ADVANCES IN MATHEMATICS 70, 135–155 (1988)

# Vector Spaces of Matrices of Low Rank

DAVID EISENBUD*

*Department of Mathematics, Brandeis University, Waltham, Massachusetts 02254*

AND

JOE HARRIS*

*Department of Mathematics, Harvard University, Cambridge, Massachusetts 02138*

In this paper we study vector spaces of matrices, all of whose elements have rank at most a given number. The problem of classifying such spaces is roughly equivalent to the problem of classifying certain torsion-free sheaves on projective spaces. We solve this problem in case the sheaf in question has first Chern class equal to 1; the characterization of the vector spaces of matrices of rank $\leqslant 3$ due to M. D. Atkinson (*J. Austral. Math. Soc.* **34** (1983), 306–315) follows. We speculate on the situation for higher rank and Chern class. Free resolutions are used to verify some properties of the low rank examples and to produce an abundance of examples of somewhat higher rank, connected with the theory of curves. 

*Contents.* 1. Vector spaces of matrices. 2. Sheaves on projective space. 3. Free resolutions. 4. Basic families and higher ranks.

## 1. VECTOR SPACES OF MATRICES

The basic objects to be considered here are vector spaces of linear transformations, that is, a pair of vector spaces $V$ and $W$ and a linear subspace $M \subset \mathrm{Hom}(V, W)$, over an algebraically closed field. We will say that $M$ has *rank* $k$ if the maximum of the ranks of the matrices in $M$ is $k$. It is an interesting problem to describe the vector spaces $M$ for which the rank is small compared to the dimensions of $V$ and $W$; in this paper we derive the description given by Atkinson [2] of spaces of ranks 1, 2, and 3 from a more general result, which we state in terms of the chern classes of a certain associated vector bundle.

A problem with a slightly different flavor is that of determining the rank $k$ spaces $M$ of maximal or near the maximal dimension for such a space,

*The authors are grateful to the NSF for partial support.



0001-8708/88 $7.50

which is $vk$. This problem was, so far as we know, first considered by Dieudonné [6], and, in contrast to the problem considered here, a good deal is now known; see Beasley [4] for the sharpest results and an excellent survey. The reason that this problem does not include the problem described above is that many maximal spaces of matrices of rank $k$ have dimension far smaller than the maximum dimension of a space of matrices of rank $k$, which is $vk$. One refinement to this line of investigation not mentioned by Beasley is that in which one looks for large-dimensional spaces of matries all of whose nonzero elements have rank exactly $k$; for results and a survey in this direction see, for example, Westwick [13]. (Some results of this type can also be deduced from the theorems of Bruns [5].) We are grateful to R. Guralnick for pointing out these references to us.

The treatment of the classification theorem in this section will be elementary and matrix-theoretic. In the next section we will reformulate the problem in terms of sheaves on projective spaces, and state and prove a somewhat more general theorem. In the third section we introduce a technique from finite free resolutions to verify the properties of some of our examples, and use it to complete the proof of the classification theorems from Section 1. In the fourth section we speculate on the situation for higher rank and Chern class, and show how to produce large families of "basic" examples by using projective embeddings of curves.

The work of this paper was done before we became aware of the prior work of Atkinson [2] proving the classification theorem for spaces of matrices of rank $\leqslant 3$. We hope that the application of the sheaf-theoretic methods from algebraic geometry presented here will lead to further progress in this interesting and rather little developed area.

We will say that a space of linear transformations $M$ is *equivalent* to a space of $\dim V \times \dim W$ matrices if they correspond after a choice of bases of $V$ and $W$, and we say that two spaces of matrices are *equivalent* if they correspond under a change of bases.

The description of vector spaces of transformations of rank 1 is classical, and easy: the elements of any such vector space $M$ must have either a common kernel $V' \subset V$ of codimension 1, or a common image $W' \subset W$ of dimension 1—that is, $M$ is equivalent to a subspace either of the space of matrices of the form

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & * \\ 0 & 0 & \cdots & 0 & * \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & \cdots & 0 & * \end{pmatrix} \tag{1}$$

or of its "transpose," the space of matrices of the form

$$\begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 0 & 0 & & 0 \\ * & * & \cdots & * \end{pmatrix}. \tag{2}$$

The trivial generalization of these is the space of maps having images contained in a fixed $k$-dimensional subspace of $W$, and the space of maps having kernels which contain a fixed $(\dim V-k)$-dimensional subspace of $V$. To avoid these and related trivialities, *we will henceforward assume that $M$ is nondegenerate, in the sense that the kernels of elements of $M$ intersect in* 0 *and the images of the elements of $M$ generate $W$.* This assumption has the effect of eliminating from consideration spaces equivalent to spaces of matrices having rows or columns of zeros in common.

There is also a less trivial generalization of the types of low-rank spaces given in (1) and (2). Suppose that for some subspaces $V' \subset V$ and $W' \subset W$ of codimension $k_1$ and dimension $k_2$, respectively, every map in $M$ maps $V'$ into $W'$. It is easy to see that the rank of $M$ is at most $\operatorname{codim} V' + \dim W'$. If equality holds, we will call $M$ a *compression space*, because its rank comes from the fact that its transformations "compress" $V'$ into $W'$. For example, in rank 2, taking $k_1 = k_2 = 1$, we get a compression space equivalent to the vector space of matrices of the form

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & * \\ 0 & 0 & \cdots & 0 & * \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & & 0 & * \\ * & * & \cdots & * & * \end{pmatrix}. \tag{3}$$

In general, a compression space of rank $k$ is one which is equivalent to a space of $(\dim V) \times (\dim W)$ matrices having a common $v_1 \times w_1$ block of zeros with $(\dim V - v_1) + (\dim W - w_1) = k$, the largest possible value.

The classical result on rank 1 spaces already mentioned is equivalent to the statement that every space of maps of rank 1 is a compression space (a generalization is proven in Proposition 2.1). But the naive hope that this might remain true for higher ranks is quickly dashed. For example, consider the space of $3\times 3$ skew-symmetric matrices,

$$\begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}. \tag{4}$$

Since all skew-symmetric maps have even rank, this family has rank 2.

Example (4) belongs to the sequence of examples with which most of this note will be concerned. Given any (abstract) vector space $M$, of dimension $m$, say, we may use the multiplication in the exterior algebra to realize $M$ as a space of maps of rank $(\dim M)-1$ from $M$ to $\Lambda^2 M$. For $\dim M=3$, the first nontrivial case, we get the space of $3\times 3$ skew-symmetric maps (note that $M^*=\Lambda^2 M$ in the 3-dimensional case), while if $\dim M=4$, $M$ is equivalent to the space of matrices of the form

$$\begin{pmatrix} b & c & d & 0 & 0 & 0 \\ -a & 0 & 0 & c & d & 0 \\ 0 & -a & 0 & -b & 0 & d \\ 0 & 0 & -a & 0 & -b & -c \end{pmatrix}. \tag{5}$$

(This notation is to be interpreted as specifying a parametrization of the space of matrices considered. It also specifies a matrix of linear forms, and thus a linear transformation of free modules over a polynomial ring in $a, b, c, d$. This second interpretation will be used in Sections 3 and 4.)

Of course we could use any (graded) algebra instead of the exterior algebra, and there are other ways of generalizing the construction of (5) as well—so many in fact that the classification problem seems hopeless for high rank. We will discuss this a little further in Section 4.

Since *transposition* gives an isomorphism $\mathrm{Hom}(V, W)=\mathrm{Hom}(W^*, V^*)$ which preserves ranks, every rank $k$ space gives rise to another, its *transpose*. Of course the transpose of a compression space is again a compression space, and the transpose of the space (4) is equivalent to (4) itself, but this is obviously no longer true for the space (5).

From a space of matrices of a given rank, we may manufacture a space of matrices of higher rank by adding some rows or columns of arbitrary entries. Thus, for example, the rank 2 space of $3\times 3$ skew-symmetric matrices gives rise to the rank 3 spaces of matrices

$$\begin{pmatrix} 0 & a & b & * \\ -a & 0 & c & * \\ -b & -c & 0 & * \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{6}$$

More generally, if $V' \subset V$ and $W' \subset W$ are subspaces, and $M' \subset \mathrm{Hom}(V', W/W')$ is any space of maps of rank $k'$, then the space $M \subset \mathrm{Hom}(V, W)$ of maps that induce maps in $M'$ has rank

$$k = k' + \operatorname{codim} V' + \dim W'.$$

To deal with this phenomenon, we need some definitions. Given $V' \subset V$ and $W' \subset W$ we write $\pi = \pi_{V'W'}$ for the *projection* map from $\mathrm{Hom}(V, W)$ to $\mathrm{Hom}(V', W/W')$ sending a map $V \to W$ to the composition

$$V' \subsetneq V \to W \twoheadrightarrow W/W'.$$

We say that a space $M \subset \mathrm{Hom}(V, W)$ is *primitive* if there are no subspaces $V' \subset V$ and $W' \subset W$ with $(V', W') \neq (V, W)$ such that

$$\operatorname{rank}(\pi^{-1}(\pi(M))) = \operatorname{rank}(M) \qquad \text{with } \pi = \pi_{V', W'}. \tag{7}$$

Proposition 3.2 gives a computationally effective criterion for primitivity. If $M$ is not primitive, there will exist $V'$ and $W'$ satisfying (7) such that $M' = \pi_{V', M'}(M)$ is primitive; we call $M'$ a *primitive part* of $M$.

A projection map $\pi$ corresponds, in terms of suitable bases, to taking submatrices. Thus $M$ fails to be primitive if $M$ is equivalent to a vector space of matrices in such a way that some space of submatrices $M'$ accounts entirely for the low rank of $M$; that is, $M$ is a subspace of a vector space $\tilde{M}$ of matrices having the same rank as $M$ and looking like

$$\left(\begin{array}{cc|ccccc} * & * & * & * & * & * & * \\ \hline * & * & & & & & \\ * & * & & & & & \\ * & * & & & M' & & \\ * & * & & & & & \\ * & * & & & & & \end{array}\right). \tag{8}$$

Reviewing the examples already given, a compression space (such as (1), (2), or (3)) is exactly a space whose primitive part is zero; in particular, there are no primitive spaces of rank 1. The space (4) of skew-symmetric $3 \times 3$ matrices is primitive, as indeed are all the spaces which like (4) and (5) are of the form $M \subset \mathrm{Hom}(M, \Lambda^2 M)$, and a primitive part of the space (6) is given by the first three rows. The primitivity of all these can be verified easily using Proposition 3.2.

One warning: the primitive part of a space of maps is not in general unique. For example, the fact that the matrix

$$\begin{pmatrix} 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \\ 0 & 0 & * & * \\ * & * & * & * \end{pmatrix} \tag{9}$$

has rank 3 is attributable to either the top left $2 \times 3$ submatrix or the top left $3 \times 2$ submatrix.

The idea of "taking submatrices" is also useful for constructing families of maps of rank $\leqslant k$ from a given rank $k$ family. Thus if $M \subset \operatorname{Hom}(V, W)$ has rank $k$ and $V' \subset V$, $W' \subset W$ are subspaces, then $\pi_{V', W'}(M) \subset \operatorname{Hom}(V', W/W')$ is a space of some rank $\leqslant k$; examples are given in Theorem 1.2 below.

It turns out that these constructions are enough to produce all spaces of linear transformations of ranks $\leqslant 3$, and that we can list the results explicitly. First we state the result for rank $\leqslant 2$.

THEOREM 1.1. *A space of matrices of rank $\leqslant 2$ is either a compression space or is primitive, in which case it is the space* (4) *of $3 \times 3$ skew-symmetric matrices.*

In rank 3 there are two complications: projections of the space (5) appear, and there are imprimitive spaces which are not compression spaces, obtained by adding a row or column to example (4), as in (6). First we treat the primitive case:

THEOREM 1.2. *A primitive rank 3 space of matrices is equivalent either to* (5) *or its transpose or to one of the following four projections of* (5) *and their transposes, which are themselves primitive and pairwise inequivalent:*

$$\begin{pmatrix} -b & -c & -d & 0 & 0 \\ a & 0 & 0 & -c & -d \\ -d & a & 0 & b & 0 \\ c & 0 & a & 0 & b \end{pmatrix} \tag{10}$$

$$\begin{pmatrix} -b & -c & -d & 0 & 0 \\ a & 0 & 0 & -c & -d \\ 0 & a & 0 & b & 0 \\ 0 & 0 & a & 0 & b \end{pmatrix} \tag{11}$$

$$\begin{pmatrix} c & d & 0 & 0 \\ 0 & 0 & c & d \\ -a & 0 & -b & 0 \\ 0 & -a & 0 & -b \end{pmatrix} \tag{12}$$

$$\begin{pmatrix} -b & -d & 0 & 0 \\ a & 0 & -c & -d \\ -d & 0 & b & 0 \\ c & a & 0 & b \end{pmatrix}. \tag{13}$$

We postpone the proofs until Section 3.

Of course it is easy to derive a list of all nondegenerate spaces of rank $\leqslant 3$:

COROLLARY 1.3. *A nondegenerate space of matrices of rank* 3 *is either a compression space, or primitive, or has rank* 3 *and primitive part the space of* $3\times 3$ *skew-symmetric matrices, so that it is of the form given by* (6) *or its transpose.*

One consequence is that the only families of "large" matrices of rank $\leqslant 3$ are the compression families:

COROLLARY 1.4. *Let* $M \subset \operatorname{Hom}(V, W)$ *be a nondegenerate space of rank* $k$, *and set* $s=\min(\dim V, \dim W)$. *If*

$$k=1$$

*or*

$$k=2 \quad \textit{and} \quad s\geqslant 4$$

*or*

$$k=3 \quad \textit{and} \quad s\geqslant 5$$

*then* $M$ *is a compression space.*

It should be mentioned that a (much weaker) result of this type is known to hold for all $k$; Beasley [4] proves that, with notation of the corollary, if $s\geqslant vk-v+w+k$, then $M$ is a compression space. Given the result of Corollary 1.4, however, it is reasonable to ask whether there exists a bound *independent of* $v$ *and* $w$. The extension of the corollary to higher ranks is further discussed in Section 4.

In the course of the proof of Theorem 1.2 it will be necessary to analyze the equivalence classes of projections of the spaces $M \subset \mathrm{Hom}(M, \Lambda^2 M)$ which have rank $= \dim M - 1$. This is done by means of the following result:

PROPOSITION 1.5. *Let $W' \subset \Lambda^2 M$ be a subspace and, considering $M$ as a space of maps $M \subset \mathrm{Hom}(M, \Lambda^2 M)$ via the multiplication map of the exterior algebra, let $M' = \pi_{M, W'}(M)$. Write $m = \dim M$, and suppose that* rank *$M' = m-1$, the rank of $M$. We have:*

(i) *$\pi_{M, W'}: M \to M'$ is an isomorphism.*

(ii) *If $W'' \subset \Lambda^2 M$ is another subspace, then the spaces of maps $\pi_{M, W'}(M)$ and $\pi_{M, W''}(M)$ are equivalent if and only if $W'$ and $W''$ are conjugate under the natural action of $Gl(M)$ on $\Lambda^2 M$.*

*Proof.* (i) If $x \in M$ and $\pi_{M, W'}(x) = 0$, then $x \wedge M \subset W'$. But then for any $y \in M/x$, the kernel of the composite $- \wedge y$: $M \to \Lambda^2 M \to (\Lambda^2 M)(W'$ contains both $x$ and $y$, so that rank $\pi_{M, W'}(M) \leqslant m-2$, a contradiction.

(ii) Suppose first that $W'$ and $W''$ are conjugate by $\Lambda^2\alpha$, for some $\alpha \in Gl(M)$. For each $x \in M$ we have a commutative diagram

$$\begin{array}{ccccc} M & \xrightarrow{\wedge x} & \Lambda^2 M & \longrightarrow & (\Lambda^2 M)/W' \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \Lambda^2\alpha} & & \downarrow{\scriptstyle \beta} \\ M & \xrightarrow{\wedge \alpha(x)} & \Lambda^2 M & \longrightarrow & (\Lambda^2 M)/W'', \end{array}$$

where $\beta$ is induced by $\Lambda^2\alpha$, so $(\alpha, \beta)$ defines an equivalence.

Conversely, suppose that $\alpha: M \to M$, $\beta: (\Lambda^2 M)/W' \to (\Lambda^2 M)/W''$ are any maps defining an equivalence between $M' = \pi_{M, W'}(M)$ and $M'' = \pi_{M, W''}(M)$. We must show that $W'' = \Lambda^2\alpha(W')$. First we prove that the map $\gamma$ defined on $M$ by the diagram

$$\begin{array}{ccc} M & \hookrightarrow & \mathrm{Hom}(M, (\Lambda^2 M)/W') \\ \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \mathrm{Hom}(\alpha^{-1}, \beta)} \\ M & \hookrightarrow & \mathrm{Hom}(M, (\Lambda^2 M)/W'') \end{array}$$

is the same as $\alpha$, at least up to a scalar multiple. Indeed, because of the hypothesis that rank $M' = m-1$, if $x \in M = M'$ is generic, then ker $x$ is the subspace of $M$ generated by $x$, and $x$ is up to a scalar the only element of $M'$ with this kernel. But $\ker \gamma(x) = \ker \beta x \alpha^{-1} = \alpha(\ker(x)$ is the subspace generated by $\alpha(x)$. Since $x$ was generic, $\alpha(x)$ is too, so the only transformations in $M''$ with this kernel are scalar multiples of $\alpha(x) \in M = M''$. This proves that for each generic $x$ there is a scalar $r_x$ such that $\gamma(x) = r_x \alpha(x)$. Since both $\alpha$ and $\gamma$ are linear transformations, this implies that for some scalar $r$, $\gamma = r\alpha$ as claimed.

From the commutativity of the previous diagram, with $\gamma = \alpha$, follows the commutativity of the "adjoint" diagram

$$\begin{tikzcd} M \otimes M \arrow[r] \arrow[d, "\alpha \otimes \alpha"'] & (\Lambda^2 M)/W' \arrow[d, "\beta"] \\ M \otimes M \arrow[r] & (\Lambda^2 M)/W'', \end{tikzcd}$$

and since the horizontal maps factor through

$$\begin{tikzcd} \Lambda^2 M \arrow[d, "\Lambda^2 \alpha"'] \\ \Lambda^2 M, \end{tikzcd}$$

we are done. ∎

Here is a possible application: Mirollo in his thesis [12], has used the classical description of vector spaces of matrices of rank 1 to classify certain subvarieties of the Grassmannian. Briefly, we say that a subvariety of the Grassmannian $G(k, W)$ has *rank* $k$ if its tangent space at a general point $\Lambda \in G(k, W)$, viewed as a vector space of maps from $\Lambda$ to $W/\Lambda$, has rank $k$; Mirollo finds that any rank 1 subvariety of $G(k, n)$ of dimension 2 or more must in fact be a subvariety of the Schubert cycle of planes containing a fixed $(k-1)$-plane or contained in a fixed $(k+1)$-plane (see Griffiths and Harris [9] for a statement for curves of arbitrary rank). It seems likely that techniques similar to Mirollo's will show in general that a subvariety of $G(k, n)$ whose tangent space at each point is a compression space will likewise be contained in a Schubert cycle corresponding to the compression data. It may be hoped that the results of the present paper could be used to give a complete description of subvarieties of the Grassmannian of ranks 2 and 3.

## 2. Sheaves on Projective Space

In this section we will analyze a vector space $M \subset \mathrm{Hom}(V, W)$ by using the tools of algebraic geometry. To do this, we pass to the projective space

$\mathbb{P} = \mathbb{P}M$ of one-dimensional subspaces of $M$. We then have a map of sheaves

$$\varphi' \colon V \otimes \mathcal{O}_{\mathbb{P}}(-1) \to W \otimes \mathcal{O}_{\mathbb{P}}$$

sending a vector $v \otimes \lambda A$, in the fiber of the vector bundle $V \otimes \mathcal{O}_{\mathbb{P}}(-1)$ over the point of $\mathbb{P}M$ corresponding to the space spanned by $A \in M$, to the vector $\lambda \cdot A(v) \in W$; more conveniently for what follows, we can twist by $\mathcal{O}_{\mathbb{P}}(1)$ to obtain

$$\varphi_M \colon V \otimes \mathcal{O}_{\mathbb{P}} \to W \otimes \mathcal{O}_{\mathbb{P}}(1).$$

Observe that the map $\varphi_M$ carries all the data of the space $M$: taking global sections we get the map $V \to W \otimes M^*$ adjoint to the inclusion $M \to \operatorname{Hom}(V, W) = V^* \otimes W$.

To analyze $\varphi_M$, we introduce the image sheaves $\mathcal{E}_M := \operatorname{Im}(\varphi_M)$, and $\mathcal{F}_M := \operatorname{Im}(\varphi_M{*}(1) \colon W^* \otimes \mathcal{O}_{\mathbb{P}} \to V^* \otimes \mathcal{O}_{\mathbb{P}}(1))$, which are torsion-free sheaves of rank $k = \operatorname{rank}(M)$, and try to describe them. Note that $\mathcal{E}_M$ and $\mathcal{F}_M$ nearly determine the data of the map $\varphi$, and hence the original vector space $M$ of maps: since $M$ is nondegenerate, $V$ and $W^*$ are subspaces of the spaces of global sections of $\mathcal{E}_M$ and $\mathcal{F}_M$, so that $M$ is a projection of the vector space of maps associated to the composite

$$H^0(\xi_M) \otimes \mathcal{O}_{\mathbb{P}} \to \mathcal{E}_M \to H^0(\mathcal{F}_M)^* \otimes \mathcal{O}_{\mathbb{P}}(1).$$

$\mathcal{E}_M$ and $\mathcal{F}_M$ are related by the fact that the double dual of each is the dual of the other, twisted by $\mathcal{O}_{\mathbb{P}}(1)$. Each is a subsheaf, generated by global sections, of its double dual.

The sheaves $\mathcal{E}_M$ and $\mathcal{F}_M$ are caught between two constraints: they are more positive than trivial bundles, but less positive than trivial bundles twisted by $\mathcal{O}_{\mathbb{P}}(1)$. Further, both are generated by global sections. As we will see below, these constraints are sometimes enough to determine them.

To see how the geometry of $\mathcal{E}_M$ and $\mathcal{F}_M$ relates to the space $M$, one might consider first the simplest possibility, where $\mathcal{E}_M$ is just a direct sum of rank 1 sheaves. It turns out that in this case $M$ is also of the simplest type, a compression space. In fact it is enough for $\mathcal{E}^{**}$ to be a direct sum of rank 1 sheaves:

PROPOSITION 2.1. *The following conditions are equivalent:*

(i) *$M$ is a compression space.*

(ii) *$\mathcal{E}^{**}$ is a direct sum of rank 1 sheaves (necessarily copies of $\mathcal{O}_{\mathbb{P}}$ and $\mathcal{O}_{\mathbb{P}}(1)$).*

(iii) *$\mathcal{E}_M$ and $\mathcal{F}_M$ have as direct summands trivial vector bundles of ranks $k_1$ and $k_2$ with $k_1 + k_2 = \operatorname{rank} M$.*

*Proof.* Let $k = \operatorname{rank} M$.

(i) $\Rightarrow$ (iii) Let $V' \subset V$, $W' \subset W$ be subspaces such that the maps in $M$ carry $V'$ into $W'$ and $\operatorname{codim} V' + \dim W' = k$. If $V''$ is a complement of $V'$ in $V$, then $\varphi_M$ maps $V'' \otimes \mathcal{O}$ monomorphically to $(W/W') \otimes \mathcal{O}_{\mathbb{P}}(1)$—else the rank of $\mathscr{E}_M$ would be $<k$. Thus $\varphi_M(V'' \otimes \mathcal{O})$ is a trivial vector bundle, which does not intersect $\varphi_M(V' \otimes \mathcal{O})$. Since together the subsheaves generate $\mathscr{E}_M$, the desired conclusion follows for $\mathscr{E}_M$, the trivial summand having rank equal to the codimension of $V'$. Applying the same argument to the transpose of $M$, we are done.

(iii) $\Rightarrow$ (ii) The existence of a trivial summand of $\mathscr{E}_M$ implies the existence of a trivial summand of $\mathscr{E}^{**}$ of the same rank. But $\mathscr{E}^{**}$ is also $\mathscr{F}_M^*(1)$, so it also has as summand a direct sum of copies of $\mathcal{O}_{\mathbb{P}}(1)$ of the same rank as the trivial summand of $\mathscr{F}_M$. Since by (iii) these ranks add up to the rank of $M$, which is the rank of $\mathscr{E}_M$, we have proven (ii).

(ii) $\Rightarrow$ (i) Since a rank one reflexive sheaf on projective space is locally free, $\mathscr{E}^{**}$ is a sum of line bundles. Since both $\mathscr{E}^{**}$ and $\mathscr{E}^*(1)$ have subsheaves of the same rank generated by global sections, we see that the line bundles must be copies of $\mathcal{O}_{\mathbb{P}}$ and $\mathcal{O}_{\mathbb{P}}(1)$. In particular it is generated by its global sections. It is not hard to show that if $M$ is the projection of a space of maps $M_1$ of the same rank as $M$, and if $M_1$ is a compression space, then $M$ is too. Thus we may assume that $\mathscr{E}_M = \mathscr{E}^{**}$ and that $V = H^0(\mathscr{E}_M)$. The hypothesis on $\mathscr{E}_M$ implies the corresponding one for $\mathscr{F}_M$, so we may make a similar assumption there as well.

Writing $\mathscr{E}'$ for a maximal summand of $\mathscr{E}_M$ which is a direct sum of copies of $\mathcal{O}_{\mathbb{P}}$, we may take $V'$ to be the kernel of the map induced on global sections by the composite map

$$H^0(\mathscr{E}_M) \otimes \mathcal{O}_{\mathbb{P}} \to \mathscr{E}_M \to \mathscr{E}',$$

and define $W'$ dually. One checks easily that these spaces have the properties that appear in the definition of compression spaces. ∎

Since every torsion-free sheaf on $\mathbb{P}^1$ is a direct sum of line bundles, we get:

COROLLARY 2.2. *Every 2-dimensional vector space of maps is a compression space.*

Classsically, this corollary is part of Weierstrass' theory of matrix pencils; see, for example, Gantmacher [8, Vol. 2, Chap 12].

Similar ideas lead to a characterization of primitivity which we will use in the proof of the main theorems:

PROPOSITION 2.3. *The following conditions are equivalent:*

(i) *$M$ is not primitive.*

(ii) *$\mathscr{E}^{**}$ or $\mathscr{F}^{**}$ has a rank 1 summand.*

(iii) *$\mathscr{E}_M$ or $\mathscr{F}_M$ has $\mathcal{O}_{\mathbb{P}}$ as a summand.*

*Proof.* (i) $\Rightarrow$ (iii) If rank $M = \dim V$ then $\mathscr{E}_M \cong \mathcal{O}_{\mathbb{P}} \otimes V$, and similarly for $W$, so we may assume that rank $M < \min(\dim V, \dim W)$.

By symmetry it suffices to assume that for some proper subspace $V' \subset V$, we have rank $M = \operatorname{rank} \pi^{-1}\pi(M)$, where $\pi = \pi_{V', W}$, and prove under this assumption that $\mathscr{E}_M$ has $\mathcal{O}_{\mathbb{P}}$ as a summand. We may clearly replace $V'$ with any larger space, and we may therefore assume that it has codimension 1. We have

$$\begin{aligned}\operatorname{rank} M &= \operatorname{rank} \pi^{-1}\pi(M)\\ &= 1 + \operatorname{rank} \varphi_M(V' \otimes \mathcal{O}_{\mathbb{P}}),\end{aligned}$$

and it follows that if $V''$ is any complement of $V'$ in $V$, then $\varphi_M(V'' \otimes \mathcal{O}_{\mathbb{P}}) \cong \mathcal{O}_{\mathbb{P}}$ is a summand of $\mathscr{E}_M$.

(iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) The first of these implications is trivial. Assume (ii) is satisfied. Since rank 1 reflexive sheaves on $\mathbb{P}$ are locally free, and both $\mathscr{E}^{**}$ and $\mathscr{F}^{**}$ contain sheaves of the same rank as themselves that are generated by global sections, the rank 1 summand must be a nonnegative line bundle; since both are contained in a sum of copies of $\mathcal{O}_{\mathbb{P}}(1)$, this line bundle must be $\mathcal{O}_{\mathbb{P}}$ or $\mathcal{O}_{\mathbb{P}}(1)$. But if $\mathscr{E}^{**}$ contains $\mathcal{O}_{\mathbb{P}}(1)$, then $\mathscr{F}^{**}$ contains $\mathcal{O}_{\mathbb{P}}$, so we may assume that the summand is $\mathcal{O}_{\mathbb{P}}$, and by symmetry we may assume that it is a summand of $\mathscr{E}^{**}$. The composite map

$$V \otimes \mathcal{O}_{\mathbb{P}} \twoheadrightarrow \mathscr{E}_M \to \mathscr{E}^{**} \twoheadrightarrow \mathcal{O}_{\mathbb{P}}$$

induces a map on global sections which cannot be zero, since $\mathscr{E}_M$ and $\mathscr{E}^{**}$ have the same rank, so its kernel $V'$ is distinct from $V$. It is easy to check that the pair $V'$, $W$ satisfies the conditions for showing that $M$ is not primitive. ∎

In the general case, the key invariant of $\mathscr{E}_M$ is its first Chern class $c_1(\mathscr{E}_M)$, which we view as an integer $d$ by writing $c_1(\mathscr{E}_M) = d \cdot c_1(\mathcal{O}_{\mathbb{P}}(1))$. Since $\mathscr{E}_M$ is generated by its global sections we have $d \geqslant 0$, and if $d$ were zero the sheaf $\mathscr{E}_M$ would be a trivial vector bundle and $M$ would be a compression space by Proposition 2.1, so we will assume $d \geqslant 1$. Applying this as well to $\mathscr{F}_M$, whose Chern class is

$$c_1(\mathscr{F}_M) \leqslant c_1(\mathscr{E}^*(1)) = k \cdot c_1(\mathcal{O}_{\mathbb{P}}(1)) - c_1(\mathscr{E}),$$

we deduce that $d \leqslant k-1$; and replacing $M$ by its transpose if necessary we may assume that

$$1 \leqslant d \leqslant [k/2], \tag{14}$$

where $[k/2]$ denotes the greatest integer less than or equal to $k/2$.
If now $k \leqslant 3$ we get $d=1$, and then the situation is simple:

THEOREM 2.4. *If $M \subset \mathrm{Hom}(V, W)$ satisfies $c_1(\mathscr{E}_M)=1$, and $\mathcal{O}_{\mathbb{P}}$ is not a summand of $\mathscr{E}_M$, then $\mathscr{E}_M$ is either $\mathcal{O}_{\mathbb{P}}(1)$ or the universal quotient bundle $Q$ on $\mathbb{P}$ defined by the tautological sequence*

$$0 \to \mathcal{O}_{\mathbb{P}}(-1) \to M \otimes \mathcal{O}_{\mathbb{P}} \to Q \to 1. \tag{15}$$

It is interesting to note that $\mathscr{E}_M$ is automatically a vector bundle in this case.

COUNTEREXAMPLE. The theorem fails if we allow $\mathscr{E}_M$ to have $\mathcal{O}_{\mathbb{P}}$ as summand. For example, if we take $M$ to be given by

$$\begin{pmatrix} a & b & 0 \\ 0 & 0 & c \\ 0 & 0 & d \end{pmatrix}$$

then $\mathscr{E}_M$ is the direct sum of $\mathcal{O}_{\mathbb{P}}$ and the product of $\mathcal{O}_{\mathbb{P}}(1)$ with the ideal of a projective line in $\mathbb{P}$—not a vector bundle.

COROLLARY 2.5. *If $M$ is primitive and $c_1(\mathscr{E}_M)=1$, then $M$ is a space of*

$$\dim M \times n \qquad \left(\dim M \leqslant n \leqslant \binom{\dim M}{2}\right)$$

*matrices of rank* $\dim M - 1$ *derived from the space*

$$M \subset \mathrm{Hom}(M, \Lambda^2 M)$$

*associated to the multiplication map of the exterior algebra by projection from a subspace of $\Lambda^2 M$.*

*Proof of Corollary* 2.5. By combining Theorem 2.4 with Proposition 2.3 we get $\mathscr{E}_M = Q$. The first map of (15) is also the first map of the Koszul complex

$$0 \to \mathcal{O}(-1) \to M \otimes \mathcal{O} \to \Lambda^2 M \otimes \mathcal{O}(1) \to \cdots,$$

so $\mathscr{E}_M$ is also the image of $M \otimes \mathcal{O} \to \Lambda^2 M \otimes \mathcal{O}(1)$. Identifying $H^0\mathcal{O}_{\mathbb{P}}(1)$ with $M^*$, and choosing dual bases $\{y_i\}$ of $M$ and $\{x_i\}$ of $H^0\mathcal{O}_{\mathbb{P}}(1)$, this last map is given on global sections by

$$y \mapsto \sum_i (y_i \wedge y) \otimes x_i \qquad \text{for} \quad y \in M = H^0(M \otimes \mathcal{O}).$$

In the fiber above a point of $\mathbb{P}$ corresponding to an element $z \in M$, we get the map

$$y \mapsto \sum_i (x_i(z) \cdot y_i) \wedge y = z \wedge y,$$

which is the multiplication map of the Koszul complex. It follows that $M$ is represented by a projection of the space $M \subset \mathrm{Hom}(M, \Lambda^2 M)$ corresponding to this multiplication map. That is, $M$ is obtained by composing the multiplication map with an inclusion $V \subset M$ and a projection $\Lambda^2 M \to W$. But the rank of $M$ and the rank of the space of maps corresponding to the multiplication map are equal, and the rank of the latter is dim $M - 1$, so if $V \neq M$ or dim $W \leqslant$ dim $M$, then $M$ would not be primitive. ∎

*Proof of Theorem* 2.4. Let $v$ be the dimension of $V$. We can define a rational map $\Psi$ of $\mathbb{P}$ to the Grassmannian $G = G(v-k, V)$ of $(v-k)$-dimensional subspaces of $V$ by sending a general point $p$ of P to the kernel of the map $\varphi$ at $p$. Away from the (codimension $\geqslant 2$) subvariety $\Sigma \subset \mathbb{P}$, where $\mathscr{E}_M$ is not locally free, $\mathscr{E}_M$ is the pullback of the universal quotient bundle $Q_G$ on $G$. Since $c_1 \mathscr{E}_M = 1$, it follows that the map $\Psi$ carries any line in $\mathbb{P}$ not meeting $\Sigma$ to a line in $G$ (that is, a line under the Plücker embedding of $G$). $\Psi$ thus maps $\mathbb{P}$ linearly onto a subspace $\Lambda \subset G$ which is a linear space in the Plücker embedding. Consequently we may obtain $\Psi$ as a linear projection of $\mathbb{P}$ onto a smaller projective space $\mathbb{P}'$, followed by an embedding of that projective space as a linear space in $G$.

To exploit this we use the classical description of the linear spaces lying on $G$: any such space consists either of a subspaces of the set of $(V-k)$-planes containing a fixed $(v-k-1)$-plane or a subspace of the set of planes lying in a fixed $(v-k+1)$-plane (this may be proved, for example, by the argument of Griffiths and Harris [9, p.757]. We consider these cases in turn.

If the linear subspace $\mathbb{P}' \subset G$ consists of planes containing a fixed $(v-k-1)$-plane, then there is a fixed $(v-k-1)$-dimensional subspace $V'$ of $V$ that is in the kernel of every element of $M$. Factoring out $V'$ if necessary, we can assume that the dimension of $V$ is $k+1$. Since the kernel $\mathscr{K}$ of the map from $\mathcal{O}_{\mathbb{P}} \otimes V$ to $\mathscr{E}_M$ is a second syzygy, it is reflexive, and since it is of rank 1 it is thus locally free. Its chern class is of course $-1$, so

$\mathscr{K} \cong \mathscr{O}_{\mathbb{P}}(-1)$. Since $M^* = H^0\mathscr{O}_{\mathbb{P}}(1)$, the inclusion $\mathscr{K} \to \mathscr{O}_{\mathbb{P}} \otimes V$ must factor through $\mathscr{O}_{\mathbb{P}} \otimes M$, and since $\mathscr{K}$ is not contained in a trivial summand of $\mathscr{O}_{\mathbb{P}} \otimes V$, the induced map $M \to V$ is onto.

Consider the dual sequence

$$\mathscr{E}_M \to \mathscr{O}_{\mathbb{P}} \otimes V^* \to \mathscr{O}_{\mathbb{P}}(1).$$

Because $\Lambda^2 V^*$ generates the kernel of

$$H^0(\mathscr{O}_{\mathbb{P}} \otimes V^*(1)) \to H^0(\mathscr{O}_{\mathbb{P}}(2)),$$

we may factor the map $\varphi M^*$ through the Koszul complex map

$$\mathscr{O}_{\mathbb{P}} \otimes \Lambda^2 V^*(-1) \to \mathscr{O}_{\mathbb{P}} \otimes V^*,$$

and dualizing again we see that the map $M \to \mathrm{Hom}(V, W)$ is the composition of the epimorphism $M \to V$, the inclusion $V \subset \mathrm{Hom}(V, \Lambda^2 V)$, and a projection $\mathrm{Hom}(V, \Lambda^2 V) \to \mathrm{Hom}(V, W)$. Since this composite is an inclusion by hypothesis, we must have $M = V$, so that $\mathscr{E}_M = \operatorname{coker} \mathscr{O}_{\mathbb{P}}(-1) \to \mathscr{O}_{\mathbb{P}} \otimes M$ as required.

Suppose on the other hand that $\mathbb{P}' \subset G$ consists of $(v-k)$-planes contained in a fixed $(v-k+1)$-plane $V'$. Let $V''$ be a complement to $V'$ in $V$. Since ranks may be measured on an open set, we see that the ranks of the images in $\mathscr{E}_M$ of $\mathscr{O}_{\mathbb{P}} \otimes V'$ and $\mathscr{O}_{\mathbb{P}} \otimes V''$ are 1 and $k-1$, respectively, and in particular that the second is isomorphic to the trivial bundle $\mathscr{O}_{\mathbb{P}} \otimes V''$. Since $\mathscr{E}_M$ is torsion free of rank $k$, it is the direct sum of these two subsheaves, and from our hypothesis that $\mathscr{E}_M$ does not admit $\mathscr{O}_{\mathbb{P}}$ as a summand, it now follows that $\mathscr{O}_{\mathbb{P}} \otimes V'' = 0$, so rank $\mathscr{E}_M = 1$.

Since rank 1 reflexive sheaves are free, we see that $\mathscr{E}_M^{**} \cong \mathscr{O}_{\mathbb{P}}(1)$. It follows that we may write $\mathscr{O}_{\mathbb{P}}(1) \otimes W$ as $\mathscr{E}_M^{**} \oplus \mathscr{O}_{\mathbb{P}}(1) \otimes W'$ in such a way that the induced map $\mathscr{E}_M \to \mathscr{O}_{\mathbb{P}}(1) \otimes W'$ is zero. Thus we may assume from the outset that $W$ is 1-dimensional, so that $M$ is a projection of the space of maps corresponding to the natural map of sheaves

$$\mathscr{O}_{\mathbb{P}} \otimes H^0\mathscr{O}_{\mathbb{P}}(1) \to \mathscr{O}_{\mathbb{P}}(1). \qquad (*)$$

But this map of sheaves corresponds to the family

$$M \subset \mathrm{Hom}(M^*, W) \cong M.$$

If the projection were proper, then $M$ would not be included in $\mathrm{Hom}(V, W)$, contradicting our hypothesis. It follows that $M$ corresponds to $(*)$ itself, so $\mathscr{E}_M \cong \mathscr{O}_{\mathbb{P}}(1)$ as claimed. ∎

## 3. Free Resolutions

Proposition 2.3 gives a simple and effective method for determining whether a given space of matrices is primitive. The map $\varphi_M$ may be represented, after a choice of bases, as a matrix of linear forms over the polynomial ring $S := F[M^*]$, where $F$ is the ground field; indeed, this is the same matrix of linear forms that we have been habitually using to write down parametrizations of $M$, such as those in (10)–(13). We may regard this matrix as a map of graded free modules over $S$, $f_M = \Gamma_*(\varphi_M)$. We write $E_M$ for the image of $f_M$, and $F_M$ for the image of $f^*(1)$. We may interpret Propositions 2.1 and 2.3 as statements about $E_M$ and $F_M$ by virtue of the following simple result:

Lemma 3.1. *If $E$ is a graded $S$-module generated by elements of degree 0, and $\mathscr{E}$ is the corresponding sheaf on $\mathbb{P}$, then the folowing are equivalent:*

(i) *$E$ has $S$ as a direct summand.*

(ii) *$E^{**}$ has $S$ as a direct summand.*

(iii) *$\mathscr{E}$ has $\mathcal{O}_{\mathbb{P}}$ as a direct summand.*

(iv) *$\mathscr{E}^{**}$ has $\mathcal{O}_{\mathbb{P}}$ as a direct summand.*

*Proof.* (i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iii) are obvious, and (ii) $\Leftrightarrow$ (iv) is also immediate because $\mathscr{E}^{**}$ is the sheafification of $E^{**}$ and $E^{**} = \Gamma_* \mathscr{E}^*$. Thus we need only prove (ii) $\Rightarrow$ (i). If $E^{**} \twoheadrightarrow S$ is an epimorphism, then since $E$ has the same rank as $E^{**}$, the induced map $E \to S$ cannot be zero. Since $E$ is generated in degree 0, this map must even be nonzero in degree 0, whence it is onto. ∎

With this in hand we give the criterion of primitivity:

Proposition 3.2. *Let $A$ be the kernel of $f_M$, and let $B$ be the largest submodule of the minimal first syzygy of $A^*(1)$ which is generated in degree 0. Similarly, let $A'$ be the kernel of $f_{M^*}(1)_\perp$ and let $B'$ be the largest submodule of the minimal first syzygy of $A'^*(1)$ which is generated in degree 0. $M$ is primitive if and only if the rank of $B$ and the rank of $B'$ are both equal to the rank of $M$.*

*Remark.* The computations necessary to apply this result are sometimes difficult to do by hand; but computer programs such as "Macaulay" of Bayer and Stillman [3] make them very quick in many cases of interest.

*Proof.* By Proposition 2.3 and Lemma 3.1, it is enough to show that the image of $f_M$ has a free summand if and only if rank $B \neq$ rank $M$. This is clear; the module $A$ is insensitive to the presence of the free summand. ∎

Of course the same ideas, applied using Proposition 2.1 in place of Proposition 2.3, yield an effective criterion for a space to be a compression space. We leave the formulation to the reader.

We can now give the proof of the classification theorem from Section 1.

*Proof of Theorems* 1.1 *and* 1.2. Theorem 1.1 follows immediately from Theorem 2.4 and Proposition 2.3. As for Theorem 1.2, Corollary 2.5 and Proposition 1.5 reduce the problem to classifying the subspaces $W'$ of $\Lambda^2 M$ under the action of $Gl(M)$ in the case where $\dim M = 4$ and $\dim W' = 1$ or 2, and to considering the corresponding spaces of linear transformations.

The necessary classification of subspaces amounts to the classification of lines and points in $\mathbb{P}^5 = \mathbb{P}(\Lambda^2 M)$ under the action of the group $PGl(M)$ of automorphisms of the Grassmannian $G \subset \mathbb{P}^5$ of lines in $\mathbb{P}^3$. This is easy, and the result is well known: there are two orbits of points, those off and those on $G$, and easy computation shows that these correspond respectively to the families of matrices (10) and (11). There are three orbits of lines in $\mathbb{P}^5$, those transverse to $G$, those tangent to $G$ but not lying in $G$, and those contained in $G$. The first two correspond to the examples (12) and (13). The example corresponding to a line in $G$ may be represented by the matrix

$$\begin{pmatrix} d & 0 & 0 & 0 \\ 0 & c & d & 0 \\ 0 & -b & 0 & d \\ -a & 0 & -b & -c \end{pmatrix},$$

which is not primitive because the rank of the family corresponding to the last three columns is 2. One checks using the criterion of Proposition 3.2 that the families (10)–(13) are primitive.

To finish the proof, we must show that the given families are pairwise inequivalent. Proposition 1.5 implies that this is so for (10)–(13), and a simple consideration of the dimensions of source and target spaces eliminates all other possible coincidences except for a possible coincidence of one of (12) and (13) with the transpose of itself or the other. Now if two spaces $M$ and $M'$ are equivalent, then the two maps $f_M$ and $f_{M'}$ are isomorphic up to a homogeneous automorphism of $S$. If $M$ is the transpose of either (12) or (13) then the kernel of $f_M$ is the tautological map $S(-1) \to M \otimes S$, which is stable under homogeneous automorphisms, but one checks that the kernels of the maps corresponding to (12) and (13) both have the form $S(-2) \to M \otimes S$. This concludes the proof. ∎

## 4. Basic Families and Higher Ranks

To make the classification problem as easy as possible, one seeks to eliminate from consideration any family of matrices which can be derived from another family in some simple way. For example, one might consider only primitive families. We will eliminate still more.

We first make three further definitions. We will say that a space of matrices $M \subset \mathrm{Hom}(V, W)$ is *strongly indecomposable* if it is not a projection of a family of the same rank and dimension which is *split*, that is, of the form

$$\left(\begin{array}{c|c} M' & 0 \\ \hline 0 & M'' \end{array}\right).$$

For example if $M$ has the form

$$\left(\begin{array}{c|c} M' & M'' \end{array}\right)$$

or

$$\left(\begin{array}{c} M' \\ \hline M'' \end{array}\right)$$

with rank $M = \text{rank } M' + \text{rank } M''$ then $M$ is not strongly indecomposable. In particular, if $M$ is strongly indecomposable then it must be primitive, and $\mathscr{E}_M$ must be indecomposable.

We will say that $M$ is unexpandable if it is not nontrivially a projection of a family of maps of the same rank. Thus if $M$ is not expandable then $\mathscr{E}_M$ must be the subsheaf of $\mathscr{E}^{**}$ generated by $V = H^0\mathscr{E}^{**}$, and similarly for $W$ and $\mathscr{F}_M$.

We further say that $M$ is *unliftable* if it is not a proper subspace of a family of the same rank in $\mathrm{Hom}(V, W)$. If $M$ is strongly indecomposable, unexpandable, and unliftable, we say that $M$ is *basic*. The results above imply that the basic spaces of rank $\leqslant 3$ are precisely the three spaces

$M \subset \mathrm{Hom}(M, \Lambda^2 M)$ for dim $M = 0$, 3, and 4, and the transpose of the last of these.

The reader may easily check that $M$ is strongly indecomposable iff $E_M$ and $F_M$ are indecomposable, unexpandable iff both $f_M$ and $f_M^*(1)$ are the linear parts of kernels of maps of graded free $F[M^*]$-modules, and unliftable iff $f_M$ is "weakly rigid" in the sense that given a vector space $N$ of which $M$ is a summand and a map $f_N$ over $F[N^*]$ which reduces to $f_M$ via the epimorphism $F[N^*] \twoheadrightarrow F[M^*]$, $f_N$ must be isomorphic to $f_M \otimes F[N^*]$ via the inclusion $F[M^*] \subsetneq F[N^*]$.

What are the basic families of higher rank? It may be that the problem is tractable if the rank is not much higher than 3, or the Chern class is not much higher than 1. For example, if the rank of the family is 4, then by (14) above we may assume that $c_1 \mathscr{E}_M \leqslant 2$, and Theorem 2.4 takes care of the case of Chern class 1, so we may assume that $c_1 \mathscr{E}_M = 2$. *We know only one such basic family: the family of all* $5 \times 5$ *skew-symmetric matrices*, though we have no proof that there are no others.

On the other hand, "linear maps" such as $f_M$, of relatively low rank, arise naturally when one considers free resolutions over $F[M^*]$, and they arise this way in such profusion that it seems as if the classification problem will become intractable for high rank or Chern class.

As a first example, the spaces $M \subset \mathrm{Hom}(M, \Lambda^2 M)$ considered above may be seen in the free resolution context as coming from the first map in a Koszul complex, the resolution of the homogeneous maximal ideal of $F[M^*]$. But it is easy to construct many more. For example, if $C$ is a curve embedded in $\mathbb{P}^r$ by a complete linear series of high degree compared to its genus, then the homogeneous ideal of $C$ is generated by a large number, say $w$, of quadrics, and the relations among these quadrics are generated by linear relations, say $v$ of them. Thus the second map in the free resolution of the homogeneous coordinate ring of $C$ yields as $(r+1)$-dimensional family of $w \times v$ matrices of rank $w-1$. Since an initial segment of the resolution will consist entirely of linear maps, we will also get families of lower rank compared to $v$ and $w$ from this source. Since the homogeneous coordinate ring of $C$ is Cohen–Macaulay, the dual of the resolution is again a resolution, from which it follows that the families so constructed are strongly indecomposable and unexpandable. Further, such a family is liftable if $C$ is the hyperplane section of a surface other than the cone over $C$. Thus the following suggests that there will be many basic examples:

THEOREM 4.1. *Suppose that $C \subset \mathbb{P}^r$ is a smooth curve embedded by a complete nonspecial linear series. If the degree of $C$ is $> 4g+5$ or if $C$ is of general moduli and genus $\geqslant 23$, then $C$ is not the hyperplane section of any surface in $\mathbb{P}^{r+1}$ except for the cone over $C$.*

*Proof Sketch.* Either hypothesis implies that if $S \subset \mathbb{P}^{r+1}$ is a surface whose hyperplane section is $C$, then $S$ is projectively ruled; that is, $S$ is the image of the projectivization $\tilde{S}$ of a vector bundle $\mathscr{E}$ of rank 2 on $C$, mapped to $\mathbb{P}^{r+1}$ by $\mathcal{O}_{\tilde{S}}(1)$. To see this use the theory of Hartshorne [11] if the degree of $C$ is $> 4g+5$, or the theory of Harris and Mumford [10] and Eisenbud and Harris [7] in the other case. From the inclusions $C \subset S \subset \mathbb{P}^{r+1}$ we get an exact sequence

$$0 \to \mathscr{M} \to \mathscr{E} \to \mathscr{L} \to 0,$$

where $\mathscr{L}$ is the line bundle associated to the embedding of $C$ and deg $\mathscr{M} = 0$. Since $h^0\mathscr{L} = r+1$ while $h^0\mathscr{E} \geqslant r+2$ it follows that $\mathscr{M} = \mathcal{O}_C$ and that the coboundary map $H^0\mathscr{L} \to H^1\mathscr{M}$ is zero. But this map is the cup product with the corresponding extension class in $H^1(\mathscr{L}^{-1})$. By Arbarello *et al* [1, exc. III.B.6], the multiplication map $H^0\mathscr{L} \otimes H^0\mathscr{K} \to H^0(\mathscr{L} \otimes \mathscr{K})$, where $\mathscr{K}$ is the canonical sheaf on $C$, is onto, so dually the map $H^1(\mathscr{L}^{-1}) \to \mathrm{Hom}(H^0\mathscr{L}, H^1\mathcal{O})$, sending an extension class of the associated coboundary map, is injective. Thus the extension class above is zero. Since now $\mathscr{E} \cong \mathcal{O} \oplus \mathscr{L}$, $S$ is a cone over $C$ as required. ∎

Although this profusion of families suggests that the classification problem is intractable in general, one might still hope that an analogue of Corollary 1.4 might hold in the general case. Is it true, for example that for each $k$ there is a bound $B_k$ such that there is no strongly decomposable space of rank $k$ with dim $V$ and dim $W$ both $\geqslant B_k$?

## References


1. E. Arbarello, M. Cornalba, P. A. Griffiths, and J. Harris, "Geometry of Algebraic Curves," Springer-Verlag, New York, 1985.
2. M. D. Atkinson, Primitive spaces of matrices of bounded rank, II, *J. Austral. Math. Soc.* **34** (1983), 306–315.
3. D. Bayer and M. Stillman, *Macaulay*, a computer system for computing in commutative algebra and algebraic geometry. Available from the authors for the Macintosh, Vax, Sun, and many other computers (1986).
4. L. B. Beasley, Null spaces of spaces of matrices of bounded rank, *in* "Current Trends *in* Matrix Theory" (R. Grone and F. Uhlig, Eds), North-Holland, Amsterdam, 1987.
5. W. Bruns, The Eisenbud–Evans generalized principal ideal theorem and determinantal ideals; *Proc. Amer. Math. Soc.* **83** (1981), 19–24.
6. J. Dieudonné, Sur une généralisation du groupe orthogonal a quatre variables, *Arch. Math.* **1** (1948), 282–287.
7. D. Eisenbud and J. Harris, The Kodaira dimension of the moduli space of curves of genus $\geqslant 23$, *Invent. Math.* **90** (1987), 359–387.
8. F. R. Gantmacher, "Theory of Matrices," Chelsea, New York, 1960.

9. P. A. Griffiths and J. Harris, Algebraic geometry and local differential geometry, *Ann. Sci. École Norm. Sup.* **12** (1979), 355–432.
10. J. Harris and D. Mumford, On the Kodaira dimension of the moduli space of curves, *Invent. Math.* **67** (1982), 23–86.
11. R. Hartshorne, Curves with high self-intersection on algebraic surfaces, *Inst. Hautes Études Sci. Publ. Math.* **36** (1969), 111–125.
12. R. Mirollo, "Rank Conditions on Subvarieties of Grassmannians," Thesis, Harvard University, 1985.
13. R. Westwick, Spaces of matrices of fixed rank, *Linear Multilinear Algebra* **20** (1987), 171–174.