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Remarks on Points in a Projective Space

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Introduction.

In this paper we will survey some results and conjectures on the free resolutions of ideals of sets of points in projective r-space. For general sets of points, there are conjectures of Lorenzini. A weaker statement, relevant to sets containing between $r+1$ and $2r$ points has been proved by Green and Lazarsfeld [**GL2**], and they conjecture a necessary and sufficient condition on the set of points for the weaker statement to hold. As Green has noted, a part of their conjecture follows from a conjecture of ours (with Mike Stillman) on linear syzygies [**EKS**], and we explain this connection and the consequences of the known cases of the linear syzygy conjecture; in particular the conjecture holds for $r \leq 4$. We also extend the result of Green-Lazarsfeld to deal with some larger sets of points.

One special case of the Green-Lazarsfeld conjecture says that if X is a set of $2r$ points in $\mathbb{P}^r$ such that no $2k+1$ of them lie in a k-plane, then the homogeneous ideal of X is generated by quadrics. We give a proof of this part of the conjecture, independent of the linear syzygy conjecture, for $r \leq 4$. Using a result in Matroid Theory due to J. Edmonds we prove a corresponding result, but only "scheme-theoretically", for sets of dr points, and forms of degree d, for any d and r.

We work over an algebraically closed field K. Let $X \subset \mathbb{P}^r_K = \mathbb{P}^r$ be a set of (distinct and reduced) points, not contained in any hyperplane. We denote by $S = \oplus_{d \geq 0} S_d = K[x_0, \ldots, x_r]$ the homogeneous coordinate ring of $\mathbb{P}^r$ and by $|X|$ the cardinality of X. We say that X imposes independent conditions on forms of degree $d \geq 1$ if the following holds:

(i) If $|X| \leq \dim_K S_d$, then $I(X)_d = \dim_K S_d - |X|$.

(ii) If $|X| \geq \dim_K S_d$, then no $(\dim_K S_d)$-points of X lie on a hypersurface of degree d.

Let $I = I(X)$ be the homogeneous ideal of X and let $E_\bullet$ be a minimal graded free resolution of I over S:

$$0 \to E_r \to E_{r-1} \to \cdots \to E_2 \to E_1 \to S \to S/I \to 0,$$

where $E_i = \oplus_{j\geq 1}\{S(-i-j)\otimes \mathrm{Tor}_i^S(S/I,K)_{i+j}\}$. We deal with the question of when the first few terms of $E_\bullet$ are as simple as possible and we extend the property (N_p) of Green and Lazarsfeld [**GL1**] as follows: If $C(r+d-1,r) \leq |X| < C(r+d,r)$, where $C(a,b)$ denote the binomial coefficient $\binom{a}{b}$, we will say that X satisfies $(N_{d,p})$ (for $0 \leq p \leq r$) if X imposes independent conditions on forms of degree d and, if $p > 0$, $\mathrm{Tor}_i^R(S/I(X),K)_j = 0$ for $j \geq d+i$ and $i \leq p$. Thus X satisfies $(N_{d,1})$ if the ideal of X is generated by forms of degree d, X satisfies $(N_{d,2})$ if in addition all the relations among these generators are linear, etc. We note that $(N_{2,p})$ is the property (N_p) of Green and Lazarsfeld [**GL1**].

This paper concerns the following result and conjecture of Green and Lazarsfeld [**GL2**]:

THEOREM (Green-Lazarsfeld). *Let $X \subset \mathbb{P}^r$ be a set of $(r+1)+(r-p)$ $(1 \leq p \leq r)$ points in linear general position, i.e. no $r+1$ lying on a hyperplane. Then X satisfies $(N_{2,p})$.*

CONJECTURE 1 (Green-Lazarsfeld). *Let $X \subset \mathbb{P}^r$ be a set of $(r+1)+(r-p)$ $(1 \leq p \leq r)$ points. If X fails to satisfy $(N_{2,p})$, then there is an integer $k \leq r$, and a subset $Y \subset X$ consisting of at least $2k+2-p$ points, such that*

(a) *Y is contained in a linear subspace $\mathbb{P}^k \subset \mathbb{P}^r$, and*
(b) *$(N_{2,p})$ fails for Y in $\mathbb{P}^k$.*

M. Green has proved that Conjecture 1(a) is a consequence of the following conjecture on linear syzygies made by us in collaboration with M. Stillman: A graded S-module $M = \oplus_{d\geq t}M_d$, $M_t \neq 0$, is said to have a k-th linear syzygy if $\mathrm{Tor}_k^S(M,K)_{k+t} \neq 0$.

CONJECTURE 2 (Linear Syzygy Conjecture). *Let $M = \oplus_{d\geq t}M_d$, $M_t \neq 0$, be a graded S-module. Let $\mathcal{R}$ denote the kernel of the map $S_1\otimes M_t \to M_{t+1}$ and let $\mathcal{R}_r$ denote the rank r locus of $\mathcal{R}$ (here an element of $\mathcal{R}$ is viewed as a linear transformation from M_t^* to S_1). If M has a linear k-th syzygy, then $\mathcal{R}$ satisfies:*

(i) *if $\dim_K M_t \leq k$ then $\dim \mathcal{R}_1 \geq k$, and*
(ii) *if $\dim_K M_t \geq k$ then $\dim \mathcal{R}_{m-k+1} \geq k$,*

where dim *denotes the dimension as an affine variety and $m = \dim_k M_t$.*

Some cases of Linear Syzygy Conjecture are known [**EKS**] and Conjecture 1(a) for $r \leq 4$ or $p \geq r-2$ follows from these cases. In Section 1 we give a modification of Green's argument to prove that Conjecture 2 implies Conjecture 1(a).

In Section 2 we give a direct argument for Conjecture 1 when $r \leq 4$ and $p = 1$ (part (b) of Conjecture 1 follows from part (a) in this case).

In Section 3 we use a result of J. Edmonds [**Ed**] to prove the following two theorems:

A form of degree d is called multilinear if it is a product of linear forms.

THEOREM 1. *Let X be a set of points in $\mathbb{P}^r$, and let $d \geq 2$ be an integer. If, for all $k \geq 1$, no $dk+1$ of the points of X lie in a projective k-plane, then X is scheme-theoretically the intersection of multilinear forms of degree d.*

THEOREM 2. *Let X be a set of points in $\mathbb{P}^r$, and let $d \geq 2$ be an integer. If, for all $k \geq 1$, no $dk+2$ of the points of X lie in a projective k-plane, then X impose independent conditions on forms of degree d; in fact there is a multilinear form of degree d containing any subset consisting of all but one of the points, but missing the last.*

Theorem 2 in case $d = 2$ is Conjecture 1(a) for the case $p = 0$. This case was also proved by Green and Lazarsfeld.

Since Edmonds's paper is somewhat obscurely published, and since his argument is very elegant, we will reproduce it in Section 3 (with minor modifications to clarify one point).

In Section 4 we extend the argument of Green and Lazarsfeld to generalize their theorem above to:

THEOREM 3. *Let $X \subset \mathbb{P}^r$ be a set of $\binom{r+d}{d} + (r-p)$ points $(0 \leq p \leq r)$ imposing independent condition on forms of degree d. Then*

(i) *X imposes independent conditions on forms of degree $(d+1)$, and*
(ii) *$\mathrm{Tor}_i{}^S(S/I, K)_j = 0$, for all $1 \leq i \leq p$ and $j \neq i+d$.*

The Green-Lazarsfeld theorem and conjectures should be contrasted with the best plausible conjectures for general points in $\mathbb{P}^r$, which have been worked out by Lorenzini [**L**]. Let M be a finitely generated graded module over S; there is a natural approximation β_{ij} to the graded Betti numbers $b_{ij} = \dim_{\mathrm{K}} \mathrm{Tor}^S{}_i(M, K)_j$ which can be computed in terms of the Hilbert function $H(M,t) = \dim_{\mathrm{K}} M_t$ of M. These are given as follows: there is a unique function $\phi : \mathbb{Z} \to \mathbb{N}$ such that $\phi(j) = 0$ for $j \ll 0$ and integers $\beta_{ij} \geq 0$ such that

(1) $H(M,t) = \sum_k (-1)^{\phi(k)} \beta_{\phi(k)k} C(r+t+1-k, r)$

(2) $\phi(j) \leq \phi(j+1) \leq \phi(j)+1$
(3) $\beta_{ij} = 0$ unless $i = \phi(j)$
(4) if $\phi(j) \neq \phi(j-1)$, then $\beta_{\phi(j),j} > 0$.

ϕ and the β_{ij} may be constructed inductively as follows: Supposing that $H(M,t) = 0$ for $t \leq t_0$, we set $\phi(t) = 0$ and all $\beta_{it} = 0$ for $t \leq t_0$. Having defined $\phi(t)$ and for all $t \leq$ some t_1, we define

$$\phi(t_1+1) = \begin{cases} \phi(t_1) & \text{if } (-1)^{\phi(t_1)}\{H(M,t_1+1) \\ & \qquad -\sum_k(-1)^{\phi(k)}\beta_{\phi(k)k}C(r+t_1+1-k,r)\} \geq 0 \\ \phi(t_1)+1 & \text{if not.} \end{cases}$$

and $\beta_{\phi(t_1+1),t_1+1} = \left|H(M,t_1+1) - \sum_k(-1)^{\phi(k)}\beta_{\phi(k)k}C(r+t_1+1-k,r)\right|$ and of course $\beta_{i,t_1+1} = 0$ for $i \neq \phi(t_1+1)$.

The reader may check that this is unique. Note however that it is easy to produce Hilbert functions for which the β_{ij} defined above cannot be equal to the b_{ij}. For example, let M be two copies of K in degrees 0 and 2, i.e. $M = M_0 \oplus M_2$ and $M_0 = M_2 = K$. One checks that $\beta_{02} = 0$ but $b_{02} = 1$ and M is the unique module whose Hilbert function is $H(t)$, where $H(t) = 1$ if $t = 0$ or 2 and 0 otherwise. However, in the case where M is the homogeneous coordinate ring of a general set of points in $\mathbb{P}^r$, and in many other geometric situations, such problems do not arise, and it is natural to conjecture that $b_{ij} = \beta_{ij}$. It would be very interesting to have general conditions under which this conjecture is plausible.

Lorenzini has worked out the numbers β_{ij} explicitly for the case of the homogeneous coordinate ring of a set of points in general position. Geramita and Maroscia [**GM**] have shown that the $b_{ij} = \beta_{ij}$ for general sets of points in $\mathbb{P}^3$. An easily stated consequence of her computation is:

General Points Conjecture. *If X is a general set of points in $\mathbb{P}^r$ with $C(r+d-1,r) \leq |X| < C(r+d,r)$, then X satisfies $N_{d,p}$ $(0 \leq p \leq r)$ iff*

$$(d/d+p+1)\,C(r+d,r) < |X| \leq (d/d+p)\,C(r+d,r).$$

The conclusion of this conjecture is much stronger than that given in the Theorem of Green-Lazarsfeld in the case $d = 2$ or by our Theorem 3 in case $d \geq 3$. For example, the the conjecture suggests that 10 general points in $\mathbb{P}^4$ satisfy $N_{2,1} = N_1$ (which is true since the hyperplane section of a general canonical curve in $\mathbb{P}^5$ clearly satisfies N_1), whereas the Green-Lazarsfeld theorem says only that 8 or fewer general points satisfy N_1. On the other hand, the hypothesis of the Green-Lazarsfeld theorem, or our Theorem 3, that the points impose independent conditions on forms of degree d, is also much weaker than the hypothesis of generality, and with this weaker hypotheses, the theorems are (at least sometimes) sharp.

For example, the Green Lazarsfeld theorem says that 7 or fewer points in linearly general position in $\mathbb{P}^3$ impose independent conditions on quadrics, while actually any number of general points have this property. However, if we choose 8 points on a twisted cubic curve, then they will be in linearly general position (no 4 on a plane) but will not impose independent conditions on quadrics, as every quadric containing 7 of the points contains the twisted cubic.

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1. The Linear Syzygy Conjecture implies part (a) of the Green-Lazarsfeld Conjecture.

The result of this section was first proved by Mark Green (unpublished). We give a simplification of his proof.

Let X be as in Conjecture 1 and let $I = I(X)$. Let $R = S/I$ and let m denote the irrelevant maximal ideal R_d. Let $\omega = \omega_R$ denote the dualizing module $\mathrm{Ext}^r_S(R, S(-r-1))$. Let x be a linear form which does not vanish at any point of X. For each point P of X, choose a homogeneous coordinates so that $x(P) = 1$. We write $\mathrm{Hom}_K(R, K)$ for $\otimes_d \mathrm{Hom}_K(R_d, K)$, the graded dual of R.

LEMMA 1. *There is an exact sequence* $0 \to \omega \to \oplus_{P\in X} K[x, x^{-1}]P \xrightarrow{\phi} \mathrm{Hom}_K(R, K) \to 0$, *where* ϕ *is defined by* $\phi(x^d P)(r) = r(P)$ *if degree* $r = d$ *and 0 otherwise.*

REMARK: The above sequence can presumably be derived from the local cohomology exact sequence associated to the inclusion of the punctured cone U over X into the cone CX over X: $(H^0_m(\omega) = 0 \to H_0(CX, \omega) \to H^0(U, \omega) \to H^1_m(\omega) \to H^1(CX, \omega) = 0$. Here one can identify $H^0(U, \omega)$ with $\oplus_{P\in X} K[x, x^{-1}]P$, since U is a disjoint union of punctured lines, one for each point P of X, and one can identify $H^1_m(\omega)$ with

$$\mathrm{Hom}_R(\mathrm{Hom}_R(\omega, \omega), E_R(K)) \simeq \mathrm{Hom}_R(R, E_R(K)) \simeq \mathrm{Hom}_K(R, K),$$

$E_R(K)$ being the injective envelope of K as an R-module. However, there are so many identifications in this interpretation that we found it simpler to give a direct proof.

PROOF: We first prove that ϕ is onto. Since $\mathrm{Hom}_K(R,K)$ is generated by its elements of large negative degree, it suffices to show that ϕ is onto in degree $-n$ for large n. Since X imposes independent conditions on forms of degree n (n large), ϕ is one-to-one in degree $-n$. Since $\dim_K\{\oplus_{P\in X} K[x,x^{-1}]P\}_{-n} = |X| = \dim_K \mathrm{Hom}_K(R,K)_{-n}$, ϕ is onto in degree $-n$ for all large n.

We now complete our proof by proving: for any exact sequence

$$(*) \qquad 0 \to M \to \oplus_{P\in X} K[x,x^{-1}]P \to \mathrm{Hom}_K(R,K) \to 0,$$

$M \simeq \omega$.

To prove this we use the fact that $M \simeq \omega$ iff M is torsion-free and, for some non-zero divisor x in R, $M/xM(1) \simeq \omega_{R/xR}$, the dualizing module of R/xR. (One may check ($\Leftarrow$) as follows: Since M is torsion-free, $\mathrm{Ext}^1_R(M,\omega) \simeq \mathrm{Hom}_K(H^0_m(M),K) = 0$ and the map $\mathrm{Hom}(M,\omega) \to \mathrm{Hom}(M,\omega/x\omega)$ is onto. Hence we can lift the map $M \to M/xM \simeq \omega_{R/xR}(-1) \simeq \omega/x\omega$ to ω and Nakayama's lemma, with the torsion-freeness of M, shows that this lifting is an isomorphism.)

Let x be a non-zero divisor. M is clearly torsion-free and we only need to check that $M/xM \simeq \omega_{R/xR}(-1)$. Since the multiplication by x gives an isomorphism of $\oplus_{P\in X} K[x,x^{-1}]P$, $\mathrm{Tor}^R_i(\oplus_{P\in X} K[x,x^{-1}]P, R/xR) = 0$ for all $i \geq 0$. Thus from the long exact sequence of Tor modules associated with the exact sequence (*), we obtain $\mathrm{Tor}^R_1(\mathrm{Hom}_K(R,K), R/xR) \simeq M/xM$. But $\mathrm{Tor}^R_1(\mathrm{Hom}_K(R,K), R/xR) \simeq \mathrm{Hom}_R(R/xR, \mathrm{Hom}_K(R,K)(-1)) \simeq \mathrm{Hom}_R(R/xR, E_R(K))(-1) \simeq E_{R/xR}(K)(-1) \simeq \omega_{R/xR}(-1)$ and we are done. (Here $E_{R/xR}$ denotes the injective envelope of R/xR). □

We say that a homogeneous ring S/I is n-regular if $\mathrm{Tor}^S_i(S/I,K)_{i+j} = 0$ for all i and all $j \geq n$.

LEMMA 2. *Let $X \subset \mathbb{P}^r$ be a set of points with $|X| \leq \dim_K S_d$. If X imposes independent conditions on forms of degree d, then*

(i) *$S/I(X)$ is $(d+1)$-regular, and*

(ii) *$\mathrm{Tor}^S_i(S/I(X),K)_{i+d} \neq 0$ if and only if $\omega_{S/I(X)}$ has a $(r-i)$-th linear syzygy.*

PROOF: (i) is well known (and easy to prove: just note that modulo a general linear form $I(X)$ contains the $(d+1)$-rst power of the maximal ideal.) For (ii) let $E_\bullet$ be a minimal graded free resolution of $S/I(X)$ over S. Since $S/I(X)$ is Cohen-Macaulay, $\mathrm{Hom}(E_\bullet, S(-r-1))$ is a minimal graded resolution of $\omega_{S/I(X)}$. Since $S/I(X)$ is $(d+1)$-regular, the conclusion follows. □

Suppose now that X fails to satisfy (N_p), so that $\mathrm{Tor}_p^S(S/I, K)_{p+2} \neq 0$ and, by Lemma 2, ω has a $(r-p)$-th linear syzygy. Let $\mathcal{R}$ denote the kernel of the map $S_1 \otimes \omega_{-1} \to \omega_0$ and $\mathcal{R}_1$ its rank 1 locus. By Linear Syzygy Conjecture,

$$(**) \qquad \dim \mathcal{R}_1 \geq r - p.$$

We can describe $\mathcal{R}_1$ explicitly from the exact sequence of Lemma 1. For a subset Y of X, let

$$B(Y) = \left\{ \sum c_P\, x^{-1} P \in \omega_{-1} \mid c_P = 0, \text{ for all } P \text{ not in } Y \right\},$$
$$L(Y) = \{ y \in S_1 \mid y(P) = 0, \text{ for all } P \in Y \}, \text{ and}$$
$$s(Y) = \text{(projective) dimension of the linear space in } \mathbb{P}^r \text{ spanned by } Y.$$

Let $y \otimes a \in \mathcal{R}_1$, where $y \in S_1$ and $a = \sum c_P\, x^{-1} P \in \omega_{-1}$. Then $ya = 0$ in ω and from the exact sequence in Lemma 1, $x(P) = 0$ whenever $c_P \neq 0$. Let $Y = \{P \in X \mid c_P \neq 0\}$. Then $y \otimes a \in L(Y) \otimes B(Y)$ and hence

$$\mathcal{R}_1 = \cup_{Y \subset X} \{L(Y) \otimes B(Y)\}_1,$$

where $\{L(Y) \otimes B(Y)\}_1$ denotes the rank 1 locus of $L(Y) \otimes B(Y)$. Since $\dim_K L(Y) = r - s(Y)$ and $\dim_K B(Y) = |Y| - s(Y) - 1$,

$$\dim\{L(Y) \otimes B(Y)\}_1 = r + |Y| - 2s(Y) - 2.$$

Hence, for some $Y \subset X$, $\dim \mathcal{R}_1 = \dim\{L(Y) \otimes B(Y)\}_1 = r + |Y| - 2s(Y) - 2$. Thus $|Y| \geq 2s(Y) + 2 - p$ by (**) and this is what we wanted to prove for Conjecture 1(a).

2. $2r$ Points in $\mathbb{P}^r$.

In this section we show that Conjecture 1 holds if $p = 1$ and $r \leq 4$. Because an ideal of $2r+1$ points in $\mathbb{P}^r$ with $r \leq 3$ is never generated by quadrics, part (b) of Conjecture 1 follows from part (a) in the case $r \leq 4$. Let $X \subset \mathbb{P}^r_K = \mathbb{P}^r$ be a set of $2r$ points such that, for all $k \geq 1$, no $2k+1$ points of X lie in a projective k-plane. For part (a), we must show that if $r \leq 4$, X satisfies $(N_{2,1})$, i.e. $I = I(X)$ is generated by quadrics.

Suppose that I is not generated by quadrics and let J denote the ideal generated by the quadrics in I. We will show that $V(J)$ has a positive dimension and this will contradict Theorem 1 with $d = 1$. Since $V(J)$ has a positive dimension if and only if a general hyper plane meets $V(J)$, it will be enough to show that the height of the ideal $J + x_0 S$ doesn't exceed r for all general linear form x_0 of S.

We will use the following notation:

$$\begin{aligned} R &= S/I \\ T &= S/x_0 S \\ \bar{R} &= R/x_0 R \\ \omega = \omega_R &= \mathrm{Ext}^r_S(R, S(-r-1)) \\ \bar{\omega} = \omega_{\bar{R}} &= \mathrm{Ext}^{r+1}_S(\bar{R}, S(-r-1)) = \mathrm{Ext}^r_T(\bar{R}, T(-r)). \end{aligned}$$

Since X imposes independent conditions on quadrics by Theorem 2, $\dim_K I_2 = (1/2)r(r-1)$ and $\bar{R} = \bar{R}_0 \oplus \bar{R}_1 \oplus \bar{R}_2$ with $\dim_K \bar{R}_2 = r-1$. By duality, $\bar{\omega} = \mathrm{Hom}_K(\bar{R}, K) = \bar{\omega}_{-2} \oplus \bar{\omega}_{-1} \oplus \bar{\omega}_0$. Since I is not generated by quadrics, $\mathrm{Tor}^S_1(R, K)_3 \neq 0$ and ω has a $(r-1)$-st linear syzygy by Lemma 2 of Section 1. Since $\bar{\omega} = (\omega/x\omega)(1)$, $\bar{\omega}$ also has a $(r-1)$-th linear syzygy over T. Let $\{x_1, \ldots, x_r\}$ be elements of S which form a basis in T. Using the Koszul resolution of K, we obtain

$$\mathrm{Tor}^T_{r-1}(\bar{\omega}, K)_{r-3} \simeq \mathrm{Ker}(\wedge^{r-1} T_1 \otimes \bar{\omega}_{-2} \xrightarrow{\partial} \wedge^{r-2} T_1 \otimes \bar{\omega}_{-1})$$

and a nonzero element of $\mathrm{Tor}^T_{r-1}(\bar{\omega}, K)_{r-3}$ can be expressed as

$$a = \sum e_i \otimes a_i, \text{ where } e_i = x_1 \wedge \cdots \wedge x_{i-1} \wedge x_{i+1} \wedge \cdots \wedge x_r.$$

Since $\partial(a) = \sum \pm e_{ij}(x_i\, a_j - x_j\, a_i)$, where $e_{ij} = x_1 \wedge \cdots \wedge x_{i-1} \wedge x_{i+1} \wedge \cdots \wedge x_{j-1} \wedge x_{j+1} \wedge \cdots \wedge x_r$, the 2×2 minors of the matrix

$$A = \begin{pmatrix} x_1 & \ldots & x_r \\ a_1 & \ldots & a_r \end{pmatrix}$$

are zero in $\bar{\omega}$. Since $\dim_K \bar{\omega}_{-2} = r-1$, we may change variables and assume $a_1 = 0$.

PROPOSITION. *Let W denote the subspace of $\bar{\omega}_{-2}$ spanned by $\{a_2, \ldots, a_{r-1}\}$*

(a) $\dim_K W < r-1$.

(b) *If $\dim_K W = 1$, then the ideal $(x_0, x_1, \ldots, x_{r-1})$ contains J.*

(c) *If $\dim_K W = r-2$, then the ideal (x_0, x_1, x_2) contains $(2r-2)$-dimensional subspace of I_2, where I_2 is the vector space of quadrics in I.*

PROOF: (a) Because $a_1 = 0$ we get $x_1\, a_i = 0$ for all $2 \le i \le r$, or equivalently, $a_i(x_1\, \bar{R}_1) = 0$. Suppose that $\dim_K W = r-1$. Since $a_2, \ldots, a_r$ span the dual of $\bar{R}_2$, $x_1\, \bar{R}_1 = 0$ and we can choose L_i in S_1 such that $x_1\, x_i - x_0\, L_i \in I$, for $1 \le i \le r$. Let

$$B = \begin{pmatrix} x_0\, x_1 & \ldots & x_r \\ x_1\, L_1 & \ldots & L_r \end{pmatrix}$$

Since any $2 \times (r+1)$ matrix of linear forms in $r+1$ variables can be transformed by row and column operations to make at least one entry 0 (see for example [**Ei**]), we can put B into the form

$$\begin{pmatrix} x_0 & \ldots & x_s & x_{s+1} & \ldots & x_r \\ L'_0 & \ldots & L'_s & 0 & \ldots & 0 \end{pmatrix},$$

for suitable s with $0 \le s < r$. Let $M = V(L'_0, \ldots, L'_s)$ and $N = V(x_{s+1}, \ldots, x_r)$. Since the 2×2 minors of B are contained in I, we get $X \subset M \cup N$. Since $\dim M + \dim N = r-1$, one of M and N, say M, must contain at least $2(\dim_K M) + 1$ points of X which contradicts our assumption.

Let $\mu : R_1 \otimes R_1 \to R_2$ be the multiplication map. To prove (b), we may assume that $a_1 = \cdots = a_{r-1} = 0$. Then $(x_1, \ldots, x_{r-1})a_r = 0$ in $\bar{\omega}$ and $\dim_K \mu((x_1, \ldots, x_{r-1}) \otimes R_1) \le r-2$. Hence $(x_0, x_1, \ldots, x_{r-1})$ contains $\{((1/2)r(r+1)-1)-(r-2)\}$-dimensional subspace of quadrics of I_2. But $((1/2)r(r+1)-1)-(r-2) = \dim_K I_2$ and we are done. To prove (c), we may assume that $a_1 = a_2 = 0$. Then $(x_1, x_2)(a_3, \ldots, a_r) = 0$ in $\bar{\omega}$ and $\dim_K \mu((x_1, x_2) \otimes R_1) \le 1$. Hence (x_0, x_1, x_2) contains $((2r-1)-1) = (2r-2)$-dimensional subspace of quadrics of I_2. □

We recall that we are trying to prove that the height of $(J + x_0 S) \le r$, $(r \le 4)$. (a) and (b) of the Proposition above prove the case when $r = 3$ and the case when $r = 4$ and $\dim_K W = 1$. It remains to check the case when $r = 4$ and $\dim_K W = 2$. Since (x_0, x_1, x_2) contains 6-dimensional subspace of quadrics of I_2, by (c) of the Proposition, and $\dim_K I_2 = 7$, (x_0, x_1, x_2)+(remaining quadric) is an ideal of height ≤ 4 which contains $J + x_0 S$.

3. Points Cut Out by Multilinear Forms of Degree d.

Theorem 1 and Theorem 2 follow easily from a result of Jack Edmonds [**Ed**]:

THEOREM E (Edmonds). *Let B be a set of points in projective space and let $d \geq 2$ be an integer. B may be divided into d disjoint sets of linearly independent points if and only if, for all $k \geq 0$, no $dk + d + 1$ of the points of B lie in a projective k-plane.*

In Theorem E we do not assume that the points of B are distinct–indeed they will not be in our applications. However, if either condition of the Theorem is satisfied, it is evident that no more than d of the points can be coincident at any one point.

The proof of Theorem E gives a little more information than we have stated: if we are given d distinct independent sets $A_i \subset A$ such that A_i has n_i elements, then there exists a decomposition of A into d disjoint subsets of independent vectors such that the i-th set has at least n_i elements. It would be nice to know under what circumstances one could guarantee a decomposition into independent sets corresponding to a given numerical decomposition of the number of points into d parts.

PROOF OF THEOREM E: For any subset C of B we write span C for the set of elements of B which are linearly dependent on the elements of C and rank C for the affine dimension of the linear space spanned by the points in C. It is easy to check that the first condition given in Theorem E is equivalent to the statement that for every subset C of B we have $(1/d)|C| \leq \operatorname{rank} C$.

Suppose we are given d (possibly empty) disjoint subsets B_i of B, each consisting of independent vectors. Let $B' = B - B_i$. If for some i the span of B_i does not contain some element x of B', then we can add x to B_i, preserving independence and completing the proof. Thus we may assume that for every i span of B_i contains B'. We will give a procedure for exchanging elements of various B_i for elements in B' in such a way as to change this situation. This will prove the Theorem.

Let $S_1 = \operatorname{span} B_1$. By our hypothesis we have $S_1 \neq \cup_{1 \le i \le d} (S_1 \cap B_i)$. We will inductively define a strictly decreasing sequence of sets S_i, and a sequence of indices $m(i)$; in general, having defined S_{i-1}, we will define $m(i)$ and S_i iff S_{i-1} meets B', so that

$$S_{i-1} \neq \bigcup_1^d (S_{i-1} \cap B_i).$$

If this inequality is satisfied then for some index j we have $|S_{i-1} \cap B_j| < (1/d)|S_{i-1}|$, and choosing such a j we set $m(i) = j$ and $S_i = \operatorname{span}(S_{i-1} \cap B_{m(i)})$.

It is obvious from the definition that the sequence of S_i is weakly decreasing, but in fact the hypothesis of Theorem E gives the last of the string of inequalities rank $S_i = |S_{i-1} \cap B_{m(i)}| < (1/d)|S_{i-1}| \le \operatorname{rank} S_{i-1}$, so in fact the sequence is strictly decreasing.

Let h be the smallest number such that S_h does not contain B', and let $x \in B'$ be an element outside S_h. By hypothesis, $\{x\} \cup B_{m(h)}$ is a dependent set, and we let C be a minimal dependent subset, necessarily containing x. Let $k \le h$ be the smallest index such that S_k does not contain C, and choose an element $y \in C$, $y \notin S_h$.

We will replace x by y in $B_{m(h)}$,obtaining a new collection of disjoint subsets

$$B'_i = \begin{cases} B_i & \text{if } i \neq m(h); \\ B_{m(h)} \cup \{y\} - \{x\} & \text{if } i = m(h). \end{cases}$$

We claim that if we now proceed as before, constructing a sequence of sets $S'_i = \operatorname{span}(S'_{i-1} \cap B'_{m(i)})$, then for $i \le k$ we will have $S_i = S'_i$ so that in particular the defining inequality $|S'_{i-1} \cap B'_{m(i)}| < (1/d)|S'_{i-1}|$ will hold in this range. We prove this inductively: the case $i = 1$ being a consequence of the fact that the spans of $B_{m(h)}$ and $B'_{m(h)}$ both contain x and y, and thus coincide. We may thus assume that $i > 1$ and that $S'_{i-1} = S_{i-1}$. If $m(i) \neq m(h)$ then the desired inequality is immediate. If on the other hand $m(i) = m(h)$, then since $S_{i-1} \supset C$, we see that $S_{i-1} \cap B_{m(h)}$ and $S_{i-1} \cap B'_{m(h)}$ differ only in that the first does not contain x while the second does not contain y; since both contain the rest of C, they have equal spans $S_i = S'_i$ as required.

Finally, we claim that $k < h$; by induction, this will complete the proof. If on the contrary $k = h$, then by the equalities just established, $C \subset S_{h-i}$. But then $C - \{x\} \subset B_{m(h)}$, and $x \in \operatorname{span} C - \{x\}$, so $x \in S_h$, contradicting the definition of h. $\square$

PROOF OF THEOREM 1 FROM THEOREM E: We must show that the points of X are separated by multilinear forms of degree d from points not in X and from infinitely near points of X. To do this, it suffices, by adding some points in general position if necessary, to prove the Theorem in the case that X spans a projective r-space and contains exactly $d(r+1)+1$ points.

Note that if B is any set obtained from X by adding d points, then the hypothesis of Theorem 1 implies that the set B will satisfy the conditions of Theorem E, except in the case where all d points coincide with one of the points of X (the exception is essentially caused by the fact that in Theorem E we allow all $k \geq 0$, whereas in Theorem 1 k is constrained to be ≥ 1).

First, to prove that X is set-theoretically cut out by multilinear forms of degree d, let P be a point not in X, and let B be X with the point P adjoined d times. By Theorem E, B can be divided into d independent sets, and of course each of these will have $r+1$ elements. Clearly, each must contain one copy of P. Dropping these d copies of P, each of the resulting sets will span a hyperplane of $\mathbb{P}^r$, and these hyperplanes cannot contain P. Thus their union is a multilinear forms of degree d containing X but not containing P.

Finally, to show that X can be separated from an infinitely near point at $P \in X$, we let Q be a point distinct from P, but lying on the line through P and the infinitely near point. Let B be the result of adjoining Q and $d-1$ copies of P to X. Again by Theorem E, B is the union of d independent sets. Since again B contains a total of d copies of P, each of these sets must contain P, and in addition one — say B_1 — contains Q. Dropping Q from B_1 and dropping P from each of the other sets, we obtain d hyperplanes; exactly one of these hyperplanes, corresponding to B_1, contains P, and that hyperplane does not contain Q, so the union of the hyperplanes contains X but not the given infinitely near point at P. □

PROOF OF THEOREM 2 FROM THEOREM E: Adding some generally situated points if necessary, it suffices to prove Theorem 2 in the case where the number of points in X is $dr+1$, the maximum possible. For $P \in X$, we wish to construct a multilinear forms of degree d containing all the points of X except P. To this end add $d-1$ copies of P to X, obtaining a set B to which Theorem E may be applied. If we divide the $d(r+1)$ points of B into d independent sets, then each will contain a copy of P. The remaining points in each set span a hyperplane of $\mathbb{P}^r$, and the union of these hyperplanes is the desired form of degree d. □

We give here a very simple proof, due to Joe Harris, of a weakening of Theorem 2 in the case $d = 2$.

PROPOSITION. *Let X be a set of points in projective space. If, for all $k \geq 1$, no $2k+2$ of the points of X lie in a projective k-plane, then the points of X impose independent conditions on quadrics.*

PROOF: By induction, every proper subset of X impose independent condition on quadrics, so if X did not, then every quadric containing all but at most one element of X would contain X. It thus suffices to find a quadric containing all but exactly one element of X.

Suppose that the span of X is r-dimensional, so in particular $|X| \leq 2r+1$, and let $Y \subset X$ be a set of $r+1$ independent elements. The residual set $X - Y$ contains at most r elements, and is thus contained in a hyperplane H_1. If H_2 is the hyperplane spanned by the elements of Y besides P, then $H_1 \cap H_2$ is the desired quadric. □

4. $\binom{r+d}{d} + (r-p)$ points in $\mathbb{P}^r$.

In this section we prove Theorem 3 using descending induction on p. The proof uses the ideas of the proof in [**GL2**].

Let $p = r$. Then $|X| = \dim_K S_d$ and S/I is $(d+1)$-regular. Since $I_d = 0$, $\operatorname{Tor}_i^S(S/I, K)_j = 0$, for all i and all $j < i+d$ and (ii) follows. To prove (i), let P be a point of X. We want to find a form G of degree $d+1$ such that $G = 0$ on $X - \{P\}$ and $G(P) \neq 0$. We can do this by first finding such a form of degree d and then multiplying it by a general linear form.

Now let $p < r$ and let $X' = X - \{\text{point}\}$. Then by our induction hypothesis, X' imposes independent conditions on $(d+1)$-forms and $I(X')$ is generated by forms of degree $d+1$. Hence $I(X)_{d+1} \neq I(X')_{d+1}$ and this proves (i).

To prove (ii) it will suffice to show, by Lemma 2 in Section 1, that $\operatorname{Tor}_p^S(S/I, K)_{p+d+1} = 0$. Let $Y = \{P_0, \ldots, P_r\}$ be a subset of X in linear general position and let $\{x_0, \ldots, x_r\}$ be a basis of S_1 such that $x_i(P_i) = \delta_{ij}$. Let Q be the ideal of Y and J the ideal of $X - Y$. Then $I = I(X) = Q \cap J$ and we have an exact sequence $0 \to (S/I) \to (S/J) \oplus (S/Q) \to (S/J+Q) \to 0$. Since X imposes independent conditions on $(d+1)$-forms, the map $(S/I)_{d'} \to (S/J)_{d'} \oplus (S/Q)_{d'}$ is an isomorphism for all $d' \geq d+1$.

We use the Koszul resolution of K to compute Tor. We have the following diagram with obvious maps (see Figure 1). Since $|X-Y| \le \dim_K S_d$, S/J is $(d+1)$-regular and $\operatorname{Tor}_p^S(S/J,K)_{p+d+1}=0$. Since $\operatorname{Tor}_p^S(S/J,K)_{p+d+1}$ is the homology of

$$\left(\wedge^{p+1} S_1 \otimes (S/J)_d \to \wedge^p S_1 \otimes (S/J)_{d+1} \to \wedge^{p-1} S_1 \otimes (S/J)_{d+2}\right),$$

$\pi(\operatorname{Im}\delta) \supset \operatorname{Ker}\partial_2$. Thus it suffice to prove that $\operatorname{Ker}\partial_1 \subset \operatorname{Im}\delta$. Since Q is generated by $\{x_i\, x_j \mid 0 \le i \ne j \le r\}$, $\operatorname{Ker}\partial_1$ is generated by

$$\left\{x_{i_1} \wedge \cdots \wedge x_{i_p} \otimes x_j^{d+1} \mid 0 \le i_1 < \cdots < i_p \le r, j \notin \{i_1, \ldots, i_p\}\right\}.$$

Fix $x_{i_1} \wedge \cdots \wedge x_{i_p} \otimes x_j^{d+1}$ and let $F = \sum c_i\, x_i \pmod{Q}$ be a form of degree d such that $F=0$ on $(X-Y) \cup \{P_{i_1}, \ldots, P_{i_p}\}$. Since X imposes independent conditions on forms of degree d and $\left|(X-Y) \cup \{P_{i_1}, \ldots, P_{i_p}\}\right| = \dim_K S_{d-1}$, $F(P_j) \ne 0$ (and hence $c_j \ne 0$) for all $j \notin \{i_1, \ldots, i_p\}$. Thus

$$\delta(x_{i_1} \wedge \cdots \wedge x_{i_p} \wedge x_j \otimes F) = e_{i_1} \wedge \cdots \wedge e_{i_p} \otimes x_j\, F = x_{i_p} \wedge \cdots \wedge x_{i_p} \otimes c_j\, x_j^{d+1}$$

and we are done. □

Remark: It follows from Theorem 3 that if a set of $(r + \dim_K S_d)$ points imposes independent conditions on forms of degree d, then it imposes independent conditions on forms of degree $d+1$ also.

$$
\begin{array}{ccccc}
\wedge^{p+1} S_1 \otimes (S/J)_d & \longrightarrow & \wedge^{p} S_1 \otimes (S/J)_{d+1} & \xrightarrow{\partial_2} & \wedge^{p-1} S_1 \otimes (S/J)_{d+2} \\
\uparrow & & \uparrow \pi & & \uparrow \pi \\
 & & \left\{\wedge^{p} S_1 \otimes (S/Q)_{d+1}\right\} & \xrightarrow{\partial_1} & \left\{\wedge^{p-1} S_1 \otimes (S/Q)_{d+2}\right\} \\
 & & \oplus\left\{\wedge^{p} S_1 \otimes (S/J)_{d+1}\right\} & & \oplus\left\{\wedge^{p-1} S_1 \otimes (S/J)_{d+2}\right\} \\
 & & \uparrow & & \uparrow \\
\wedge^{p+1} S_1 \otimes (S/J \cap Q)_d & \xrightarrow{\delta} & \wedge^{p} S_1 \otimes (S/J \cap Q)_{d+1} & \xrightarrow{\partial=\partial_1+\partial_2} & \wedge^{p-1} S_1 \otimes (S/J \cap Q)_{d+2}
\end{array}
$$

Figure 1

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