is equidimensional and connected in codimension one. More generally, for any pair of nondecreasing sequences $\alpha = (\alpha_0, \ldots, \alpha_r)$ and $\beta = (\beta_0, \ldots, \beta_r)$ let $\mathscr{G}^r_d(p, q; \alpha, \beta)$ be the variety parametrizing g^r_d's on smooth two-pointed curves $(C; p, q)$ having ramification sequences $\alpha^L_i(p) \geqq \alpha_i$ and $\alpha^L_i(q) \geqq \beta_i$. By the same argument, it may be seen that the irreducibility of this family follows from the analogous statement that the simplicial complex of all chains of Schubert cycles on $\mathbb{G}(r, g-d+2r)$ *contained in the Schubert cycle* τ *and containing the Schubert cycle* μ is equidimensional and connected in codimension one, where for some n and m

with $n-m=g-d+r$, τ and μ are the cycles

$$\tau=\sigma_{m-\alpha_0,\ m-\alpha_1,\ \ldots,\ m-\alpha_r}$$

and

$$\mu=\sigma_{\beta_r+n,\ \beta_{r-1}+n,\ \ldots,\ \beta_0+n}.$$

The proof given in [E-H1] of this fact in case $\alpha=\beta=0$ applies verbatim in the present circumstances. ■

Note that this lemma is a special case of the more general.

LEMMA (3.6). — *Let g, r and d be integers, and $\underline{\alpha}=(\alpha_0, \ldots, \alpha_r)$ and $\underline{\beta}=(\beta_0, \ldots, \beta_r)$ sequences of integers such that the adjusted Brill-Noether number*

$$\rho'=g-(r+1)(g-d+r)-|\underline{\alpha}|-|\underline{\beta}|=0.$$

Let $\mathscr{M}^0_{g,\,2}$ be the moduli of two-pointed smooth automorphism-free curves of genus $g-2$, and let $\Psi \to \mathscr{C}$ be the cover whose fiber over $(C_0, p, q)\in\mathscr{M}^0_{g,\,2}$ is the set of linear series g^r_d's on C_0 with ramification sequences $\underline{\alpha}$ at p and $\underline{\beta}$ at q. Then the monodromy on the points of a general fiber of Ψ over $\mathscr{M}^0_{g,\,2}$ is transitive, that is, Ψ has a unique irreducible component dominating $\mathscr{M}^0_{g,\,2}$.

The proof of this is exactly the same as that of the special case $\underline{\alpha}=0$, $\underline{\beta}=(0, 1, 2, \ldots, 2)$ above. We should note that while there is no reason to limit the statement to two-pointed curves—there is an obvious analogue for linear series with adjusted Brill-Noether number $\rho'=0$ on k-pointed curves—it appears that our techniques will go no further than the two-pointed case. The general case remains, thus, a conjecture.

4. Proof of Theorem (1.2)

As in the preceding case, let us start by fixing notation. We will be given integers g, r and d and a sequence $\underline{\alpha}=(\alpha_0, \ldots, \alpha_r)$; unless otherwise stated we will assume that the adjusted Brill-Noether number $\rho'=g-(r+1)(g-d+r)-\Sigma\alpha_i=-1$. We will denote by $\mathscr{C}^r_{g,\,d}(\underline{\alpha})\subset\mathscr{C}_g$ the locus of pointed curves (C, p) of genus g possessing a linear system of degree d and dimension r having ramification at least $\underline{\alpha}$ at p, and by $\overline{\mathscr{C}}^r_{g,\,d}$ its closure in $\overline{\mathscr{C}}_g$. As in the previous sections, $\mathscr{G}^r_d(\underline{\alpha})$ will denote the variety of triples $(C, p; \mathscr{D})$ where (C, p) is a pointed curve of genus g without automorphisms and $\mathscr{D}$ a linear system of degree d and dimension r having ramification at least $\underline{\alpha}$ at p. If $\mathscr{C}\to B$ is a family of stable pointed curves of compact type with section $\sigma: B\to\mathscr{C}$, with B local, then we further define

$$\widetilde{\mathscr{G}}^r_d(\mathscr{C}/B, \sigma, \underline{\alpha})\to B$$

to be the associated family of crude limit linear series. Thus a point of $\widetilde{\mathscr{G}}^r_d(\mathscr{C}/B, \sigma, \underline{\alpha})$ is essentially a triple $(C, p, \mathscr{D})$ where (C, p) is a stable pointed curve of compact type—a fiber $C=C_b$ of $\mathscr{C}\to B$ whith point $p=\sigma(b)$—and $\mathscr{D}$ is a *limit linear series* of degree d and genus g on C with the appropriate ramification at p. If $B^0\subset B$ is the germ of the set of points of B corresponding to smooth pointed curves without automorphisms, and $\mathscr{C}^0\to B^0$ is the restriction of $\mathscr{C}\to B$, then we define $\overline{\mathscr{G}}^r_d(\mathscr{C}/B, \sigma, \underline{\alpha})$ to be the closure of $\mathscr{G}^r_d(\mathscr{C}^0/B^0, \sigma, \underline{\alpha})$ in $\tilde{\mathscr{G}}^r_d(\mathscr{C}/B, \sigma, \underline{\alpha})$. Applying this to the Kuranishi space of a stable pointed curve we can define, a variety $\overline{\mathscr{G}}^r_d(\underline{\alpha})$ which maps naturally to the intersection of the closure $\overline{\mathscr{C}}^r_{g,\,d}(\underline{\alpha})$ with the open set $\mathscr{U}\subset\overline{\mathscr{C}}_g$ of pointed curves of compact type.

With this out of the way, the proof of Theorem (1.2) decomposes into two steps in the same way as that of Theorem (1.1): we will first exhibit a locus in $\overline{\mathscr{C}}_g$ that must lie in any codimension 1 component of the closure $\overline{\mathscr{C}}^r_{g,\,d}(\underline{\alpha})$ of $\mathscr{C}^r_{g,\,d}(\underline{\alpha})$, and then show that at most one component of $\mathscr{G}^r_d(\underline{\alpha})$ passes above this locus. The latter half of the argument is essentially isomorphic to the corresponding part of the proof of Theorem (1.1), but the first step is different: it is *not* the case that all components of $\overline{\mathscr{C}}^r_{g,\,d}(\underline{\alpha})$ must have similar classes in $\mathrm{Pic}(\overline{\mathscr{C}}_g)$, as it was in the corresponding case quoted from [E-H2] in the course of the proof of Lemma 3.1. Instead, we have the following analog of the result from [E-H2], which is the main result of this section:

THEOREM (4.1). — *Let* D *be any divisor in* $\overline{\mathscr{C}}_g$, *not supported on the boundary* $\overline{\mathscr{C}}_g-\mathscr{C}_g$, *and such that for a general point* (C, p) *in the support of* D *there exists a linear series* g^r_d *on* C *having ramification* $\underline{\alpha}$ *at* p *with* $g-(r+1)(g-d+r)-\Sigma\alpha_i\leqq -1$. *Then the class of* D *is a linear combination*

$$(\bigstar)\qquad D\sim\mu.B+\nu.W$$

where

$$B\sim(g+3).\lambda-\frac{g+1}{6}.\delta_0-\sum_{i=1}^{g-1} i(g-i).\delta_i$$

and

$$W\sim -\lambda+\frac{g(g+1)}{2}\omega-\sum_{i=1}^{g-1}\frac{(g-i)(g-i+1)}{2}\delta_i.$$

Further, if D *is effective, then the coefficients* μ *and* ν *are non-negative.*

Remark. — B here is just the pullback of the class of any divisor in $\overline{\mathscr{M}}_g$ supported on the locus of curves possessing linear series with negative Brill-Noether number, while W is the class of the locus of Weierstrass points [Cuk]. The first example of a divisor D whose class is a nontrivial linear combination of these two is the divisor in $\overline{\mathscr{C}}_4$ that is the closure of the locus of ramification points of g^1_3's on smooth curves of genus 4; its class is $(6/5)\,B+(2/5)\,W$.

Proof. — Suppose the class of D is given by

$$D\sim a\lambda+b\omega-\Sigma c_i\delta_i.$$

We will show that

$$c_0=\frac{g+1}{6(g+3)}a+\frac{1}{3g(g+3)}b$$

and

$$c_i=\frac{i(g-i)}{g+3}a+\left(\frac{(g-i)(g-i-1)}{g(g-1)}+\frac{4i(g-i)}{g(g-1)(g+3)}\right)b$$

for $i\geqq 1$, which together are equivalent to the first statement of the Theorem. We prove these by considering the restrictions of D to each of 2 loci in $\overline{\mathscr{C}}_g$.

The first locus we look at is a part of the universal curve over the locus in $\overline{\mathscr{M}}_g$ of curves with g elliptic tails. Specifically, let P_g denote the moduli space of stable g-pointed rational curves, and fix once and for all g pointed elliptic curves (E_i, q_i). We then have a map

$$k:\quad \mathrm{P}_{g+1}\to\overline{\mathscr{C}}_g$$

obtained by associating to the $(g+1)$-pointed stable curve $(\mathrm{C}_0, p_1, \ldots, p_{g+1})$ the pointed curve (C, p) where C is the curve obtained by attaching the g elliptic tails (E_j, q_j) to C_0 at the points $p_1, \ldots, p_g$ and p is the image of p_{g+1} in the quotient. For $\alpha=1, \ldots, g-2$, let ε_α be the divisor which is the closure in P_{g+1} of the locus of $(\mathrm{C}_0, p_1, \ldots, p_{g+1})$ where C_0 has two components, with p_{g+1} and exactly α of the points $p_1, \ldots, p_g$ lying on one and the remaining of the points p_j lying on the other; as observed in [E-H2] these divisors will be independent in $\mathrm{Pic}(\mathrm{P}_{g+1})$.

We want now to describe the pullback map k^*. First of all, we have, clearly,

$$k^*(\delta_\alpha)=\varepsilon_\alpha \qquad \text{for}\quad \alpha=1, \ldots, g-2$$

and also, as in [E-H2],

$$k^*\lambda=k^*\delta_0=0.$$

To evaluate the pullbacks of δ_{g-1} and ω, we use the set-up of [E-H2, § 3]: we let $\pi:\mathscr{C}\to\mathrm{B}$ be any family of stable $(g+1)$-pointed curves with sections $\sigma_1, \ldots, \sigma_{g+1}:\mathrm{B}\to\mathscr{C}$; let $\overline{\pi}:\overline{\mathscr{C}}\to\mathrm{B}$ be the family obtained by blowing down, in every fiber of π, the components not meeting the image of σ_{g+1} and $\overline{\sigma}_i$ the corresponding sections (such a blow-down may be obtained, for example, as the image of the map defined by a high multiple of the relative divisor which is the image of σ_{g+1}). We then have

$$k^*(\delta_{g-1})=\pi_*\sum_{i=1}^{g}\sigma_1(\mathrm{B})^2$$

and

$$k^*(\omega)=-\pi_*\,\sigma_{g+1}(\mathrm{B})^2.$$

As in [E-H2] we can evaluate these in terms of the ε_α by using the basic relation

$$\bar{\sigma}_i(\mathrm{B})^2+\bar{\sigma}_j(\mathrm{B})^2=-2\,\bar{\sigma}_i(\mathrm{B})\,.\,\bar{\sigma}_j(\mathrm{B}).$$

We first sum this over all $1 \leqq i<j \leqq g$ to obtain

$$\begin{aligned}(g-1)\,\bar{\pi}_*\sum_{i=1}^{g}\bar{\sigma}_i(\mathrm{B})^2&=2\,\bar{\pi}_*\sum_{1\leqq i<j\leqq g}\bar{\sigma}_i(\mathrm{B})\,.\,\bar{\sigma}_j(\mathrm{B})\\&=\sum_{i=2}^{g-2} i\,(i-1)\,\varepsilon_i\end{aligned}$$

and hence

$$\begin{aligned}k^*(\delta_{g-1})&=\pi_*\sum_{i=1}^{g}\sigma_i(\mathrm{B})^2\\&=\frac{1}{g-1}\sum_{i=2}^{g-2} i\,(i-1)\,\varepsilon_i-\sum_{i=1}^{g-2} i\,\varepsilon_i\\&=\sum_{i=1}^{g-2}\frac{i\,(g-i)}{g-1}\varepsilon_i\end{aligned}$$

Next, we sum the basic relations above for $i=g+1$ and $j=1,\ldots,g$ to obtain

$$\begin{aligned}g\,\bar{\pi}_*\,\bar{\sigma}_{g+1}(\mathrm{B})^2+\bar{\pi}_*\sum_{i=1}^{g}\bar{\sigma}_i(\mathrm{B})^2&=2\,\bar{\pi}_*\sum_{i=1}^{g}\bar{\sigma}_i(\mathrm{B})\,.\,\bar{\sigma}_{g+1}(\mathrm{B})\\&=2\sum_{i=1}^{g} i\,\varepsilon_i\end{aligned}$$

so

$$\begin{aligned}g\,\pi_*\,\sigma_{g+1}(\mathrm{B})^2+\pi_*\sum_{i=1}^{g}\sigma_1(\mathrm{B})^2&=2\sum_{i=1}^{g} i\,\varepsilon_i-\sum_{i=1}^{g}(g+i)\,\varepsilon_i\\&=\sum_{i=1}^{g}(i-g)\,\varepsilon_i\end{aligned}$$

and

$$\begin{aligned}k^*(\omega)&=-\pi_*\,\sigma_{g+1}(\mathrm{B})^2\\&=\frac{1}{g}\left(\sum_{i=1}^{g}(g-i)\,\varepsilon_i+\sum_{i=1}^{g}\frac{i\,(g-i)}{g-1}\varepsilon_i\right)\\&=\sum_{i=1}^{g}\frac{(g-i)\,(g-i+1)}{g\,(g-1)}\varepsilon_i.\end{aligned}$$

Thus, a divisor D on $\overline{\mathscr{C}}_g$ with class $a.\lambda+b.\omega-\Sigma c_\alpha \delta_\alpha$ would pull back to a divisor with class

$$i^*(\mathrm{D}) \sim \sum_{i=1}^{g-2}\left(b\frac{(g-i)(g-i-1)}{g(g-1)}+c_{g-1}\frac{i(g-i)}{g-1}-c_i\right)\varepsilon_i$$

On the other hand, if D is any divisor in $\overline{\mathscr{C}}_g$ supported on the closure of the locus of pointed curves admitting a linear series with adjusted Brill-Noether number -1, then the support of D will be disjoint from the image of i. We have thus proved:

LEMMA (4.2). — *Let* D *be any divisor on* $\overline{\mathscr{C}}_g$ *satisfying the hypotheses of Lemma* (4.1), *and suppose that* $\mathrm{D}\sim a.\lambda+b.\omega-\Sigma c_\alpha \delta_\alpha$. *Then for* $i\geqq 1$,

$$c_i=\frac{(g-i)(g-i-1)}{g(g-1)}b+\frac{i(g-i)}{g-1}c_{g-1}$$

Next, we consider a map from the moduli space $\overline{\mathscr{C}}_2$ of pointed curves of genus two to $\overline{\mathscr{C}}_g$ analogous to the map j to $\overline{\mathscr{M}}_g$ described in [E-H2]. We fix a general curve C_0 of genus $g-2$ and two general points p, $q\in \mathrm{C}_0$; and then to any pointed curve (B, r) of genus two we associate the stable pointed curve $\mathrm{C}=\mathrm{C}_0\cup \mathrm{B}/q\sim r$ obtained by identifying q and r, and taking the image of p in the quotient as the marked point. The pullback map j^* on divisors is then given by

$$\begin{gathered}
j^*\lambda=\lambda\\
j^*\omega=0\\
j^*\delta_0=\delta_0\\
j^*\delta_\alpha=0,\qquad \text{for}\quad \alpha=1,2,\ldots,g-3\\
j^*\delta_{g-2}=-\omega;
\end{gathered}$$

and

$$j^*\delta_{g-1}=\delta_1.$$

Suppose now that (C, p) is any pointed curve in the image of j where $\mathrm{C}=\mathrm{C}_0\cup\mathrm{B}$, and that $\mathscr{D}$ is a limit g_d^r on C with negative adjusted Brill-Noether number with respect to p. C_0, p and q being generic, the adjusted Brill-Noether number of any linear series on C_0 with respect to the two points p and q will be non negative by Theorem (1.1) of [E-H2]. By the additivity of the adjusted Brill-Noether number, then, the aspect on B of $\mathscr{D}$ must have negative adjusted Brill-Noether number with respect to r; *i. e.*, by Lemma (3.2) above the pair (B, r) must lie in the closure of the locus $\mathrm{W}\subset\overline{\mathscr{C}}_2$ of Weierstrass points. By the argument for Theorem 2.1 of [E-H2], we deduce:

LEMMA (4.3). — *Let* D *be any divisor on* $\overline{\mathscr{C}}_g$ *satisfying the hypotheses of Theorem* (4.1), *and suppose that* $\mathrm{D}\sim a.\lambda+b.\omega-\Sigma c_\alpha\delta_\alpha$. *Then*

$$a=5c_{g-1}-2c_{g-2};$$

and

$$c_0 = \frac{c_{g-1}}{2} - \frac{c_{g-2}}{6}.$$

Combining Lemmas (4.2) and (4.3) we can solve for the coefficients c_i in terms of a and b, thereby establishing the first statement of Theorem (4.1).

We can also obtain information about D by restricting, for example, to the variety $\overline{\mathscr{C}}_3$ of pointed curves of genus 3. Thus, we fix a general curve C'_0 of genus $g-3$ and two general points $p, q \in C'_0$, and define a map

$$m: \quad \overline{\mathscr{C}}_3 \to \overline{\mathscr{C}}_g$$

by sending a pointed curve (B, r) of genus 3 to the curve $C = C'_0 \cup B/q \sim r$, with marked point the image of p. The pullback $m^* D$ of any divisor in $\overline{\mathscr{C}}_g$ satisfying the conditions of Theorem (4.1) will be supported on the union of two loci: the closure of the locus B_3 of curves (B, r) with B smooth and hyperelliptic, and the closure of the locus W_3 of pairs (B, r) with B smooth and r a Weierstrass point of B. The pullback class will thus be a linear combination of the classes of

$$B_3 \sim 9\lambda - \delta_0 - 3\delta_1 - 3\delta_2$$

and

$$W_3 \sim -\lambda + 6\omega - 3\delta_1 - \delta_2$$

(*see* [H1] and [Cuk] respectively for these computations). Now, in the Picard group of $\overline{\mathscr{C}}_3$ the classes λ, ω, δ_0, δ_1 and δ_2 are independent; and under the pullback we have

$$m^* \lambda = \lambda$$

$$m^* \omega = 0$$

$$m^* \delta_0 = \delta_0$$

$$m^* \delta_i = 0, \qquad \text{for} \quad i = 1, 2, \ldots, g-4$$

$$m^* \delta_{g-3} = -\omega$$

$$m^* \delta_{g-2} = \delta_1$$

and

$$m^* \delta_{g-1} = \delta_2.$$

Making the relevant computation, we see that *we get exactly the same information about the class of a divisor satisfying the hypotheses of* (4.1) *from this map as we did from the map j*. Explicitly, by comparing coefficients of ω and δ_0, we see that if the pullback

$$(\bigstar\bigstar) \qquad m^* D \sim \zeta . B_3 + \xi . W_3$$

then the divisor D must be equivalent to the linear combination

$$\mathrm{D}\sim\frac{6\zeta}{g+1}\mathrm{B}+\left(\xi-\frac{3(g-3)\zeta}{g+1}\right)\mathrm{W}.$$

But on the basis of this last computation, we can say a little more if D is *effective.* To begin with, D will then have nonnegative intersection number with a fiber of the map $\pi:\overline{\mathscr{C}}_g\to\overline{\mathscr{M}}_g$; but since the divisor class B has intersection number zero with this fiber, this amounts to saying that the coefficient ν in the expression ($\bigstar$) of D as a linear combination of the divisor classes B and W is nonnegative. Moreover, if D is effective then, since its support cannot contain the boundary component Δ_3 and so will not contain the image of m, the pullback $m^*\mathrm{D}$ will be as well. Inasmuch as, as we remarked, $k^*\mathrm{D}$ is supported on the union of the two loci B_3 and W_3 and is effective, it follows that both the coefficients of B_3 in the expression ($\bigstar\bigstar$) for the pullback will be nonnegative. This implies that the coefficient μ in ($\bigstar$) will be nonnegative as well, completing the proof of Theorem 4.1.

Let us return now to examining the variety $\overline{\mathscr{C}}^r_{g,d}(\underline{\alpha})\subset\overline{\mathscr{C}}_g$. The rest of the argument follows very much as in the preceding section. Let C_0 be a general curve of genus $g-2$ as above, $\tilde{p}$ and q two general points on C_0, and let B be a general curve of genus 2 and r any point of B. We consider then the stable pointed curve $(\mathrm{C}, p)\in\overline{\mathscr{C}}_g$ where

$$\mathrm{C}=\mathrm{C}_0\cup\mathrm{B}/q\sim r$$

and p is the image of the point $\tilde{p}$ in the quotient. We claim then that

LEMMA (4.4). — *Any component of* $\overline{\mathscr{C}}^r_{g,d}(\underline{\alpha})$ *having codimension one in* $\overline{\mathscr{M}}_g$ *contains all pointed curves of the form* C *for which* r *is a Weierstrass point.*

Proof. — Fix C_0, $\tilde{p}$, q and B and let $\Gamma=\{(\mathrm{C}_r, p)\}\subset\Delta_{g-2}\subset\overline{\mathscr{C}}_g$ be the family of pointed curves obtained by identifying a variable point r on the curve B with a fixed general point q on C. As before, the degrees of all the divisors λ, ω and δ_i for $i\neq g-2$ are zero on Γ, while $\deg_\Gamma(\delta_{g-2})=-2$. It follows that both divisor classes B and W have positive intersection with Γ, so that any component Ψ of $\overline{\mathscr{C}}^r_{g,d}(\underline{\alpha})$ will met Γ. But as we have seen such a component can met Γ only at the points $(\mathrm{C}_{r,p})\in\Gamma$ corresponding to Weierstrass points r of B. Since the set $\mathrm{W}\subset\Delta_{g-2}$ of stable pointed curves obtained by attaching a 2-pointed curve of genus $g-2$ to a curve of genus 2 at a Weierstrass point of the latter is an irreducible codimension 1 locus of Δ_{g-2}, it follows that Ψ must contain the locus of such curves. ■

Next, consider what $\mathscr{G}^r_d(\underline{\alpha})$ looks like over the point $(\mathrm{C}, p)\in\overline{\mathscr{C}}_g$. By Lemma (3.2) above, we see that any limit linear series $\mathscr{D}$ on C with ramification $\underline{\alpha}$ at p will have to have aspect $\mathscr{D}_\mathrm{B}=|(r+2)p|+(d-r-2)p$ on B; and hence its aspect on C_0 will be a g^r_d with ramification $\underline{\alpha}$ at p and $(0, 1, 2, \ldots, 2)$ at q. Moreover, by the analogue of Lemma (3.4) we see that $\overline{\mathscr{G}}^r_d(\underline{\alpha})$ is smooth over (C, p), so that the branches of $\overline{\mathscr{C}}^r_{g,d}(\underline{\alpha})$ at (C, p) correspond to such linear series on C_0.

Now, as before we let the curve C vary in Δ_2 by fixing the pair (B, p) and letting the pair (C_0, p, q) vary in moduli space of two-pointed curves of genus $g-2$. We claim that the branches of $\overline{\mathscr{C}}^r_{g,d}(\underline{\alpha})$ over C are in this way permuted transitively, *i.e.*, that the branches of $\overline{\mathscr{C}}^r_{g,d}(\underline{\alpha})$ passing over C belong to a *unique* irreducible component of $\overline{\mathscr{C}}^r_{g,d}(\underline{\alpha})$; this follows from the general form of Lemma (3.6) and we may deduce Theorem (1.2) as a consequence. ■

REFERENCES

[ACGH] E. Arbarello, M. Cornalba, P. Griffiths and J. Harris, *Geometry of Algebraic Curves*, Vol. 1, Springer-Verlag, New York, 1985, Vol. 2 (to appear).

[Cuk] F. Cukierman, *Ph. D. Thesis*, Brown University, 1987.

[E1] L. Ein, *Hilbert Scheme of Smooth Space Curves* (*Ann. Sci. Éc. Norm. Sup.*, Vol. 19, 1986, pp. 469-478).

[E2] L. Ein, *Irreducibility of the Hilbert Scheme of Smooth Space Curves* (in *Algebraic Geometry*, Bowdoin, 1985, S. Block Ed., *Proc. Symp. Pure Math.*, Vol. 46, 1987, pp. 83-87).

[E-H1] D. Eisenbud and J. Harris, *Irreducibility and Monodromy of Some Families of Linear Series* (*Ann. scient. Éc. Norm. Sup.*, Vol. 20, 1987, pp. 65-87).

[E-H2] D. Eisenbud and J. Harris, *The Kodaira Dimension of the Moduli Space of Curves of Genus* ≧ 23 (*Invent. Math.*, Vol. 90, 1987, pp. 359-387).

[E-H3] D. Eisenbud and J. Harris, *Limit Linear Series: Basic Theory* (*Invent. Math.*, Vol. 85, 1986, pp. 337-371).

[E-H4] D. Eisenbud and J. Harris, *Divisors on General Curves and Cuspidal Rational Curves* (*Invent. Math.*, Vol. 74, 1983, pp. 371-418).

[E-H5] D. Eisenbud and J. Harris, *A Simpler Proof of the Gieseker-Petri Theorem on Special Divisors* (*Invent. Math.*, Vol. 74, 1983, pp. 269-280).

[E-H6] D. Eisenbud and J. Harris, *Existence, Decomposition and Limits of Weierstrass Points* (*Invent. Math.*, Vol. 87, 1987, pp. 495-515).

[F-L] W. Fulton and R. Lazarsfeld, *On the Connectedness of Degeneracy Loci and Special Divisors* (*Acta Math.*, Vol. 146, 1981, pp. 271-283).

[H1] J. Harris, *Curves and their Moduli* (*Proceedings of the A.M.S. Summer Institute in Algebraic Geometry*, Bowdoin 1985).

[H2] J. Harris and D. Eisenbud, *Curves in Projective Space*, Les Presses de l'Université de Montréal, 1982.

[H-M] J. Harris and D. Mumford, *On the Kodaira Dimension of the Moduli Space of Curves* (*Invent. Math.*, Vol. 67, 1982, pp. 23-86).

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