Math. Ann. 280, 347–352 (1988)

Mathematische Annalen

The Classification of Homogeneous Cohen-Macaulay Rings of Finite Representation Type

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In this paper we classify the homogeneous Cohen-Macaulay rings which are of finite representation type, that is, the Cohen-Macaulay rings which are positively graded and generated in degree 1, with an algebraically closed field of characteristic 0 in degree 0, and which have, up to isomorphisms and shifts in the grading, only a finite number of indecomposable maximal Cohen-Macaulay modules (MCM-modules).

Our main contribution is to show that a homogeneous Cohen-Macaulay ring of finite representation type and dimension $\geqq 2$ must have "minimal multiplicity". Putting this together with previous results of Auslander, Auslander-Reiten, Buchweitz-Greuel-Schreyer, Greuel-Knörrer, and Solberg, we obtain:

Theorem. *Let R be a homogeneous Cohen-Macaulay ring. R is of finite representation type, if and only if R is isomorphic to one of the following rings:*
Arbitrary dimension:

$$K[X_1,\dots,X_n], \quad K[X_1,\dots,X_n]/(X_1^2+\dots+X_n^2), \quad n\geqq 1$$

Dimension 0:

$$K[X]/(X^m), \quad m\geqq 1$$

Dimension 1:

$$K[X_1,X_2]/(X_1X_2), \quad K[X_1,X_2]\,(X_1X_2(X_1+X_2))$$

or

$$K[X_1,X_2,X_3]/\det_2\begin{bmatrix} X_1, X_2, X_3 \\ X_2, X_3, X_1\end{bmatrix}$$

Dimension 2:

$$K[X_1,\dots,X_{n+1}]/\det_2\begin{bmatrix} X_1,\dots,X_n \\ X_2,\dots,X_{n+1}\end{bmatrix}, \quad n\geqq 2$$

Dimension 3:

$$K[X_1,\dots,X_5]/\det_2\begin{bmatrix} X_1, X_2, X_4 \\ X_2, X_3, X_5 \end{bmatrix}$$

or

$$K[X_1,\dots,X_6]/\det_2(\mathrm{Sym}\, 3\times 3)$$

The examples described algebraically in the theorem may also be described geometrically: Leaving aside the zero dimensional examples ("thick points" contained in a line), all the given rings are reduced, so it is convenient to describe them as the homogeneous coordinate rings of algebraic sets in the projective space. These, with their dimensions as projective varieties (one less than the dimensions of the corresponding rings), are:

Dimension 0: 3 or fewer distinct points in $\mathbb{P}^2$.

Dimension 1: A curve of degree n in $\mathbb{P}^n$. (These curves are all isomorphic to $\mathbb{P}^1$, and any two examples in $\mathbb{P}^n$ are equivalent. They are called rational normal curves.)

Dimension 2. The Veronese embedding of $\mathbb{P}^2$ in $\mathbb{P}^5$ or the rational normal scroll $S(1,2)$ in $\mathbb{P}^4$. [The second of these is an embedding of the projectivization of the vector bundle $\mathcal{O}_{\mathbb{P}^1}(1)\oplus\mathcal{O}_{\mathbb{P}^1}(2)$.]

Arbitrary dimension: $\mathbb{P}^n$ itself, and smooth quadric hypersurfaces.

In particular, aside from the hypersurface rings and the zero dimensional rings, there is just one infinite family, corresponding to the rational normal curves, and 5 exceptional examples.

Our proof that this list includes all examples is based on the following Theorems A and B.

The power series $H_R(t)=\sum_i(\dim_K R_i)t^i$ is called the *Hilbert-series* of R. It can be written as $H_R(t)=P(t)/(1-t)^d$, where $d=\dim R$, and $P(t)\in\mathbb{Z}[t]$.

Let $y_1,\dots,y_d$ be a regular sequence of elements of degree 1 in R, and let

$$\bar{R}=R/(y_1,\dots,y_d)R,$$

then

$H_{\bar{R}}(t)=H_R(t)(1-t)^d=P(t)$. So we see that $H_{\bar{R}}(t)$ doesn't depend on the particular chosen regular sequence $y_1,\dots,y_d$.

Sally calls in [12] the ring R a *stretched* CM-ring if $\dim_K\bar{R}_i\leqq 1$ for $i\geqq 2$.

Theorem A. *Homogeneous Cohen-Macaulay rings of finite representation type are stretched.*

The other result that we shall need is

Theorem B. *Let B be a homogeneous Cohen-Macaulay domain, and let $\bar{R}$ be the graded artinian ring obtained by reducing modulo a maximal regular sequence of elements of degree* 1. *If there exists a degree* 1 *element of $\bar{R}$ which is in the socle of $\bar{R}$, then the square of the maximal ideal of $\bar{R}$ is* 0 *(that is, R has minimal multiplicity).*

One can easily show that this result is equivalent to the statement that if C is an irreducible, reduced arithmetically Cohen-Macaulay curve in $\mathbb{P}^r$ of arithmetic genus >0, then the module $\sum_{v} H^0\omega_C(v)$ is generated in degree $v \leqq 0$ over the homogeneous coordinate ring of C. Actually rather more is true: if C is smooth, and L is a line bundle generated by its global sections, with $h^0L \geqq 3$, then a result of Mark Green, Theorem 4.b.2 of [7] says that $\sum H^0(L^n \otimes \omega_C)$ is generated, as module over $\sum H^0(L^n)$, by elements of degree $\leqq 0$. The techniques used by Green can be extended to cover at least the locally Gorenstein curves. We will however give a direct algebraic proof.

We first derive our Classification Theorem from A and B: We may assume that R is not the polynomial ring. If $\dim(R)=0$, then $R \simeq K[X]/(X^m)$, as follows from [9]. If $\dim(R)=1$, then Greuel and Knörrer [8] have shown that R must dominate a simple hypersurface singularity. The only graded rings among these are the one-dimensional rings which appear in the theorem.

Now we assume that $\dim(R) \geqq 2$. If R is Gorenstein, then by [9]

$$R \simeq K[X_1, \ldots, X_n]/(f)$$

is a hypersurface ring, and according to [4], Theorem C the degree of f is 2. In [10] it is even shown that if $\deg f > 2$, then R admits graded indecomposable MCM-modules of arbitrary high rank. Thus $\deg f = 2$, and

$$R \simeq K[X_1, \ldots, X_n]/(X_1^2 + \ldots + X_m^2)$$

with $m \leqq n$. If $m=n$, then Knörrer's result [11] implies that R is of finite representation type. If, on the other hand, $m<n$, then the plane $X_1 = \ldots = X_m = 0$ belongs to the singular locus of the quadric, and R cannot be of finite representation type as follows from the graded version of a theorem of Auslander [2]: If $R = K[X_1, \ldots, X_n]/I$ is a graded CM-ring of finite representation type, then R has an isolated singularity, which means that R_P is regular for all (homogeneous) prime ideals which are different from the irrelevant maximal ideal of R.

In particular, if $\dim(R) \geqq 2$, then by Serre's criterion and Auslander's result above R must be a (normal) domain, and we may thus assume that R is a graded non-Gorenstein stretched Cohen-Macaulay domain. Since it is stretched and of type >1, the artinian ring obtained by reducing modulo a regular sequence of elements of degree 1 must have a socle element of degree 1. Hence Theorem B implies that R is a homogeneous Cohen-Macaulay domain of minimal multiplicity.

The homogeneous domains of minimal multiplicity have been classified by Bertini. This classification can be found for instance in the article [6] of Eisenbud and Harris.

Auslander and Reiten [3] have shown that, aside from $K[X_1, X_2, X_3]^{(2)}$, the only rings in Bertini's list which are of finite representation type are the Veronese rings $K[X_1, X_2]^{(m)}$, $m \geqq 2$, and the ring

$$K[X_1, X_2, X_3, X_4, X_5]/(X_1^2 - X_2X_3, X_2X_5 - X_1X_4, X_1X_5 - X_3X_4).$$

That the ring $K[X_1, X_2, X_3]^{(2)}$ is also of finite representation type has been shown with similar methods by Solberg in [13].

Proof of Theorem A. Assuming that R is not stretched, we show that R is of infinite representation type. We choose a regular sequence $y_1, \ldots, y_d \in R$ of forms of degree 1, and set $\bar{R} = R/(y_1, \ldots, y_d)$. Since R is not stretched, there exists an integer

$c>1$ such that $\dim_K \bar{R}_c \geqq 2$. If $\bar{y}\in\bar{R}_c$, we write $I(\bar{y})$ for the ideal which is generated by $y_1,\dots,y_d$ and y, where $y\in R_c$ is a representative of $\bar{y}$ in R.

Consider the d-th syzygy-module $\Omega^d_R(R/I(\bar{y}))$ of the R-module $R/I(\bar{y})$. $\Omega^d_R(R/I(\bar{y}))$ is a graded MCM-module. We claim:

1) There exists an integer N such that $\operatorname{rank}\Omega^d_R(R/I(\bar{y}))\leqq N$ for all $\bar{y}\in\bar{R}_c$.

2) Up to a scalar multiple, there exists a unique homogeneous generator $e\in\Omega^d_R(R/I(\bar{y}))$ of degree d [and all other homogeneous generators of $\Omega^d_R(R/I(\bar{y}))$ have a higher degree].

3) Let $\bar{e}$ be the image of e in $\Omega^d_R(R/I(\bar{y}))/(y_1,\dots,y_d)\Omega^d_R(R/I(\bar{y}))$ then $\operatorname{Ann}_R\bar{e}=I(\bar{y})$.

These three assertions imply that R is of infinite representation type. In fact, suppose $\Omega^d_R(R/I(\bar{y}))\simeq\Omega^d_R(R/I(\bar{y}'))$ for $\bar{y},\bar{y}'\in\bar{R}_c$, then

$$\Omega^d_R(R/I(\bar{y}))/(y_1,\dots,y_d)\Omega^d_R(R/I(\bar{y}))\simeq\Omega^d_R(R/I(\bar{y}'))/(y_1,\dots,y_d)\Omega^d_R(R/I(\bar{y}')),$$

and under this isomorphism $R\bar{e}$ is mapped isomorphically to $R\bar{e}'$, where e (respectively e') is the "unique" element of least degree in $\Omega^d_R(R/I(\bar{y}))$ [respectively $\Omega^d_R(R/I(\bar{y}'))$]. It follows from 3) that $I(\bar{y})=I(\bar{y}')$, and this implies $(\bar{y})=(\bar{y}')$. But since $\dim_K\bar{R}_c\geqq 2$, there are infinitely many ideals $(\bar{y})$, $\bar{y}\in\bar{R}_c$, and thus 1) implies that R is of infinite representation type.

As $\bar{y}$ varies, the length of $R/I(\bar{y})$ is bounded by a number N_1. Therefore $R/I(\bar{y})$ admits a filtration of length at most N_1 in which all factors are isomorphic to K. It follows that $\operatorname{rank}\Omega^d_R(R/I(\bar{y}))$ can be estimated by $N:=\operatorname{rank}\Omega^d_R(K)\cdot N_1$.

To prove 2) we consider a minimal free homogeneous R-resolution $F_.$ of $R/I(\bar{y})$, and the Koszul complex $K_.$ of the regular sequence $y_1,\dots,y_d$, which is a free resolution of $R/(y_1,\dots,y_d)$. The epimorphism $R/(y_1,\dots,y_d)\to R/I(\bar{y})$ induces a homogeneous comparison map $\varphi: K_.\to F_.$ of complexes.

Note that $K_i\simeq R(-i)^{b_i}$, while $F_i\simeq\bigoplus_j R(-d_{ij})$ with $d_{ij}\geqq i$. We claim that the induced maps $\bar{\varphi}_i: K_i/m_R K_i\to F_i/m_R F_i$ are all injective. This is a special case of a general and well-known principle, and may be proved as follows by induction on i. For $i=0$, the assertion is trivial. Now let $i>0$, and $\bar{x}\in K_i/m_R K_i$ with $\bar{\varphi}_i(\bar{x})=0$. We choose a homogeneous element $x\in K_i$, whose residue class $\bmod\, m$ is $\bar{x}$. We have $\deg\varphi_i(x)=i$, since $\deg x=i$. So $\varphi_i(x)$ belongs to a homogeneous basis of F_i (since all elements in F_i have degree $\geqq i$), or $\varphi_i(x)=0$. The first case leads to a contradiction since $\bar{\varphi}_i(\bar{x})=0$. So we must have that $\varphi_i(x)=0$. Consider the commutative diagram

$$\begin{array}{ccc} K_i & \xrightarrow{\varphi_i} & F_i \\ {\scriptstyle\partial'_i}\downarrow & & \downarrow{\scriptstyle\partial_i} \\ K_{i-1} & \xrightarrow[\varphi_{i-1}]{} & F_{i-1} \end{array}$$

φ_{i-1} is injective since by induction hypothesis $\bar{\varphi}_{i-1}$ is injective. It follows that $\varphi_{i-1}(\partial'_1(x))=\partial_i(\varphi_i(x))=0$, and hence $\partial'_i(x)=0$. However, since $\ker\partial'_i\subseteqq mK_i$, we get $\bar{x}=0$.

Thus we may consider $K_.$ as a subcomplex of $F_.$, and for all i we can write $F_i=K_i\oplus F'_i$, where F'_i is free.

Next we show by induction on i that all homogeneous basis elements of F'_i are of degree $>i$. $F'_0=0$, and $F'_1=R(-c)$, so let $i>1$, and suppose that there exists an element $x\in F'_i$, $x\neq 0$, $\deg x=i$. Then $\partial_i(x)=a+b$, where $a\in m_R K_{i-1}$ and $b\in m_R F'_{i-1}$, and $\deg a=\deg b=i$. Since by induction hypothesis all elements of

F'_{i-1} have degree $\geqq i$, it follows that $b=0$, and so $\partial_i(x) \in K_{i-1}$. The cycle $\partial_i(x)$ is a boundary in $K_.$ since $i-1>0$. Thus there exists $x' \in K_i$ of degree i such that $\partial_i(x') = \partial_i(x)$. The element $z = x' - x \in \ker \partial_i \subseteq \mathfrak{m}_R F_i$ is of degree i. But all elements $\neq 0$ in $\mathfrak{m}_R F_i$ have degree $>i$, therefore $z=0$. This implies that $x = x' \in K_i$, a contradiction.

From our description of $F_.$, assertion 2) follows at once since we have a minimal presentation $K_d \oplus F'_d \to \Omega^d(R/I(\bar{y})) \to 0$, where $K_d \simeq R(-d)$, and where the homogeneous generators of F'_d all have degree $>d$.

To prove 3) let $\bar{F}_. = F_./(y_1, \dots, y_d)F_.$ and observe that the d-th homology $H_d(\bar{F}_.)$ is isomorphic to the d-th homology $H_d(y_1, \dots, y_d; R/I(\bar{y}))$ of the Koszul complex of the sequence $y_1, \dots, y_d$ with values in $R/I(\bar{y})$. Since the y_i annihilate $R/I(\bar{y})$, we have

$$H_d(y_1, \dots, y_d; R/I(\bar{y})) \simeq R/I(\bar{y})(-d).$$

Therefore the inclusion $0 \to \Omega^d_R(R/I(\bar{y})) \to F_{d-1}$ induces an exact sequence

$$0 \to R/I(\bar{y})(-d) \to \Omega^d_R(R/I(\bar{y}))/(y_1, \dots, y_d)\Omega^d_R(R/I(\bar{y})) \xrightarrow{\alpha} \bar{F}_{d-1}.$$

Since $e \in \Omega^d_R(R/I(\bar{y}))$ corresponds to the generator of K_d, it is clear that $\alpha(\bar{e})=0$, and it follows from the exact sequence that $R\bar{e} \simeq R/I(\bar{y})(-d)$.

Proof of Theorem B. The condition on $\bar{R}$ is independent of the choice of the maximal regular sequence of degree 1 elements of R. Thus we may assume that the regular sequence is chosen generally, and this has the consequence that $\bar{R}$ may be written as S/x_0, where S is the homogeneous coordinate ring of a reduced set of points of $\mathbb{P}^n$ in "linearly general position", that is, no $k+2$ of which are contained in a k-plane, for any $k<n$. This statement follows from the "General Position Theorem" on p. 109 in [1]. Of course, by our definition of n, these points lie in no hyperplane, so there are at least $n+1$ of them. The conclusion of the theorem is equivalent to the statement that there are exactly $n+1$.

Let $x_1 \in S_1$ be a representative for the given socle element of $\bar{R}$, and let $x_1, \dots, x_n$ be the remainder of the basis of S_1. Lifting the relations $x_1 x_i = 0$, which hold in $\bar{R}$, to S, we get the relations $x_1 x_i = s_i x_0$, where the s_i are linear forms in $x_0, \dots, x_n$. From these relations we obtain

$$x_0(x_i s_j - x_j s_i) = x_i(x_0 s_j - x_j x_1) - x_j(x_0 s_i - x_i x_1) = 0$$

for $1 \leqq i < j \leqq n$, and since x_0 is a nonzerodivisor in S, we get further relations $x_i s_j - x_j s_i$, for $i<j$. So we see that the ideal defining S contains the 2×2 minors of a $2 \times (n+1)$ matrix in the form

$$\begin{bmatrix} x_0, x_1, \dots, x_n \\ x_1, s_1, \dots, s_n \end{bmatrix},$$

where the s_i are linear forms in $x_0, \dots, x_n$. Let $s_i = \sum_{j=0}^{n} a_{ij} x_j$, $i=1, \dots, n$, $(s_0 = x_1)$. The matrix $A = (a_{ij})$ has an eigenvector $(c_0, \dots, c_n)$ with eigenvalue $\lambda \neq 0$, and we assume that $c_n = 1$. Hence we may apply the following row and column operations:

$$\begin{bmatrix} x_0, x_1, \dots, x_n \\ s_0, s_1, \dots, s_n \end{bmatrix} \to \begin{bmatrix} x_0, \dots, x_{n-1}, c_0 x_0 + c_1 x_1 + \dots + x_n \\ s_0, \dots, s_{n-1}, c_0 s_0 + c_1 s_1 + \dots + s_n \end{bmatrix}$$

$$\to \begin{bmatrix} x_0, \dots, x_{n-1}, & c_0 x_0 + c_1 x_1 + \dots + x_n \\ s_0 - \lambda x_0, \dots, s_{n-1} - \lambda x_{n-1}, & 0 \end{bmatrix}.$$

After further such operations we may assume that the first row still consists of independent variables, while the second row consists of some number $t<n+1$ of independent linear forms followed by $n+1-t$ zeros. For a different and more general form of this argument see [5].

The ideal defining S thus contains the product of the ideal generated by the $n+1-t$ independent linear forms in the new first row and an ideal generated by t independent linear forms – that is, the set of points of which S is the homogeneous coordinate ring lies on the union of an $n-t$-plane and a $t-1$-plane. Since it consisted of points in linearly general position, the total number of these points is at most $(n-t+1)+t=n+1$, so S is the homogeneous coordinate ring of $n+1$ independent points in n-space, and we are done.

The results of this paper suggest the following problem: If R is a complete local Cohen-Macaulay ring of finite representation type and dimension $\geqq 2$, must R have minimal multiplicity?

Acknowledgement. We would like to thank Idun Reiten for valuable comments during the preparation of this work.

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Received June 11, 1987