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Varieties cut out by quadrics: Scheme-theoretic versus homogeneous generation of ideals

by

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Contents

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Abstract

In this note we consider cases in which a curve in $\mathbb{P}^r$ which is scheme theoretically the intersection of quadrics necessarily has homogeneous ideal generated by quadrics. The first case in which this does not happen is for a general elliptic octic in $\mathbb{P}^5$; we give a proof of this using the surjectivity of the period map for K3 surfaces.

Introduction

Several important results in the theory of projective curves assert that a given class of curves has homogeneous ideal generated by quadrics. Such for example is the case of a canonically embedded curve (Noether's Theorem) or a curve embedded by complete linear series of high degree compared to the genus of the curve. Because direct geometric techniques are available, these results are generally easier to prove scheme theoretically -- in algebraic language, it is easier to prove the weaker statement that the ideal generated by the quadratic and linear forms vanishing on the curve agrees with the ideal of the curve up to an "irrelevant" component.

This reflection gives rise to the wish that there should be some principal saying that, under suitable hypotheses, a curve cut out scheme theoretically by quadrics has ideal generated by quadratic forms (one can imagine much more general statements, but perhaps it is well not to be too greedy.) This paper is an exploration of the extent to which such a principal may exist.

The positive results are roughly as follows: For curves on 2-dimensional rational normal scrolls, always the easiest to study, the principal is true in an extremely strong form, without further hypotheses, and even stays true if we replace quadrics by forms of higher degree (section 1). It remains true for all curves in $\mathbb{P}^r$ with $r \leq 4$ (section 2), but it cannot be extended to forms of higher degree, even in $\mathbb{P}^3$ (section 3). It is also true for projectively normal curves in $\mathbb{P}^r$ which lie on projectively normal K3 surfaces cut out by quadrics; this includes in particular all projectively normal curves in $\mathbb{P}^5$ (section 2). These last results are proved by combining liaison techniques with a sort of general position result, Lemma 2.7, which asserts that the canonical module of the homogeneous coordinate ring of an irreducible projectively Cohen-Macaulay curve is generated in degree 0.

On the other hand, the principal fails already for some non projectively normal curves in $\mathbb{P}^5$. The example of smallest degree is the general elliptic octic in $\mathbb{P}^5$, which is, as we show, cut out scheme theoretically by 5 quadrics, though its homogeneous ideal requires two additional cubic generators (section 5). The example is constructed, following the attack of Mori [1984], by exploiting the surjectivity of the period map for K3 surfaces to first construct the K3 surface in $\mathbb{P}^5$ which will be the intersection of 3 general quadrics containing C. After the fact, we discovered an explicit example as well, which however we can only verify by computer, using the program Macaulay of Bayer and Stillman [1986].

We see from the example of the elliptic octic that some additional hypotheses on C will be necessary in general. Perhaps the most salient possibility in this direction, supported by the results in $\mathbb{P}^5$ and on K3 surfaces, is that projective normality might suffice:

Problem: Let $C \subset \mathbb{P}^r$ be a projectively normal curve which is scheme theoretically cut out by quadrics. Is the homogeneous ideal of C necessarily generated by forms of degree ≤ 2 ?

One warning note should be sounded: We will show in section 4, following an idea of Harris, that a general set of d points in $\mathbb{P}^r$ is scheme-theoretically the intersection of quadrics, but has homogeneous ideal not generated by quadrics, if

$$\frac{2}{3}\left[\begin{matrix} r+2 \\ 2 \end{matrix}\right] < d \leq \left[\begin{matrix} r+1 \\ 2 \end{matrix}\right]$$

and these inequalities are satisfiable as soon as $r \geq 5$. As sets of points are always aritmetically Cohen-Macaulay, this example shows that the analogue of the problem has a negative solution for sets of points. Now if one of these bad sets of points were the hyperplane section of a projectively normal curve that was cut out scheme-theoretically by quadrics, then the solution to the problem above would be negative. At least in the extremal case

$$d = \binom{r+1}{2},$$

(which is the only one to occur in $\mathbb{P}^5$) we prove in Proposition 4.4 that no such curve can exist. (The related problem of whether the general set of d points in $\mathbb{P}^r$ is the hyperplane section of a curve, or for that matter of a projectively normal curve, seems open.)

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1) Curves on rational normal scrolls

For the background on rational normal scrolls necessary for this section, the reader may consult, for example, the book of Hartshorne [1977] (Ch.5 sect. 4) and the paper of Eisenbud and Harris [1987].

Theorem 1.1: *Let* $C \subset \mathbb{P}^r$ *be a curve. If* C *is scheme theoretically cut out by hypersurfaces of degree* e, *and if* C *is contained in some two-dimensional rational normal scroll in* $\mathbb{P}^r$, *then the homogeneous ideal of* C *is generated by forms of degree* $\le e$.

Remark: We allow the possibility that the scroll may be singular, and require of C only that it be a purely 1-dimensional subscheme.

Theorem 1.1 may be applied in several situations. For example, if $C \subset \mathbb{P}^r$ is any hyperelliptic curve (including the cases of genera 0 and 1) embedded by a complete linear series, then, as is well-known, the union of the lines joining points of C which correspond under the hyperelliptic involution is a rational normal scroll, and Theorem 1.1 applies. It also applies to all nondegenerate curves of degree $\le r+1$ in $\mathbb{P}^r$:

Proposition 1.2: *If* $C \subset \mathbb{P}^r$ *is a smooth connected curve of degree* $\le r+1$, *not contained in a hyperplane, then* C *is contained in a rational normal scroll of dimension 2.*

The proof of this more or less well-known fact, which we shall sketch below, goes via the following lemma, itself almost a special case:

Lemma 1.3: *If* $D \subset \mathbb{P}^{r+1}$ *is the rational normal curve of degree* $r+1$, *and* L *is a line meeting* D *in (at least) a point, then* $C = D \cup L$ *is contained in a 2-dimensional rational normal scroll.*

Remark: There is a unique such scroll with L as a ruling.

Proof of Lemma 1.3 (sketch): Let $p \in D \cap L$, and let q be any other point of L. We may write the equations of D as the 2×2 minors of a $2\times(r+1)$ matrix M of linear forms, and after row operations we may assume that all the forms in the first row of M vanish at p. After column operations we may further assume that the first r entries of the first row vanish at q. The ideal of 2×2 minors of the $2\times r$ matrix consisting of the first r columns of M is now the homogeneous ideal of the desired scroll.

Proof of Proposition 1.2 (sketch): By Clifford's Theorem the embedding of C must be nonspecial, and then by the Riemann-Roch Theorem the genus of C must be either 0 or 1. Further, in case the genus is 1 the embedding is complete, and the discussion immediately preceding the Proposition may be applied (or see the paper of Eisenbud, Koh, and Stillman [1986] for a fairly explicit view of the (all quadratic) generators of the homogeneous ideal of such a curve.) The same arguments (in an even more trivial version) handle the case where the embedding is complete and the genus is 0.

There remains the case of a smooth rational curve of degree r+1 in $\mathbb{P}^r$. Such a curve C is the projection of a rational normal curve D in $\mathbb{P}^{r+1}$ from a point p off the curve. Taking L to be any line joining p to a point of C, we see from Lemma 1.3 that $D \cup L$ lies on a scroll, whose projection will be a scroll containing C (in fact this argument shows that C lies on a 1-parameter family of rational normal scrolls.)

We now turn to the proof of Theorem 1.1. The case $e=1$ being trivial, we henceforward assume that $e \geq 2$.

We will actually prove a sharper result, for which we need more notation: Let S be the hypothesized scroll containing C. By definition, S is the image of a projectivised vector bundle $S' = \mathbb{P}(\mathcal{E})$, where $\mathcal{E}$ is a rank 2 vector bundle on $\mathbb{P}^1$, under the map $\varphi: \mathbb{P}(\mathcal{E}) \to \mathbb{P}^r$ induced by the complete linear series $|H|$ associated to the tautological divisor H on

$\mathbb{P}(\mathcal{E})$. Let C' be the divisor on S' which is the total transform of C. Let F be the fiber of the natural projection $\mathbb{P}(\mathcal{E}) \rightarrow \mathbb{P}^1$. The classes of H and F generate the Picard group of S', so we may write $C' \sim aH+bF$ for some integers a and b. Since C' is effective we have $a = C'.F \geq 0$.

Since S, the image of S' under $|H|$, is assumed 2-dimensional, we may write $\mathcal{E} \cong \mathcal{O}_{\mathbb{P}_1}(c) \oplus \mathcal{O}_{\mathbb{P}_1}(d)$ with $0 \leq c \leq d$. We write $C_0 \sim H-dF$ for the effective irreducible divisor which is the section of the natural projection $\mathbb{P}(\mathcal{E}) \rightarrow \mathbb{P}^1$ corresponding to the quotient $\mathcal{E} \rightarrow \mathcal{O}_{\mathbb{P}_1}(c)$.

With this notation established, we can state the sharper version:

<u>Theorem 1.1 bis:</u> <u>With notation as above, the following are equivalent:</u>

i) <u>The homogeneous ideal of</u> C <u>is generated by forms of degree</u> $\leq$ e.

ii) C <u>is contained in some hypersurface of degree</u> e <u>not containing</u> S, <u>and in a neighborhood of some point of</u> $\varphi(C_0)$, C <u>is cut out scheme theoretically by hypersurfaces of degree</u> e.

iii) $e \geq a$ <u>and</u> $(e-a)c \geq b$.

<u>Remark:</u> If there is a hypersurface of degree e containing C but not containing $\varphi(C_0)$, then condition ii) is satisfied.

<u>Proof of Theorem 1.1 bis:</u> Condition i) trivially implies condition ii).

Suppose that condition ii) is satisfied. It follows that the linear series $|eH-C'|$ does not have C_0 as a base component. Intersecting with F and C_0 we see that $e-a \geq 0$ and $b \leq (e-a)c$ as required for iii).

Finally, suppose that iii) is satisfied. To prove i) it is enough, since the homogeneous ideal of S is generated by quadrics and $e \geq 2$, to show that the multiplication map $H^0\mathcal{J}_{C/S}(e) \otimes H^0\mathcal{O}_S(k) \rightarrow H^0\mathcal{J}_{C/S}(e+k)$ is an epimorphism for every $k \geq 0$. Since condition iii) implies the corresponding condition for larger values of e, we may restrict ourselves to the case $k=1$. Writing R for the "residual" divisor $eH-C'$, and using the fact that $\mathcal{I}_{C/S}(e) = \varphi_* \mathcal{O}_{S'}(eH-C')$, we must show that the multiplication map

$$H^0\mathcal{O}_{S'}(R)\otimes H^0\mathcal{O}_{S'}(H) \to H^0\mathcal{O}_{S'}(R+H) \tag{1}$$

is onto.

Now it is easy to compute groups of global sections of line bundles on S'; one has, in general, a natural identification

$$H^0(S', \mathcal{O}_{S'}(mH+nF)) = H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}_1}(n) \otimes \mathrm{Sym}_m(\mathcal{O}_{\mathbb{P}_1}(c)\oplus \mathcal{O}_{\mathbb{P}_1}(d))).$$

In terms of this (1) becomes the natural multiplication map

$$\Big[\bigoplus_{\substack{i+j=e-a\\ i,j\geq 0}} H^0 \mathcal{O}_{\mathbb{P}_1}(ic+jd-b) \Big] \otimes \big[H^0 \mathcal{O}_{\mathbb{P}_1}(c) \oplus H^0\mathcal{O}_{\mathbb{P}_1}(d)\big]$$

$$\to \bigoplus_{\substack{i+j=e-a+1\\ i,j\geq 0}} H^0 \mathcal{O}_{\mathbb{P}_1}(ic+jd-b) \quad ,$$

and one easily checks that this is onto because of the inequalities in iii).

2) <u>Curves in</u> $\mathbb{P}^4$ <u>and</u> $\mathbb{P}^5$

It is easy to check that every curve in $\mathbb{P}^2$ or $\mathbb{P}^3$ which is scheme theoretically the intersection of quadrics has homogeneous ideal generated by quadrics. We prove here the corresponding result for all smooth curves in $\mathbb{P}^4$, for all projectively normal curves in $\mathbb{P}^5$ and for many projectively normal curves in higher dimensional spaces; it becomes false for arbitrary smooth curves in $\mathbb{P}^5$, as shown by the example in section 5 below. First, in $\mathbb{P}^4$:

<u>Theorem 2.1:</u> *If* $C \subset \mathbb{P}^4$ *be a smooth irreducible curve which is scheme theoretically cut out by quadrics, then the homogeneous ideal of* C *is generated by quadrics*.

Our sharpest positive result in higher dimensional spaces is:

<u>Theorem 2.2:</u> *Suppose that* $C \subset \mathbb{P}^r$ *is a projectively normal curve which lies on a projectively normal K3 surface* S *cut out by quadrics. If there is a reduced irreducible curve* C' *on* S *such that* $C+C'$ *is linearly equivalent to twice the hyperplane section on* S, *and* C *is contained in some quadric not containing* C' *(or equivalently the arithmetic genus of* C' *is* >0) *then the homogeneous ideal of C is generated by quadrics.*

From this we get:

<u>Corollary 2.3:</u> *If* $C \subset \mathbb{P}^5$ *is a projectively normal curve which is the scheme-theoretic intersection of quadrics, then the homogeneous ideal of* C *is generated by quadrics.*

To prove these results we will use the residual curve C' to C in the complete intersection of $r-1$ general quadrics containing C. The following result shows us (in a more general setting) what we can expect of C'. Its first statement will become relevant in section 5. (Of course the same statement would hold for complete intersections of hypersurfaces of higher degree.) The complete intersection of 3 quadrics in $\mathbb{P}^5$ is a K3 surface, so it also provides the link between Theorem 2.2 and Corollary 2.3.

Proposition 2.4: If $C \subset \mathbb{P}^r$ *is a smooth curve which is the scheme theoretic intersection of hypersurfaces of degree* 2, *then a general set of* $r-2$ *quadrics containing C meet in a smooth surface, and a general set of* $r-1$ *such quadrics meet in a curve of the form* $C \cup C'$, *where* C' *is a smooth (possibly empty or disconnected) curve meeting* C *in ordinary double points.*

Further, if C' *is not empty then writing* d, d' *and* g, g' *for the degrees and genera of* C *and* C' *we have*

$$d' = 2^{r-1} - d$$
$$g' = g + (r-3)(2^{r-2}-d),$$

and C *meets* C' *in* $(r-3)d - 2g + 2$ *points.*

If C *is projectively normal, then so is* C', *and in particular* C' *is then irreducible.*

(Here we have taken the genus of C', in case C' is disconnected, to be the sum of the genera of the components minus the number of components plus 1, as usual.)

The curve C' of Proposition 2.4 is *linked* to C by D in the sense of the theory of liaison (see for example Peskine-Szpiro [1973].)

As an immediate consequence, we see that it is extremely rare for a curve in $\mathbb{P}^r$ to be the scheme-theoretic intersection of exactly r quadrics:

Corollary 2.5: If $C \subset \mathbb{P}^r$ *is a curve which is scheme-theoretically the intersection of* r *linearly independent quadrics, then the degree* d *and genus* g *of* C *satisfy*

$$g = (r-1)d/2 + 1 - 2^{r-1} .$$

In fact, the conditions on C of interest to us are easy to express in terms of the geometry of C', even in a more general situation:

Proposition 2.6: *Let* C *and* C' *be (non-empty) locally Cohen-Macaulay curves which are linked by a complete intersection of quadrics (respectively by a complete intersection of a quadric and a projectively normal K3 surface cut out by quadrics). We have:*

i) C *is nondegenerate iff* $\omega_{C'}(4-r)$ *(resp.* $\omega_{C'}(-1)$ *) has no global sections.*

ii) C *is cut out scheme theoretically by quadrics iff* $\omega_{C'}(5-r)$ (resp. $\omega_{C'}$) *is generated by global sections. The ideal of C contains exactly* $h^0\omega_{C'}(5-r) + r - 1$ *(resp.* $h^0\omega_{C'} + r - 1$ *) independent quadrics.*

iii) *The homogeneous ideal of* C *is generated by quadrics iff the multiplication maps* $H^0\omega_{C'}(5-r) \otimes H^0\mathcal{O}_{C'}(k) \to H^0\omega_{C'}(5-r+k)$ (resp. $H^0\omega_{C'} \otimes H^0\mathcal{O}_{C'}(k) \to H^0\omega_{C'}(k)$) *are surjective for all* $k>0$.

Theorem 2.2 follows easily from Proposition 2.6 and the following result, essentially a consequence of Green's "$K_{p,1}$ Theorem" (Theorem 4.b.2 of Green [1984], modified to work for singular curves by the technique of Eisenbud, Koh, and Stillman [1986], for example):

Theorem 2.7: *If* C *is a reduced and irreducible locally Gorenstein curve of arithmetic genus* >0, *and L is a line bundle generated by its global sections with* $h^0(L) \geq 3$, *then the module*

$$\sum H^0(\omega \otimes L^n)$$

is generated, over the ring

$$\sum H^0 L^n$$

by elements of degree ≤ 0.

Proof of Proposition 2.4: Bertini's Theorem shows that neither S nor D have singularities away from C. Let $\mathcal{N}_C{}^* = \mathcal{I}_C/\mathcal{I}_C{}^2$ be the conormal bundle of C, and let $V = H^0\mathcal{I}_C(2)$. Our hypothesis implies that V generates $\mathcal{N}_C{}^*(2)$ on C. Since $\mathcal{N}_C{}^*$ is a vector bundle of rank $r-1$, it follows that $r-2$ general sections will be everywhere independent on C, and thus the intersection S of the corresponding hypersurfaces will have no singularities along C, proving the first statement.

To prove the second statement, fix S as above. It suffices to show that a general element of V induces a section of the line bundle $\mathcal{N}_{C/S}{}^*$ having only simple zeros. But by our hypothesis V generates $\mathcal{N}_{C/S}{}^*$, so again by a version of Bertini's Theorem a general section vanishes on a reduced set of points, as desired.

Finally, the formulas for the numerical characters of C' and $C \cap C'$ follow at once if we put together the facts that, because D is a complete intersection of $r-1$ quadrics, its canonical bundle is the restriction of $\mathcal{O}_{\mathbb{P}_r}(r-3)$, and the restriction of its canonical bundle to C or C' is the canonical bundle of the smaller curve twisted by the divisor on that curve which is the sum of the points of $C \cap C'$. (These formulas are also special cases of exc. 9.1.12 of Fulton [1984].)

Proof of Corollary 2.5: The first $r-1$ of the hypersurfaces cutting out C meet in $C \cup C'$, so the r^{th} must meet C' in exactly the $(r-3)d - 2g + 2$ points of intersection. Thus this number must be twice the degree of C'. The formulas of Proposition 2.2 and simple arithmetic now yield the desired result.

Proof of Proposition 2.6: First, let $D = C \cup C'$ be the complete intersection of quadrics. Write $\mathcal{I}_C$ for the ideal sheaf of C in $\mathbb{P}^r$, and similarly for C' and D. Since the canonical bundle on D is given by $\omega_D = \mathcal{O}_D(r-3)$, we have by the theory of liaison that

$$\begin{aligned}\mathcal{I}_C/\mathcal{I}_D &= (\mathcal{I}_D:\mathcal{I}_{C'})/\mathcal{I}_D \\ &= \mathrm{Hom}(\mathcal{O}_{C'}, \mathcal{O}_D) \\ &= \mathrm{Hom}(\mathcal{O}_{C'}, \omega_D)(3-r) \\ &= \omega_{C'}(3-r).\end{aligned}$$

Further, since D is projectively normal, we have for each integer k an exact sequence

$$0 \to H^0\, \mathcal{I}_D(k) \to H^0\, \mathcal{I}_C(k) \to H^0\, \omega_{C'}(k+3-r) \to 0\,.$$

The statements of the proposition now follow at once.

If D is instead the complete intersection of a projectively normal K3 surface S cut out by quadrics and a quadric hypersurface then, since D is again projectively Gorenstein, exactly the same argument applies, using now $\omega_D = (\omega_S \otimes \mathcal{O}_S(D))|_D = \mathcal{O}_D(2)$.

Proof of Theorem 2.1: We adopt the notation of Proposition 2.1, and let $i = (r-3)d - 2g + 2$, the number of points of intersection of C and C'.

We leave the cases $r \le 3$ to the reader. Consider first the case where $r=4$. A theorem of Castelnuovo [1893] (or see Mattuck [1964]) asserts that if C is any curve of degree d and genus g embedded by a complete series, then the homogeneous ideal of C is generated by quadrics if $d \ge 2g+2$, so we may ignore these cases. Further, if $i > 2d'$, then any quadric containing C contains a component of C' as well, contradicting our assumptions. These remarks, together with the ideas of section 1 suffice. We run quickly through the possible degrees d, assuming that C is not contained in a hyperplane, so that $d \ge 4$:

- $d=8$: C is a complete intersection.
- $d=7$: We have $d'=1$, so C' is a line and $g'=0$, whence $i=3>2d'$, so all the quadrics containing C contain C' as well.
- $d=6$: Here $d'=2$, so $g'=0$ or -1. If $g'=-1$ then we see again $i>2d'$, and we are done as before. If $g'=0$, then $g=2$,

$d \geq 2g+2$, and Castelnuovo's theorem applies to show that the homogeneous ideal of C is generated by quadrics.

$d \leq 5$: C is contained in a rational normal scroll by Proposition 1.2, so we are done by Theorem 1.1.

Proof of Theorem 2.2: Immediate from Proposition 2.6 and Theorem 2.7.

3) Determinantal Constructions

While it seems to be difficult to construct varieties scheme-theoretically but not arithmetcally cut out by quadrics, there is no difficulty in making examples if one admits equations of higher degree. Perhaps the simplest example is that of 18 general points in $\mathbb{P}^2$; the points are cut out scheme-theoretically by 3 quintics, but their homogeneous ideal requires in addition a sextic generator (this turns out to be the example of lowest degree in $\mathbb{P}^2$).

A general technique produces this and many other examples: Let A be a $p\times q$ matrix with $p\leq q$, filled with a $p\times p$ block A_2 of general quadratic forms and a $p\times(q-p)$ block A_1 of linear forms over a polynomial ring in r+1 variables $k[x_0,x_1,\dots,x_r]$.

$$A = \begin{pmatrix} \boxed{\begin{matrix} A_2 \\ \deg 2 \end{matrix}} & \boxed{\begin{matrix} A_1 \\ \deg 1 \end{matrix}} \end{pmatrix}$$

Proposition 3.1: If the entries of A_1 *generate the ideal* $(x_0,x_1,\dots,x_r)$, *then the ideal of all* $p\times p$ *minors of* A *defines the same scheme as the ideal of all* $p\times p$ *minors of* A *except the determinant of* A_2. *In particular, if* $p(q-p) \geq r+1 \geq q-p+2$ (*respectively* $\geq q-p+3$) *and* A *is chosen as generically as possible, then the* $p\times p$ *minors of* A *cut out a nonsingular (respectively nonsingular and irreducible) scheme of codimension* $q-p+1$ *which is scheme-theoretically but not arithmetically cut out by equations of degree* $< 2p$.

The case of 18 general points in the plane is obtained by taking $p=3$, $q=4$; if instead we take $p=4$, $q=5$, $r=3$, we get a smooth irreducible curve in $\mathbb{P}^3$, of degree 32 and genus 109, cut out scheme-theoretically by 4 forms of degree 7, whose homogeneous ideal requires an additional generator of degree 8.

Proof. We need only prove the first statement, as the second follows by considering the generic case and applying Bertini's Theorem. Considering the relations among the minors given by the rows of $p\times(p+1)$ submatrices containing A_2, we see however that $(x_0,x_1,\ldots,x_r)\cdot\det(A_2)$ is contained in the ideal generated by the $p\times p$ minors of A other than A_2.

4) General sets of points

The ideas of this section were suggested to us by Joe Harris.

Theorem 4.1: *If* Γ *is a general set of* d *points in* $\mathbb{P}^r$ *with*

$$\frac{2}{3}\binom{r+2}{2} < d \le \binom{r+1}{2}$$

then Γ *is scheme theoretically but not homogeneously cut out by quadrics. In particular, a variety consisting of* 15 *general points in* $\mathbb{P}^5$ *is scheme theoretically but not homogeneously the interesection of quadrics.*

This result follows easily from a general fact, of interest in its own right:

Theorem 4.2: *Let* $X \subset \mathbb{P}^r$ *be a reduced irreducible variety, not contained in a hyperplane, and let* $\Gamma \subset X$ *be a general set of* d *points. If* $d \le \operatorname{codim} X$, *then* Γ *is the scheme theoretic intersection of the linear space it spans with* X.

Corollary 4.3: *If* Γ *is a general set of* d *points in* $\mathbb{P}^r$, *then* Γ *is scheme theoretically cut out by forms of degree* e *as long as*

$$d \le \binom{r+e}{r} - 1 - r.$$

Proof of Corollary 4.3: Apply Theorem 4.2 to $\mathbb{P}^r$ embedded by the e^{th} Veronese mapping.

Theorem 4.1 suggests a way to look for interesting examples: if a set of points in $\mathbb{P}^r$ satisfying Theorem 4.1 could be "lifted" to a projectively normal curve in $\mathbb{P}^{r+1}$, scheme theoretically cut out by quadrics, then that curve would not have homogeneous ideal generated by quadrics. It is not known whether a general set of points in $\mathbb{P}^r$ lifts at all to a curve in $\mathbb{P}^{r+1}$. We will prove a weak non-lifting result, which at least rules out the possibility of finding an example of a curve in $\mathbb{P}^6$ in this way.

Proposition 4.4: Let Γ be a general set of

$$d := \binom{r+1}{2}$$

points in $\mathbb{P}^r$, with $r \geq 5$. There is no linearly normal curve in $\mathbb{P}^{r+1}$, cut out scheme theoretically by quadrics, whose hyperplane section is Γ.

We now turn to the proofs:

Proof of Theorem 4.1: Γ is scheme theoretically cut out by quadrics by virtue of Corollary 4.3. Thus we need only show that the homogeneous ideal of Γ is not generated by quadrics.

Since the points of Γ are general, the space of quadrics vanishing on Γ has dimension exactly

$$\binom{r+2}{2} - d \ ,$$

while the dimension of the space of cubics vanishing on Γ is

$$\binom{r+3}{3} - d \ ,$$

which, as elementary computation shows, is $>$ r+1 times the dimension of the space of quadrics vanishing on Γ. Thus the ideal generated by the quadrics in the ideal of Γ does not contain all the cubics in the ideal of Γ. The second statement of the Theorem follows trivially from the first.

Proof of Theorem 4.2: It is enough to show that projection of X from (the linear span of) any d-1 of the points is birational; for then it will be smooth at the last point, which is the desired conclusion. For this it is inductively enough to show that projection from any general point is birational, that is, that not every secant of X is a multisecant as long as the codimension of X is at least 2. By taking hyperplane sections, we may assume that X is a curve. The result then follows from the fact that the general hyperplane section of an irreducible curve is a set of points in linearly general position.

Proof of Proposition 4.4: One checks immediatly that a linearly normal curve C whose hyperplane section is Γ lies on exactly r+1 quadrics in $\mathbb{P}^{r+1}$. Corollary 2.5 gives a formula for the genus g of C in terms of the degree and r (note that the "r" given must be replaced by r+1 in our case). If $r \geq 5$, the formula of Corollary 2.5 yields a value incompatible with the inequality $g \geq d-r-1$ coming from the linear normality of C.

5) Elliptic octic curves in $\mathbb{P}^5$

In this section we work over the complex numbers. For the necessary background on linear series on K3 surfaces the reader may consult the paper of Saint-Donat [1974]. The paper of Beauville [1985] and the first 2 sections of the paper of Mérindol [1985] provide excellent background on Hodge theory and the period morphism for K3 surfaces, and their relation to the Picard group.

Theorem 5.1: The general elliptic octic in $\mathbb{P}^5$ is scheme theoretically the intersection of five quadric hypersurfaces, but its homogeneous ideal requires two generators of degree three.

Example: Having dealt with the general situation, it is pleasant, though not particularly enlightening, to be able to write down an explicit example:

Let E be the elliptic curve defined in $\mathbb{P}^2$ by the equation

$$x^3+xz^2-y^2z = 0 ,$$

and let $\varphi:E\rightarrow\mathbb{P}^5$ be the map defined by the linear series

$$x^3,\ x^2y\ ,\ xy^2,\ x^2z+y^2z\ ,\ y^3+xz^2,\ yz^2.$$

Using the computer program Macaulay of Bayer and Stillman [1986] we have shown that (in characteristic 31991 and several others) the homogeneous ideal of E in $\mathbb{P}^5$ is minimally generated by 5 quadrics and 2 cubics. The product of either of the cubic generators with any form of positive degree lies in the subideal generated by the quadrics alone,so E is scheme theoretically the intersection of the 5 quadrics. The actual equations involve so many terms that they are probably not interesting to anyone without a computer system like Macaulay to manipulate them, and with such a system they can be generated easily from the data just given, so we will not reproduce them here.

To understand our approach to Theorem 5.1, note that by

Proposition 2.2, such a curve as in the Theorem will have to lie on a smooth surface which is the complete intersection of 3 quadrics. Such a surface is a K3 surface, and we will begin by constructing a candidate for it:

Proposition 5.2: *There is a* K3 *surface whose divisor class group is of rank 2 with intersection form*

$$\begin{bmatrix} 8 & 8 \\ 8 & 0 \end{bmatrix} .$$

Let S be a K3 surface as in the Proposition, and let A, E be divisor classes on S with $A^2 = AE = 8$, $E^2 = 0$. By Riemann-Roch either A or -A and either E or -E are effective, and we may assume that A and E are. Evidently both are numerically effective and primitive in Pic S, so by Theorem 5 of Mori [1984], and the fact that every intersection number on S is divisible by 8, $|A|$ is very ample and $|E|$ is base point free. Again by Riemann-Roch and the results 2.2 and 7.2 of Saint-Donat [1974] the image of S under $|A|$ is a complete intersection of 3 quadrics in $\mathbb{P}^5$. By Proposition 2.6 of Saint-Donat [1974] the general member of $|E|$ is a smooth elliptic curve, which we may as well assume was E to start with. (Remark on references: The results used here were proved in Characteristic 0 by Mayer [1972]; the cited paper of Saint-Donat extends them to Characteristic p, while the paper of Mori summarizes some of them in a form that is convenient for us.) With this notation we will show:

Theorem 5.3: *Let* S, A, E *be a* K3 *surface and divisors as above. The complete linear series* $|A|$, *restricted to* E, *embeds* E *as an elliptic octic in* $\mathbb{P}^5$ *which is scheme theoretically the intersection of five quadric hypersurfaces, but whose homogeneous ideal requires two generators of degree three*.

Proposition 5.2 follows easily from the surjectivity of the period morphism for K3 surfaces, via Corollary 1.9 of Morrison [1984]. We sketch the required ideas, which, with Theorem 5.4, certainly belong to the folklore:

We write H for the integral lattice with quadratic form represented by the matrix

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

the "hyperbolic plane", and E_8 for the negative definite quadratic form with Dynkin diagram E_8, so that for a K3 surface S we have

$$H^2(S, \mathbb{Z}) = 3H \oplus 2E_8.$$

We write V for this integral lattice.

We will say that an integral lattice L with quadratic form is <u>a K3 lattice</u> if it can be realized as the Picard group of a K3 surface with the intersection form. Of course if L is a K3 lattice then, because of the index theorem, L must satisfy the <u>index condition</u> that $\mathbb{R}\otimes L$ does not contain a 2-dimensional positive definite subspace. Also, L must be <u>embeddable</u> in the sense that L can be embedded in V in such a way that the underlying abelian group is a direct summand (it will be the intersection of V with the 1,1 forms in $H^2(S, \mathbb{C})$. We will say that L is <u>nondegenerate</u> if the induced bilinear form on L corresponds to an injection of L into its dual lattice.

<u>Theorem 5.4:</u> L <u>is a K3 lattice if and only if</u> L <u>can be embedded in</u> V <u>in such a way that the underlying abelian group of L is a direct summand and</u> $\mathbb{R}\otimes L^{\perp}$ <u>contains a 2-dimensional positive definite form.</u>

<u>Corollary 5.5 (Morrison [1984] Cor.1.9.i)</u>: <u>If L is nondegenerate, then L is a K3 lattice if and only if L is embeddable and satisfies the index condition.</u>

We begin the proofs with the results on surfaces:

<u>Proof of Theorem 5.4:</u> The Hodge Theorem and the surjectivity of the period morphism for K3 surfaces imply that L is a K3 lattice iff it can

be written as a sublattice of V in such a way that there exists a vector $\omega \in \mathbb{C}\otimes V$ with

$$\omega^2 = 0\ ,$$
$$\omega\overline{\omega} > 0\ , \text{ and}$$
$$L = (\mathbb{C}\omega \oplus \mathbb{C}\overline{\omega})^{\perp} \cap V\ .$$

In particular, if L is a K3 lattice, then L is embeddable. Further,since the conditions $\omega^2 = 0$ and $\omega\overline{\omega} > 0$ are equivalent to the conditions $(\mathrm{Re}\ \omega).(\mathrm{Im}\ \omega) = 0$ and $(\mathrm{Re}\ \omega)^2 = (\mathrm{Im}\ \omega)^2 > 0$, we see that $\mathbb{R}\otimes L^{\perp}$ contains the positive definite space spanned by Re ω and Im ω.

Conversely, suppose that L is embeddable in V in our sense, and so that $\mathbb{R}\otimes L^{\perp}$ contains a positive definite space, spanned by vectors α and β, say. Multiplying by a real factor, we may assume $\alpha^2=\beta^2$. Let $\omega' = \alpha+i\beta \in \mathbb{C}\otimes V$, so that

$$(\omega')^2=0\ ,$$
$$\omega'\,\overline{\omega}' > 0, \text{ and}$$
$$L \subset (\mathbb{C}\omega' \oplus \mathbb{C}\overline{\omega}')^{\perp} \cap V\ .$$

We will finish the proof by perturbing ω' in such a way as to preserve the first two relations and achieve equality in the third.

The second of the three relations is preserved under all small perturbations of ω', so we may ignore it. The first and third, thought of as conditions on ω', define a complex quadric hypersurface Q in $\mathbb{C}\otimes L^{\perp}$. Suppose $x \in V - L$. Because L is a direct summand of V as an abelian group, we have $L = V\cap((\mathbb{C}\otimes L^{\perp})^{\perp})$, so the hyperplane $(\mathbb{C}\otimes x)^{\perp}$ meets $\mathbb{C}\otimes L^{\perp}$ properly. By our hypothesis, $\mathbb{R}\otimes L^{\perp}$ contains a positive definite plane D, and the intersection of Q with $\mathbb{C}\otimes D$ is then the union of 2 distinct lines. Thus Q is not a double plane, so the hyperplane $(\mathbb{C}\otimes x)^{\perp}$ meets Q in a proper subvariety.

There are only countably many $x \in V - L$, so the complement of the union of all the $Q\cap(\mathbb{C}\otimes x)^{\perp}$ is dense in Q, and we may approximate ω' by an element ω in this set, which will have the desired properties.

Proof of Corollary 5.5: In the nondegenerate case, if L is embedded in V, then $\mathbb{R}\otimes L$ is an orthogonal direct summand of $\mathbb{R}\otimes V$. But $\mathbb{R}\otimes V$ has signature (3,19), so the dimensions of the maximal positive definite subspaces of $\mathbb{R}\otimes L$ and $\mathbb{R}\otimes L^{\perp}$ add up to 3.

Proof of Proposition 5.2: Note that the lattice in Proposition 5.2 is nondegenerate and satisfies the index condition (in fact $\mathbb{R}\otimes L$ is a hyperbolic plane), so that by Corollary 5.5 it is enough to embed it suitably. In fact it can be embedded already in $H\oplus H$ in the desired sense: taking a basis e_1,f_1,e_2,f_2 of $H\oplus H$ with $(e_1f_1) = (e_2f_2) = 1$ and all other products 0, elementary considerations lead to the choice of generators

$$E = e_1$$
$$A = e_1+8f_1+e_2-4f_2$$

for a direct summand with the required induced quadratic form.

Proof of Theorem 5.3: Regard $E \subset S$ as embedded by $|A|$ in $\mathbb{P}^5$. To show that E is scheme theoretically the intersection of quadrics it suffices, since S is already the complete intersection of quadrics, to show that the residual divisor $R=2A-E$ moves in a linear series without base points. Note that the basis $\{A,R\}$ of Pic S satisfies the same numerical conditions as $\{A,E\}$, and $-R$ cannot be effective since $(-R)A<0$, so R is effective and thus base-point free by the same argument that shows E is. Since $R^2=0$ it follows that $|R|$ is one dimensional, so E is the scheme theoretic intersection of two quadrics and S, that is, of five quadrics, as claimed.

To show that the homogeneous ideal of E requires two generators of degree 3 we must show that the multiplication map

$$H^0\mathcal{O}_{\mathbb{P}_5}(1) \otimes H^0\mathcal{I}_C(2) \rightarrow H^0\mathcal{I}_C(3)$$

has 2-dimensional cokernel. Since S is the complete intersection of quadrics, the restriction of this map to S,

$$H^0\mathcal{O}_S(A) \otimes H^0(R) \rightarrow H^0(3A-E),$$

has the same cokernel. Since $|R|$ is a base point free pencil we have an exact sequence

$$0 \rightarrow \mathcal{O}_S(A-R) \rightarrow \mathcal{O}_S(A) \otimes H^0(R) \rightarrow \mathcal{O}_S(3A-E) \rightarrow 0,$$

from which we see that the above cokernel is $H^1\ \mathcal{O}_S(A-R)$. Since $A(A-R)=0$ we see that neither $A-R$ nor $R-A$ is effective, so the Riemann-Roch formula yields $h^1\ \mathcal{O}_S(A-R) = 2$, as required.

Proof of Theorem 5.1: First, since the family of nondegenerate elliptic octics in $\mathbb{P}^5$ is irreducible, it follows from Theorem 5.3 that the general one will be scheme theoretically the intersection of quadrics. By Proposition 2.2 the family of pairs E,S with $E \subset S \subset \mathbb{P}^5$ and S a smooth surface which is the complete intersection of 3 quadrics (so that in particular S is a K3 surface) is irreducible , and it follows from the existence of the surface guaranteed in Proposition 5.2 that for the general such pair Pic S is generated by E and the hyperplane section A. Thus Theorem 5.3 describes the generic situation.

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