LINEAR SECTIONS OF DETERMINANTAL VARIETIES

By David Eisenbud

Introduction. In this paper we study determinantal varieties satisfying a kind of weak genericity property of a sort that occurs in several applications. Our main results are a "Resiliency Theorem" (Theorem 2.1) which says that *arbitrary* low-codimensional linear sections of such weakly generic determinantal varieties are reduced and irreducible, and a "Classification Theorem" (Theorem 5.1) listing the most degenerate of them—they are familiar varieties from classical algebraic geometry. We give an application to linear series on a projective variety; applications to free resolutions of homogeneous coordinate rings of projective varieties and to the construction of special types of Cohen-Macaulay modules will appear elsewhere.

The determinantal varieties we are interested in typically arise from a pairing

$$* \qquad \mu : A \otimes B \to C$$

of vectorspaces over an algebraically closed field, such as the multiplication may of a graded integral domain or the multiplication pairing between spaces of global sections

$$H^0(X, L_1) \otimes H^0(X, L_2) \to H^0(X, L_1 \otimes L_2)$$

of line bundles L_1, L_2 on a reduced irreducible variety X.

The sort of "good" property of μ that we require is that if $0 \neq a \in A$ and $0 \neq b \in B$, then $\mu(a \otimes b) \neq 0$; we say that μ is 1-*generic* if it satisfies this condition.

Setting $M = C^*$, $V = A^*$, and $W = B$, and assuming for simplicity

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that μ is an epimorphism, we may by taking adjoints regard μ as specifying a linear space of linear transformations

$$** \qquad M \subset \mathrm{Hom}(V, W),$$

and we say that M is 1-generic if μ is.

For each k we let M_k be the subscheme of matrices of rank $\leq k$ in M, defined as the scheme-theoretic intersection of M with the rank $\leq k$ locus in $\mathrm{Hom}(V, W)$. Changing notation if necessary we may assume $v := \dim V \geq \dim W =: w$. Our main results fall into two groups, the first having applications in algebra, the second in algebraic geometry:

1) If M is 1-generic, then not only is M_{w-1} reduced and irreducible of correct codimension (a result essentially due to Kempf [1973]) but this remains true when M is cut by $\leq w - 2$ hyperplanes. The simplest special case says that if D is the determinant of a $w \times w$ matrix of indeterminants (the case $M = \mathrm{Hom}(W, W)$) or even for example, the determinant of a "Hankel", or "Catalecticant" matrix

$$\begin{bmatrix} x_1 & x_2 & x_3 & \cdots & x_w \\ x_2 & x_3 & & \iddots & \vdots \\ x_3 & & \iddots & & \\ \vdots & \iddots & & & \\ x_w & \cdots & & & x_{2w-1} \end{bmatrix}$$

(the case where μ is the multiplication map on the space of homogeneous polynomials of degree $w - 1$ in 2 variables), then D is prime and remains prime modulo *any* $w - 2$ or fewer linear forms. This and corresponding results are also given for M_{w-k} under progressively stronger assumptions ("k-genericity") on μ or M.

Applications of this material have been made to the construction of maximal Cohen-Macaulay modules and compressed algebras (Herzog-Kühl [1987], Iarrobino), generalizing and giving an alternate approach to some of the material of Iarrobino [1984].

2) The codimension of the rank k locus in $\mathrm{Hom}(V, W)$ is $(v - k)(w - k)$, so $\mathrm{codim}_M M_k \leq (v - k)(w - k)$ for any linear space M. We give in sections 2 and 3 a number of lower bounds for $\mathrm{codim}_M M_k$. For

arbitrary M, these are stated in terms of the behavior of the determinantal varieties in the "annihilator"

$$M^{\perp} = \{\psi \in \operatorname{Hom}(W, V) \mid \operatorname{trace}(\phi\psi) = 0 \text{ for all } \phi \in M\}.$$

On the other hand, if M is 1-generic then we show that

$$*) \qquad \operatorname{codim}_M M_k \geq v + w - 1 - 2k,$$

and we get corresponding stronger results for ℓ-generic spaces.

The lower bound *) is sharp, being achieved by the Hankel matrices for every k.

In section 4 we list further examples of 1-generic spaces with equality in *), and other phenomena. Then in section 5 we prove our second main result, a Classification Theorem which describes all examples of 1-generic spaces with equality in *); most of them are spaces of Hankel matrices, which correspond to certain product embeddings of $\mathbf{P}^1$; an exceptional family corresponds to certain embeddings of rational ruled surfaces; and a final, lone exceptional case corresponds (as usual) to the Veronese surface. The algebraic descriptions of these examples correspond to a conjecture made by Craig Huneke.

The geometric applications that we have in view rest on the following observation: Suppose that $\mathcal{L}_1$ and $\mathcal{L}_2$ are line bundles on a variety X such that the linear series associated to the image C of the product map

$$\mu : H^0(X, \mathcal{L}_1) \otimes H^0(X, \mathcal{L}_2) \to H^0(X, \mathcal{L}_1 \otimes \mathcal{L}_2)$$

defines a morphism ϕ from X to the projective space of lines in C^*:

$$\phi : X \to \mathbf{P}(C^*).$$

Regarding $C^* \subset \operatorname{Hom}(H^0(X, \mathcal{L}_1), H^0(X, \mathcal{L}_2)^*)$ as a space of matrices, and writing $\mathbf{M} = \mathbf{P}(M) = \mathbf{P}(C^*)$, the image $\phi(X)$ will be contained in the rank 1 locus, $\mathbf{M}_1$. Applying the lower bound *) with $k = 1$, and setting $v = \dim H^0(X, \mathcal{L}_1)$, $w = \dim H^0(X, \mathcal{L}_2)$ we get

$$\dim C \geq v + w + \dim \phi(X) - 2.$$

If $\phi(X)$ is a curve, then this estimate coincides with the weak estimate for all 1-generic pairings

$$\dim C \geq v + w - 1,$$

noted already by Hopf [1940–41], and probably long before. As Kempf [1971] observed, this estimate, in the geometric case above, with X a curve, gives the weak form of Clifford's Theorem (omitting the characterization of extremal cases as the canonical bundle and the hyperelliptic curves; see Arbarello et al [1984] p. 108 for a proof in this style). In this setting the Classification Theorem of section 5 of this paper fills the gap to give the full version of Clifford's Theorem. In general, for varieties of higher dimension our results give a generalization of Clifford's Theorem. For example, for a surface X whose canonical image is a surface we obtain: If D and $D - K_X$ are divisors both moving in a pencil, then

$$h^1(D) \geq q_X - g(D) - 2 - D^2,$$

(where K_X is the canonical divisor, q_x the irregularity of X, and $g(D)$ the arithmetic genus of D), with equality only if the canonical image of X is the Veronese surface in $\mathbf{P}^5$ or a rational normal scroll.

The classification theorem also yields a result (Corollary 5.2 below) about the composition of linear series, which seems to be new even for curves: Somewhat restricted for simplicity, it says that if D_1 and D_2 are divisors moving in nontrivial linear systems on a variety X, and if every divisor in $|D_1 + D_2|$ is the sum of a divisor in $|D_1|$ and a divisor in $|D_2|$, then $|D_1|$ and $|D_2|$ are both multiples of the same linear pencil; that is, there is a rational function φ on X such that both D_1 and D_2 are unions of the "level sets" of φ. (I am grateful to Fernando Serrano-Garcia for pointing out to me a special case of this.)

Sometimes it turns out that $\phi(X) = \mathbf{M}_1$, or, still better, that the affine cone over $\phi(X)$ is scheme-theoretically M_1. This is rather trivially the case for example when $\mathcal{L}_1 = \mathcal{L}_2$ and X maps via $\mathcal{L}_1$ to a variety whose homogeneous ideal is generated by quadratic forms. The author, with J. Koh and M. Stillman, has shown [1988] that it is also the case when X is a curve of genus g and $\mathcal{L}_1$ and $\mathcal{L}_2$ have degrees $\geq 2g + 1$, and are distinct if both degrees are $2g + 1$.

Another source of interest in varieties like $\mathbf{P}(M)_1$ lies in their connection with syzygies, and with the conjectures of Mark Green on the free resolutions of homogeneous coordinate rings of canonical curves—see Green-Lazarsfeld [1986] for an explanation and a description of our current knowledge. We have shown that if $M \subseteq \mathrm{Hom}(V, W)$ is a 1-generic space, then the ideal of 2×2 minors defining $\mathbf{P}(M)_1$ has a $(v + w - 4)^{th}$ *syzygy* of degree $v + w - 2$, or put differently, the "linear part" of the *syzygy* chain of this ideal is as long as it is for the 2×2 minors of a generic $v \times w$ matrix. This result generalizes a result of Green and Lazarsfeld [1984], who prove it under the additional hypothesis that $\mathbf{M}_1$ contains a smooth nondegenerate subvariety of $\mathbf{M}$.

Our approach to determinantal varieties is through the canonical resolutions studied implicitly by Room [1938] and explicitly by Kempf [1971, 1973]. The main novelty introduced here is the connection between the determinantal varieties in a linear space $M \subset H := \mathrm{Hom}(V, W)$ and those in the annihilator $M^\perp$ of M in the dual space $H^* = \mathrm{Hom}(V, W)^* = \mathrm{Hom}(W, V)$; the canonical resolution of M_k is fundamentally related to the canonical resolution of $M^\perp_{w-k}$ (see section 2, especially Lemma 2.6 for this relation). Of course the connection is plausible because of the duality between the determinantal varieties in H and $H^* : H_{w-k} \subset H^*$ is the closure of the set of linear functionals vanishing on hyperplanes tangent to H_k at smooth points, and vice versa (the proof of this fact, for which see Proposition 1.7 below, is almost immediate, though the result seems not widely known).

Two related papers require note: Merle and Giusti [1982] study linear sections of determinantal varieties by *coordinate* planes—that is, spaces $M \subset H$ corresponding to a matrix of linear forms whose $(i, j)^{\text{th}}$ entry is either a variable x_{ij} or 0, and give combinatorial formulas for the heights of certain ideals of minors. Their results and our Theorem 2.1 have in common some (very) special cases.

A result which seems to orient Theorem 2.1 in, perhaps, a more important direction, is that of Zak: *Every* hyperplane section of a *smooth* variety X in $\mathbf{P}^n$ such that $2 \dim X \geq n + 2$ is reduced, irreducible, and normal (see Fulton-Lazarsfeld [1981] for an exposition). Unlike our Theorem 2.1, Zak's result does not extend to plane sections of higher codimension, and indeed does not even *apply* to determinantal varieties, because of the requirement of smoothness. It would be quite interesting to know whether there is a broad class of singular varieties, including determinantal varieties, to which Zak's result could be extended.

Finally, we mention the existence of our [1987] which, containing simply a very special case of Theorem 2.1, is intended as an introduction to the present work.

I am grateful to Craig Huneke for a number of interesting discussions about 2×2 minors; to Jürgen Herzog for reawakening my interest in the problem after I had given it up, and for discussions about the maximal minors of Hankel matrices; and more globally to David Buchsbaum and Joe Harris for teaching me, algebraically and geometrically, about determinantal varieties.

1. Basic notions. We fix throughout vectorspaces V and W of finite dimensions $v \geq w$ over an algebraically closed field F, and write

$$H = \mathrm{Hom}(V, W).$$

We let $M \subset H$ be a subspace of dimension m, and for each $k = 0, \ldots, m$ we write M_k for the locus of maps in M of rank $\leq k$, regarded as the scheme-theoretic intersection of M with H_k, the variety of all maps of rank $\leq k$. We write $\mathbf{M}$ and $\mathbf{H}$ for the projective space of lines in M and H respectively, and $\mathbf{M}_k$, $\mathbf{H}_k$ for the corresponding projective schemes.

We identify the dual space H^* to H as

$$H^* = \mathrm{Hom}(W, V),$$

the natural pairing being $\langle \phi, \psi \rangle = \mathrm{Trace}\, \phi\psi = \mathrm{Trace}\, \psi\phi$. We write

$$M^\perp = \{\psi \in \mathrm{Hom}(W, V) \mid \langle \phi, \psi \rangle = 0 \text{ for all } \phi \in M\}$$

for the annihilator of M in H^*, and $\mathbf{M}^\perp$ for the corresponding projective variety.

We may think of a parametrization of M as being given, after choice of bases for M, V, and W, by a $w \times v$ matrix of linear forms L_{ij} in m variables $x_1, \ldots, x_m$

$$L = (L_{ij})$$

$$L_{ij} = L_{ij}(x_1, \ldots, x_m)$$

such that the L_{ij} span the space M^* of all linear forms in the x_i, and we say that L is a matrix associated to M, or that M is the space associated to L.

For $\ell = 0, \ldots, w$ we write

$$I_\ell(M) \subset \mathrm{Sym}(M^*) = k[x_1, \ldots, x_m]$$

for the ideal generated by the $\ell \times \ell$ minors of a matrix associated to M—the result is immediately seen to be independent of the choices of bases in V and W.

The *generic* matrix, whose entries are all indeterminates corresponds to the subspace $M = H$. Since H_k is the scheme defined by the ideal $I_{k+1}(H)$ generated by the $k+1 \times k+1$ minors of the generic matrix and since we know that this ideal is prime (see for example DeConcini-Eisenbud-Procesi [1980]) it follows from the definitions that M_k is the scheme defined by the ideal $I_{k+1}(M)$.

Of course we can also describe M by giving the corresponding pairing

$$\mu : V \otimes W^* = (\mathrm{Hom}(V, W))^* \twoheadrightarrow M^*,$$

as was done in the introduction. Unfortunately, all three descriptions are useful!

The basic notion of this paper is described by the following:

Proposition-Definition 1.1. We say that the space $M \subset \mathrm{Hom}(V, W)$, or an associated $w \times v$ matrix L of linear forms, or pairing μ is *k-generic* for some integer $1 \leq k \leq w$ if the following equivalent conditions hold:

1) $(M^\perp)_k = 0$.

2) Even after arbitrary invertible row and column operations, any k of the linear forms L_{ij} in L are linearly independent.

3) The kernel of μ does not contain any sums of k or fewer pure tensors $a \otimes b$.

Sketch of proof of equivalence. 1) $\Leftrightarrow$ 3): The kernel of μ is $M^\perp$; and a matrix $\psi \in \mathrm{Hom}(W, V) = V \otimes W^*$ has rank $\leq k$ iff it can be written as a sum of k pure tensors.

3) $\Leftrightarrow$ 2) The L_{ij}, as elements of $M^* \subset \mathrm{Hom}(W, V)$, correspond to pure tensors in $W^* \otimes V$, and of course this is not changed by invertible row and column operations (invertible transformations of V and W). A dependence relation among k of the L_{ij} is thus a linear combination of k pure tensors which vanishes on M, that is, which is in $M^\perp$. □

From characterizations 2 and 3, the following useful remark is obvious:

PROPOSITION 1.2. *Let* $W_1 \subset W$, $V_1 \subset V$ *be subspaces, and let* $\pi_{V_1,W_1} : \mathrm{Hom}(V, W) \to \mathrm{Hom}(V_1, W/W_1)$ *be the projection. If* $M \subset \mathrm{Hom}(V, W)$ *is* k*-generic, then* $\pi_{V_1W_1}(M) \subset \mathrm{Hom}(V_1, W/W_1)$ *is* k*-generic.*

Also, from characterization 1, we get the "obvious" bound:

PROPOSITION 1.3. *If* $M \subset \mathrm{Hom}(V, W)$ *is* k*-generic, then* $\dim M \geq k(v + w - k)$; *if* M *is* k*-generic, then any sufficiently general subspace of dimension* $\geq k(v + w - k)$ *is* k*-generic.*

Remark. Since $\dim M = \mathrm{codim}_M M_0$, we might try to generalize Proposition 1.3 to a lower bound on $\mathrm{codim}_M M_\ell$ for a k-generic space M and every ℓ. This will be done in Proposition 3.2, below.

Proof. We have of course $\dim M^\perp = vw - m$. On the other hand, $\mathbf{H}_k$ is a projective variety of codimension $(v - k)(w - k)$, so if M is 1-generic then $vw - m - 1 = \dim \mathbf{M}^\perp < (v - k)(w - k)$, whence the desired estimate. The second statement follows because if $\mathbf{M}^\perp \cap \mathbf{H}_k = \emptyset$, then the same is true for general spaces containing $\mathbf{M}^\perp$, so long as their dimension is small enough. □

We note that Proposition 1.3 fails spectacularly over nonalgebraically closed fields; in fact, this failure has considerable importance in the theory of immersions of real projective spaces and vectorfields on spheres; see Bott-Gitler-James [1972] pp. 140–144 for a survey. The simplest counterexample is the 1-generic pairing of real vectorspaces:

$$\text{multiplication: } \mathbf{C} \otimes_{\mathbf{R}} \mathbf{C} \to \mathbf{C}.$$

Examples. The reader may wish to keep the following examples in mind:

1) The only w-generic family is $M = H$, corresponding to the generic matrix of linear forms, or the identity pairing $V \otimes W \to V \otimes W$. This is obvious from characterization 1 or 3, surprisingly nonobvious from characterization 2.

2) If $V = W$, then the space of matrices of trace 0 is $w - 1$ generic. It is, up to an obvious notion of equivalence, the only $w - 1$ generic family beside H in this case.

3) The space of Hankel matrices, that is the space with corresponding matrix of linear forms the "catalecticant" matrix

$$\mathrm{Cat}(v, w) = \begin{bmatrix} x_1 & x_2 & x_3 & \cdots & x_v \\ x_2 & x_3 & & & \vdots \\ x_3 & & & & \\ \vdots & & & & \\ x_w & & & & x_{v+w-1} \end{bmatrix}$$

is 1-generic. This is best seen from the fact that it corresponds to the multiplication pairing between $W^* = F[s, t]_{w-1}$, the space of linear forms of degree $w - 1$, and $V = F[s, t]_{v-1}$ into $M^* = F[s, t]_{v+w-2}$. This example is treated at length in section 4.

1-generic matrices in geometry. We conclude this section with an explanation of the geometric significance of varieties of the form $\mathbf{M}_1$.

Recall that a rational map of a scheme X to a projective space $\mathbf{P}(N) = \mathbf{N}$ of lines in N corresponds to a line bundle $\mathcal{L}$ on an open set of X (the pullback of $\mathcal{O}_{\mathbf{P}(N)}(1)$), a space of global sections $U \subset H^0(X, \mathcal{L})$ and an inclusion $i : U^* \hookrightarrow N$. The datum $L = (\mathcal{L}, U, i)$ is called a *linear series* on X; we write p_L for the associated map.

Suppose that we can write $\mathcal{L} = \mathcal{L}_1 \otimes \mathcal{L}_2$ for some line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ on X, and that $U_\lambda \subset H^0(X, \mathcal{L}_i)$ are such that the image C of $U_1 \otimes U_2 \to H^0(X, \mathcal{L})$ is contained in U. Writing $M = C^*$ as in the introduction, we have

$$N \supset U^* \twoheadrightarrow M \subset \mathrm{Hom}(U_1, U_2^*).$$

Thus a subscheme such as $\mathbf{M}_1$ in $\mathbf{M} = \mathbf{P}(M)$ gives rise to a cone in $\mathbf{P}(U^*) \subset \mathbf{N}$, whose vertex is $\mathbf{P}(\ker U^* \to M)$.

PROPOSITION 1.4. *The image $p_L(X)$ of X in $\mathbf{N}$ is contained in the cone over $\mathbf{M}_1$; that is, the homogeneous ideal of $\overline{p_L(X)}$ contains the* 2×2 *minors of a matrix of linear forms corresponding to $U_1 \otimes U_2 \to U$.*

Proof. We must show that the 2×2 minors of a matrix of linear forms associated to $\mathbf{M}$ vanish on $p_L(X)$, or equivalently, that their pullbacks via p_L vanish on X. But after pulling back to X, and restricting to a small open set X', the sections of $\mathcal{L}_1$ and $\mathcal{L}_2$ may be identified with functions on X', and a typical 2×2 submatrix has the form

$$\begin{pmatrix} a_1b_1 & a_1b_2 \\ a_2b_1 & a_2b_2 \end{pmatrix},$$

where $a_1, a_2 \in U_1$ and $b_1, b_2 \in U_2$. The determinant of this submatrix vanishes by the commutative law. □

Remark. It is perhaps also interesting to give a direct argument, at least set-theoretically. If $x \in X$, and we identify the fiber $\mathcal{L}_x$ of $\mathcal{L}$ at x with the ground-field, then $p_L(x)$ takes an element $u_1 \in U_1$ to a functional on U_2 sending $u_2 \in U_2$ to $p_L(x)(u_1)(u_2) = u_1(x) \otimes u_2(x) \in \mathcal{L}_x$ (note that $p_L(x)$ is only well defined up to a scalar by this procedure, as it should be). We see that $p_L(x)$ has rank ≤ 1, as claimed, because its kernel contains the set of $u_1 \in U_1$ with $u_1(x) = 0$, and this has codimension ≤ 1 because $\mathcal{L}_1$ is a line bundle. □

The special case of most interest is the following:

COROLLARY 1.5. *Suppose X is a reduced and irreducible scheme embedded in $\mathbf{P}(H^0(X, \mathcal{L})^*) = \mathbf{N}$ by the complete linear series $(\mathcal{L}, H^0(X, \mathcal{L}))$. Let $D \subset X$ be a Cartier divisor. If v is the codimension of the linear span of D in $\mathbf{N}$ and w the (affine) dimension of the linear equivalence class in which D moves, then the homogeneous ideal of X contains the 2×2 minors of a 1-generic $v \times w$ matrix of linear forms.*

Proof. We apply the proposition, with

$$\mathcal{L}_1 = \mathcal{O}_X(D), \qquad U_1 = H^0(\mathcal{L}_1),$$

and

$$\mathcal{L}_2 = \mathcal{L}(-D), \qquad U_2 = H^0(\mathcal{L}_2).$$

The pairing $U_1 \otimes U_2 \to U$ is 1-generic in this case because X is reduced and irreducible. □

Conversely, suppose that the homogeneous ideal of a subscheme $X \subset \mathbf{P}^r$ contains the ideal of 2×2 minors of a $v \times w$ matrix

$$L = (L_{ij})$$

of linear forms $L_{ij}(x_0, \ldots, x_r)$. Restricting everything to the linear span of X, we assume that X is nondegenerate (not contained in a hyperplane), and we also assume that the rows and columns of L are linearly independent.

Note that if X is reduced and irreducible then L must be 1-generic, since else, after transforming L so that some entry is 0, we get a quadric of the form

$$xy = \det\begin{pmatrix} x & 0 \\ z & y \end{pmatrix}$$

vanishing on X which is impossible.

PROPOSITION 1.6. *For any subscheme X as above there are rational maps $X \to \mathbf{P}^{v-1}$ and $X \to \mathbf{P}^{w-1}$ corresponding to linear series $(\mathcal{L}_1, U_1)$ and $(\mathcal{L}_2, U_2)$ from which the matrix (L_{ij}) is derived as in Proposition* 1.4.

Proof. If C is the linear span of the L_{ij}, then the ideal of 2×2 minors of (L_{ij}) defines a cone over $\mathbf{P}(C^*) = \mathbf{M}$, and we may replace X by its (rational) image in $\mathbf{M}$. But (L_{ij}) may be thought of as arising from the generic $v \times w$ matrix by specialization; that is $\mathbf{M} \subset \mathbf{M} = \mathbf{P}(\mathrm{Hom}(V, W)) = \mathbf{P}(V \otimes W^*)$ for suitable V and W, and the 2×2 minors of (L_{ij}) define

$$\mathbf{M}_1 = \mathbf{M} \cap \mathbf{H}_1 \subset \mathbf{H}_1 = \mathbf{P}(V) \times \mathbf{P}(W^*).$$

Thus we obtain rational maps $X \to \mathbf{P}(V)$ and $X \to \mathbf{P}(W^*)$, and the rest is routine. □

One "reason" for the close relation between the determinantal varieties in $\mathbf{M}$ and those in $M^{\perp}$ seems to be the duality of the determinantal varieties in $\mathrm{Hom}(V, W)$ and $\mathrm{Hom}(V, W)^*$. Although it is hard to believe that it has not been observed before, I was unable to find a reference for this fact, so I include the proof. Recall that a hyperplane $L \subset \mathbf{P}^r$ is said to be tangent to a variety $X \subset \mathbf{P}^r$ at a point $p \in X$ if L contains the tangent plane to X at p. The *dual* variety to X is then the closure in $\mathbf{P}^{r*}$ of the set of all hyperplanes tangent to X at smomth points, regarded as a subvariety of the dual projective space $\mathbf{P}^{r*}$.

PROPOSITION 1.7. *Let* $\mathbf{H}$ *be the projective space of lines in* Hom(V, W), *and let* $\mathbf{H}^*$ *be the dual projective space, identified with the space of lines in* Hom(W, V). *If* $\varphi \in H_k$ *has* rank k, *then* $\psi \in H^*$ *corresponds to a hyperplane of* $\mathbf{H}$ *containing the tangent space to* H_k *at* φ iff $\varphi\psi = \psi\varphi = 0$. *The dual variety to* $\mathbf{H}_k$ *is thus* $\mathbf{H}^*_{w-k}$, *the projective variety of (lines of) maps of rank* $\leq w - k$.

Proof. The second statement follows easily from the first since, given a transformation $\psi \in H^*$, there exists a transformation φ of rank $= k$ such that $\varphi\psi = \psi\varphi = 0$ iff rank $\psi \leq w - k$.

Suppose rank $\varphi = k$. The tangent space to H_k at φ may be identified (see section 2, below) with the sum of the subspaces Hom$(V/\ker \varphi, W)$ and Hom$(V, \operatorname{im} \varphi)$ of H. Thus ψ is orthogonal to this tangent space iff $\psi \in$ Hom(W, V) induces 0 both in Hom$(W, V/\ker \varphi)$ and in Hom$(\operatorname{im} \varphi, V)$; that is, iff $\psi(\operatorname{im} \varphi) = 0$ and $\psi(W) \subset \ker \varphi$. These last two conditions are equivalent to $\psi\varphi = 0$ and $\varphi\psi = 0$ respectively, giving the desired result. □

2. Resiliency: dimension and irreducibility. We use notation as in section 1.

As was shown by Igusa, Kempf, and others, the determinantal varieties H are reduced and irreducible (that is, $I_{k+1}(H)$ is prime) of codimension $(v - k)(w - k)$, and have rational singularities, so that they are in particular normal and Cohen-Macaulay. If $M \subseteq H$, then we say that M meets H_k *properly* if

$$\operatorname{codim}_M M_k = \operatorname{codim}_H H_k = (v - k)(w - k),$$

in which case M_k is Cohen-Macaulay.

Our first main result is that if M' is a $(w - k)$-generic space, then not only M' itself, but also *all* its subspaces of low codimension inherit good properties from H_k:

THEOREM 2.1. *If* $M' \subseteq H$ *is a* $(w - k)$*-generic space, and* $M \subseteq M'$ *is an arbitrary subspace then*:

1) *If* $\operatorname{codim}_{M'} M \leq k$, *then* M *meets* H_k *properly*.

2) *If* $\operatorname{codim}_{M'} M \leq k - 1$, *then* M_k *is reduced and irreducible (that is,* $I_{k+1}(M)$ *is prime)*.

3) *If* $k < w - 1$ *and* $\operatorname{codim}_{M'} M \leq k - 2$, *then* M_k *is normal*.

We conjecture that part 3) holds without the hypothesis $k < w - 1$. We will show in section 3 that the singular locus of M_k is contained in the union of M_{k-1} and the set

$$N_2^0 := \{\phi \in M_k - M_{k-1} | \operatorname{codim}_M\{\psi \in M | \psi V \subseteq \phi V\} < v(w - k)\},$$

which has codimension at least $k - \operatorname{codim}_{M'} M$, so it would be enough to show that if $\operatorname{codim}_{M'} M \leq k - 2$ then M_{k-1} has codimension at least 2 in M_k.

It is easy to see that part 1 is actually a consequence of part 2, but we will prove part 1 first, in this section, by means of a more general formula for the dimension of a determinantal variety and deduce parts 2 and 3 from part 1 by means of an irreducibility criterion and the above description of the singular locus. First we give some consequences.

COROLLARY 2.2. *Let $L = (L_{ij})$ be a $(w - k)$-generic matrix of linear forms in variables $x_1, x_2, \ldots, x_m$. Let $K_1, K_2, \ldots, K_c$ be arbitrary linear forms in the same variables, and let L^- be the matrix L modulo $(K_1, K_2, \ldots, K_m)$. If $c \leq k$ then the $(k + 1) \times (k + 1)$ minors of L^- are linearly independent forms of degree $k + 1$ (and thus in particular are nonzero). If $c \leq k - 1$ then each of these minors is prime.*

Proof of the Corollary. Let M be the space corresponding to L^-. The second statement is immediate from the theorem; if $I_{k+1}(L^-)$ is a prime ideal, all its generators must be prime because they all have the same degree $w + 1$. To prove the first statement we use the fact that $I_k(H)$ is perfect. By the theorem M meets H_k properly, so the minimal graded free resolution of $I_k(M)$ is the reduction modulo linear forms defining M in H of the minimal graded free resolution of $K_k(H)$. Since there are no linear relations with scalar coefficients among the minors of the generic matrix, the same must hold for L^-. □

Remarks. It is easy to see that the second conclusion of the Corollary characterizes $(w - k)$-generic families. Also, the bounds given are best possible; for example, if M' is the family of traceless 2×2 matrices, corresponding to the matrix of linear forms

$$L = \begin{pmatrix} x_1 & x_2 \\ x_3 & -x_1 \end{pmatrix}$$

then L is 1-generic, but $\det L$ modulo x_1 is reducible, and $\det L$ modulo x_3 is a square.

For any linear space $M \subset H$, and any k we set

$$\delta_k(M) = \max_{0<\ell\leq w-k} (\dim(M^{\perp}{}_{\ell}) - k\ell).$$

Clearly, if $M' \subset H$ is $(w-k)$-generic, then $\delta_k(M') = -k$. Also, if $M \subset M' \subset H$ are any linear spaces, then

$$\operatorname{codim}_{M^{\perp}} M'^{\perp} = \operatorname{codim}_{M'} M,$$

so

$$\delta_k(M) \leq \delta_k(M') + \operatorname{codim}_{M'} M.$$

Thus Part I of Theorem 2.1 and the irreducibility statement of part 2 follow from a more general result:

THEOREM 2.3. *For any $M \subset H$,*

$$\dim M_k \leq m - (v-k)(w-k) + \max(0, \delta_k(M)),$$

so that if $\delta_k(M) \leq 0$, then M meets H_k properly. Further, if $\delta_k(M) < 0$ then M_k is irreducible.

Theorem 2.1, except for the reducedness statement of part 2), follows from this by a direct computation of $\delta_k(M)$: If M' is $(w-k)$-generic, then $M'^{\perp}{}_{\ell} = 0$ for $\ell \leq w-k$; if now $\operatorname{codim}_{M'} M = c$, then $\operatorname{codim}_{M^{\perp}} M'^{\perp} = c$, so

$$\dim M^{\perp}{}_{\ell} \leq c \quad \text{for} \quad \ell \leq w-k,$$

whence

$$\delta_k(M) \leq c - k,$$

and the desired results follow by Theorem 2.3.

It is possible to give a more precise dimension formula with the same techniques:

PROPOSITION 2.4. *For any $M \subset H$,*

$$\max_{j\leq k} (j(w-k) + \dim M_{k-j})$$

$$= m - v(w-k) + \max_{\ell\leq w-k} (\ell k + \dim M^{\perp}{}_{w-k-\ell}).$$

These results are both proven via the canonical desingularizations of H_k and H^*_{w-k}. Recall that the "canonical desingularization" (actually a rational resolution (Kempf [1973])) of H_k is

$$H_k^{\sim} = \{(\phi, W') \in H \times \mathrm{Gr}(k, W) \mid \phi(V) \subset W'\},$$

where we have written $\mathrm{Gr}(k, W)$ for the Grassmannian of k-planes in W. We also consider the "dual" canonical desingularization $H^*_{w-k}{}^{\sim}$ of H^*_{w-k}, defined by

$$H^*_{w-k}{}^{\sim} = \{(\psi, W') \in H^* \times \mathrm{Gr}(k, W) \mid \psi(W') = 0\}.$$

For $M \subset H$ we write $M_k{}^{\sim}$ for the preimage of $M_k = M \cap H_k$ in $H_k{}^{\sim}$, and we similarly write $M^{\perp}{}_{w-k}{}^{\sim}$ for the preimage of $M^{\perp}{}_{w-k}$ in $H^*{}_{w-k}{}^{\sim}$.

Fibering $M_k{}^{\sim}$ over M_k we easily obtain

$$\dim M_k{}^{\sim} = \max_{j\leq k} (j(w-k) + \dim M_j),$$

and similarly for $M^{\perp}{}_{w-k}{}^{\sim}$; this is the source of the numbers in Proposition 2.4.

Note that, because of our choices, both $M_k{}^{\sim}$ and $M^{\perp}_{w-k}{}^{\sim}$ map to $\mathrm{Gr}(k, W)$. Parts of Theorem 2.3 and Proposition 2.4 follow from:

THEOREM 2.5. *For any $M \subset H$:*

1) $\dim M_k{}^{\sim} = m - v(w-k) + \dim M^{\perp}_{w-k}{}^{\sim}$.

2) *$M_k{}^{\sim}$ may be written as the union of two (possibly empty), sets,*

$$M_k{}^{\sim} = N_1 \cup N_2,$$

such that

a) *N_1 is the pullback of an open set in* Gr(k, W), *and is either empty or irreducible of dimension*

$$m - (v - k)(w - k).$$

b) *N_2 is the pullback of the complementary closed set in* Gr(k, W) *and has dimension*

$$m - (v - k)(w - k) + \delta_k(M).$$

Further, if $\delta_k(M) < 0$ then N_1 is dense in $M_k\tilde{}$.

We can now complete the proofs of all the results of this section except part 3) of Theorem 2.1 and the reducedness assertion in part 2. Modulo some arithmetic which we leave to the reader, it suffices to prove Theorem 2.5.

Note that because of the way we have chosen our resolutions, both $M_k\tilde{}$ and $M^{\perp}{}_{w-k}\tilde{}$ map to Gr(k, W). The central point is that their fibers over a k-plane $U \subset W$,

$$(M_k\tilde{})_U = \{\phi \in M \mid \phi(V) \subset U\}$$

and

$$(M^{\perp}{}_{W-k}\tilde{})_U = \{\psi \in M^{\perp} \mid \psi(U) = 0\}$$

are simply related:

LEMMA 2.6. *There is an exact sequence*

$$0 \to (M^{\perp}{}_{w-k}\tilde{})_U \to \mathrm{Hom}(W/U, V) \to M^* \to [(M_k\tilde{})_U]^* \to 0,$$

so that

$$\dim(M_k\tilde{})_U = m - v(w - k) + \dim(M^{\perp}{}_{w-k}\tilde{})_U.$$

Proof of Lemma 2.6. The left hand map sends ψ to the map induced on W/U; the right-hand map is the dual of the obvious inclusion; and the middle map is the composite $\mathrm{Hom}(W/U, V) \subset \mathrm{Hom}(W, V) = H^* \to M^*$. Exactness follows easily from the definitions. □

Proof of Theorem 2.5. Stratifying $G(k, W)$ by the dimension of the fibers of $M_k^{\sim}$ or, equivalently $M^{\perp}{}_{W-k}{}^{\sim}$, and using the last statement of Lemma 2.6, we get part 1).

For part 2, let $P_1 \subset G(k, W)$ be the set of $U \subset W$ such that $\dim(M_k^{\sim})_U = \dim M - v(w - k)$ the smallest possible value. Although the map $M_k^{\sim} \to G(k, W)$ is not proper, the equations defining it are bihomogeneous, so that it induces a proper map on an appropriate projectivization $\mathbf{M}_k^{\sim}$ of $M_k^{\sim}$, and P_1 is open. Let P_2 be the complement of P_1, and let N_1 and N_2 be the preimages of P_1 and P_2 in $M_k^{\sim}$. Since the fibers of $N_1 \to P_1$ are irreducible (linear spaces) and all of the same dimension, and since again $N_1 \to P_1$ is naturally associated to a proper map, we see that N_1 is irreducible, and has dimension

$$\dim N_1 = \dim P_1 + \dim M - v(w - k)$$

$$= m - (v - k)(w - k),$$

as required. Using Lemma 2.6 again, and the fact that $\dim M^{\perp}_{w-k}{}^{\sim} = (w - k)k + \delta_k(M)$, we obtain the required formula for the dimension of N_2.

Finally, we must show that if $\delta_k(M) < 0$ so that $\dim N_2 < \dim N_1$, then N_1 is dense; that is, no component of $M_k^{\sim}$ can have dimension $< m - (v - k)(w - k)$. But of course this follows at once since $M_k^{\sim}$ is the intersection, in the smooth space $H \times G(k, W)$, of $M \times G(k, W)$ with $H_k^{\sim}$, a subvariety whose codimension is equal to $\dim G(k, W) + (v - k)(w - k)$. □

3. Tangent spaces and singular loci. In this section we complete the proof of Theorem 2.1 and derive sharp lower bounds for the codimensions of the determinantal varieties M_k of an ℓ-generic space M. All the results are based on the following observation: Let $\phi \in H$ be a map of rank exactly k. We may identify the tangent space $T_{H_k,\phi}$ to H_k at ϕ, as usual, with

$$\{\phi' \in H \mid \phi'(\ker\phi) \subset \operatorname{im}\phi\} = \ker\{\pi_{\ker\phi,\operatorname{im}\phi} : H \to \operatorname{Hom}(V/\ker\phi, \operatorname{im}\phi\}]$$

—see for example Arbarello et al, [1984]. Thus if $\phi \in M$ has rank exactly ϕ, we have

$$T_{M_k,\phi} \subseteq M \cap T_{H_k,\phi} = \ker(\pi_{\ker\phi,\operatorname{im}\phi}|_M).$$

We will play this off against the fact that $\pi_{\ker\phi,\mathrm{im}\,\phi}$ takes ℓ-generic spaces to ℓ-generic spaces.

As a first application of this principle, we have a nonsingularity criterion which will allow us to complete the proof of Theorem 2.1:

PROPOSITION 3.1. *Let $M \subset H$ be any subspace, and let $\phi \in M$. Let*

$$M_k^{\sim} = N_1 \cup N_2$$

be the decomposition of Theorem 2.5. If ϕ has rank exactly k and $(\phi, \mathrm{im}\,\phi) \in N_1$, then M_k is nonsingular and of codimension $(v - k)(w - k)$ in M at ϕ.

Proof. $(\phi, \mathrm{im}\,\phi) \in N_1$ means that $\{\phi' \in M \mid \mathrm{im}\,\phi' \subset \mathrm{im}\,\phi\} = M \cap \ker \pi_{0,\mathrm{im}\,\phi}$ has codimension $v(w - k)$, the maximimum possible value, in M. But it then follows that

$$M \cap \ker \pi_{\ker\phi,\mathrm{im}\,\phi}$$

has codimension $(v - k)(w - k)$ in M. Since $\mathrm{codim}_H H_k = (v - k)(w - k)$, this is the maximum possible codimension of a component of M_k so M_k is nonsingular and of codimension $(v - k)(w - k)$ at ϕ, as claimed. □

Completion of the proof of Theorem 2.1. It remains to prove part 3) and the reducedness in part 2). By part 1), $M_k = M \cap H_k$ is proper. Since H_k is Cohen-Macaulay, this implies that M_k is Cohen-Macaulay, so it suffices to prove that the singular locus of M_k has codimension ≥ 1 under the hypothesis of part 2) and ≥ 2 under the hypothesis of part 3).

By Proposition 3.1, Sing M_k is contained in the union of M_{k-1} and the set N_2^0, the image of N_2 in M_k intersected with $M_k - M_{k-1}$. By Theorem 2.5,

$$\begin{aligned} \dim N_2 &= m - (v - k)(w - k) + \delta_k(M) \\ &\leq m - (v - k)(w - k) - [k - \mathrm{codim}_{M'} M], \end{aligned}$$

as desired, so it is enough to prove that M_{k-1} has codimension ≥ 1 in Case 2) and ≥ 2 in Case 3). To this end, let $W_1 \subset W$ be any 1-dimensional space, and consider the projections

$$\begin{array}{ccc} \pi_{V,W_1}: H & \longrightarrow & \mathrm{Hom}(V, W/W_1) \\ \cup & & \cup \\ M' & \longrightarrow & \bar{M}' \\ \cup & & \cup \\ M & \longrightarrow & \bar{M} \\ \cup & & \cup \\ M_{k-1} & \longrightarrow & \bar{M}_{k-1} \end{array}$$

By Proposition 1.2, $\bar{M}'$ is $w - k = (w - 1) - (k - 1)$ generic, and $\operatorname{codim}_{\bar{M}'}\bar{M} \leq \operatorname{codim}_{M'} M \leq k - 2 = (k - 1) - 1$, so by part 1 of Theorem 2.1 again,

$$\operatorname{codim}_{\bar{M}}\bar{M}_{k-1} = (v - k + 1)(w - k) = (v - k)(w - k) + (w - k).$$

The theorem being trivial if $k \geq w$, we may assume in any case that $w - k \geq 1$, while in case 3) $w - k \geq 2$. Since $\operatorname{Codim}_M M_{k-1} \geq \operatorname{codim}_{\bar{M}}\bar{M}_{k-1}$, we are done. □

As a second application of the tangent space observation, we get good *lower* bounds for the codimension of M_k when M is ℓ-generic, generalizing Proposition 1.3 and the special case of Theorem 2.1, 1), where $M = M'$:

PROPOSITION 3.2. *Let $M \subset H$ be an ℓ-generic space, and let $\phi \in M_k - M_{k-1}$ with $k \leq w - \ell$. The tangent space to M_k at ϕ satisfies*:

$$\operatorname{codim}_{T_{M,\phi}} T_{M_k,\phi} \geq \ell(v + w - 2k - \ell).$$

Proof. $\pi_{\ker\phi,\operatorname{coker}\phi}(M)$ is an ℓ-generic subspace of $\mathrm{Hom}(\ker\phi, \operatorname{coker}\phi)$; thus by Proposition 1.3 it has dimension $\geq \ell((v - k) + (w - k) - \ell) = \ell(v + w - 2k - \ell)$ as required. □

As an immediate consequence, we have:

COROLLARY 3.3. *If M is ℓ-generic and $k \leq w - \ell$, then every component of M_k has dimension $\geq \ell(v + w - 2k - \ell)$. If M_k has a component of that codimension, then its singular locus is contained in M_{k-1}.* □

4. Examples of *l*-generic spaces. In this section we describe some examples of 1-generic spaces M, emphasizing those with $\operatorname{codim} M_1 = v + w - 3$, the lowest possible value. Finally, we give a list of the values of v, w, and k for which there are only finitely many equivalence classes of k-generic subspaces in $\mathrm{Hom}(V, W)$.

From Corollary 3.3 we see that if M is 1-generic and $\operatorname{codim}_M M_1 = v + w - 3$, then $\operatorname{Sing} M_1 = M_0 = 0$, so M_1 is a cone with isolated singularity. It is thus natural to consider $\mathbf{M}_1$, which will be a smooth projective variety in $\mathbf{M}$, the space of lines in M, and we will take the projective point of view throughout this section and the next.

To establish properties of our examples, we will use the following result:

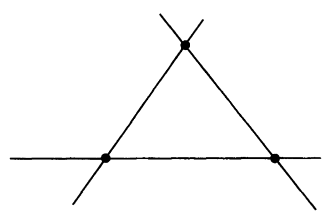
PROPOSITION 4.1. *Let* $X \subset \mathbf{P}(V^*) \times \mathbf{P}(W) \subset \mathbf{P}(V^* \otimes W) = \mathbf{H}$ *be a subscheme of the veronese embedding of* $\mathbf{P}(V^*) \times \mathbf{P}(W) = \mathbf{H}_1$. *If* $\mathbf{M} \subset \mathbf{H}$ *is the linear span of* X, *then* $\mathbf{M}$ *is* 1-*generic* iff *there do not exist hyperplanes* $\mathbf{P}(V_1^*) \subset \mathbf{P}(V^*)$ *and* $\mathbf{P}(W_1) \subset \mathbf{P}(W)$ *such that* $X \subset \mathbf{P}(V_1^*) \times \mathbf{P}(W) \cup \mathbf{P}(V^*) \times \mathbf{P}(W_1)$. *In particular, if* X *is irreducible and the images of* X *in* $\mathbf{P}(V^*)$ *and in* $\mathbf{P}(W)$ *are nondegenerate, then* $\mathbf{M}$ *is* 1-*generic*.

Proof. Let π_1, π_2 be the projections of $\mathbf{P}(V^*) \times \mathbf{P}(W)$ onto the first and second factors. The Veronese embedding corresponds to the line bundle $\mathcal{L} = \pi_1^* \mathcal{O}_{\mathbf{P}(V^*)}(1) \otimes \pi_2^* \mathcal{O}_{\mathbf{P}(W)}(1)$, and the set $\mathbf{P}(V_1^*) \times \mathbf{P}(W) \cup \mathbf{P}(V^*) \times \mathbf{P}(W_1)$ is the hyperplane section corresponding to an element of the form $\sigma \otimes \tau \in V \otimes W^* = H^0(\mathcal{L})^*$ with $V_1^* = \ker \sigma$, $W_1 = \ker \tau$. But a nonzero element ψ of $V \otimes W^*$ may be written as $\sigma \otimes \tau$ iff the corresponding element of $H^\vee = \mathbf{P}(V \otimes W^*)$ is of rank 1. Thus X, and hence $\mathbf{M}$, is contained in a set ψ, with $\psi \subset H_1^\vee$, iff X is contained in a set of the form given in the proposition. $\square$

Further, we note that if $X \subset \mathbf{P}(V^*) \times \mathbf{P}(W) = H_1 \subset H$, and $\mathbf{M}$ is the linear span of X, then $X = \mathbf{M}_1$ iff X is defined, in $\mathbf{P}(V^*) \times \mathbf{P}(W)$, by bilinear equations.

Remark. It should be noted that there are also 1-generic $\mathbf{M}$ for which $\mathbf{M}_1$ is reducible, and does not even contain an irreducible nondegenerate component. Perhaps the simplest example is built from a triangle X of 3 lines:

X:



Regarding X as embedded, by a map φ_1 as the union of 3 lines in $\mathbf{P}^2$, we may define a 1-parameter family of further maps $\varphi_\lambda : X \to \mathbf{P}^2$, for $\lambda \in \mathbf{C}^* = \mathbf{C} - \{0\}$ by taking φ_λ to be the identity on two of the lines and multiplication by λ on the third line minus the two points of intersection, identified with $\mathbf{C}^*$. If we embed X by $(\varphi_1, \varphi_\lambda) : X \to \mathbf{P}^2 \times \mathbf{P}^2$ then with notation as in Prop. 4.1, $\mathbf{M}$ will be 1-generic, although neither projection of X contains an irreducible nondegenerated subvariety. It is not hard to check in this case that $\mathbf{M}_1 = X$ iff $\lambda \neq 1$, a fact which is almost a special case of the theory of Eisenbud-Koh-Stillman [1988]. These examples were found by the author with Joe Harris, and independently, and in a different form, by Jee Koh.

We are now ready for the examples:

a) *Catalecticant matrices, and Secant varieties of rational normal curves.*

Let

$$C_{v,w} = \begin{bmatrix} x_1 & x_2 & x_3 & \cdot\cdot & x_v \\ x_2 & x_3 & \cdots & x_v & x_{v+1} \\ x_3 & & & & \vdots \\ \vdots & & & & \\ x_w & x_{w+1} & x_v & \cdots & x_{v+w-1} \end{bmatrix}$$

be the $w \times v$ catalecticant matrix (beloved of invariant theorists; see for example Grace and Young [1903]), which corresponds to a linear space of dimension $v + w - 2$

$$\mathrm{Cat}(v, w) \subset \mathbf{P}(\mathrm{Hom}(V, W)),$$

with $\dim V = v$ and $\dim W = w$, as usual.

PROPOSITION 4.2. *If v and w are ≥ 2, then $\mathbf{M} = \mathrm{Cat}(V, W)$ is 1-generic. Further $\mathbf{M}$ is spanned by $\mathbf{M}_1$, which is a rational normal curve of degree $v + w - 2$; in particular,* $\mathrm{codim}_{\mathbf{M}} \mathbf{M}_1 = v + w - 3$. *The projection maps*

$$\mathbf{P}^1 \cong \mathbf{M}_1 \subset H_1 = \mathbf{P}^{v-1} \times \mathbf{P}^{w-1} \quad \begin{matrix} \xrightarrow{\pi_1} \mathbf{P}^{v-1} \\ \xrightarrow{\pi_2} \mathbf{P}^{w-1} \end{matrix}$$

are given by the complete linear series $\mathcal{O}_{\mathbf{P}^1}(v-1)$ *and* $\mathcal{O}_{\mathbf{P}^1}(w-1)$, *so that* $\mathbf{M}_1$ *is the graph of an isomorphism between the rational normal curve of degree* $v-1$ *in* $\mathbf{P}^{v-1}$ *and a rational normal curve of degree* $w-1$ *in* $\mathbf{P}^{w-1}$. *Any such graph spans a space which is equivalent under* $\mathrm{GL}(V^*) \times \mathrm{GL}(W) \subset \mathrm{GL}(V^* \otimes W)$ *to the catalecticant space.*

Proof of Proposition 4.2. Choosing coordinates x_i on $\mathbf{P}(V^*)$, y_i on $\mathbf{P}(W)$, and $z_{ij} = x_i \otimes y_j$ on $\mathbf{P}(V^* \otimes W)$, we see that $\mathbf{M} = \mathrm{Cat}(v, w)$ is cut out in H by the equations $z_{ij} = z_{k\ell}$ if $i + j = k + \ell$, so $\mathbf{M}_1$ is cut out by these same equations, which may be thought of as the bilinear equations $x_i y_j = x_k y_\ell$ for $i + j = k + \ell$ on $\mathbf{P}(V^*) \times \mathbf{P}(W)$. From the equations

$$x_0 y_{j+1} = x_1 y_j \quad \text{and} \quad x_{k+1} y_0 = x_k y_1$$

we see that, on $\mathbf{M}_1$,

$$y_j/y_{j+1} = y_1/y_0 = x_1/x_0 = x_{k+1}/x_k$$

for all j and k, so that on the affine open set $x_0 = y_0 = 1$, $\mathbf{M}_1$ is the graph of the isomorphism of rational normal curves

$$\mathbf{P}^{v-1} \ni (1, t, \ldots, t^{v-1}) \to (1, t, \ldots, t^{w-1}) \in \mathbf{P}^{w-1}.$$

Checking similarly on the other open affines, we see that $\mathbf{M}_1$ is the graph of an isomorphism of rational normal curves as in the Proposition. It is obvious that the Veronese embedding of such a graph is a rational normal curve of degree $v + w - 2$.

Now applying Proposition 4.1, we see that the span of $\mathbf{M}_1$, and thus a fortiori $\mathbf{M}$, is 1-generic. As noted, the graph is embedded in $\mathbf{H}$ as a rational normal curve of degree $v + w - 2$, so its linear span is of dimension $v + w - 2$. Since this is the dimension of $\mathbf{M}$, we see that $\mathbf{M}$ is the linear span of $\mathbf{M}_1$, as claimed.

Finally, the equivalence of any two spaces spanned by graphs is assured because every isomorphism from a rational normal curve of degree $v - 1$ in $\mathbf{P}^{r-1}$ to another is induced by a (linear) isomorphism of $\mathbf{P}^{v-1}$, are similarly for $\mathbf{P}^{w-1}$. □

It is worth noting that the other determinantal varieties of the catelecticant space have geometric significance too:

Proposition 4.3. (Wakerling): *Let* $\mathbf{M} = \mathrm{Cat}(v, w)$ *be the catalecticant space. For each* $k = 1, \ldots, w$, $\mathbf{M}_k$ *is the reduced union of the* k*-secant* $(k - 1)$*-planes to the rational normal curve* $\mathbf{M}_1$. *(A* k*-secant* $(k - 1)$*-plane to a curve* $C \subset M$ *is a* $(k - 1)$*-dimensional plane* $P \subset \mathbf{M}$ *such that for every hyperplane* $\mathbf{M}' \supset P$, *the set* $\mathbf{M}' \cap C$ *contains at most* $\deg C - k$ *points not in* $P \cap C$. *This includes all those* $(k - 1)$ *planes meeting* C *in* $\geq k$ *points as well as their limits, which meet* C *in* $\geq k$ *points "properly counted".)*

In particular, $\mathrm{codim}_{\mathbf{M}} \mathbf{M}_k = v + w - 1 - 2k$, the minimum possible value for a 1-generic space. The proof will show even that the M_k are reduced—that is, all the ideals of minors are prime.

Remark. I learned of this result from Alan Adler, who found it, asserted without proof or reference, in an apparently unpublished manuscript of Wakerling gathering dust in the Stacks of the Mathematics library in Berkeley. Since I am not aware of any published proof, I sketch one:

Proof of Proposition 4.3. By a result of Gruson and Peskine [1982, Lemma 2.3] the ideal of $(k + 1) \times (k + 1)$ minors of $C_{v,w}$ is the same as that of $C_{v+(w-k-1),k+1}$; so it suffices to treat the case $k = w - 1$. Of course

$$\mathrm{Cat}(v, w)_{w-1} = \bigcup_{\lambda \in \mathbf{P}(W^*)} \{\phi \in \mathrm{Cat}(v, w) \mid \lambda\phi = 0\},$$

and each of the terms in brackets is a plane of codimension v by 1-genericity, and thus of projective dimension $w - 2 = k - 1$. It is thus enough to show that these are the k-secant planes.

Let $\lambda = (\lambda_0, \ldots, \lambda_{w-1})$ in projective coordinates dual to the coordinates on $\mathbf{P}W$ for which $\mathrm{Cat}(v, w)$ is represented by a matrix of linear forms $C_{v,w}$. We may associate to λ the $k = w - 1$ points of $\mathbf{P}^1$ which are the homogeneous roots of the polynomial

$$\lambda_0 s^{w-1} + \lambda_1 s^{w-2} t + \cdots + \lambda_{w-1} t^{w-1}.$$

Elementary computation shows that the plane

$$\{\phi \in \mathrm{Cat}(v, w) \mid \lambda\phi = 0\}$$

contains these $w - 1$ points in the rational normal curves $\mathbf{P}^1 \ni (s, t) \mapsto$

$(s^{v+w-2}, s^{v+w-3}t, \ldots) \in \mathbf{P}^{v+w-2}$. It follows that they each occur with correct multiplicity by degenerating from the case where they all occur with multiplicity 1.

That $\mathrm{Cat}(v, w)_{w-1}$ is reduced (even normal) follows because $\mathrm{Cat}(v, w)$ is 1-generic.

b) *Symmetric matrices*

Another way of producing 1-generic families, in case $v = w$, is to choose an identification of V with W^* and take $\mathbf{M} = \mathrm{Sym}(v)$ to be the projective space of symmetric matrices with respect to this identification. As is well-known, we have

$$\operatorname{codim}_{\mathbf{M}} \mathbf{M}_k = \binom{v-k+1}{2}.$$

Thus, in this family of examples, $\operatorname{codim} M_1 = v + w - 3 = 2v - 3$ iff $v = 3$. Of course for any v, $\mathbf{M}_1$ is the diagonal embedding of $\mathbf{P}^{v-1}$ in $H_1 = \mathbf{P}^{v-1} \times \mathbf{P}^{v-1}$. We have

PROPOSITION 4.4. *Let $S = \mathbf{P}^{v-1}$ be embedded in $\mathbf{P}^{v-1} \times \mathbf{P}^{v-1} = \mathbf{H}_1$ as the diagonal. If $\mathbf{M}$ is the span of S, then $\mathbf{M} = \mathrm{Sym}(v)$, $\mathbf{M}$ is 1-generic, and $\mathbf{M}_1 = S$.*

Proof of Proposition 4.4. By Proposition 4.1, M is 1-generic. Further, since S is cut out in $\mathbf{P}^{v-1} \times \mathbf{P}^{v-1}$ by the bilinear equations $x_iy_j = x_jy_i$, or $Z_{ij} = Z_{ji}$ in terms of natural coordinates on $\mathbf{H} = \mathbf{P}(V^* \otimes V^*)$, we see that $S = \mathbf{M}_1$ and $\mathbf{M} \subset \mathrm{Sym}(v)$. To show that $\mathbf{M} = \mathrm{Sym}(v)$ it is enough to show that no linear form is contained in the radical of the ideal I generated by the 2×2 minors of a generic symmetric matrix. But it is a well-known fact that I is prime. The easiest way of checking what we need is to note that if Z_{ij} are the elements of the generic symmetric matrix, then $k[Z_{ij}]/I$ maps to the even-degree part of the polynomial ring $k[x_1, \ldots, x_v]$ by $Z_{ij} \to x_ix_j$, and this map (which is actually an isomorphism) obviously sends no linear from to 0. □

c) *The case $w = 2$: Scrollar spaces.*

If $w = 2$ and $\mathbf{M}$ is any 1-generic space, then since $w - 1 = 1$, Theorem 2.1 applies. Thus we see that automatically $\mathbf{M}_1$ is smooth, $I_2(M)$ is prime, so that $\mathbf{M}_1$ is nondegenerate in $\mathbf{M}$, and, since $\mathbf{H}_1 \cap \mathbf{M} = \mathbf{M}_1$ is proper, $\mathbf{M}_1$ is arithmetically Cohen-Macaulay, and of degree equal to the degree of $\mathbf{H}_1$, which is $v + w - 2 = \operatorname{codim}_{\mathbf{M}} \mathbf{M}_1 + 1$.

Now for any nondegenerate projective variety $T \subset \mathbf{M}$ of codimension c we have degree $T \geq c + 1$, and the examples for which equality holds are classified by a famous theorem of Del Pezzo [1896], who proved it for surfaces, and Bertini [1907]; see Eisenbud-Harris [1987] for a modern account. The Del Pezzo-Bertini Theorem says that such a variety is either: $\mathbf{P}^2$ embedded in itself; or the Veronese surface, which is $\mathbf{P}^2$ embedded in $\mathbf{P}^5$ by the complete system of conics; or a nonsingular quadric hypersurface; or a *rational normal scroll*—that is a projectivised vector bundle over $\mathbf{P}^1$ of the form

$$S = S(a_1, \ldots, a_d) = \mathbf{P}(\mathcal{O}_{\mathbf{P}^1}(a_1) \otimes \cdots \otimes \mathcal{O}_{\mathbf{P}^1}(a_d)) \xrightarrow{\pi} \mathbf{P}^1,$$

with

$$1 \leq a_1 \leq \cdots \leq a_d, \qquad d = \dim S,$$

embedded by the complete linear series associated to $\mathcal{O}_S(1)$, the tautological line bundle on the projectivised vector bundle S. The degree of S in this embedding is $\Sigma_1^d a_i$.

Of these varieties of minimal degree, the only ones that admit nontrivial maps to $\mathbf{P}^1(=\mathbf{P}(W)$ if $w = 2)$ are the scrolls, and it turns out that these really do appear in 1-generic spaces. To give the embedding explicitly, and for later use, we recall that the Picard group of $S = S(a_1, \ldots, a_d)$ is generated by the line bundles $\mathcal{O}_S(1)$ and $\pi^*\mathcal{O}_{\mathbf{P}^1}(1)$; if $d > 1$ it is free on these two generators, while if $d = 1$, $S = S(a_1)$, we have $\mathcal{O}_S(1) = \pi^*\mathcal{O}_{\mathbf{P}^1}(a_1)$. We recall as well that $\pi_*\mathcal{O}_S(1) \cong \bigoplus \mathcal{O}_{\mathbf{P}^1}(a_i)$. All these facts are very special cases of general remarks about projectivized vectorbundles; see for example Hartshorne [1977] Chapter II section 7 and Chapter V section 2, for some of the basic theory.

PROPOSITION 4.5. *Given $S = S(a_1, \ldots, a_d)$ as above, with $v = \Sigma a_i$, $1 \leq a_1 \leq \cdots \leq a_d$, S may be embedded in $\mathbf{H}_1 \cong \mathbf{P}^{v-1} \times \mathbf{P}^1$ by the complete linear series $\mathcal{O}_S(1) \otimes \pi^*\mathcal{O}_{\mathbf{P}^1}(-1)$ in the first factor and the complete linear series $\pi^*\mathcal{O}_{\mathbf{P}^1}(1)$ in the second factor. If we write $\mathbf{M} = \mathbf{M}(a_1, \ldots, a_d)$ for the linear span in $\mathbf{H}$ of S then we have:*

i) $\mathbf{M}$ *is* 1-*generic*
ii) $\dim \mathbf{M} = \nu - 1 + d$
iii) $\mathbf{M}_1 = S$

We call $\mathbf{M} = \mathbf{M}(a_1, \ldots, a_d)$ *a scrollar space of transformations.*

Remark. In more down-to-earth but less invariant terms, the scrollar space $\mathbf{M}(a_1, \ldots, a_d)$ may be described by giving a corresponding matrix of linear forms involving $m + 1 = \nu + d$ variables:

$$\left[\begin{array}{ccc|ccc|cc} x_{1,0} & \cdots & x_{1,a_1-1} & x_{2,0} & \cdots & x_{2,a_2-1} & & x_{d,a_d-1} \\ & & & & & & \cdots & \\ & & & & & & & \\ x_{1,1} & \cdots & x_{1,a_1} & x_{2,1} & \cdots & x_{2,a_2} & & x_{d,a_d} \end{array}\right]$$

Proof of Proposition 4.5.
Since each $a_i \geq 1$ the multiplication map

$$[H^0(\mathcal{L}_1) := (\oplus H^0\mathcal{O}_{\mathbf{P}^1}(a_i - 1))] \otimes [H^0(\mathcal{L}_2) := H^0\mathcal{O}_{\mathbf{P}^1}(1)]$$
$$\to [H^0(\mathcal{L}_1 \otimes \mathcal{L}_2) = H^0\mathcal{O}_S(1) = \oplus H^0\mathcal{O}_{\mathbf{P}^1}(a_i)]$$

is obviously onto, so the embedding of S in H corresponds to the complete linear series $(\mathcal{L}, L = H^0\mathcal{L})$ with $\mathcal{L} = \mathcal{O}_S(1)$. It follows that the span $\mathbf{M}$ of S has

$$\dim \mathbf{M} = h^0(\mathcal{O}_S(1)) - 1 = \sum_1^d (a_i + 1) - 1$$
$$= \nu + d - 1,$$

and that S is embedded in $\mathbf{M}$. Since the two projections of S to $\mathbf{P}^{\nu-1}$ and $\mathbf{P}^1$ are by construction nondegenerate, it follows from Proposition 4.1 that $\mathbf{M}$ is 1-generic. But then by the remarks at the beginning of this section, $\mathbf{M}_1$ is an irreducible and reduced variety of dimension

$$\dim \mathbf{M}_1 = \dim \mathbf{M} - \operatorname{codim} \mathbf{M}_1$$
$$= (\nu + d - 1) - (\nu - 1)$$
$$= d,$$

and since $\mathbf{M}_1 \supseteq S$, a variety of the same dimension, $\mathbf{M}_1 = S$. This completes the proof. □

d) *Cases with finite classification*

It is interesting to ask for what triples v, w, k there are only finitely many equivalence classes of k-generic matrices. The cases $k = w$ and $w = 1$, $k = 0$ are trivial such examples. Also, if $v = w$, $k = w - 1$ then by Proposition 1.3 $\dim \mathbf{M} \geq \dim \mathbf{H} - 1$, so $\mathbf{M}^\perp$ is a point $\{\psi\}$, with ψ nonsingular. Thus the only equivalence classes are those of H itself and of the transformations having trace 0 with respect to some fixed identification of V and W.

At least when the ground field is algebraically closed, there is only one more family of cases, the rational normal scrolls.

PROPOSITION 4.6. *Suppose* $0 < k < w$. *The set of equivalence classes of k-generic spaces* $\mathbf{M} \subset \mathbf{H}$ *is finite if and only if either*

i) $v = w = k + 1$,
ii) $w = 2$, $k = 1$.

Remark. As noted, there is a unique example in type i). The case of type ii) is the Scrollar case already treated.

Proof of Proposition 4.6. Let $m = k(v + w - k) - 1$, the smallest dimension of a k-generic space. Tedious computation shows that the Grassmann variety

$$\mathrm{Gr}(m, \mathbf{P}^{vw-1})$$

has dimension > then the dimension of $\mathrm{PGL}(V) \times \mathrm{PGL}(W)$ (so that the number of orbits must be infinite) except in the cases described by the proposition *and* the case $v = w + 1$, $k = w - 1$. In this last case $M^\perp$ is a line in the space $H^\vee - H^\vee_{w-1}$. It is easy to see directly that for $w > 2$ there are infinitely many orbits of such lines (a representative line is spanned by a pair of matrices

$$\begin{bmatrix} 1 & 0 & \cdots & & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & \\ & & & 1 & 0 \end{bmatrix}, \qquad \left[\begin{array}{ccc|c} 0 & \cdots & 0 & 1 \\ \hline & & & 0 \\ & A & & \vdots \\ & & & 0 \end{array}\right],$$

and the ratios of the $w-1$ eigenvalues of the matrix

$$\begin{pmatrix} 0 \cdots 0 \\ A \end{pmatrix}$$

are invariants of the orbit).

5. Classification of the 1-generic spaces M with codim $M_1 \leqslant v + w - 3$. The following is the second main result of this paper; it was first conjectured by Craig Huneke:

THEOREM 5.1. *If* $\mathbf{M} \subset \mathbf{H} = \mathbf{P}(\mathrm{Hom}(V, W))$ *is a 1-generic space with* $\mathrm{codim}_{\mathbf{M}} \mathbf{M}_1 \leq v + w - 3$, *then*

i) *if* $\dim \mathbf{M}_1 = 1$, *then* $\mathbf{M}$ *is equivalent to the catalecticant space* $\mathrm{Cat}(v, w)$. *Else either*

ii) $\dim \mathbf{M}_1 = 2$, $v = w = 3$, *and* $\mathbf{M}$ *is the space of symmetric matrices with respect to some identification of* V^* *and* W (*any two such are of course equivalent*); *or*

iii) $w = 2$, *and* $\mathbf{M}$ *is equivalent to a unique scrollar space* $\mathbf{M}(a_1, \ldots, a_d)$ *with* $1 \leq a_1 \leq \cdots \leq a_d$, $\Sigma_1^d a_i = v$, *and* $d = \dim \mathbf{M}_1$.

COROLLARY 5.2. *Let X be a reduced irreducible projective variety, and let* $(\mathcal{L}_i, V_i)$, $i = 1, 2, 3$, *be linear series with* $\dim V_i \geq 2$ *such that* $\mathcal{L}_3 = \mathcal{L}_1 \otimes \mathcal{L}_2$, *and* $V_3 \supset \mathrm{im}(V_1 \otimes V_2)$. *The natural map*

$$|V_1| \times |V_2| \to |V_3|$$

is onto iff (*up to base locus*) *there exists a pencil* $(\mathcal{L}, V)$ *such that*

$$\mathcal{L}_i = \mathcal{L}^{\otimes(\dim V_i - 1)}, \qquad V_i = \mathrm{Sym}_{(\dim V_i - 1)}(V);$$

that is, iff *each* $(\mathcal{L}_i, V_i)$ *is the composite of the pencil and the complete series of degree* $\dim V_i - 1$ *on* $\mathbf{P}^1$.

Proof. Suppose that

$$|V_1| \times |V_2| \to |V_3|$$

is onto. It follows at once that $\dim V_3 \leqslant \dim V_1 + \dim V_2 - 1$, and since $V_1 \otimes V_2 \to V_3$ is 1-generic, the opposite equality must hold as well. The image of X under the rational map φ_{V_3} associated to $(\mathcal{L}_3, V_3)$ must have dimension at least 1, and is contained in the rank 1 locus associated to the pairing $V_1 \otimes V_2 \to V_3$. Since codim $\varphi_{V_3}(X) \subset |V_3|^\vee$ is thus $\leq$ dim V_1 + dim $V_2 - 3$, we must have equality by Corollary 3.3, and the rank $= 1$ locus of the pairing must also have this minimal codimension, and dimension 1. By Theorem 5.1, the image of X is then the rational normal curve in $|V_3|$, and the pairing is the catalecticant pairing, whence the identification of the $(\mathcal{L}_i, V_i)$.

The converse assertion is immediate. □

Remark. The scrollar space $\mathbf{M}(v)$ is the catalecticant space $\mathrm{Cat}(v, 2)$. This coincidence will serve to start an induction on w for the proof of i).

Proof of Theorem 5.1. Of course by Corollary 3.3 the codimension of every component of $\mathbf{M}_1$ in $\mathbf{M}$ is $\geq v + w - 3$, so the hypothesis is equivalent to the hypothesis that there exists a component S of $\mathbf{M}_1$ of codimension $= v + 2 - 3$. By Corollary 3.3, such a component must be smooth.

We will prove the theorem by analyzing the projection maps

$$S \subset \mathbf{M}_1 = \mathbf{M} \cap \mathbf{H}_1 \subset \mathbf{H}_1 = \mathbf{P}(V^*) \times \mathbf{P}(W) \quad \begin{matrix} \xrightarrow{\pi_1} \mathbf{P}(V^*) \\ \xrightarrow{\pi_2} \mathbf{P}(W) \end{matrix},$$

on S and we write, throughout, $(\mathcal{L}_i, L_i \subset H^0(\mathcal{L}_i))$ for the linear series on S defining the maps $\pi_i|_S$. Since $\mathbf{P}(V^*) \times \mathbf{P}(W)$ is embedded in $\mathbf{H}$ by the complete linear series $\pi_1^*\mathcal{O}_{\mathbf{P}(V^*)}(1) \otimes \pi_2^*\mathcal{O}_{\mathbf{P}(W)}(1)$, S will be embedded in $\mathbf{H}$ by the series

$$\mathcal{L} = (\mathcal{L}_1 \otimes \mathcal{L}_2, L \subset H^0(\mathcal{L})),$$

where L is the image of the multiplication map $L_1 \otimes L_2 \to H^0(\mathcal{L}_1) \otimes H^0(\mathcal{L}_2) \to H^0(\mathcal{L})$.

We begin with the case $w = 2$, and prove iii). As explained at the beginning of section 4c, $\mathbf{M}_1$ is in this case reduced and irreducible, and spans $\mathbf{M}$. By Proposition 4.1, both π_1 and π_2 are nondegenerate maps on $\mathbf{M}_1$. As noted in the beginning of section 4c, $\mathbf{M}_1$ is a smooth variety of minimal degree in $\mathbf{M}$, whose degree is v. So $\mathbf{M}_1 \subset \mathbf{M}$ is either the Veronese surface or a rational normal scroll $S(a_1, \ldots, a_d)$ of degree $\Sigma a_i = v$ or a smooth quadric hypersurface; of these the first is impossible since the Veronese surface, being isomorphic to $\mathbf{P}^2$, admits no nontrivial map to $\mathbf{P}^1$, and the last, which is possible only if $v = 2$ is subsumed in the first since then $\dim \mathbf{M} \leq \dim \mathbf{H} = 3$, and the quadrics in $\mathbf{P}^3$ and $\mathbf{P}^2$ are also scrolls.

Suppose then that $\mathbf{M}_1 \cong S = S(a_1, \ldots, a_d)$. Since $\mathcal{L}_1 \otimes \mathcal{L}_2$ gives the embedding of S as a scroll in $\mathbf{M}$, and both have at least two sections, we see that we may write one (or in case $d = 1$ both) of the $\mathcal{L}_i$ in the form $\pi^*\mathcal{O}_{\mathbf{P}^1}(a)$, with $a > 0$, and the other has the form $\mathcal{O}_S(1) \otimes \pi^*\mathcal{O}_{\mathbf{P}^1}(-a)$.

Using the fact that $\mathcal{O}_S(1) \otimes \pi^*\mathcal{O}_{\mathbf{P}^1}(-a)$ must be generated by global sections, and taking π_*, we see that

$$\left(\bigoplus_1^d \mathcal{O}_{\mathbf{P}^1}(a_i)\right) \otimes \mathcal{O}_{\mathbf{P}^1}(-a)$$

is generated by global sections, whence $a_i \geq a$ for all i.

We now wish to show that $\mathcal{L}_2$ may be taken to have the form $\pi^*\mathcal{O}_{\mathbf{P}^1}(a)$. Suppose not, so that in particular $d \geq 2$. But then $\mathcal{L}_1$ has the form $\pi^*\mathcal{O}_{\mathbf{P}^1}(a)$ so

$$\sum_1^d a_i = v \leq h^0\mathcal{L}_1 = a + 1.$$

Since each a_i is $\geq a$ and $d \geq 2$ we see that $d = 2$, $a = a_1 = a_2 = 1$, $S = S(1, 1) = \mathbf{P}^1 \times \mathbf{P}^1 \subset \mathbf{P}^3$. Now $S(1, 1)$ is a projectivized vector bundle in two ways corresponding to the two projections π, π' to $\mathbf{P}^1$, and

$$\mathcal{O}_S(1) = \pi^*\mathcal{O}_{\mathbf{P}^1}(1) \otimes \pi'^*\mathcal{O}_{\mathbf{P}^1}(1),$$

so in this case $\mathcal{L}_2$ has the form $\pi'^*\mathcal{O}_{\mathbf{P}^1}(1)$. Replacing π with π', we get the desired form even in this case, and we write

$$\mathcal{L}_1 = \mathcal{O}_S(1) \otimes \pi^*\mathcal{O}_{\mathbf{P}^1}(-a), \qquad \mathcal{L}_2 = \pi^*\mathcal{O}_{P^1}(a), \qquad a > 0.$$

From the nondegeneracy of π_1 we get

$$\begin{aligned}\sum_1^d a_i = v &\leq h^0\mathcal{L}_1 \\ &= h^0(\oplus\, \mathcal{O}_{\mathbf{P}^1}(a_i - a)) \\ &= \Sigma\,(a_i - a + 1),\end{aligned}$$

since $a_i - a \geq 0$. It follows that $a = 1$, so $\mathcal{L}_1 = \mathcal{O}_S(1) \otimes \pi^*\mathcal{O}_{\mathbf{P}^1}(-1)$, $\mathcal{L}_2 = \pi^*\mathcal{O}_{\mathbf{P}^1}(1)$ and that $L_1 = H^0(\mathcal{L}_1)$. Further, since $\dim L_2 \geq 2$, and $h^0(\pi^*\mathcal{O}_{\mathbf{P}^1}(1)) = 2$, we get $L_2 = H^0\mathcal{L}_2$. This completes the proof of Theorem 5.1 and case $w = 2$.

We now turn to part i) and assume $\dim S = 1 = \dim \mathbf{M}_1$, $\dim \mathbf{M} = v + w - 2$. We use the description of $\mathrm{Cat}(v, w)$ given in Proposition 4.2, so that we wish to show that

$$\mathbf{M}_1 = S;$$

$$\mathbf{M}_1 \text{ is spanned by } S;$$

$$S \cong \mathbf{P}^1;$$

and that

$$(\mathcal{L}_1, V_1) = (\mathcal{O}_{\mathbf{P}^1}(v - 1), H^0\mathcal{O}_{\mathbf{P}^1}(v - 1))$$

and

$$(\mathcal{L}_1, V_2) = (\mathcal{O}_{\mathbf{P}^1}(w - 1), H^0\mathcal{O}_{\mathbf{P}^1}(w - 1)),$$

are the complete linear series associated to $\mathcal{O}_{\mathbf{P}^1}(v - 1)$ and $\mathcal{O}_{\mathbf{P}^1}(w - 1)$.

We do induction on $w \geq 2$. If $w = 2$, then Theorem 1.4 gives exactly the desired conclusion, since the only curve of the form $S(a_1, \ldots, a_d)$, $\Sigma\, a_i = v$ is the rational normal curve $S(v)$, with $\mathcal{O}_S(1) = \mathcal{O}_{\mathbf{P}^1}(v)$.

We first note that $\mathbf{M}_1$ cannot contain a line (and in particular S is not a line). Indeed, let (L_{ij}) be a matrix of linear forms in $v + w - 1$ variables x_i corresponding to $\mathbf{M}$. If M_1 contained the line $Z_1 = \cdots = Z_{v+w-3} = 0$, where the Z_j are independent linear forms in the x_i, then 2×2 minors of (L_{ij}) would be contained in the ideal $(Z_1, \ldots, Z_{v+w-3})$, so that modulo $Z_1, \ldots, Z_{v+w-3}$, (L_{ij}) would be equivalent either to a matrix with only one nonzero row or a matrix with only one nonzero column (note that this argument is special to the 2×2 minors of a matrix of linear forms!). But this implies that after some (scalar) row and column operations, some $v \times (w - 1)$ submatrix of (L_{ij}) or some $(v - 1) \times w$ submatrix of (L_{ij}) will contain only linear forms in the $Z_1, \ldots, Z_{v+w-3}$, and will thus correspond to a $v + w - 4$ dimensional 1-generic space of $v \times (w - 1)$ or $(v - 1) \times w$ matrices. Such a space cannot exist by Proposition 1.3, so $\mathbf{M}_1$ contains no lines, as claimed.

Consider again the two projections π_i on $\mathbf{H}_1$. If $x \in \mathbf{P}(V^*)$ or $\mathbf{P}(W)$ then $\pi_i^{-1}(x)$ is a linear subspace of $\mathbf{H}_1$, so the scheme $\mathbf{M}_1 \cap \pi_i^{-1}(x) = \mathbf{M} \cap \pi_i^{-1}(x)$ is a linear space which must be a point by our previous remarks. Thus the projections π_i are both 1 to 1 on $\mathbf{M}_1$, with reduced points as scheme-theoretic fibers.

Choose a point $p \in S$, and let $W_1 \subset W$ be the 1-dimensional space corresponding to the point $\pi_2(p)$. Consider, with notation as in Proposition 1.2, the projection $\pi_{V,W_1} : \mathbf{H} \to \bar{\mathbf{H}} = \mathbf{P}(\mathrm{Hom}(V, W/W_1))$, which is the projection from the subspace $L = \mathbf{P}\,\mathrm{Hom}(V, W_1) \subset \mathbf{H}$. Since L meets S in a single, reduced point by the argument above, the curve $\bar{S} := \pi_{V,W_1}(S) \cap \pi_{V,W_1}(\mathbf{M})\,\bar{\mathbf{H}}_1$ has degree exactly one less than that of S. $\bar{\mathbf{M}} := \pi_{V,W_1}(\mathbf{M})$ is 1-generic by Proposition 1.2; by induction then, $\bar{S} = \bar{\mathbf{M}} \cap \bar{\mathbf{H}}_1$ is a rational normal curve of degree $v + (w - 1) - 2$ in $\bar{\mathbf{M}}$. It follows that the degree of S is $v + w - 2$, so the span of S has dimension $v + w - 2 = \dim \mathbf{M}$. Thus $\mathbf{M}$ is the span of S, and S is a rational normal curve in $\mathbf{M}$.

Further, we have a commutative diagram of (rational) maps:

$$
\begin{array}{ccccc}
 & \nearrow^{\pi_1} & \mathbf{P}(V^*) & \xrightarrow{\quad = \quad} & \mathbf{P}(V^*) \\
p \in S \subset \mathbf{H}_1 & & \xrightarrow{\qquad \pi_{V,W_1} \qquad} & \bar{\mathbf{H}}_1 \supset & \bar{S} \cong \mathbf{P}^1 \quad (\bar{\pi}_1 \nearrow,\ \bar{\pi}_2 \searrow) \\
 & \searrow_{\pi_2} & \mathbf{P}(W) & \xrightarrow[\text{projection from } \pi_2(p)]{} & \mathbf{P}(W/W_1)
\end{array}
$$

and $\bar{\pi}_1$ and $\bar{\pi}_2$ are given on $\bar{S}$ by $\mathcal{O}_{\mathbf{P}^1}(v)$ and $\mathcal{O}_{\mathbf{P}^1}(w - 1)$ respectively. It follows that π_1 and π_2 are given on S by the desired maps.

It remains to show that $\mathbf{M}_1$ contains only S. Suppose on the contrary that $q \in \mathbf{M}_1 - S$. Of course $q \notin L$, since $L \cap M = \{p\}$, so $\pi_{V,W_1}(q)$ is well defined. Since $\pi_1^{-1}(\pi_1(q)) \cap \mathbf{M} \subset \mathbf{M}_1$ is a linear space, and $\mathbf{M}_1$ contains no lines, it consists of q alone. Thus by the commutativity of the above diagram, $\pi_{V,W_1}(q) \notin \bar{S}$. However $\pi_{V,W_1}(q) \in \bar{\mathbf{M}} \cap \bar{\mathbf{H}}_1$, which is $\bar{S}$ by induction, contradicting our hypothesis $q \notin S$. This completes the proof of i).

To finish the proof, we suppose that $w > 2$ and $\dim \mathbf{M}_1 \geq 2$ so that $\dim \mathbf{M} \geq v + w - 1$. We must show that we are in case ii).

First suppose $\dim \mathbf{M}_1 = 2$, $\dim \mathbf{M} = v + w - 1$. Since $\dim \mathbf{M} > v + w - 2$, if $\mathbf{H}' \subset \mathbf{H}$ is a general hyperplane, then $\mathbf{M}' = \mathbf{H}' \cap \mathbf{M} \subset \mathbf{H}$ will again be 1-generic, and $\mathbf{M}_1'$, which is $\mathbf{H}' \cap \mathbf{M}_1$, will have dimension 1, so we may apply part i); we see that $\mathbf{M}_1' \cong \mathbf{P}^1$ is a rational normal curve projecting isomorphically via π_1 and π_2 to rational normal curves in $\mathbf{P}(V^*)$ and $\mathbf{P}(W)$. Of course $\mathbf{M}_1' = \mathbf{H}' \cap S = \mathbf{H}' \cap M_1$. It follows that S is a nondegenerate surface of degree $v + w - 2$ in $\mathbf{M} \cong \mathbf{P}^{v+w-1}$. By the Del

Pezzo-Bertini theorem, since S is smooth, S is either a rational normal scroll $S(a_1, a_2)$ with $1 \leq a_1 < a_2$, say, and $a_1 + a_2 = \deg S = v + w - 2$, or else $v + w - 1 = 5$ and S is the Veronese surface.

Now suppose that S is the Veronese surface. The restrictions of the series $(\mathcal{L}_i, L_i)$ to $\mathbf{M}_1' = \mathbf{P}^1$ are the complete series $\mathcal{O}_{\mathbf{P}^1}(v-1)$ and $\mathcal{O}_{\mathbf{P}^1}(w-1)$, so $\dim L_1 = v$ and $\dim L_2 = w$. On the other hand $S \cong \mathbf{P}^2$, so $\mathcal{L}_i \cong \mathcal{O}_{\mathbf{P}^2}(a_i)$ for some a_i, and $\mathcal{L}_1 \otimes \mathcal{L}_2 = \mathcal{O}_{\mathbf{P}^2}(a_1 + a_2) \cong \mathcal{O}_{\mathbf{P}^2}(2)$, since S is embedded in H by conics. It follows that $a_1 = a_2 = 1$, and since $v \geq w \geq 3$, we get $v = w = 3$ and $L_i = H^0\mathcal{O}_{\mathbf{P}^2}(1)$, that is, $(\mathcal{L}_i, L_i)$ is the complete series of lines, mapping S isomorphically to $\mathbf{P}^2$ as required.

If S is not the Veronese, then $S = S(a_1, a_2)$ as above. Let $\pi : S \to \mathbf{P}_1$ be the structure map of S as a projectivised vectorbundle. As in the proof of Proposition 4.5 we may assume that

$$\mathcal{L}_{i_1} \cong \mathcal{O}_S(1) \otimes \pi^*\mathcal{O}_{\mathbf{P}^1}(-a)$$

$$\mathcal{L}_{i_2} \cong \pi^*\mathcal{O}_{\mathbf{P}^1}(a)$$

for some $a \geq v - 1$, $\{i_1, i_2\} = \{1, 2\}$.

Since the restriction of $\mathcal{L}_{i_1}$ to the hyperplane section $\mathbf{M}_1' \cong \mathbf{P}^1$ is $\mathcal{O}_{\mathbf{P}^1}(v-1)$ or $\mathcal{O}_{\mathbf{P}^1}(w-1)$ depending on whether $i_1 = 1$ or 2, we get $a = w - 1$ or $v - 1$ respectively in these two cases.

Since $\mathcal{L}_2$ is generated by global sections, the same goes for

$$\pi_*\mathcal{L}_2 \cong \mathcal{O}_{\mathbf{P}^1}(a_1 - a) \oplus \mathcal{O}_{\mathbf{P}^2}(a_2 - a),$$

so we get $a \leq a_1 \leq a_2$. Since $a + a_2 = (v-1) + (w-1)$, and $w \leq v$, we see that $a = w - 1$ and $i_1 = 1$, $i_2 = 2$.

Now the hyperplane $\mathbf{H}'$ meets each fiber of π transversely in a single point, so $\pi|_{\mathbf{M}_1'} : \mathbf{M}_1' \to \mathbf{P}^1$ is an isomorphism.

Since the linear system $(\mathcal{L}_1, L_1)$ induces the complete linear system of degree $v - 1$ on $\mathbf{M}_1'$, we see, taking π_*, that the induced map

$$\mathcal{O}_{\mathbf{P}^1}(a_1 - a) \oplus \mathcal{O}_{\mathbf{P}^1}(a_2 - a) = \pi_*\mathcal{L}_2 \to \pi_*(\mathcal{L}_2|_{\mathbf{M}_1'}) = \mathcal{O}_{\mathbf{P}^1}(v-1)$$

is onto on global sections. Identifying the global sections of $\mathcal{O}_{\mathbf{P}^1}(n)$ with forms of degree n in 2 variables, the above map is given by multiplication on $\mathcal{O}_{\mathbf{P}^1}(a_i - a)$ by a form f_i of degree $v - 1 - a_i + a = v - 1 - a_i + a =$

$v + w - 2 - a_i$. But quite generally, if two forms of degrees $n_1, n_2 > 0$ generate all forms of degrees n, then $n \geq n_1 + n_2 - 1$; so we get

$$v - 1 \geq (v + w - 2 - a_1) + (v + w - 2 - a_2) - 1$$

or, using $a_1 + a_2 = v + w - 2$,

$$2 \geq w,$$

contradicting our hypothesis. It follows that this case does not occur, completing the proof in case $w > 2$, $\dim M_1 = 2$.

Finally, suppose $\dim \mathbf{M}_1 \geq 3$. Cutting with hyperplanes we may assume that $\dim \mathbf{M}_1 = 3$, and that the hyperplane section of S is the Veronese surface. It is well-known that the Veronese is *not* the hyperplane section of a smooth variety, so we are done. (Proof sketch: If it were, the variety would be a scroll by the del Pezzo-Bertini theorem. But the scroll contains (many) linear spaces of codimension 1, whose intersections with the hyperplane would be lines on the Veronese. However the curves on the Veronese all have even degree—they are duple embeddings of plane curves.) □

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