DETERMINANTAL EQUATIONS FOR CURVES OF HIGH DEGREE

By David Eisenbud,* Jee Koh* and Michael Stillman*

Introduction. Let $C \subset \mathbf{P}^r$ be a curve of degree d and arithmetic genus g. We show that if $d \geq 4g + 2$, then the homogeneous ideal of C is generated by the 2×2 minors of a matrix of linear forms—in infinitely many ways if $g > 0$. We give examples of the determinantal representations of elliptic and hyperelliptic curves and some others. In an appendix with Joe Harris we describe all reduced irreducible curves C and torsion free sheaves $\mathfrak{F}$ on them such that $2(h^0(\mathfrak{F}) - 1) = \deg \mathfrak{F}$, the situation of equality in Clifford's Theorem; there is one "new" family, where $\mathfrak{F}$ is not locally free.

We work over an algebraically closed field. Let X be a scheme. Suppose that

$$\mathcal{L}_1 = (L_1, V_1)$$

$$\mathcal{L}_2 = (L_2, V_2)$$

are two linear series on X (that is, L_i are line bundles on X, and $V_i \subset H^0(X, L_i)$ are finite dimensional vector spaces). Let

$$\mathcal{L}_1\mathcal{L}_2 = (L_1 \otimes L_2, V = \operatorname{im}(\mu\colon V_1 \otimes V_2 \to H^0(L_1 \otimes L_2))$$

be the product series, where $\mu\colon V_1 \otimes V_2 \to H^0(L_1 \otimes L_2)$ is the multiplication map. Let $\varphi_{\mathcal{L}_1\mathcal{L}_2}$ be the rational map from X to $\mathbf{P}(V)$ associated to $\mathcal{L}_1\mathcal{L}_2$.

The elements of V are linear forms on $\mathbf{P}(V)$, so if we choose bases $\{e_i\}$ and $\{f_j\}$ of V_1 and V_2, respectively, then the matrix

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$$M = (\mu(e_i \otimes f_j))$$

is a matrix of linear forms. We write $I_2(M) = I_2(\mathcal{L}_1, \mathcal{L}_2)$ for the ideal generated by the 2×2 minors of M in the homogeneous coordinate ring $S = \operatorname{Sym}(V)$ of $\mathbf{P}(V)$, and $I(\varphi_{\mathcal{L}_1\mathcal{L}_2}(X))$ for the (maximal) homogeneous ideal of $\varphi_{\mathcal{L}_1\mathcal{L}_2}(X)$. Of course $I_2(\mathcal{L}_1, \mathcal{L}_2)$ is independent of the choices of bases made above. We have, in general,

$$I_2(\mathcal{L}_1, \mathcal{L}_2) \subset I(\varphi_{\mathcal{L}_1\mathcal{L}_2}(X)).$$

(Proof: On a sufficiently fine affine cover we may identify the e_i and f_j with elements of O_X; the fact that the 2×2 minors of M are zero in O_X is then just the commutative law!)

In this paper we shall be concerned with cases where the above inequality becomes an equality; we say that $(\mathcal{L}_1, \mathcal{L}_2)$ is **determinantally presented** if

$$I_2(\mathcal{L}_1, \mathcal{L}_2) = I(\varphi_{\mathcal{L}_1\mathcal{L}_2}(X)).$$

To simplify the notation, if L is a line bundle, we write L again for the complete linear series $(L, H^0(L))$. In the special case $L_1 = L_2 = L$, we may use the following result of Castelnuovo [3] (See also Fujita [6], Green [7], Mattuck [11], Mumford [12] and St. Donat [15]).

THEOREM. *Let C be a reduced, irreducible curve of genus g. If L is a line bundle on C of degree $\geq 2g + 2$, then $I(\varphi_L(C))$ is generated by quadrics.*

Here and throughout this paper, genus means arithmetic genus.

The bound $2g + 2$ is best possible.

The main result of this paper is the following extension of Castelnuovo's Theorem:

THEOREM 1. *Let C be a reduced, irreducible curve of genus g. If L_1 and L_2 are line bundles on C of degree $\geq 2g + 1$, nonisomorphic if $g > 0$ and both have degree $2g + 1$, then (L_1, L_2) is determinantally presented.*

Since a line bundle of degree $4g + 2$ can be split into 2 nonisomorphic line bundles of degree $2g + 1$ in a g-dimensional family of ways, we see for example that any curve of genus > 0, embedded by a complete linear series of degree $4g + 2$ in $\mathbf{P}^{3g+2}$ has equations which may be realized as the 2×2 minors of a $(g + 2) \times (g + 2)$ matrix of linear forms in infinitely many ways! Further, it is not hard to see that the determinantal representation associated to (L_1, L_2) determines the bundles L_1 and L_2 (as the images of the matrix of linear forms and its dual, restricted to the curve), so these "infinitely many ways" are really associated to different orbits of $\mathrm{GL}(g+2) \times \mathrm{GL}(g+2)$, acting on the space of matrices of linear forms.

For the proof of Theorem 1 we use techniques developed by Mark Green [7] (and already used by him to reprove and extend Castelnuovo's Theorem in a related direction). The crucial point is to note that a certain module is presented by a matrix of linear forms; in fact the technique shows that under suitable hypotheses this module has a minimal free resolution whose first k stages consist of matrices of linear forms.

In general, we will say that a graded module M over a graded ring S has **linear free resolution to stage k**, if its minimal free resolution has the form

$$\cdots \to \oplus_{p \geq k+1} S^{n_{k+1,p}}(-p) \to S^{n_k}(-k) \to \cdots \to S^{n_1}(-1) \to S^{n_0} \to M \to 0,$$

and we say that M has **linear free resolution** if this is true for every k.

THEOREM 2. *Let L_1 be a base point free line bundle on the reduced irreducible curve C of genus g, and let L_2 be any line bundle. Let*

$$S = \mathrm{Sym}\, H^0(L_1),$$

$$M = \bigoplus_{\nu \geq 0} H^\circ(L_1{}^\nu L_2).$$

M has a linear free resolution over S if and only if $H^1(L_1{}^{-1}L_2) = 0$. It has a linear resolution to stage k if $\deg L_2 \geq 2g + k$ and either

$$\deg L_1 + \deg L_2 \geq 4g + 2k + 1,$$

or

$$\deg L_1 + \deg L_2 = 4g + 2k,$$

but $L_1{}^{-1}L_2 \neq O_C, \omega_C$, or, if C is hyperelliptic, any other multiple of the $g^1{}_2$.

The condition "C hyperelliptic" at the end of the Theorem is to be interpreted strictly; that is $g \geq 2$ and C admits a 2-1 morphism to $\mathbf{P}^1$ (In particular C is locally Gorenstein, since any double cover of a smooth curve is locally planar).

In case $L_1 = L_2$ and C is smooth, this result is equivalent to Theorem 4.a.1 of Green [7]. The case $k = 0$ is Green's "Explicit H^0 Lemma" 4.e.4 of the same paper. Allowing $L_1 \neq L_2$ is an easy extension (which happily improves the bound on degrees!). Dropping the smoothness hypothesis, on the other hand, seems to require a different proof of the Duality Theorem of Green [7]. We give an algebraic proof adapted to the case at hand.

The second assertion is related to the method of proof of Lemma 1.7 of Gruson, Lazarsfeld, Peskine [8] as pointed out to us by the referee.

In the first section of this paper we give two elementary results—belonging, perhaps, to folklore—explaining the geometry and prevalence of ideals of 2×2 minors.

The proofs of Theorems 1 and 2 above occupy Section 2.

Section 3 contains various explicit examples of Theorems 1 and 2; the well-known "catalecticant" matrices that arise in case $g = 0$, and matrices for certain pairs of line bundles on an elliptic curve. One sees the modulus of the elliptic curve easily in the matrix; the lowest degree case of this was already recognized by Hurwitz [10]. Many examples of determinantal varieties coming from catalecticants with some rows and columns deleted appear in this section. It would be interesting to know more about such loci.

We are grateful to Klaus Hulek for a discussion of the determinantal representation of an elliptic curve, in an early stage of this work; in particular, he pointed out Hurwitz' work to us. Also, we have profited greatly from numerous conversations with Joe Harris; in particular, he pointed out to us that determinants and secants are related as in Proposition (1.3).

The work reported here was strongly influenced by the use of the computer algebra program "Macaulay" by David Bayer and Michael Stillman [1]; we were led to conjecture Theorem 1 by examples it worked out for us.

1. Geometry and 2×2 minors; elementary results. We sketch two results that "explain" the appearance of 2×2 minors, and a result showing a relation to secant loci.

PROPOSITION 1.1. *Let $X \subset \mathbf{P}^r = \mathbf{P}(V)$ be a nondegenerate (that is, $H^0(O_{\mathbf{P}^r}(1)) \to H^0(O_X(1))$ is a monomorphism) subscheme, and let $L = O_X(1)$. There is a 1-1 correspondence between*

(1) *$p \times q$ matrices of linear forms*

$$A = (\ell_{ij}) \qquad \ell_{ij} \in W, \qquad i = 1, \ldots, p, \qquad j = 1, \ldots, q$$

such that $I_2(A) \subset I(X)$, *up to the action of* $\mathrm{GL}(p) \times \mathrm{GL}(q)$, *and* $X - V(I_1(A))$ *is dense in* X, *and*

(2) *Ordered pairs of linear series*

$$(L_1, V_1), \qquad (L_2, V_2)$$

defined on dense sets $U \subset X$ *such that* $L_1 \otimes L_2 = L|_U$, $V_1 \otimes V_2 \to H^0(L)$ *has image contained in* V, *such that* $\dim V_1 = p$, $\dim V_2 = q$, *and* V_i *generates* L_i *on* U.

Proof. The correspondence is given by associating to (L_1, V_1), (L_2, V_2) a matrix of linear forms associated to the pairing $V_1 \otimes V_2 \to V$, and to a matrix of linear forms A the series on $U = X - V(I_1(A))$:

$$L_1 = \mathrm{im}(A\colon O_X{}^p \to O_X(1)^q)|_U$$

$$V_1 = \mathrm{im}\, H^0 O_X{}^p \to H^0 L_1$$

$$L_2 = \mathrm{im}(A^{\mathrm{tr}}\colon O_X{}^q \to O_X(1)^p)|_U$$

$$V_2 = \mathrm{im}\, H^0 O_X{}^q \to H^0 L_2.$$

The proof is left to the reader. □

Next we have an elementary result showing that a suitable multiply of *any* projective embedding of a scheme has ideal generated by the 2×2 minors of a (symmetric) matrix of linear forms. (Compare Mumford [12]).

PROPOSITION 1.2. *Let* $X \subset \mathbf{P}^r$ *be a subscheme, let* $d \geq 2$ *be an integer, and let* $X \to \mathbf{P}^N$ *be the associated d-uple embedding.*

(1) *If* $I(X)$ *is generated by forms of degree* $\leq d$ *then* $I(i_d(X))$ *is generated by* $I(i_d(\mathbf{P}^r))$ (*which in turn is generated by quadrics*), *and linear forms.*

(2) *Further,* $I(X)$ *is generated scheme-theoretically by forms of degree* $\leq d$ *if and only if* $I(i_d(X))$ *is generated scheme-theoretically by* $I(i_d(\mathbf{P}^r))$ *and linear forms.*

Proof. This follows from the fact that the homogeneous ideal of the d-uple embedding of $\mathbf{P}^r$ is generated by quadrics, which are 2×2 minors of a generic symmetric $(r + 1) \times (r + 1)$ matrix in case $d = 2$. □

COROLLARY. *Let* $X \subset \mathbf{P}^r$ *be a subscheme, and let* $\mathcal{L} = (O_X(1), H^0(O_{\mathbf{P}^r}(1))|_X)$.

(1) *If* $(\mathcal{L}, \mathcal{L})$ *is determinantally presented, then* $I(X)$ *is generated scheme-theoretically by forms of degree* ≤ 2.

(2) *If* $I(X)$ *is generated by forms of degree* ≤ 2, *then* $(\mathcal{L}, \mathcal{L})$ *is determinantally presented.*

Proof. Notice that $\varphi_{L^2}(X) = i_2(X)$, and that $I(i_2(\mathbf{P}^r))$ is generated by the 2×2 minors of a generic symmetric $(r+1) \times (r+1)$ matrix. Thus, $(\mathcal{L}, \mathcal{L})$ is determinantally presented if and only if $I(i_2(X))$ is generated by $I(i_2(\mathbf{P}^r))$ and linear forms. The corollary now follows from Proposition (1.2). □

PROPOSITION 1.3. *Let* $A = (\ell_{ij})$ *be a matrix of linear forms over* $S(W)$, *and let* $X = V(I_2(A))$ *be the rank* 1 *locus of* A. *Every* $(k+1)$*-secant* k*-plane to* X *is contained in* $V(I_{k+2}(A))$.

A $(k+1)$-secant k-plane is a k-plane in $\mathbf{P}(W)$ that meets X in $k+1$ distinct points, or a limit of such; $\mathrm{Sec}_K(X)$ is the reduced union of the $(k+1)$-secant k-planes to X.

Proof. (Pointed out to us by J. Harris) A sum of $\leq k+1$ matrices of rank 1 has rank $\leq k+1$. □

Remark. Let C and L_1, L_2 be as in Theorem 1. Let $A = A(L_1, L_2)$, and $X = \varphi_{L_1 \otimes L_2}(C)$. Then $I_2(A) = I(\mathrm{Sec}_0(X))$ by Theorem 1 and for all k, $I_{k+2}(A) \subset I(\mathrm{Sec}_k(X))$ by Proposition (1.3). We conjecture that there exists a good constant k_0 depending only on g and the $\deg L_i$ such that

$$(*) \qquad I_{k+2}(A) = I(\mathrm{Sec}_k(X)) \quad \text{for} \quad k \leq k_0.$$

In the case of $\mathbf{P}^1$, a result of Wakerling (see Eisenbud [4] for a proof) says that (*) holds for all $k \leq \min\{\deg L_i\} - 1$.

On the other hand, for $g \geq 1$ there is a necessary bound on k:

$$g \leq (\deg L_1 - 1 - g - k)(\deg L_2 - 1 - g - k)$$

obtained by comparing bounds on the codimensions of $\mathrm{Sec}_k(X)$ and $V(I_{k+2}(A))$. In particular for $g = 1$, the maximal size minors cannot define the corresponding secant variety.

For the 1-cuspidal rational curve X of arithmetic genus 1, if $\deg L_1 = 4$ and $\deg L_2 = 5$, then using *Macaulay* [1] we found that $I_3(A)$ and the ideal of $\mathrm{Sec}_1(X)$ are generated by 30 cubics. Since $I(\mathrm{Sec}_1(X)) \supseteq I_3(A)$, the two loci must coincide. In case $\deg L_1 = \deg L_2 = 5$, both $I(\mathrm{Sec}_1(X))$ and $I_3(A)$ are generated by 50 cubics, and so $I(\mathrm{Sec}_1(X)) = I_3(A)$.

2. Proofs of the Main Theorems.

Proof of Theorem 1 *from Theorem* 2. We use notation as in the introduction, and in addition we write

$$L_{12} = L_1 \otimes L_2, \qquad L_{112} = L_1 \otimes L_1 \otimes L_2, \qquad \text{etc.}$$

$$V_{12} = H^0 L_{12}, \qquad V_{112} = H^0 L_{112}, \qquad \text{etc.}$$

Note that by Theorem 2 the multiplication map $V_1 \otimes V_2 \to H^0 L_{12}$ is onto, so V_{12} is the same as the V in the definition of $\mathcal{L}_1 \mathcal{L}_2$ in the introduction. By Castelnuovo's Theorem [3] (or indeed by Theorem 2 applied with $L_1 = L_2$), the ideal of C in $\mathbf{P}(V_{12})$ is generated by a vector space I_2 of quadrics, and we must show that these quadrics are spanned by appropriate 2×2 minors.

Consider the diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & K_2 & \longrightarrow & K_1 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \wedge^2 V_1 \otimes \wedge^2 V_2 & \longrightarrow & \mathrm{Sym}_2(V_1 \otimes V_2) & \longrightarrow & \mathrm{Sym}_2(V_1) \otimes \mathrm{Sym}_2(V_2) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \varphi_1} & & \\
0 & \longrightarrow & I_2 & \longrightarrow & \mathrm{Sym}_2(V_{12}) & \longrightarrow & V_{1212} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Here the exact sequence in the bottom row arises simply from the definition of I_2. In the middle row the map

$$\wedge^2 V_1 \otimes \wedge^2 V_2 \longrightarrow \mathrm{Sym}_2(V_1 \otimes V_2)$$

is given by

$$e_1 \wedge f_1 \otimes e_2 \wedge f_2 \longrightarrow (e_1 \otimes e_2) \cdot (f_1 \otimes f_2) - (e_1 \otimes f_2) \cdot (f_1 \otimes e_2),$$

the inclusion of $\wedge^2 V_1 \otimes \wedge^2 V_2$ as the span of the 2×2 minors of the generic matrix, familiar from representation theory, while

$$\mathrm{Sym}_2(V_1 \otimes V_2) \longrightarrow \mathrm{Sym}_2(V_1) \otimes \mathrm{Sym}_2(V_2)$$

is given by

$$(e_1 \otimes e_2) \cdot (f_1 \otimes f_2) \longrightarrow e_1 f_1 \otimes e_2 f_2.$$

The maps φ_1 and φ_2 are induced by the multiplication maps $V_1 \otimes V_2 \to V_{12}$, and φ is the map they induce. K_1 and K_2 are defined to be the kernels of φ_1 and φ_2, while ψ: $K_2 \to K_1$ is the map induced between them.

The conclusion of Theorem 1 is the statement that φ is onto. An obvious diagram chase shows φ is onto if and only if ψ is onto, and this is what we will prove.

Let

$$R(V_1, V_2) = \ker V_1 \otimes V_2 \longrightarrow V_{12}$$

$$R(V_{11}, V_2) = \ker V_{11} \otimes V_2 \longrightarrow V_{112}, \qquad \text{etc.}, \ldots$$

and

$$\bar{R}(V_1) = \ker \mathrm{Sym}_2 V_1 \longrightarrow V_{11}$$

$$\bar{R}(V_2) = \ker \mathrm{Sym}_2 V_2 \longrightarrow V_{22}.$$

Since φ_2 is the symmetric square of the epimorphism $V_1 \otimes V_2 \to V_{12}$, its kernel is the image of

$$V_1 \otimes V_2 \otimes R(V_1, V_2) \longrightarrow V_1 \otimes V_2 \otimes V_1 \otimes V_2 \longrightarrow \mathrm{Sym}_2(V_1 \otimes V_2).$$

On the other hand, φ_1 may be factored as

$$\mathrm{Sym}_2 V_1 \otimes \mathrm{Sym}_2 V_2 \longrightarrow V_{11} \otimes V_{22} \longrightarrow V_{1212},$$

so K_1 is the sum of the images of

$$K'_1 = \bar{R}(V_1) \otimes V_2 \otimes V_2$$

and

$$V_1 \otimes V_1 \otimes \bar{R}(V_2),$$

and the pullback to $\mathrm{Sym}_2\, V_1 \otimes \mathrm{Sym}_2\, V_2$ of

$$K''_1 = R(V_{11}, V_{22}) \subset V_{11} \otimes V_{22}.$$

$$\begin{tikzcd}
K'_1 \arrow[d, equal] & & & & K''_1 \arrow[d, equal] \\
(\bar{R}(V_1) \otimes V_2 \otimes V_2) \oplus (V_1 \otimes V_1 \otimes \bar{R}(V_2)) \arrow[dr] & & & & R(V_{11}, V_{22}) \arrow[dl] \\
& \mathrm{Sym}_2(V_1) \otimes \mathrm{Sym}_2(V_2) \arrow[rr] \arrow[dr] & & V_{11} \otimes V_{22} \arrow[dl] & \\
& & V_{1212} \arrow[dl] \arrow[dr] & & \\
& 0 & & 0 &
\end{tikzcd}$$

It suffices by symmetry to show that the image of $V_1 \otimes V_2 \otimes R(V_1, V_2)$ in $\mathrm{Sym}_2\, V_1 \otimes \mathrm{Sym}_2\, V_2$ contains K'_1 and that the image of $V_1 \otimes V_2 \otimes R(V_1, V_2)$ in $V_{11} \otimes V_{22}$ contains K''_1.

By Theorem 2 applied to L_2 and $L_1{}^2$ in place of L_1 and L_2, the relations on

$$\bigoplus_{\nu \geq 0} H^0(L_1{}^2 \otimes L_2{}^\nu),$$

regarded as a $\mathrm{Sym}(V_2)$-module, are generated by the linear relations, which are given by $R(V_{11}, V_2)$. On the other hand, the quadratic relations, which are the kernel of the composite map

$$(\mathrm{Sym}_2\, V_2) \otimes V_{11} \longrightarrow V_{11} \otimes V_{22} \longrightarrow V_{1122}$$

evidently map onto $R(V_{11}, V_{22})$, so the natural map

$$V_2 \otimes R(V_{11}, V_2) \longrightarrow R(V_{11}, V_{22}) = K''_1$$

is an epimorphism.

Thus it suffices to show that

$$V_1 \otimes R(V_1, V_2)$$

maps by the natural map onto

$$R(V_{11}, V_2),$$

and that its image under

$$V_1 \otimes V_1 \otimes V_2 \longrightarrow \mathrm{Sym}_2(V_1) \otimes V_2$$

contains $\bar{R}(V_1) \otimes V_2$.

Applying Theorem 2 again, this time to L_1 and L_2, we see that $R(V_1, V_2)$, the span of the linear relations on $\bigoplus_{\nu \geq 0} H^0(L_2 \otimes L_1{}^{\nu})$ regarded as a Sym $H^0(L_1)$-module, generate all the relations; in particular

$$V_1 \otimes R(V_1, V_2)$$

maps by the natural map onto the quadratic relations, which are the kernel of the composite map

$$(\mathrm{Sym}_2 V_2) \otimes V_2 \longrightarrow V_{11} \otimes V_2 \longrightarrow V_{112}.$$

But as before, this kernel is generated by $\bar{R}(V_1) \otimes V_2$ and the pullback of $R(V_{11}, V_2)$, so we are done. □

Proof of Theorem 2. To say that M has linear free resolution to stage k means that

$$(*) \qquad \mathrm{Tor}_j{}^S(M, S/S_+)_l = 0 \quad \text{for all} \quad j \leq k, l > j,$$

where we write $S_+ = \bigoplus_{\nu \geq 0} \mathrm{Sym}_\nu\, H^0(L_1)$ for the maximal homogeneous ideal of S. It is easy to see that this last formulation is equivalent to the same statement (*) for the (possibly) larger module $M' = \bigoplus_{\nu \in \mathbf{Z}} H^0(L_1{}^{\nu} \otimes L_2)$ in place of M. We accordingly change our notation and write M for this larger module.

We now use the following two results, which were proved by Green [7] under somewhat different hypotheses.

THEOREM 3. (*Duality*) *Let C be a locally Cohen-Macaulay curve, (L_1, V) a linear series without base points,* $\dim V = r + 1$, *and L_2 any line bundle. Let*

$$S = \operatorname{Sym} V$$

$$M = \bigoplus_{\nu \in \mathbf{Z}} H^0(L_1{}^\nu \otimes L_2).$$

If ω_C is the canonical sheaf of C, and

$$N = \bigoplus_{\nu \in \mathbf{Z}} H^0(L_1{}^\nu \otimes L_2{}^{-1} \otimes \omega_C),$$

then there is a natural isomorphism

$$\operatorname{Tor}_j{}^S(M, S/S_+)_l \cong (\operatorname{Tor}^S{}_{r-1-j}(N, S/S_+)_{r+1-l})^*.$$

We give a proof at the end of this section.

THEOREM 4. (*Vanishing*) *Let*

$$F = \bigoplus_{\nu \geq 0} F_\nu$$

be a graded $\operatorname{Sym}(V)$*-module. If F is torsion free as a module over some factor ring* $\operatorname{Sym}(V)/Q$ *such that Q is a prime containing no linear forms, then*

$$\operatorname{Tor}_m{}^S(F, S/S_+)_n = 0$$

whenever $m \geq \dim F_0$ and $n \leq m$.

Although Green proves this theorem under other hypotheses, his proof applies without change.

CONJECTURE. *The same conclusion should hold whenever F is a module such that no element of F_0 is annihilated by a linear form.*

To complete the proof of Theorem 2, we must, by Theorem 3, prove that

$$\operatorname{Tor}_m{}^S(N(1), S/S_+)_n = 0,$$

where N is, as in Theorem 3,

$$N = \bigoplus_{\nu\in\mathbf{Z}} H^0(L_1{}^\nu \otimes L_2{}^{-1} \otimes \omega_C),$$

for all $n \leq m$ and all m if $H^1(L_1{}^{-1} \otimes L_2) = 0$; and for all $n \leq m$ and $m \geq r - 1 - k$ under the hypotheses in the second statement of Theorem 2.

Since $H^1(L_1{}^{-1} \otimes L_2) = H^0(L_1 \otimes L_2{}^{-1} \otimes \omega_C)$, the first statement is immediate. For the second, applying Theorem 4 to the module $N(1)$, we see that it suffices to show that

$$h^0(L_1 \otimes L_2{}^{-1} \otimes \omega_C) \leq h^0(L_1) - 2 - k,$$

or, better, that

$$(**) \qquad h^0(L_1 \otimes L_2{}^{-1} \otimes \omega_C) \leq (\deg L_1 - g + 1) - 2 - k.$$

Since $\deg L_2 \geq 2g + k$, we have $\deg L_2{}^{-1} \otimes \omega_C \leq -2 - k$, so we are done if $H^1(L_1 \otimes L_2{}^{-1} \otimes \omega_C) = 0$. Since $H^0(L_1 \otimes L_2{}^{-1} \otimes \omega_C) \cong H^1(L_1{}^{-1} \otimes L_2)$, $H^0(L_1 \otimes L_2{}^{-1} \otimes \omega_C) = 0$ implies that M has a linear resolution. In the case when $h^0(L_1 \otimes L_2{}^{-1} \otimes \omega_C) \neq 0$ and $h^1(L_1 \otimes L_2{}^{-1} \otimes \omega_C) \neq 0$ we may apply Clifford's Theorem to the line bundle $L_1 \otimes L_2{}^{-1} \otimes \omega_C$ (this is a special case of what is proved in the appendix) and get

$$\begin{aligned} h^0(L_1 \otimes L_2{}^{-1} \otimes \omega_C) &\leq (\deg L_1 \otimes L_2{}^{-1} \otimes \omega_C)/2 + 1 \\ &= (\deg L_1 - \deg L_2)/2 + g, \end{aligned}$$

in general, with equality only in the cases $L_1{}^{-1}L_2 = O_C$ or ω_C, or, in case C is hyperelliptic, $L_1{}^{-1}L_2$ is a multiple of the $g^1{}_2$. The inequality (**) now follows at once from the conditions given in Theorem 2. $\square$

We now turn to the proof of Theorem 3. Green's proof, which does not require C to be 1-dimensional, unfortunately uses smoothness, essentially to identify the dualizing module. It seems likely that the proof outlined below for our case would generalize to the higher dimensional case as well, with suitable hypotheses as in Green's version on the cohomology of twists of L, to assure that M behaves enough like a Cohen-Macaulay module.

Proof of Theorem 3. Since (L_1, V) is base-point free, M and N are both Cohen-Macaulay modules of dimension 2 over S (sketch: it is enough to do the case of M. Let $V_1 \subset V$ be a 2-dimensional base point free subspace. The obvious "Koszul" complexes

$$0 \longrightarrow \wedge^2 V_1 \otimes L_1{}^{\nu-2}L_2 \longrightarrow V_1 \otimes L_1{}^{\nu-1}L_2 \longrightarrow L_1{}^{\nu}L_2 \longrightarrow 0$$

are exact, and the direct sum of their global sections, over all ν, is the Koszul complex of V_1 tensored with M). Further, $\mathrm{Tor}_j{}^S(M, S/S_+)$ is naturally dual to $\mathrm{Ext}_S{}^j(M, S/S_+)$, and since M is Cohen-Macaulay, this is the same as $\mathrm{Tor}^S{}_{r-1-j}(\mathrm{Ext}_S{}^{r-1}(M, S), S/S_+)$. Thus it suffices to show that

$$\mathrm{Ext}_S{}^{r-1}(M, S) = N(r+1),$$

or equivalently

$$\mathrm{Ext}_S{}^{r-1}(M, S(-r-1)) = N.$$

But by local duality

$$\begin{aligned}\mathrm{Ext}_S{}^{r-1}(M, S(-r-1)) &= H^2{}_{S_+}(M)' \\ &= \bigoplus_{\nu\in\mathbf{Z}} H^1(\mathbf{P}(V), M^{\sim}(\nu)),\end{aligned}$$

where $M^{\sim}$ is the sheaf associated to M, and by Grothendieck-Serre duality, this is N. □

3. Examples. (a) The case of $\mathbf{P}^1$. We first consider the case of $\mathbf{P}^1$ where the matrix $A(L_1, L_2)$ and the resolutions of $\bigoplus_{n\geq 0} H^0(L_2 \otimes L_1{}^n)$ over Sym $H^0(L_1)$ can be expressed explicitly. Write $O = O_{\mathbf{P}^1}$ and identify $\bigoplus_{n\geq 0} H^0(O(n)) \cong k[s, t]$. We choose bases for $H^0(O(a))$, $H^0(O(b))$, and $H^0(O(a+b))$ so that the i^{th} basis vector in each case $x_i = s^i t^n$ for appropriate n. An easy calculation shows that

$$A(O(a), O(b)) = \mathrm{Cat}(a+1, b+1),$$

where $\mathrm{Cat}(v, w)$ denotes the $v \times w$ catalecticant matrix, that is

$$\mathrm{Cat}(v, w) = (x_{i+j})_{0\le i\le v-1, 0\le j\le w-1}$$

$$= \begin{pmatrix} x_0 & x_1 & x_2 & \cdots & x_{w-1} \\ x_1 & x_2 & & \cdots & x_w \\ x_2 & \cdots & & & x_{w+1} \\ \cdots & & & & \\ x_{v-1} & x_v & x_{v+1} & \cdots & x_{v+w-2} \end{pmatrix}$$

That $I_2(\mathrm{Cat}(a+1, b+1))$ is the homogeneous ideal of the rational normal curve, as in the conclusion of Theorem 1, is well-known in case $a = 1$, and follows in general because $I_2(\mathrm{Cat}(a+1, b+1)) = I_2(\mathrm{Cat}(2, a+b))$. (In fact,

$$I_k(\mathrm{Cat}(a+1, b+1)) = I_k(\mathrm{Cat}(k, a+b+2-k))$$

in general; see Gruson, Peskine [9]).

Let $S = \mathrm{Sym}\, H^0(O(a)) \cong k[x_0, \ldots, x_a]$, $a \ge 1$, and $M_b = \bigoplus_{\nu\ge 0} H^0(O(b+a\nu))$, $b \ge 1$. By Theorem 2, we know that M_b is linearly presented. It follows from this and a dimension computation that the presentation comes from

$$0 \longrightarrow H^0(O(a-1)) \otimes H^0(O(b-1)) \otimes \wedge^2 H^0(O(1))$$
$$\longrightarrow H^0(O(a)) \otimes H^0(O(b)) \longrightarrow H^0(O(a+b)) \longrightarrow 0,$$

where $H^0(O(b))$ is the degree 0 part of M_b, and $H^0(O(a))$ is the degree 1 part of the ring S.

To identify M_b with something we know, let $F = H^0(O(a-1)) \otimes_k S$, $G = H^0(O(1)) \otimes_k S$, and $\varphi\colon F \to G$ be defined by

$$\begin{pmatrix} x_0 & x_2 & \cdots & x_{a-1} \\ -x_1 & -x_2 & \cdots & -x_a \end{pmatrix}$$

Then we have an exact sequence

$$F \otimes \mathrm{Sym}_{b-1} G \longrightarrow \mathrm{Sym}_b\, G \longrightarrow \mathrm{Sym}_b(\mathrm{Coker}\, \varphi) \longrightarrow 0.$$

Note that Coker $\varphi = M_1$, and the associated sheaf to M_b is isomorphic to $\mathrm{Sym}_b(M_1\tilde{\ })$. Thus there is a natural map $\mathrm{Sym}_b M_1 \to M_b$. This is an isomorphism since both modules have the same presentation. Thus $M_b \cong \mathrm{Sym}_b(\mathrm{Coker}\,\varphi)$. The resolution of M_b over S given below is well-known (see Bruns [2] for details). To save space we write S_iG to denote the i^{th} symmetric power of G:

(i) $b \geq a$.

$$0 \longrightarrow (\wedge^a F \otimes S_{b-a}G)(-a) \longrightarrow (\wedge^{a-1} F \otimes S_{b-a+1}G)(-a+1) \longrightarrow \cdots$$

$$\longrightarrow (F \otimes S_{b-1}G)(-1) \longrightarrow S_bG \longrightarrow M_b \longrightarrow 0.$$

(ii) $b = a - 1$.

$$0 \longrightarrow (\wedge^{a-1} F \otimes S_0G)(-a+1) \longrightarrow (\wedge^{a-2} F \otimes S_1G)(-a+2) \longrightarrow \cdots$$

$$\longrightarrow (F \otimes S_{a-2}G)(-1) \longrightarrow S_{a-1}G \longrightarrow M_{a-1} \longrightarrow 0.$$

(iii) $0 \leq b \leq a - 2$.

$$0 \to (\wedge^a F \otimes (S_{a-b-2}G)^*)(-a) \to (\wedge^{a-1} F \otimes (S_{a-b-3}G)^*)(-a+1) \to \cdots$$

$$\to (\wedge^{b+2} F \otimes (S_0G)^*)(-b-2) \to (\wedge^b F \otimes S_0G)(-b) \to \cdots$$

$$\to (F \otimes S_{b-1}G)(-1) \to S_bG \to M_b \to 0.$$

Remark. It follows from Theorem 2 that the resolutions in cases (i) and (ii) are linear ($H^0(O(b-a)) = 0$) and the resolution in case (iii) is linear to stage b. The lengths of these resolutions are to be expected from the fact that M_b is supported on a curve and M_b is Cohen-Macaulay (i.e. $M_b = \oplus_{\nu \in \mathbf{Z}} H^0(O(b + a\nu)))$ if and only if $b \leq a - 1$.

(b) Monomial curves. Theorem 1 also enables us to find the equations of an affine monomial curve. Recall that a monomial curve is the image of a map $\psi: \mathbf{A}^1 \to \mathbf{A}^r \subset \mathbf{P}^r$ defined by $\psi(t) = (1, t^{n_1}, \ldots, t^{n_r})$, where $n_1 < \cdots < n_r$ are positive integers with no common factors other than 1. Let $X = \overline{\psi(\mathbf{A}^1)}$ in $\mathbf{P}^r$ and $P = \psi(0) \in X$. Then nP defines a line bundle on X if and only if $n \in \Gamma$, where Γ denotes the sub-semigroup of the nonnegative integers generated by $\{n_1, \ldots, n_r\}$. We write O for O_X, and for each $n \in \Gamma$, we choose

$$\{w_\gamma = t^\gamma | \gamma \in \Gamma, \gamma \leq n\} \text{ for a basis of } H^0(O(nP)).$$

Let a and b be integers such that $O(aP)$ and $O(bP)$ are line bundles satisfying the hypotheses of Theorem 1, e.g. $a \geq 2g + 1$ and $b \geq 2g + 2$ where g denotes the arithmetic genus of X. Note that every such a and b are in Γ.

Then with respect to our choice of bases,

$$A = A(O(aP), O(bP)) = (w_{\alpha+\beta})_{\{\alpha\in\Gamma|\alpha\leq a\},\{\beta\in\Gamma|\beta\leq b\}}.$$

We note here that the matrix $(w_{\alpha+\beta})$ can be obtained from $\mathrm{Cat}(a + 1, b + 1)$ by deleting all the i^{th} rows with $i \notin \Gamma$ and all the j^{th} columns with $j \notin \Gamma$. This type of matrix arises in other examples we consider later and we call it a "deleted Catalecticant" matrix.

Let $\varphi: X \to \mathbf{P}(H^0(O((a + b)P)))$ be the embedding corresponding to $O((a + b)P)$. Then $I(\varphi(X)) = I_2(A)$ by Theorem 1. If we let $\bar{A}$ denote A with w_0 replaced by 1,

$$I((\varphi \circ \psi)(\mathbf{A}^1)) = I_2(\bar{A}) \subset k[\{w_\gamma | \gamma \in \Gamma, 0 < \gamma \leq a + b\}].$$

Since $(\varphi \circ \psi)^*: k[\{w_\gamma\}] \to k[t]$ is given by $w_\gamma \to t^\gamma$ and $\{n_1, \ldots, n_r\}$ generate Γ, we can choose a surjective map $f: k[\{w_\gamma\}] \to k[x_1, \ldots, x_r]$ so that $(\varphi \circ \psi)^* = \psi^* \circ f$. Hence

$$I(\psi(\mathbf{A}^1)) = \mathrm{Ker}\, \psi^* = f(\mathrm{Ker}(\varphi \circ \psi)^*) = f(I_2(\bar{A})) = I_2(f(\bar{A})).$$

Example. Consider the monomial curve which is the image of the map $\psi: \mathbf{A}^1 \to \mathbf{A}^3 \subset \mathbf{P}^3$ defined by $\psi(t) = (1, t^2, t^4, t^5)$. Let $X = \overline{\psi(\mathbf{A}^1)}$, a projective monomial curve of arithmetic genus 2 in $\mathbf{P}^3$. The line bundles $L_1 = O(5P)$ and $L_2 = O(6P)$ on X satisfy the hypotheses of Theorem 1. The matrix $\bar{A} = \bar{A}(L_1, L_2)$ is then

$$\begin{pmatrix} 1 & t^2 & t^4 & t^5 & t^6 \\ t^2 & t^4 & t^6 & t^7 & t^8 \\ t^4 & t^6 & t^8 & t^9 & t^{10} \\ t^5 & t^7 & t^9 & t^{10} & t^{11} \end{pmatrix}.$$

To find the equations for $\psi(\mathbf{A}^1)$, we choose a surjective map

$$f\colon k[t^2, t^4, t^5, t^6, \ldots, t^{11}] \longrightarrow k[a, b, c]$$

with $f(t^2) = a$, $f(t^4) = b$, and $f(t^5) = c$. Note that we cannot set $f(t^4) = a^2$, since f must be surjective. We choose also

$$f(t^6) = a^3, \quad f(t^7) = ac, \quad f(t^8) = a^4,$$
$$f(t^9) = a^2c, \quad f(t^{10}) = a^5, \quad f(t^{11}) = a^3c.$$

Then

$$f(\bar{A}) = \begin{bmatrix} 1 & a & b & c & a^3 \\ a & b & a^3 & ac & a^4 \\ b & a^3 & a^4 & a^2c & a^5 \\ c & ac & a^2c & a^5 & a^3c \end{bmatrix}.$$

The resulting equations for $\psi(\mathbf{A}^1)$ are the 2×2 minors of this matrix. In this example, this ideal of minors is minimally generated by $(b - a^2, a^5 - c^2)$.

Note that if we instead replace one occurrence ($(4, 4)^{\text{th}}$ entry) of t^{10} with c^2, $a^5 - c^2$ is not in the resulting ideal of 2×2 minors, since f is not a well-defined function. However, if we change both occurrences, in effect changing $f(t^{10}) = c^2$, then $a^5 - c^2$ is in the corresponding ideal.

(c) Elliptic and hyperelliptic curves. Let C be a smooth curve of genus g, and let P be a point of C such that $h^0(O_C(2P)) = 2$ (so that C is either elliptic or hyperelliptic). Then for a suitable choice of t, x, y,

$$\bigoplus_{n\geq 0} H^0(O_C(nP)) \cong k[t, x, y]/(y^2 - x(x - t^2)(x - \lambda_1 t^2) \cdots (x - \lambda_{2g-1}t^2)),$$

where $\deg t = 1$, $\deg x = 2$, $\deg y = 2g + 1$, and $\lambda_i \neq 0, 1$ are all distinct. For each $n \geq 0$, we choose

$$\{x_i = x^i t^{n-2i} \mid 0 \leq i \leq n/2\}$$
$$\cup \{y_j = yx^j t^{n-2g-1-2j} \mid 0 \leq j \leq (n - 2g - 1)/2\}$$

for a basis of $H^0(O_C(nP))$. Write $x(x-t^2)(x-\lambda_1 t^2)\cdots(x-\lambda_{2g-1}t^2)$ as

$$x^{2g+1}+c_{2g}x^{2g}t^2+\cdots+c_1xt^{4g}$$

so that

$$(yx^jt^{a-2g-1-2j})(yx^{j'}t^{b-2g-1-2j'})=x^{j+j'}(x^{2g+1}+c_{2g}x^{2g}t^2+\cdots+c_1xt^{4g}).$$

With respect to this choice of bases,

$A(O_C(aP),\, O_C(bP))$

$$=\left(\begin{array}{c|c}(x_{i+i'}) & (y_{i+j'})\\ \hline (y_{j+i'}) & (x_{j+j'+2g+1}+c_{2g}x_{j+j'+2g}+\cdots+c_1x_{j+j'+1})\end{array}\right)$$

where $0\le i\le[a/2]$, $0\le j\le[(a-2g-1)/2]$, $0\le i'\le[b/2]$, and $0\le j'\le[(b-2g-1)/2]$.

We note that each block of the above matrix is catalecticant and it can be obtained from $\mathrm{Cat}([a/2]+[b/2]+[(a-2g-1)/2],\,[a/2]+[b/2]+[(b-2g-1)/2])$ by first deleting $[b/2]-1$ rows and $[a/2]-1$ columns in the middle and then specializing the variables in the bottom right block (the bottom right block contains one new variable when a and b are both odd).

To give an example showing the necessity of the condition $L_1\ne L_2$ in Theorem 1 when $\deg L_1=\deg L_2$ and $g>0$, we let P be any point on an elliptic curve E. Then $A(O_E(3P),\,O_E(3P))$ is the 3×3 symmetric matrix whose 2×2 minors define the Veronese surface in $\mathbf{P}^5$.

Except for those cases discussed above, there does not seem to be a natural candidate for bases of $H^0(L_1)$, $H^0(L_2)$, and $H^0(L_1\otimes L_2)$ making $A(L_1, L_2)$ nice. However, in some cases we may get a reasonable approximation $\mathrm{IN}(A(L_1,L_2))$ of $A(L_1,L_2)$ by considering vanishing sequences. Let P be a smooth point of a reduced and irreducible curve C and let $\nu_P(.)$ denote the order of vanishing at P. Choose a basis $\{e_{n_i}\}$ of $H^0(L_1)$ such that $\nu_P(e_{n_i})=n_i$, where $\{n_i|0\le n_1<\cdots<n_{h^0(L_1)}\le\deg L_1\}$ is the vanishing sequence of L_1 at P. Choose bases $\{f_{n_j}\}$ of $H^0(L_2)$ and $\{x_{n_l}\}$ of $H^0(L_1\otimes L_2)$ in the same way. Since P is a smooth point,

$$\mu(e_{n_i}\otimes f_{n_j})=c_{ij}x_{n_i+n_j}+\text{(terms with higher order of vanishing)},$$

for some nonzero constants c_{ij}. Now $\mathrm{IN}(A(L_1,L_2))$ is defined to be the matrix $(x_{n_i+n_j})$ of initial terms disregarding the nonzero coefficients c_{ij}. This is a deleted catalecticant matrix. Because polynomials with independent initial forms are independent, the number of linearly independent 2×2 minors of $\mathrm{IN}(A(L_1, L_2))$ does not exceed that of $A(L_1, L_2)$. (We first confirmed Theorem 1 in the case of an elliptic curve by finding enough linearly independent 2×2 minors of $\mathrm{IN}(A(L_1, L_2))$).

If P is a point of an elliptic curve E, then the vanishing sequence of $O_E(aP)$ at P is $\{0, 1, \ldots, a-2, a\}$ and the vanishing sequence of $O_E(bQ)$ at P is $\{0, 1, \ldots, b-1\}$ provided that $bQ \not\sim bP$. Hence $\mathrm{IN}(A(O_E(aP), O_E(bP)))$ is the catalecticant $\mathrm{Cat}(a+1, b+1)$ with the a^{th} row and b^{th} column deleted; and $\mathrm{IN}(A(O_E(aP), O_E(bQ)))$ is the catalecticant $\mathrm{Cat}(a+1, b)$ with a^{th} row deleted. If P is a point of a hyperelliptic curve C of genus g such that $h^0(O_C(2P)) = 2$, then the vanishing sequence of $O_C(aP)$ at P, for $a \geq 2g+1$, is $\{0, 1, \ldots, a-2g-1, a-2g, a-2g+2, \ldots, a-2, a\}$ and $\mathrm{IN}(A(O_C(aP), O_C(bP)))$ is the catalecticant $\mathrm{Cat}(a+1, b+1)$ with appropriate g rows and g columns deleted.

It would be interesting to know more about the ideals defined by such deleted catalecticant matrices. For instance, let B be the deleted catalecticant matrix $\mathrm{IN}(A(O_C(aP), O_C(bQ)))$ where C is a smooth hyperelliptic curve of genus g, a and b are integers $\geq 2g+2$, and P and Q are points of C such that $h^0(O_C(2P)) = 2$, and $2P$ is not equivalent to $2Q$. It follows from Theorem 1 that $I_2(A(O_C(aP), O_C(bQ)))$ defines an arithmetically Cohen-Macaulay curve of degree $a+b$. On the other hand $I_2(B)$ is neither irreducible nor reduced ($I_2(B)$ defines a scheme consisting of a rational normal curve together with a nonreduced structure on a tangent line to this rational normal curve). However, in all of the examples which we have checked using *Macaulay* [1], $I_2(B)$ defines a degree $a+b$ arithmetically Cohen-Macaulay subscheme of dimension 1. We conjecture that this holds for all a and b. We remark that there are deleted catalecticant matrices, B, such that $I_2(B)$ defines a 1-dimensional subscheme which is not arithmetically Cohen-Macaulay.

Let $E \subset \mathbf{P}^{a-1}$ be an elliptic curve embedded by $O_E(aP)$ where P is a point on E, and $a \geq 4$. We will now describe the resolution of the homogeneous coordinate ring of E, $\Gamma_*(O_E) = \oplus_{\nu\in\mathbf{Z}} H^0(O_E(a\nu P))$ over $\mathrm{Sym}\, H^0(O_E(aP))$. The pencil $|2P|$ spans a 2-dimensional rational normal scroll X containing E (see Eisenbud, Harris [5]). Let $\pi: S \to \mathbf{P}^1$ be the corresponding $\mathbf{P}^1$-bundle and let $O_S(1)$ be the bundle corresponding to the map $S \to X \subset \mathbf{P}^{a-1}$. It turns out that E pulls back to a divisor on S with

$$O_S(E) = O_S(2) \otimes \pi^* O_{\mathbf{P}^1}(4-a).$$

One checks that $H^1(S, O_S(-E) \otimes O_S(\nu)) = 0$ for all $\nu \in \mathbf{Z}$, and thus

$$0 \longrightarrow \Gamma_*(O_S(-E)) \longrightarrow \Gamma_*(O_S) \longrightarrow \Gamma_*(O_E) \longrightarrow 0$$

is exact. Thus the mapping cone of the resolutions of $\Gamma_*(O_S(-E))$ and $\Gamma_*(O_S)$ gives a (not necessarily minimal) resolution of $\Gamma_*(O_E)$. However the resolutions of $\Gamma_*(O_S(-E))$ and $\Gamma_*(O_S)$ are known explicitly (see Schreyer [13] for the method) and the minimality of the above mapping cone follows simply from the degrees of the free modules in these resolutions. Notice that the graded betti numbers of $\Gamma_*(O_E)$ depend only on a and are independent of the moduli of E.

Unfortunately, we do not know how to exhibit the resolutions of the modules M that appear in Theorem 2, though it is possible to compute examples one by one, using the Macaulay program (Bayer, Stillman [1]).

As an example of the resolution of $M_b = \bigoplus_{\nu \geq 0} H^0 O_E((b + a\nu)P)$ over $\operatorname{Sym} H^0(O(aP))$, let $\Gamma_*(O_E) = k[t, x, y]/(y^2 - x(x - t^2)(x + t^2))$, and $a = b = 5$, and $S = \operatorname{Sym} H^0(O(5P))$. One obtains

$$0 \longrightarrow S(-4) \longrightarrow \begin{matrix} S(-3)^5 \\ \oplus \\ S(-4) \end{matrix} \longrightarrow S(-2)^{15} \longrightarrow S(-1)^{15} \longrightarrow S^5 \longrightarrow M_5 \longrightarrow 0$$

for the graded betti numbers of M_5 over S.

Appendix (with J. Harris): **Clifford's Theorem for singular curves.** In this appendix we prove a generalized form of Clifford's Theorem. The Theorem generalizes readily to the case when C is singular and $\mathfrak{F}$ is a line bundle. Perhaps surprisingly, there is a "new" class of torsion free sheaves of rank 1 satisfying Clifford's equality (case (c) in the second part of the Theorem below).

THEOREM A. *(Clifford) Let C be a reduced, irreducible curve of arithmetic genus g, and let $\mathfrak{F}$ be a torsion free sheaf of rank* 1 *on C, such that $h^0(\mathfrak{F}) \geq 1$, $h^1(\mathfrak{F}) \geq 1$. Then*

$$(1) \qquad 2(h^0(\mathfrak{F}) - 1) \leq \deg \mathfrak{F},$$

and

(2) *equality holds if and only if either*

(a) *$h^0(\mathfrak{F}) = 1$ and $\mathfrak{F} = O_C$; or $h^1(\mathfrak{F}) = 1$ and $\mathfrak{F} = \omega_C$, or $h^0(\mathfrak{F}) \geq 2$, $h^1(\mathfrak{F}) \geq 2$, and either*

(b) *C is hyperelliptic* (*i.e. there exists a morphism from C to $\mathbf{P}^1$ of degree* 2, *and $g \geq 2$; in particular C is locally Gorenstein*) *and $\mathfrak{F}$ is a multiple of the $g^1{}_2$, in particular $\mathfrak{F}$ is a line bundle, or*

(c) *The normalization of C is $\mathbf{P}^1$, C has exactly one singular point, x, and the conductor at x, $c_x = m_x$, the maximal ideal of $O_{C,x}$. Also $\mathfrak{F} = O_C\langle 1, t, \ldots, t^d\rangle$, where $d \leq g - 1$, and $(1, t, \ldots, t^d) = H^0(\mathbf{P}^1, O(d))$.*

Remark. The sheaves $\mathfrak{F} = O_C\langle 1, t, \ldots, t^d\rangle$ occurring in case (c) of the Theorem are not locally free, unless $d = 0$, in which case $\mathfrak{F} = O_C$.

We can identify all such C occurring in case (c) of the Theorem explicitly. Let $\pi\colon \mathbf{P}^1 \to C$ be the normalization. Let $D = \pi^{-1}(x) = \Sigma_{i=1..r}\, n_iP_i$, where $P_i \neq \infty$ for all i. D is a divisor of degree $g + 1$, since $c_x = m_x$. Let $f(s, t) = (t - P_1s)^{n_1} \cdots (t - P_rs)^{n_r}$. Then $C \cong \operatorname{Proj} k[s^{2g+1}, s^gf(s, t), s^{g-1}tf(s, t), \ldots, t^gf(s, t)]$ (See Serre [14] for details). Since $C \subset \mathbf{P}^{g+1}$ has degree $2g + 1$, C is arithmetically Cohen-Macaulay by Fujita [6]. Further $C \subset S \subset \mathbf{P}^{g+1}$, where S is the cone over a rational normal curve of degree g in $\mathbf{P}^g$. The vertex of S is the singular point, x of C and blowing up S at x desingularizes C in one step.

If C is reduced and irreducible, we will let $\pi\colon C' \to C$ be the normalization of C. We write O for O_C, and O' for $\pi_*O_{C'}$, the integral closure of O. The conductor, c, of π is defined to be $c = \operatorname{Ann}_O(O'/O)$. c can be written as $\pi_*(c')$, where c' is a sheaf on C'. $\omega = \omega_C$ is the dualizing sheaf on C. ω is a torsion free rank 1 sheaf on C.

The degree of a torsion free sheaf $\mathfrak{F}$ of rank 1 on C is defined to be

$$\deg(\mathfrak{F}) = \chi(\mathfrak{F}) - \chi(O_C).$$

Thus Clifford's inequality (1) is equivalent to $h^0(\mathfrak{F}) + h^1(\mathfrak{F}) \leq g + 1$.

Proof of part (1) *of Clifford's Theorem.* Since $\mathfrak{F}$ is torsion free, the pairing

$$\psi\colon H^0(\mathfrak{F}) \otimes \operatorname{Hom}(\mathfrak{F}, \omega) \longrightarrow H^0(\omega)$$

is nondegenerate (i.e. $\psi(a \otimes b) = 0$ implies that $a = 0$, or $b = 0$). Hence

$$g = h^0(\omega) \geq h^0(\mathfrak{F}) + \dim \operatorname{Hom}(\mathfrak{F}, \omega) - 1,$$

see for example Eisenbud [4]. Clifford's inequality now follows since $H^1(\mathfrak{F}) \cong \mathrm{Hom}(\mathfrak{F}, \omega)$. $\square$

The above proof shows that if $\mathfrak{F}$ satisfies $h^0(\mathfrak{F}) + h^1(\mathfrak{F}) = g + 1$, then $\psi\colon H^0(\mathfrak{F}) \otimes \mathrm{Hom}(\mathfrak{F}, \omega) \to H^0(\omega)$ is surjective.

To handle the case of equality in (1), we need more precise information about the relationship of torsion free sheaves on C and the corresponding line bundles on C'. If $\mathfrak{F}$ is a torsion free sheaf of rank 1 on C, we embed $\mathfrak{F}$ once and for all in $K(C)$, so that $\mathfrak{F}O' \cong \pi^*(\mathfrak{F})/\text{torsion}$ is a line bundle on C'.

LEMMA 1. *Let $\mathfrak{F}$ be a torsion free sheaf of rank 1 on C. There exists a line bundle $L \subset \mathfrak{F}$ so that $LO' = \mathfrak{F}O'$. Thus $\mathfrak{F}$ is a line bundle if and only if* $\deg_C(\mathfrak{F}) = \deg_{C'}(\mathfrak{F}O')$.

Proof. At each singular point p of C, choose $f_p \in \mathfrak{F}_p$ so that $\nu_q(f_p) = \min\{\nu_q(g_p) | g_p \in \mathfrak{F}_p\}$, for every $q \in C'$ with $\pi(q) = p$. Define for each open set $U \subset C$,

$$L(U) = \{\varphi \in \mathfrak{F}(U) | \varphi_p \in f_p O_p \subset \mathfrak{F}_p \text{ for each singular point } p \in U\}.$$

L is a line bundle on C with the desired properties. For the second statement, if L is a line bundle, then $\deg_C(L) = \deg_{C'}(LO')$, since both are equal to $-\Sigma_{q \in C'} \nu_q(L)$, where $\nu_q(L) = \min\{\nu_q(g_p) | g_p \in \mathfrak{F}_p$, where $p = \pi(q)\}$. Then if $L \subset \mathfrak{F}$ is the line bundle constructed above,

$$\begin{aligned} \deg_C \mathfrak{F} &= \deg L + \mathrm{length}(\mathfrak{F}/L) \\ &= \deg(LO') + \mathrm{length}(\mathfrak{F}/L) \\ &= \deg(\mathfrak{F}O') + \mathrm{length}(\mathfrak{F}/L), \end{aligned}$$

and so $\mathfrak{F}$ is a line bundle if and only if $\deg_C(\mathfrak{F}) = \deg_{C'}(\mathfrak{F}O')$. $\square$

LEMMA 2. *Let C be a reduced, irreducible curve of genus g. Let $\mathrm{length}(O'/O) = \delta$. Then*

$$\deg_{C'}(\omega O') = (g - 1) + (g - \delta) + (\mathit{length}(O/c) - 1).$$

Proof. Using the exact sequence

$$0 \longrightarrow \omega \longrightarrow \pi_*(\omega O') \longrightarrow \pi_*(\omega O')/\omega \longrightarrow 0,$$

we obtain

$$\deg_C \pi_*(\omega O') = \deg_C \omega + \operatorname{length}(\pi_*(\omega O')/\omega).$$

Since $\deg_{C'}(L) = \deg_C(\pi_* L) - \delta$ for any line bundle L on C',

$$\deg_{C'}(\omega O') = 2g - 2 - \delta + \operatorname{length}(\pi_*(\omega O')/\omega).$$

But $\operatorname{length}(\pi_*\omega O'/\omega) = \operatorname{length}(O/c)$, because of the following duality. If $Q \in C$, then

$$\omega_Q = \left\{\alpha \in \omega_{C'} \otimes K(C) \,\middle|\, \sum_{P \to Q} \operatorname{Res}_P(f\alpha) = 0 \text{ for all } f \in O_Q\right\}$$

(see Serre [14]). Define a pairing $\psi\colon \omega_Q O'_Q/\omega_Q \otimes O_Q/c_Q \to k$ by

$$\psi(f \otimes \alpha) = \sum_{P \to Q} \operatorname{Res}_P(f\alpha).$$

One easily checks that this is well-defined. ψ is a perfect pairing because if $\alpha \in \omega_Q O'_Q$ is such that $\psi(f \otimes \alpha) = 0$ for all $f \in O_Q$, then $f \in c_Q$ by definition. Furthermore, if $f \in O_Q$ is such that $\psi(f \otimes \alpha) = 0$ for all $\alpha \in \psi_Q O'_Q$, then $fO'_Q\omega_Q \subset \omega_Q$ and so $fO'_Q \subset \operatorname{End}(\omega_Q) = O_Q$, and $f \in c_Q$. □

COROLLARY 2. *Let C be a reduced, irreducible curve of genus g. Then*

$$\deg_{C'}(\omega O') \geq g - 1$$

and $\deg_{C'}(\omega O') = g - 1$ *if and only if* $C' \cong \mathbf{P}^1$, *C has exactly one singular point x, and $c_x = m_x$.*

LEMMA 3. *Let $\mathfrak{F}$ be a torsion free sheaf of rank 1 on C, such that $\mathfrak{F}O' = O_{C'}(D)$. Then*

$$c(D) = c\mathfrak{F} \subset \mathfrak{F} \subset \pi_*(\mathfrak{F}O') = \pi_*(O_{C'}(D)).$$

COROLLARY 3. *Suppose $C' = \mathbf{P}^1$, and $\deg_{\mathbf{P}^1}(\mathfrak{F}O') = d \leq g - 1$. Then $H^0(\mathfrak{F})$ surjects onto $\mathfrak{F}/c\mathfrak{F}$ if and only if $h^1(\mathfrak{F}) = g - d$, in which case $H^0(\mathfrak{F}) \cong \mathfrak{F}/c\mathfrak{F}$.*

Proof. With notation as in Lemma 3, we have an exact sequence

$$0 \longrightarrow c(D) \longrightarrow \mathfrak{F} \longrightarrow \mathfrak{F}/c(D) \longrightarrow 0,$$

from which we obtain

$$0 \to H^0(c(D)) \to H^0(\mathfrak{F}) \to \mathfrak{F}/c(D) \to H^1(c(D)) \to H^1(\mathfrak{F}) \to 0.$$

However $H^i(c(D)) = H^i(\mathbf{P}^1, c'(D))$ and $c'(D) = O_{\mathbf{P}^1}(d - g - 1)$. Thus $h^0(c(D)) = 0$, and $h^1(c(D)) = g - d$, and the corollary follows at once. $\square$

Proof of part (2) *of Clifford's Theorem.* Suppose now that $\mathfrak{F}$ is a torsion free sheaf of rank 1 on C satisfying Clifford's equality: $2(h^0(\mathfrak{F}) - 1) = \deg \mathfrak{F}$. As before, this is equivalent to $h^0(\mathfrak{F}) + h^1(\mathfrak{F}) = g + 1$. By symmetry and duality $\mathfrak{F}$ satisfies this equality if and only if $\mathcal{H}om(\mathfrak{F}, \omega)$ does. Also, any such $\mathfrak{F}$ must be generated by global sections, since if $\mathfrak{F}' = H^0(\mathfrak{F}) \cdot O \subset \mathfrak{F}$ is the subsheaf generated by the global sections of $\mathfrak{F}$, then $\deg(\mathfrak{F}') \le \deg(\mathfrak{F})$, $h^0(\mathfrak{F}') = h^0(\mathfrak{F})$, and $h^1(\mathfrak{F}') \ge h^1(\mathfrak{F}) \ge 1$, thus by part (1) of Clifford's Theorem, $\mathfrak{F}' = \mathfrak{F}$. In particular, ω and $\mathcal{H}om(\mathfrak{F}, \omega)$ are generated by global sections.

If $h^0(\mathfrak{F}) = 1$, then $\deg \mathfrak{F} = 0$, and $\mathfrak{F} = O_C$. If $h^1(\mathfrak{F}) = 1$, then $h^0(\mathcal{H}om(\mathfrak{F}, \omega)) = 1$ and $\deg \mathcal{H}om(\mathfrak{F}, \omega) = 0$, so $\mathcal{H}om(\mathfrak{F}, \omega) = O_C$, and $\mathfrak{F} = \omega_C$.

For the remainder of the proof, we assume that $h^0(\mathfrak{F}) \ge 2$ and $h^1(\mathfrak{F}) \ge 2$. Using the global sections of ω, we obtain a rational map

$$\varphi_\omega \colon C - \cdots \to \mathbf{P}^{g-1}.$$

Since ω is generated by global sections, $\omega O'$ is a line bundle on C' without base points. Thus $\omega O'$ defines a morphism to $\mathbf{P}^{g-1}$:

$$\begin{tikzcd} C \arrow[r, dashed, "\varphi_\omega"] & \mathbf{P}^{g-1} \\ C' \arrow[u, "\pi"] \arrow[ur, "\varphi_{\omega O'}"'] & \end{tikzcd}$$

We identify the image of C', $\overline{\varphi_\omega(C)} \subset \mathbf{P}^{g-1}$ using Eisenbud's classification of 1-generic matrices of minimal codimension, which we now describe.

Let $\psi\colon V \otimes W \to T$ be a surjective nondegenerate pairing of vector spaces. Let $\dim V = v \geq \dim W = w$, and $\dim T = t$. We say that ψ is represented by a $v \times w$ matrix A with entries in T if for some choice of bases $\{e_i\}$ of V and $\{f_j\}$ of W,

$$A_{ij} = \psi(e_i \otimes f_j).$$

The 2×2 minors of this matrix generate an ideal $I_2(A) \subset \operatorname{Sym} T$.

THEOREM B. (*Eisenbud* [4, *Theorem* 5.1]) *Let* $\psi: V \otimes W \to T$ *be a surjective nondegenerate pairing as above. Let* ψ *be represented by the matrix A. If* $htI_2(A) \leq v + w - 3$, *then*

(1) *If* $htI_2(A) = t - 2$, ψ *can be represented by the* $v \times w$ *catalecticant matrix* $\operatorname{Cat}(v, w)$ (*see Section* 3) (*in this case* $t = v + w - 1$), *else, either*

(2) $htI_2(A) = 3$, $v = w = 3$, $t = 6$, *and* ψ *can be represented by the* 3×3 *generic symmetric matrix* (*in this case* $t = v + w$), *or*

(3) $w = 2$, *and there is* $1 \leq a_1 \leq \cdots \leq a_d$, *with* $\Sigma a_i = v$, *where* $d = t - htI_2(A)$ *such that* ψ *is represented by the matrix*

$$\begin{pmatrix} x_{1,0} & x_{1,1} & \cdots & x_{1,a_1-1} & \Big| \cdots \Big| & x_{d,0} & x_{d,1} & \cdots & x_{d,a_d-1} \\ x_{1,1} & x_{1,2} & \cdots & x_{1,a_1} & \Big| \cdots \Big| & x_{d,1} & x_{d,2} & \cdots & x_{d,a_d} \end{pmatrix}$$

(*in this case*, $t = v + w + (d - 2)$).

In our case, let $\psi: H^0(\mathfrak{F}) \otimes \operatorname{Hom}(\mathfrak{F}, \omega) \to H^0(\omega)$. ψ is surjective and nondegenerate. If ψ is represented by a matrix A, then $I_2(A) \subset I(\varphi_\omega(C))$ and $htI_2(A) \leq htI(\varphi_\omega(C)) = g - 2 = h^0(\mathfrak{F}) + h^1(\mathfrak{F}) - 3$. By Theorem B, ψ can be represented by the catalecticant matrix $\operatorname{Cat}(h^0(\mathfrak{F}), h^1(\mathfrak{F}))$ because only case (1), and case (3) with $d = 1$ satisfy

$$h^0(\omega) = g = h^0(\mathfrak{F}) + h^1(\mathfrak{F}) - 1.$$

Therefore, $I_2(A)$ is the ideal of the rational normal curve of degree $g - 1$ in $\mathbf{P}^{g-1}$, and so the image of C' is the rational normal curve of degree $g - 1$.

By Lemma 2, $g - 1 \leq \deg_{C'}(\omega O') \leq 2g - 2$, and so $\varphi_{\omega O'}$ is either an isomorphism or a degree 2 map.

Case 1. $\varphi_{\omega O'}: C' \to \mathbf{P}^1$ is a degree 2 map.

In this case, $\omega O'$ has degree $2g - 2$. Therefore ω is a line bundle on C by Lemma 1 and so $\varphi_\omega: C \to \mathbf{P}^{g-1}$ is a morphism of degree 2. Therefore, C is hyperelliptic as desired.

To identify $\mathfrak{F}$ first notice that $\mathfrak{F}$ and $\mathcal{H}om(\mathfrak{F}, \omega)$ are line bundles, since if $\mathfrak{F}$ and $\mathcal{G}$ are torsion free sheaves such that $\mathfrak{F} \otimes \mathcal{G}$ maps onto a line bundle, then $\mathfrak{F}$ and $\mathcal{G}$ must be locally free.

Let $\deg \mathfrak{F} = 2e$, with $2 \leq e \leq g - 2$. We may write $\mathfrak{F} = O_C(D)$, where D is effective and supported on the smooth points of C. In addition, since D imposes e conditions on ω ($h^0(\omega(-D)) = h^0(\omega \otimes \mathfrak{F}^{-1}) = h^1(\mathfrak{F}) = g - e$), we can write

$$D = \sum_{i=1..e} x_i + \sum_{i=1..e} y_i,$$

where $h^0(\omega(-\Sigma_{i=1..e}\, x_i)) = g - e$, i.e. the points x_i impose independent conditions on the sections of ω. Therefore $\{\varphi_\omega(y_1), \ldots, \varphi_\omega(y_e)\} \subset \{\varphi_\omega(x_1), \ldots, \varphi_\omega(x_e)\}$. Since φ_ω is 2-1, and $\varphi_\omega(x_1), \ldots, \varphi_\omega(x_e)$ are all distinct, D must consist of e fibers of the map φ_ω, and therefore $\mathfrak{F} = eg^1{}_2$.

Case 2. $\varphi_{\omega O'}: C' \to \mathbf{P}^1$ is an isomorphism.

In this case $\deg(\omega O') = g - 1$, and by Corollary 2, C has the properties specified in part (c) of the Theorem. We must now identify $\mathfrak{F}$.

Let $d = \deg(\mathfrak{F}O')$. $H^1(\mathfrak{F}) = \mathrm{Hom}(\mathfrak{F}, \omega) \neq 0$, $\mathrm{Hom}(\mathfrak{F}O', \omega O') \neq 0$, whence $d \leq \deg(\omega O') = g - 1$. Since $\mathfrak{F}$ is generated by global sections, the natural map $H^0(\mathfrak{F}) \to \mathfrak{F}/c\mathfrak{F}$ is a surjection, and by Corollary 3, $h^1(\mathfrak{F}) = g - d$. Since $h^0(\mathfrak{F}) + h^1(\mathfrak{F}) = g + 1$, $h^0(\mathfrak{F}) = d + 1$. But

$$H^0(\mathfrak{F}) \subset H^0(\pi_*(\mathfrak{F}O')) = H^0(\mathbf{P}^1, O(d)).$$

Thus $\mathfrak{F} = O_C\langle 1, t, \ldots, t^d\rangle$, where $(1, t, \ldots, t^d)$ generate $H^0(\mathbf{P}^1, O(d))$.

Conversely, suppose that $\mathfrak{F} = O_C\langle 1, t, \ldots, t^d\rangle$, where $(1, t, \ldots, t^d)$ generate $H^0(\mathbf{P}^1, O(d))$, and $d \leq g - 1$. By construction $\mathfrak{F}$ is generated by global sections, and $h^0(\mathfrak{F}) \geq d + 1$. On the other hand, $h^0(\mathfrak{F}) \leq h^0(O(d)) = d + 1$. Therefore $h^0(\mathfrak{F}) = d + 1$. By Corollary 2, $h^1(\mathfrak{F}) = g - d$, and so $\mathfrak{F}$ satisfies Clifford's equality

$$h^0(\mathfrak{F}) + h^1(\mathfrak{F}) = g + 1. \qquad \square$$

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