Cohen-Macaulay Modules on Quadrics

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with an appendix by
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Dedicated to Maurice Auslander on the occasion of his 60^{th} birthday

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Abstract This paper analyzes the graded maximal Cohen-Macaulay modules over rings of the form $R = k[x_1, \ldots, x_r]/Q$, when Q is a quadratic form defining a regular projective hypersurface, and k is an arbitrary field (the case when k is algebraically closed of characteristic $\neq 2$ is a special case of the theory developed by Knörrer [1986]). For any nonzero quadratic form Q, regular or not, the graded maximal Cohen-Macaulay R-modules define modules over the even Clifford algebra of Q, and we show that this algebra is semi-simple iff Q is regular (this is classical for char $k \neq 2$). As a result of this and other information about the Clifford algebra, we give a detailed account of the Cohen-Macaulay modules when Q is regular, identifying the number of indecomposables (2 or 3, counting R) their ranks, and the relations of duality and syzygy among them.

We also show that when Q is nonsingular every maximal Cohen-Macaulay module over the completion of R at $(x_1, \ldots, x_r)$ is the completion of a graded module.

Our approach is via the matrix factorizations of quadratic forms in the sense of Eisenbud [1980], and includes a classification of these. As an appendix we include a sketch of the relevant case of a more general theory of Buchweitz giving equivalences of certain derived categories of modules over the algebras of a Koszul pair, generalizing work of Bernstein-Gelfand-Gelfand.

Introduction:

If R is a local or graded ring, then a maximal Cohen-Macaulay (abbreviated MCM) R-module M is a finitely generated module whose depth is equal to the dimension of R. There has recently been much interest in Cohen-Macaulay rings R which have only finitely many isomorphism classes of indecomposable MCM modules, the "rings of finite CM-type"; see for example Auslander-Reiten [1986]. Two milestones of importance for this paper are 1) Auslander's result [1986] that if R is a local (or graded) ring of finite CM-Type then R has an isolated singularity and 2) Knörrer's results [1986] analyzing the Cohen-Macaulay modules over

$$k[[x_1,\ldots,x_n,y]]/f(x)+y^2$$

or

$$k[[x_1,\ldots,x_n,y,z]]/f(x)+yz$$

in terms of those over

$$k[[x_1,\ldots,x_n]]/f \; .$$

These results suggest that if Q is a quadratic form on a vector-space V over k considered as an element of $S_2(V^*)$, which is regular in the sense that

$$R = k[V^*]/Q$$

has only an isolated singularity, then the MCM modules over R should be particularly tractable; indeed, Knörrers results show that if k is algebraically closed and of characteristic not 2, then there are always 1 or 2 nonfree indecomposable MCM R-modules depending on whether $\dim V$ is odd or even; that they both have the same rank and are syzygies of one another when there are 2; and that writing

$$m = \left[\frac{\dim V}{2}\right] - 1 \; , \quad \text{([x] denoting the largest integer smaller or equal to x)}$$

the rank of their direct sum is 2^m (Note that if $R = k[x]/x^2$ or $k[x,y]/xy$, then rank R/x may reasonably be taken to be 1/2 as this formula implies.)

In this paper we derive parallel results for all fields k. (Theorem 3.1 and 3.2). As in Knörrer's case, if $\dim V$ is odd there is a unique nonfree indecomposable MCM R-module; but if $\dim V$ is even, there can be 1 or 2, depending on the discriminant or Arf invariant of Q (Theorem 3.1). Again when there are two modules they have the same rank and are syzygies of one another; but in the general case the rank of their direct sum may be any power of 2 between 2^m and

$2^{\dim V-2}$, depending on the degree of the division algebra which is the Hasse-Witt invariant of Q (except in characteristic 2, where the nullity of the bilinear form associated to Q also plays a role).

To obtain these results we use the following idea to transform the problem to a problem about modules over the even Clifford algebra, $C_0(Q)$:

If M is an MCM R-module, then the free resolution of M is known to be periodic of period 2 (Eisenbud [1980]), and the periodicity isomorphism on the level of $\mathrm{Tor}_*^R(k,M)$ is multiplication by an element $\sigma \in \mathrm{Ext}^2(k,k)$. Thus $\mathrm{Tor}_0^R(k,M) \oplus \mathrm{Tor}_1^R(k,M)$ is a module over the algebra $\mathrm{Ext}_R^*(k,k)/\langle\sigma-1\rangle$, which is isomorphic as a $\mathbb{Z}/2\mathbb{Z}$-graded algebra to the Clifford algebra $C(Q)$. Taking even degree parts, $\mathrm{Tor}_0^R(k,M) = k \otimes_R M$ becomes a module over $C_0(Q)$ in a functorial way. In fact (Theorem 2.1), for any quadratic form $Q \neq 0$ this functor induces an equivalence between the category of modules over $C_0(Q)$ and a certain subcategory of the category of MCM R-modules modulo projectives. If Q is nonsingular, we obtain in this way the whole category of graded MCM R-modules modulo projectives, and even, by Theorem 2.2, all MCM modules over the completion

$$\hat{R} = k[[V^*]]/Q .$$

To construct the inverse of the functor $M \to k \otimes_R M$, we use the idea of matrix factorization (Eisenbud [1980]). Recall that a matrix factorization of a nonzerodivisor f of a ring S is a pair of (necessarily) square matrices φ, ψ over S such that $\varphi\psi$ is f times the identity matrix, $f\cdot 1$. It follows at once that $\psi\varphi = f\cdot 1$, too. If φ, ψ is a matrix factorization of f, and S is Cohen-Macaulay, then coker φ is an MCM S/f-module. If S is regular, then every MCM S/f-module arises in this way, and the category of MCM modules mod projectives is equivalent to the category of matrix factorizations (with maps defined up to homotopy in an obvious sense). Thus, we may construct the inverse functor by constructing a matrix factorization of Q from a $C_0(Q)$-module.

If A_0 is a $C_0(Q)$-module then $A = C(Q) \otimes_{C_0(Q)} A_0 =$
$= A_0 \oplus (A_1 := C_1(Q) \otimes_{C_0(Q)} A_0)$ is a $\mathbb{Z}/2\mathbb{Z}$-graded $C(Q)$-module; in fact this construction gives an equivalence of categories iff $C_0(Q)$ is non-zero (Atiyah-Bott-Shapiro [1964]). The module structure on

A yields, via the inclusion $V \to C_o(Q)$, maps

$$V \otimes A_o \to A_1$$
$$V \otimes A_1 \to A_o$$

whose adjoints

$$A_o \to A_1 \otimes V^*$$
$$A_1 \to A_o \otimes V^*$$

define degree 1 maps of $S = k[V^*]$-modules

$$A_o \otimes S \overset{\psi}{\to} A_1 \otimes S$$
$$A_1 \otimes S \overset{\varphi}{\to} A_o \otimes S \ ;$$

it turns out that φ, ψ is a matrix factorization of Q, and $A_o \to \operatorname{coker} \varphi$ is the desired inverse to $M \to k \otimes_R M$ in the regular case.

In fact we may conveniently define the functor $M \to k \otimes M$, regarded as a module over the even Clifford algebra, in a completely elementary way using matrix factorizations, and we adapt this method in the text, below.

To use this machine, we need information about $C_o(Q)$ and its modules. This is provided in section 1 below. The case where the characteristic of k is not 2, and even the "non-defective" case in characteristic 2, are classical. Our main - and probably only - new result in this section is that Q is regular in the sense above iff $C_o(Q)$ is semi-simple (Theorem 1.1). We describe the structure of this algebra in the semi-simple case. We also describe the possibilities - which are more limited than one might at first think - over a "global" field, and we show that the Clifford algebra of a generic form in characteristic 0 is a division ring.

Closely related to this paper in several points is the noteworthy paper of Swan [1985]. In particular Swan makes use of the Clifford algebra - in a way closely related to ours - to analyze the K-theory of modules over R when Q satisfies the condition that the hypersurface defined by Q is smooth over k except at the origin. The condition of smoothness - which Swan calls nonsingularity - is somewhat more restrictive than our condition of regularity: a simple example is the form

$$x^2+ty^2$$

over $k_o(t)$, the field of rational functions in t over a field k_o of characteristic 2, which is regular but not smooth.

Many open problems about MCM modules over hypersurfaces remain. Some which seem to us particularly interesting are:

1) It seems natural to try to imitate the theory of Knörrer [1986] and consider MCM modules over a ring of the form

$$R = k[x_1,\ldots,x_n,y_1,\ldots,y_m]/f(x)+Q(y)$$

where Q is a quadratic form and k is an arbitrary field. Of course if M is an MCM R-module, then $N = M/(y_1,\ldots,y_m)M$ is an MCM module over

$$R_1 = R/(y) = k[x_1,\ldots,x_n]/f(x) .$$

Further,

$$N \otimes_{R_1} k = M \otimes_R k = (M/(x_1,\ldots,x_n)M) \otimes_{k[y]/Q(y)} k,$$

so $N \otimes k$ is naturally a $C_o(Q)$-module. Can one establish an inverse construction from some compatibility between these two structures on N ?

2) Let Q be a nonsingular quadric on V and $R = k[V^*]/Q$ as above. What R-modules have the indecomposable MCM R-modules as syzygies? For example, if $C_o(Q)$ is a division ring, then the dimension of the unique indecomposable $C_o(Q)$-module (which is $C_o(Q)$ itself!) is $2^{\dim V-1}$, and it follows that the corresponding MCM R-module is the n^{th} syzygy of the residue class field of R (for all $n \geq \dim R$). In general, the n^{th} syzygies of the residue class field always have this number of generators (and the rank $2^{\dim V-2}$), so they are always decomposable if $C_o(Q)$ is not a division ring. In the appendix (Cor. 2.) we give a necessary and sufficient condition for an MCM R-module to be a syzygy module of an R-module of finite length.

3) What is the nature of the matrix factorization of a quadric? More specifically, for Q regular, or for linear factorizations in general we know that matrix factorizations of Q correspond to modules over $C_o(Q)$ as well as to MCM R-modules. If $C_o(Q)$ is simple, so there is just one nonfree indecomposable MCM R-module, then it is easy to see that the corresponding matrix factorization can be chosen in the form $\varphi^2 = Q\cdot 1$. Can φ always be taken to be either symmetric or shew-symmetric? What determines this? Of course if A_o is the unique indecomposable $C_o(Q)$-module in this case, then A_o will be iso-

morphic to A_o^* (with action of $C_o(Q)$ via the "main involution"), so we may regard the matrix factorization as involving $\varphi : F^* \to F$ in a fairly natural way. If C_o is not simple, so $C_o(Q) = (k \times k) \otimes_k \overline{C}$, with $\overline{C}$ central simple, then as we show in Theorem 3.2, the first syzygy module N of one of the nonfree indecomposable MCM R-modules M has $k \otimes_R N$ the contagredient $C_o(Q)$-module to $k \otimes_R M$ iff $\dim V \equiv 2 (\mathrm{mod}\ 4)$, so at least in this case we may write the minimal free resolution of M naturally as

$$F^* \xrightarrow{\varphi} F \xrightarrow{\psi} F^* \xrightarrow{\varphi} F \to M .$$

Can φ and ψ be taken to be of some special form? In the case $\dim V \equiv 0(4)$, $M \cong N^*$, so φ is similar to ψ^*. When can φ and ψ be chosen so that φ <u>is</u> the transpose of ψ ?

1) Regular forms and semi-simple even Clifford algebras

In this section we present the facts about Clifford algebras. Some of the results are new in characteristic 2. The reader primarily interested in Cohen-Macaulay modules may wish to read at least the beginning of section 2, to see how the Clifford algebra arises naturally in a description of linear matrix factorizations, before this section. The text by Jacobson [1980] sect. 4.8 contains an excellent introduction to the theory of Clifford algebras. Other good references are Atiyah-Bott-Shapiro [1964], Lam [1980], and Scharlau [1985].

a) Results

Throughout, we let Q be a nonzero quadratic form on a vectorspace V of dimension r over a field k.

We regard Q as a map $Q : V \to k$ satisfying

(i) $Q(\alpha v) = \alpha^2 Q(v)$ for all $\alpha \in k$,

(ii) $(v,w)_Q = Q(v+w) - Q(v) - Q(w)$ is bilinear.

Quadratic forms may be identified with element of $S_2(V^*)$ and we will henceforth make this identification.

We will write V_{ins} for the nullspace of $(,)_Q$.

The Clifford algebra of Q is the tensor algebra $\otimes V$ modulo the relations generated by those of the form $v \otimes v - Q(v)$ for $v \in V$. Since both terms of the relations are of even degree, $C(Q)$ is $\mathbb{Z}/2\mathbb{Z}$-graded; we write $C_o(Q)$ and $C_1(Q)$ for the even and odd parts. We have $\dim C(Q) = 2^r$, $\dim C_o(Q) = 2^{r-1}$.

We will say that a quadratic form Q is regular if the projective hypersurface it defines in $\mathbb{P}(V^*)$ is regular, in the sense that all its local rings are regular local rings, or, equivalently, if the ring $R = k[V^*]/Q$ becomes regular when localized at any prime other than that generated by V^*. If char $k \neq 2$, this is equivalent to the usual definition of nonsingularity of a quadratic form.

(The condition that the projective hypersurface defined by Q be smooth over k is equivalent to regularity in characteristics other than 2, but definitely stronger in characteristic 2. Smoothness, in this sense, is equivalent in all characteristics to the condition called nonsingularity by Swan [1985]).

Regularity is a very natural condition from the point of view of the even Clifford algebra:

Theorem 1.1: The following conditions are equivalent:

1) Q is regular
2) $C_o(Q)$ is semi-simple
3) Either the nullspace V_{ins} of $(,)_Q$ is zero, or the restriction of Q to V_{ins} has the form

$$\sum_{i=1}^{n} s_i x_i^2, \quad s_1 \neq 0, \quad \text{and} \quad [k(\{(s_i/s_1)^{1/2}\}):k] = 2^{\dim V_{ins}-1}$$

Remarks: 1) If char $k \neq 2$, then $Q|V_{ins} \equiv 0$, so $V_{ins} = 0$ iff Q is regular.

2) Of course any basis for V_{ins} is orthogonal, in the sense that Q becomes diagonal; and it is easy to see that the condition in 3) is independent of the basis chosen, and is in fact equivalent to the basis-free condition:

$$[k(\{(Q(v)/Q(w))^{1/2} | v,w \in V_{ins}, Q(w) \neq 0\}) : k] = 2^{\dim V_{ins}-1}.$$

3) We will give below a direct but in characteristic 2 somewhat computional proof of the implication 1) $\Rightarrow$ 3). Our original proof was indirect but more conceptual: The equivalence of 3) and 2) is

fairly easy. Given the category equivalence described in section 2, below, the equivalence 1) $\Longleftrightarrow$ 2) follows from the Theorem of M. Auslander [1986] that a local Cohen-Macaulay ring has an isolated singularity if it has finite CM-type.

The main facts we will use about the even Clifford algebra of a regular form are summarized in the next Theorem, where we write

$\det Q = \Pi s_i$, if $Q \sim \Sigma s_i x_i^2$, char $k \neq 2$
$\operatorname{arf} Q = \Pi Q(e_i)Q(f_i)$, if char $k = 2$, $(\ ,\ )_Q$ is nongenerate,
and $\{(e_i, f_i)\}$ is a symplectic basis for V with respect to $(\ ,\ )_Q$.

As is well known, $\det Q$ yields an invariant of Q in $k^*/(k^*)^2$, and $\operatorname{arf} Q$ an invariant modulo $\{x^2+x \mid x \in k\}$.

Theorem 1.2: If $(\ ,\ )_Q$ is nondegenerate, then $C_o(Q)$ is central simple over k if $\dim V$ is odd, and else the tensor product over k of a central simple algebra with the extension

$$k[x]\big/ x^2-(-1)^{(\dim V)/2}\det Q \quad \text{if char } k \neq 2$$

$$k[x]\big/ x^2+x+\operatorname{arf} Q \qquad \text{if char } k = 2 .$$

If $(\ ,\ )_Q$ is degenerate, but Q is regular, then $C_o(Q)$ is the tensor product of a central simple algebra with the purely inseparable field extension, of degree $2^{\dim V_{ins}-1}$, given in part 3) of Theorem 1.1. In either case, the central simple factor of $C_o(q)$ is Brauer equivalent to a product of quaternion algebras over k.

Remark: Q is called non-defective if $Q|_{V_{ins}} \equiv 0$, as is true for every Q when char $k \neq 2$. The Theorems are new, if at all, only in the defective case; see the proof below.

We will also use two facts about quaternion algebras. Recall that a local field is either $\mathbb{R}$, $\mathbb{C}$, or complete with respect to a non-archimedean discrete valuation and with finite residue field; a global field is a finite algebraic extension of either the rational field, or a field of rational functions over a finite field. The following results are classical. For the convenience of the reader we will include sketches of proofs.

Theorem 1.3: If k is any local or global field, then any tensor product of quaternion algebras over k is Brauer equivalent to a quaternion algebra. In particular, the division algebra corresponding to the central simple factor described in Theorem 1.2 has dimension either 1 or 4 over k.

By contrast, a product of generic quaternion algebras is a division algebra. In terms of the even Clifford algebra, we have:

Theorem 1.4: Let $K = k(s_1,\dots,s_n)$ be a field of rational functions over a field k. If Q is the quadratic form $\sum_1^n s_i x_i^2$, then $C_o(Q)$ is a central division algebra over an extension field of K as described in 1.2.

b) Proofs and sketches

Proof of Theorem 1.1 and 1.2: First, if Q is non-defective - that is, the restriction of Q to V_{ins} is identically 0 - everything is known: For in Theorem 1.1 condition 3) becomes $V_{ins} = 0$ which by the Jacobian criterion implies nonsingularity, whereas if $V_{ins} \neq 0$, then V_{ins} is contained in the singular locus of $\{Q = 0\}$; thus 1) $\Longleftrightarrow$ 3) in this case. That $V_{ins} = 0$ implies the semi-simplicity of $C_o(Q)$ is classical; see for example Jacobson [1980] section 4.8, which also contains the conclusions of Theorem 1.2 in this non-defective case. On the other hand, even if $V_{ins} \neq 0$, then choosing a non isotropic vector v in V (which is possible since we assumed at the outset of this section that $Q \neq 0$) we see that

$$C_o(Q) \cong C(-Q(v)\circ Q|_{v^\perp})$$

(as for example in Jacobson Thm. 4.13; the hypotheses there are superfluous for this conclusion) and

$$C(-Q(v)Q|_{v^\perp}) = A \mathbin{\hat{\otimes}} C(-Q(v)Q\,|_{V_{ins}}) ,$$

when $\hat{\otimes}$ is the graded tensor product and A is the Clifford algebra of Q restricted to a complement of V_{ins}. Since

$$C(-Q(v)\cdot Q|_{V_{ins}}) = C(0|_{V_{ins}}) \cong \Lambda(V_{ins}),$$

the exterior algebra, we see this is not semi-simple and so 1) $\Longleftrightarrow$ 2) in the non-defective case.

Thus we may henceforward assume

$$\text{char } k = 2$$

$$V_{ins} \neq 0$$

$$Q|_{V_{ins}} \neq 0 .$$

We first prove the equivalence of 1) and 3) in Theorem 1.1. Choose a complement V_{sep} for V_{ins} in V, and a symplectic basis $\{(e_i,f_i)\}$ of V_{sep} with respect to the form $(\ ,\ )_Q$. If y_i,z_i are dual to e_i,f_i, we may write Q as

$$Q = \Sigma y_i z_i + \Sigma r_i y_i^2 + \Sigma t_i z_i^2 + \sum_{j=1}^{n} s_j x_j^2 \quad \text{with } n \geq 1 .$$

Now suppose condition 3) is satisfied. Passing to an affine open cover, we may assume that one of the variables, say x_1 or y_1, is 1. We must show that Q is not contained in the square of any maximal ideal N of $S = k[\underline{x},\underline{y},\underline{z}]$. For this it suffices to show that there is a derivation D of S with $D(Q) \notin N$. Suppose on the contrary that $D(Q) \in N$ for all derivations D. Now the derivations $\frac{\partial}{\partial y_i}$ and $\frac{\partial}{\partial z_i}$ take the values z_i and y_i on Q, respectively, so we need only to consider those primes N containing all y_i, z_i, and Q. In particular, we have no problem with the open set $y_1 = 1$, so we may take $x_1 = 1$.

By condition 3), the $(s_i/s_1)^{1/2}$ are 2-independent over k in the sense of, for example, Matsumura [1980] section 38, and thus the s_i/s_1 are 2-independent over k^2. It follows that there is a derivation D_j of k taking s_i/s_1 to δ_{ij} ("Kroneker δ"), and we may extend D_j to all of S so that $D_j(y_i) = D_j(z_i) = 0$ for all i. Applying D_j to $\frac{1}{s_1}Q$ we get x_j^2 modulo the y_i and z_i, for $j > 1$, so N must contain all these as well. But then N contains 1 (since $x_1 = 1$ by assumption), a contradiction. Thus the hypersurface associated to Q is regular, proving 3) $\Rightarrow$ 1).

For the converse, note that if $n < \dim V_{ins}$, then the projective hypersurface associated to Q is obviously singular, so we may assume $n = \dim V_{ins}$. Suppose that

$$[k(\{(s_i/s_1)^{1/2}\}) : k] < 2^{n-1}$$

we will find a singular maximal ideal containing all the y_i and z_i.

Replacing Q by $\frac{1}{s_1}Q$, we may suppose $s_1 = 1$. Rearranging the s_i, we may assume that

$$k(\{\sqrt{s_i}\}_{i=2,\dots,n}) = k(\{\sqrt{s_i}\}_{i=2,\dots,m})$$

for some $m < n$ such that $s_2,\dots,s_m$ are 2-independent.

To find a singular point on the hypersurface $Q = 0$, we work on the affine open set $x_{m+1} = 1$, and employ the criterion given in Matsumura [1980] Theorem 95, according to which a prime ideal $N \subset S' = k[\{y,z\},x_1,\dots,x_m,x_{m+2},\dots,x_n] = k[V^*]/(x_{m+1}-1)$, of height h, containing $Q' = Q|_{x_{m+1}=1}$ is a singular point of Spec S'/Q' if there is a set of derivations $\Delta = \{D_i\}$ and a set of elements $\{g_j\} \subset N$ such that

1) $D_iQ' \in N$ for all i.
2) Some $h \times h$ minor of the matrix
$$(D_ig_j)$$
is not contained in N.

First we define Δ. Since $s_2,\dots,s_m$ are 2-independent over k^2, the $k^2(s_2,\dots,s_m)$-vectorspace

$$\mathrm{Der}_{k^2}k^2(s_2,\dots,s_m)$$

has a basis consisting of derivations D_i; $i = 2,\dots,m$; such that

$$D_i(s_j) = \delta_{ij}$$

and we may extend these first to $\mathrm{Der}_{k^2}(k)$ and then also to derivations of S' which annihilate the variables. We take Δ to be the set

$$\Delta = \{D_2,\dots,D_m, \frac{\partial}{\partial y_i}, \frac{\partial}{\partial z_j}, \frac{\partial}{\partial x_k}; \text{ all } i,j,k\}.$$

Next we define N, by giving generators. Let N' be the ideal generated by all the y_i and z_i together with $x_{m+2},\dots,x_n$, and let N'' be the ideal generated by N' and the following polynomials $f_1,\dots,f_m$:

$$f_1 = x_1^2 + s_{m+1} + \sum_{i=2}^{m} s_i D_i(s_{m+1}) ,$$

and for $i = 2,\ldots,m$,

$$f_i = x_i^2 + D_i(s_{m+1}) .$$

Note that

$$Q' = f_1 + \sum_2^m s_i f_i \mod N' ,$$

so $Q' \in N''$. Also $D_i(Q') \equiv f_i \mod N'$, and so $D_i(Q') \in N''$ for all i.

Clearly N' is prime, but contains neither Q' nor $\Delta Q'$. N'' on the other hand contains both Q' and $\Delta Q'$ but is generally not prime. We will remedy this by adding square roots of some of the f_i to N'' to obtain the ideal N.

Rearranging $x_2,\ldots,x_m$ if necessary, we may suppose that

$$D_2(s_{m+1}),\ldots,D_l(s_{m+1})$$

are 2-independent over k^2 while

$$D_{l+1}(s_{m+1}),\ldots,D_m(s_{m+1})$$

are 2-dependent on them (the case $l = m$ is not excluded), say

$$\sqrt{D_{l+k}(s_{m+1})} = q_{l+k}(\sqrt{D_2(s_{m+1})},\ldots,\sqrt{D_l(s_{m+1})}) \quad \text{for } k = 1,\ldots,m-l$$

where the q_{l+k} are polynomials with coefficients in k.

It follows that for $k = 1,\dots,m-1$,

$f_{l+k} \equiv (x_{l+k}+q_{l+k}(x_2,\dots,x_l))^2$ modulo $N'+(f_2,\dots,f_l)$, and we set

$$g_k = f_k \ , \ k = 2,\dots,l$$

$$g_{l+k} = x_{l+k}+q_{l+k}(x_2,\dots,x_l), \ k = 1,\dots,m-1 \ .$$

Further, if $s_{m+1} + \sum_{i=2}^{m} s_i D_i(s_{m+1})$ is not 2-dependent on $D_2(s_{m+1}),\dots,D_l(s_{m+1})$ we set $g_1 = f_1$, else we write its square roots as $q_1(\sqrt{D_2(s_{m+1})},\dots,\sqrt{D_2(s_{m+1})})$ where q_1 is a polynomial with coefficients in k, and set

$$g_1 = x_1+q_1(x_2,\dots,x_l) \ .$$

We call these case I and case II, respectively.

We take $N = N'+(g_1,\dots,g_m)$. From the construction it is clear that $Q',\Delta Q'$ are in N and N is a maximal ideal of S' which thus has height equal to $\dim V-1$.

Applying $\Delta_I = \{\frac{\partial}{\partial y_i} ; \frac{\partial}{\partial z_j} ; \frac{\partial}{\partial x_{l+k}} ;$ all i,j and $k = 1,\dots,n-l\}$ to $A_I = \{y_i;z_j;g_{l+k};$ all i,j and $k = 1,\dots,n-l\}$ yields an identity matrix of size $\dim V-l-1$ in case I. In case II, $\Delta_{II} = \Delta_I \cup \{\frac{\partial}{\partial x_1}\}$ applied to $A_{II} = A_I \cup \{g_1\}$ yields an identity matrix of size $\dim V-l$. It thus suffices to show that the matrix

$(D_i g_j)$ $i = 2,\dots,m$
$j = 1,\dots,l$ in case I
$j = 2,\dots,l$ in case II

which is actually a matrix with coefficient in $k^2(s_2,\dots,s_m) \subset k \subset S'/N$, has rank equal to l in case I ($l-1$ in case II). Of course $D_i g_j = D_i(g_{j,o})$, where $g_{j,o}$ is the constant term of g_j . Now the $D_2,\dots,D_m$ form a $k^2(s_2,\dots,s_m)$-basis of $\mathrm{Der}_{k^2} k^2(s_2,\dots,s_m)$, and thus linear combinations of them induce all derivations of $k^2(g_{1,o},\dots,g_{l,o})$. The rank of the given matrix $(D_i g_j)$ is consequently the maximal number of 2-independant elements among the $g_{j,o}$, so by construction it is the desired number l in case I (or $l-1$ in case II). Thus $N/(Q)$ represents a singular point of the projective hypersurface $Q = 0$, proving 1) $\Rightarrow$ 3) in Theorem 1.1.

Continuing with the case char $k = 2$, $V_{ins} \neq 0$, $Q|V_{ins} \not\equiv 0$, we analyze $C_0(Q)$; the result will show that 2) and 3) of Theorem 1.1 are equivalent and establish the conclusions of Theorem 1.2.

Since $Q|V_{ins} \neq 0$, we may choose a non-isotropic vector $v_1 \in V_{ins}$. Taking V_1 to be any complement of v_1 in V containing a codimension 1 subspace of V_{ins}, the map

$$C_0(Q) \ni v_1 v \mapsto v \in V_1 \subset C(Q(v_1)Q|V_1)$$

induces an epimorphism of algebras which must be an isomorphism since both sides have dimension $2^{\dim V-1}$, and setting $Q_1 = Q(v_1)Q|V_1$, we need only analyze $C(Q_1)$.

Let $V_{1,ins} = V_{ins} \cap V_1$, which by construction has codimension 1 in V_{ins}, and let V_{sep} be a complement of $V_{1,ins}$ in V_1. We have $C(Q_1) = C(Q_1|_{V_{sep}}) \otimes_k C(Q_1|_{V_{1,ins}})$, (we can ignore the grading on the tensor product because we are in characteristic 2). On the other hand $(\ ,\ )_{Q_1} = Q(v_1)(\ ,\ )_Q|V_1$, so $Q_1|_{V_{sep}}$ is non-defective and nonsingular; we may now apply the results of Jacobson [1980], section 4.8, exercises, and we see that $C(Q_1|_{V_{sep}})$ is central simple over k, and is equivalent to a product of quaternion algebras.

On the other hand, since $V_{1,ins}$ possesses an orthogonal basis, we see that $C(Q_1|_{V_{1,ins}})$ is commutative. In fact, if V^*_{ins} has the basis $x_1,\ldots,x_m$ $(m \geq n)$, $Q|_{V_{ins}}$ has the form

$$Q|_{V_{ins}} = \sum_1^m s_i x_i^2 ,$$

and V_1 is the kernel of the linear form x_1, then

$$Q_1|V_{1,ins} = \sum_2^n s_1 s_i x_i^2 \sim \sum_2^n (s_i/s_1) x_i^2 ,$$

and

$$C(Q_1|V_{1,ins}) \cong k[e_2,\ldots,e_m]/(e_2^2-(s_2/s_1),\ldots,e_n^2-(s_n/s_1),e_{n+1}^2,\ldots,e_m^2),$$

so that $C(Q_1|V_{1,ins})$ is semi-simple(in fact, a field) iff $n = m$ and $s_2/s_1,\ldots,s_n/s_1$ are 2-independant, that is, iff condition 3) of Theorem 1.1.is satisfied. This completes the proof of Theorem 1.1 and Theorem 1.2.

Example: The proof of Theorem 1.1, 3) $\Rightarrow$ 1) yields an explicit construction of a singular point of $Q = 0$ if Q is not regular. Consider for example the form

$$Q = x_1^2+sx_2^2+tx_3^2+stx_4^2 ,$$

where s and t are indeterminates over a field k_o of characteristic 2. Then Q is not regular over $k = k_o(s,t)$, since

$$[k(\sqrt{s},\sqrt{t}) : k] = 4 < 2^{\dim V_{ins}-1} = 8 .$$

A singular point N of $Q = 0$ can be found on the affine open set $x_4 = 1$, since s and t are 2-independent over k^2, while $s\cdot t$ depends on s and t. Following the arguments of the proof we see that we are in case II, and that $N = (x_1+x_2x_3, x_2^2+t, x_3^2+s)$ is a singular point of the affine hypersurface $Q' = Q|_{x_4 = 1}$.

Proof of Theorem 1.3 (With thanks to B. Gross and M. Levine): It suffices to show that any product of quaternion algebras over a local or global field is split by a separable quadratic extension of that field; for such an extension is necessarily galois, and the splitting allows us to write an equivalent algebra as 4-dimensional quaternion algebra, associated to the galois 2-cocycle.

The Brauer group of any local field is $\mathbb{Q}/\mathbb{Z}$, which contains a unique element of order 2, so the Proposition is trivial in the local case.

In the global case the fact is well known, see Vignéras [1980], Theorem 3.8, but we include a proof for the reader's convenience.

Let k be a global field, and let A be a central simple k-algebra which is a product of quaternion algebras. By the Hasse principle, a field extension k'/k will split A iff its completion splits A at the finitely many valuations v such that A_v is not split already. Thus it suffices to find an extension k'/k which localizes to given extensions at finitely many places. If char $k \neq 2$, the local extensions have the form $k_v(\sqrt{s_v})$, and it is enough to choose $s \in k$ approximating each s_v (since any unit closely approximating 1 in k_v has a square root.) If char $k = 2$, the local extensions have the form $k_v[x]/x^2+x+s_v$ for $s_v \in k_v$, and it suffices to find an $s \in k$ which differs from each of the finitely many s_v by an element of

$$P(k_v) = \{x+x^2 \mid x \in k_v\}.$$

But since $t \to t+t^2$ is a continuous homomorphism from $\mathcal{O}_v^+$ to $\mathcal{O}_v^+$, and since $(1+ax)+(1+ax)^2 = ax+a^2x^2$, we see that

$$P(k_v) \supset \{x \in k_v \mid v(x) = 1\} ,$$

so again it suffices to choose s approximating each s_v.

Proof of Theorem 1.4: We have $C_o(Q) \cong C(s_1 \cdot Q|_{\ker x_1})$ as usual so it suffices to prove the corresponding facts about the whole Clifford algebra.

In $C(Q)$ we have $s_i = x_i^2$, so $C(Q)$ may be viewed as the (non-commutative) field of fractions of the skew polynomial ring $k[x_1,\ldots,x_n]$ with $x_ix_j = -x_jx_i$. A proof that such a skew polynomial ring is a domain, and has a field of fractions, can be found for instance in section 12.2 in the book of P.M. Cohen [1977].

2. Cohen-Macaulay modules and modules over the even Clifford algebra

Let k be a field, let V be a vectorspace of dimension r over k, and let Q be a quadratic form on V, so that $Q \in k[V^*] =: S$. Set $R = S/Q$, and let m be the maximal ideal generated by V^*.

We will say that a graded R-module M is <u>linear</u> if it is an MCM module admitting a graded free presentation of the form

$$R^n(-1) \xrightarrow{\varphi} R^m \to M \to 0,$$

where φ is a matrix of linear forms.

Suppose M is linear, with presentation as above. From the theory of matrix factorizations it follows at once that if M has no free summand, then $m = n$. Since the degree 1 parts R_1 and S_1 of R and S are equal, we may canonically lift φ to a matrix of linear forms over S. If (φ,ψ) is the matrix factorization corresponding to M, then since φ defines a monomorphism of free S-modules, ψ must have only linear entries as well. Thus the syzygies of linear modules are linear modules, up to a shift in degree.

We will say that a module over R_m or $\hat{R}_m$, the localization and completion of R respectively, is <u>linearizable</u> if it is isomorphic to the localization or completion of a linear module.

We next explain the fundamental construction of this paper, which gives an equivalence between linear R-modules and modules over the even Clifford algebra, $C_0(Q)$.

We have already seen that linear R-modules correspond to <u>linear matrix factorizations</u>, that is, pairs of square matrices (φ,ψ) with entries in $S_1 = V^*$ such that $\varphi\psi = \psi\varphi = Q \cdot 1$. It is with these matrix factorizations that we will work.

As we remarked in the introduction, the category of modules over $C_0(Q)$ is equivalent to the category of $\mathbb{Z}/2\mathbb{Z}$-graded modules over $C(Q)$ via

$$A_0 \mapsto C(Q) \otimes_{C_0(Q)} A_0$$

$$= A_0 \oplus C_1(Q) \otimes_{C_0(Q)} A_0$$

$$A_0 \oplus A_1 \mapsto A_0.$$

Thus we may work with $\mathbb{Z}/2\mathbb{Z}$-graded $C(Q)$-modules.

If A is a vectorspace, then a $C(Q)$-module structure on A is a linear family of maps.

$$\lambda_v : A \to A, \qquad v \in V,$$

such that $\lambda_v^2 = Q(v)$. If A is $\mathbb{Z}/2\mathbb{Z}$-graded, then each λ_v has two components

$$\varphi_v : A_1 \to A_o$$
$$\psi_v : A_o \to A_1,$$

and the condition becomes

$$\varphi_v\psi_v = \psi_v\varphi_v = Q(v) \cdot 1.$$

On the other hand, a "linear family of maps induced by V", such as φ_v is the same as a single matrix φ whose entries are linear forms in $S = k[V^*]$; and the conditions $\varphi_v\psi_v = \psi_v\varphi_v = Q(v) \cdot 1$ become

$$\varphi\psi = \psi\varphi = Q \cdot 1$$

as matrices over S. Thus we see that: <u>a linear matrix factorization over S is the same thing as a $\mathbb{Z}/2\mathbb{Z}$-graded $C(Q)$-module.</u> If the matrix factorization (φ,ψ) corresponds to the linear R-module $M = \operatorname{Coker} \varphi$, then clearly $A_o = k \otimes_R M$, and $A_1 = k \otimes_R \Omega^1(M)$, where $\Omega^1(M)$ is the first syzygy-module of M without free summands, form the corresponding $\mathbb{Z}/2\mathbb{Z}$-graded $C(Q)$-module, and $k \otimes M$ is the (underlying vectorspace of) the corresponding $C_o(Q)$-module.

It is not difficult to see that the correspondences just given are the same as the ones described via Ext and Tor in the introduction to this paper; since we will not use this fact explicitly, we will leave it to the reader.

We summarize and complete the properties of this construction:

<u>Theorem 2.1:</u> The functor

$$T : M \to T(M) = M/_{\mathfrak{m}M}$$

is an equivalence of categories from the category of linear R-modules without free summands, and maps of degree 0, to the category of modules over $C_o(Q)$, the even Clifford algebra of Q. Under this equivalence, the first syzygy module $\Omega^1 M$ corresponds to the C_o-module

$$T(\Omega^1 M) = C_1(Q) \otimes_{C_o(Q)} T(M),$$

and the dual module $M^* = Hom_R(M,R)$ corresponds to the contragredient

$$T(M^*) = Hom(C_1(Q) \otimes_{C_o(Q)} T(M),k)$$

regarded as a $C_o(Q)$-module via the main involution of $C_o(Q)$. The dimension of $T(M)$ as a k-vectorspace, which is the minimal number of generators of M, is equal to twice the rank of M (which may be a half-integer if $r = 1$ or 2).

The next two results show that when Q is regular, in the sense of section 1, the equivalence described in Theorem 1 catches the essence of the category of all MCM modules, even over the completion $\hat{R}_m$.

Theorem 2.2: If Q is regular, then every MCM R_m-(or $\hat{R}_m$-) module M is linearizable, so that $M \cong (gr_m M)_m$ (or $(\widehat{gr_m M})_m$ respectively).

This says in effect that when Q is regular, linear modules are really arbitrary MCM modules with their gradings normalized. It will be proved by treating arbitrary MCM $\hat{R}_m$-modules as high degree deformations of linear ones, by means of matrix factorizations. Morally speaking, such deformations are trivial because $Ext^1_R(M,M)$, the tangent space to the versal deformation space, is nonzero in (at most) degree -1 when M is linear, as the following result shows.

To simplify notation, we imitate the notion of Tate cohomology for a finite group. If

$$\cdots \to F_1 \to F_o \to F_1 \to F_o \to M \to 0$$

is the periodic R-free resolution of an MCM module M, we may continue it to the right as well, and set

$$\underline{Ext}^i_R(M,N) = H_i(Hom(\cdots \to F_1 \to F_o \to F_1 \to F_o \to \cdots),N)),$$

so that

$$\underline{Ext}^i_R = Ext^i_R \quad \text{for } i > 0$$

$\underline{Ext}^o_R = \underline{Hom}$, that is homomorphisms modulo those factoring through projectives,

$\underline{Ext}^{-1}_R =$ kernel of the natural map $N \otimes M^* \to Hom_R(M,N)$

and

$$\underline{Ext}^i_R(M,N) = Tor^R_{-i+1}(N,M^*) \quad \text{for } i < -1$$

Proposition 2.3: Let Q be an arbitrary nonzero quadratic form. If M and N are linear modules such that

$$\mathrm{Ext}^1_R(M,N) \quad \text{and} \quad \mathrm{Ext}^2_R(M,N)$$

have finite length, then

$$\underline{\mathrm{Ext}}^i_R(M,N)$$

is a vectorspace concentrated in degree $-i$; in particular, if Q is regular, this holds for any MCM modules.

Remarks: 1) An interesting singular example to which the Proposition applies is the case

$$R = k[x,y]/x^2 + y^2, \quad \mathrm{char}\, k = 2,$$

$$M = \mathrm{coker}\begin{pmatrix} x & y \\ y & x \end{pmatrix},$$

N any MCM module,

(since M is free on the punctured spectrum of R).

2) The technique of proof will show that for arbitrary Q and linear R-modules M and N, $\mathrm{Ext}^i_R(M,N)$ is at least generated in degree $-i$.

Proof of Theorem 2.1: The first statement of the Theorem and the identification of $T(\Omega^1 M)$ are immediate from the construction given before the Theorem.

To identify $T(M^*)$ (where M^* means $\mathrm{Hom}_R(M,R)$) note that since M is an MCM R-module,

$$M^* \cong \mathrm{Ext}^1_S(M, S(-2)),$$

so $M^* = \mathrm{coker}\ \varphi^*$. Thus the matrix factorization corresponding to M^* is (φ^*, ψ^*), and

$$T(M) = (k \underset{R}{\otimes} \Omega^1 M)^*$$

as vectorspaces.

Further, it is clear from the basic construction that each $v \in V$ acts on the $\mathbb{Z}/2\mathbb{Z}$-graded $C(Q)$-module

$$(k \underset{R}{\otimes} \Omega^1 M)^* \oplus (k \underset{R}{\otimes} M)^*$$

by the dual of its action on

$$(k \otimes_R M) \oplus (k \otimes_R \Omega^1 M).$$

Thus the $\mathbb{Z}/2\mathbb{Z}$-graded $C(Q)$-module structure corresponding to M^* is that obtained by regarding $(k \otimes_R \Omega^1 M)^* \oplus (k \otimes_R M)^*$ as a $C(Q)^{op}$-module in the natural way, and then using the isomorphism $C(Q) \to C(Q)^{op}$ which is the identity on $V \subset C(Q)$, that is, the main involution of $C(Q)$. Passing to the $C_0(Q)$-module $(k \otimes_R \Omega^1 M)^* = (C_1(Q) \otimes_{C_0(Q)} TM)^*$, we get the required statement.

It remains to prove the last statement, regarding the rank of an MCM R-module M without free summands. We use the following easy

Lemma 2.4: If k is any field, $Q \in k[x_1,\ldots,x_r]$ any quadratic form, then there exists a regular sequence of length $r - 1$ in $R = k[x_1,\ldots,x_r]/Q$ consisting of elements of degree 1.

Proof: It suffices by induction to find a degree 1 nonzerodivisor, given $r > 1$. Because S is factorial, any degree 1 zerodivisor must be a factor of Q. But $k[x_1,\ldots,x_r]$ has at least 3 distinct linear forms (mod scalars), so not all can be factors.

By the Lemma we may assume that $x_1,\ldots,x_{r-1}$ is a regular sequence modulo Q. Then

$$\operatorname{rank}_{k[x_1,\ldots,x_r]/Q} M = \frac{1}{2} \operatorname{rank}_{k[x_1,\ldots,x_{r-1}]} M$$

$$= \frac{1}{2} \dim_k M/(x_1,\ldots,x_{r-1})M'$$

which holds because M is free as a $k[x_1,\ldots,x_{r-1}]$-module.

We will complete the proof by using induction on r to show

$$M/(x_1,\ldots,x_{r-1})M$$

$$= M/(x_1,\ldots,x_r)M,$$

which is of course $T(M)$.

If $r = 1$, $R \cong k[x]/(x^2)$, and since M has no free summands, it is annihilated by x.

In the general case, any free summand of $M/x_1M =: \overline{M}$ could be lifted to a free summand of M:

$$\begin{array}{ccc}
 & & 0 \\
 & & \downarrow \\
 & & x_1 R \cong R \\
 & & \downarrow \\
M & \dashrightarrow\!\!\!\!\rightarrow & R \\
\downarrow & & \downarrow \\
M/x_1M & \twoheadrightarrow & R/x_1R \\
 & & \downarrow \\
 & & 0
\end{array}$$

because $\operatorname{Ext}^1_R(M,R) = 0$, so M/x_1M is a Cohen-Macaulay R/x_1R-module without free summands, and

$$\begin{aligned} M/(x_1,\ldots,x_{r-1})M &= \overline{M}/(x_2,\ldots,x_{r-1})\overline{M} \\ &= \overline{M}/(x_2,\ldots,x_r)\overline{M} \\ &= M/(x_1,\ldots,x_r)M \end{aligned}$$

as required. This completes the proof of Theorem 2.1.

Proof of Proposition 2.3: If Q is regular, then every localization of R at a relevant prime is regular, so M is free on the punctured spectrum, and $\underline{\operatorname{Ext}}^i_R(M,N)$ has finite length for all i. Thus it suffices to prove the first statement. We do an induction using the following well-known change of rings result:

Lemma 2.5: Let R be any graded ring, and let N be a graded R-module. If $x \in R$ is a nonzerodivisor on R and N, and M is a graded R/xR-module, then

$$\operatorname{Ext}^i_R(M,N) \cong \operatorname{Ext}^{i-1}_{R/xR}(M,N/xN\ (1).$$

Proof: The spectral sequence

$$\operatorname{Ext}^i_{R/xR}(M,\operatorname{Ext}^j_R(R/xR,N)) \Rightarrow \operatorname{Ext}^{i+j}_R(M,N)$$

degenerates, since

$$\operatorname{Ext}^*_R(R/xR,N) = \operatorname{Ext}^1_R(R/xR,N) = N/xN(1),$$

to give the desired isomorphisms.

We now continue with the proof of Proposition 2.3. $\underline{\mathrm{Ext}}^i_R(M,N)$ is a quotient of $\mathrm{Hom}_R(\Omega^i M,N)$, when $\Omega^i M$ is the i^{th} syzygy. Since $\Omega^i M$ is generated in degree i, we see that

$$\mathrm{Ext}^i_R(M,N)_p = 0 \quad \text{for } p < -i.$$

We now do induction on dim R. If dim R = 0, then either R is a field (and the result is obvious) or $R \cong k[x]/x^2$. We may clearly assume that M and N are indecomposable and without free summand, whence $M = R/xR$, $N = R/xR$, and the result follows by direct computation.

Now assume dim R > 1. By Lemma 2.4 there is an $x \in R$, which is a nonzerodivisor on R, and thus on M and N. By periodicity, it is enough to prove the Proposition when i is large. From $0 \to M(-1) \xrightarrow{x} M \to M/_{xM} \to 0$ we get the following exact sequence, the first of whose terms we identify by using Lemma 2.5:

$$\begin{array}{l} \to \mathrm{Ext}^i_R(M/xM,N) \to \mathrm{Ext}^i_R(M,N) \xrightarrow{x} \mathrm{Ext}^i_R(M,N)(1) \to \mathrm{Ext}^{i+1}_R(M/xM,N) \to \\ \quad \wr\Vert \\ \mathrm{Ext}^{i-1}_{R/xR}(M/xM,N/xN)(1) \end{array}$$

It now follows that all $\mathrm{Ext}^i_{R/x}(M/xM,N/xN)$ have finite length, so by induction

$$\mathrm{Ext}^{i-1}_{R/xR}(M/xM,N/xN)(1)$$

is a vectorspace concentrated in degree -i . Since a power of x kills all of $\mathrm{Ext}^i_R(M,N)$, we see that $\mathrm{Ext}^i_R(M,N)_p = 0$ for $p > -i$, as required.

<u>Proof of Theorem 2.2:</u> Write $\hat{R}$ and $\hat{S}$ for the completion of R and S at m. Let M be an MCM $\hat{R}$-module, with minimal free resolution

$$\cdots \xrightarrow{\bar\psi} F_1 \xrightarrow{\bar\varphi} F_o \xrightarrow{\bar\psi} F_1 \xrightarrow{\bar\varphi} F_o \to M,$$

where $\varphi\psi = \psi\varphi = Q \cdot 1$ in $\hat{S}$, and $^-$ denotes reduction mod Q. Write $\varphi = \sum_{i\geq 1} A_i$, $\psi = \sum_{i\geq 1} B_i$, where A_i and B_i are matrices of forms of degree i, in S. Taking homogeneous parts of $\varphi\psi = Q \cdot 1$, we see that

$$B_1A_1 = A_1B_1 = Q \cdot 1,$$

so that $M' := \mathrm{coker}\, A_1$ is a linear module, with resolution

$$\cdots \xrightarrow{\overline{A}_1} F_0 \xrightarrow{\overline{B}_1} F_1 \xrightarrow{\overline{A}_1} F_0 \longrightarrow M' .$$

We will produce matrices $s_i : F_0 \to F_0$, $t_i : F_1 \to F_1$, $i \geq 1$, whose elements are forms of degree i such that the matrices

$$A^{(i+1)} := (1 - t_i) \ldots (1 - t_1)A(1 - s_1) \ldots (1 - s_i)$$

satisfy

$$A^{(1)} = A$$

$$A^{(i+1)} - A_1 \equiv 0(\text{mod } m^{i+2}).$$

The matrices

$$\sigma_i = (1 - s_1) \ldots (1 - s_i)$$

$$\tau_i = (1 - t_i) \ldots (1 - t_1)$$

are congruent to 1 modulo m and form Cauchy sequences. Thus setting

$$\sigma = \lim \sigma_i, \quad \tau = \lim \tau_i$$

we see that σ and τ are invertible and

$$A_1 = \tau A \sigma,$$

so (coker A_1) $\otimes \hat{R}_m$ = coker A, and M is linearizable. Of course it follows from this that the linear module coker A_1 is isomorphic to $gr_m M$.

Suppose then that for all $j < i$ the maps s_j and t_j have been defined and satisfy the given conditions. Let $A^{(i)}$ be as above, and set

$$B^{(i)} = \sigma_{i-1}^{-1} B \tau_{i-1}^{-1}$$

$$= B_1 + B_2^{(i)} + \ldots ,$$

with $B_j^{(i)}$ a matrix of forms of degree j. We have

$$A^{(i)}B^{(i)} = Q \circ 1 = A_1 B_1 .$$

Writing $A^{(i)} = A_1 + A_{i+1}^{(i)} + \ldots$, and comparing terms of degree $i + 2$ in these products we get

$$A_1B_{i+1}^{(i)} + A_{i+1}^{(i)}B_1 = 0.$$

This equation can be interpreted as saying that $A_{i+1}^{(i)}$ and $B_{i+1}^{(i)}$ define the beginning of a map of complexes

$$(*)\qquad \begin{array}{ccccc} \to F_0 & \xrightarrow{\overline{B}_1} & F_1 & \xrightarrow{\overline{A}_1} & F_0 \\ \Big\downarrow{\scriptstyle \overline{B}_{i+1}^{(i)}} & & \Big\downarrow{\scriptstyle \overline{A}_{i+1}^{(i)}} & & \\ \to F_1 & \xrightarrow{-\overline{A}_1} & F_0 & & \end{array}$$

which in turn defines an element of

$$\mathrm{Ext}_R^1(M',M')_{i+1}$$

from a deformation-theoretic point of view, this element is the tangent vector associated to the deformation of M' which is M. Since $i \geq 1$, this group is 0 by Proposition 2.3. Thus the map of complexes in (*) is null-homotopic; that is, there exist maps of degree i, $\overline{s}_i : F_0 \to F_0$ and $\overline{t}_i : F_1 \to F_1$ such that $\overline{A}_{i+1}^{(i)} = \overline{s}_i\overline{A}_1 + \overline{A}_1\overline{t}_i$. Lifting $\overline{s}_i$ and $\overline{t}_i$ back to $S = k[V^*]$ in an arbitrary way, we get

$$A_{i+1}^{(i)} = s_iA_1 + A_1t_i + u_{i-1}Q,$$

where u_{i-1} is a matrix of forms of degree $i - 1$. Changing s_i to $(s_i + u_{i-1}B_1)$ we get s_i and t_i such that

$$A_{i+1}^{(i)} = s_iA_1 + A_1t_i,$$

so

$$A^{(i+1)} := (1 - s_i)A^{(i)}(1 - t_i) = A_1 + (\text{terms of degree} \geq i + 2)$$

as required.

3. Cohen-Macaulay modules and properties of quadratic forms

In this section we will derive explicit information about MCM modules over $R = S/Q$ from properties of Q, using the tools of section 2 and the information about Clifford algebras in section 1.

We will preserve the notation introduced at the beginning of section 2, and we assume troughout that Q is regular on V. As usual, we write V_{ins} for the nullspace of $(\, , \,)_Q$. For any R-module M, let $\Omega^1(M)$ be the first syzygy of M.

We begin with the number of MCM modules, and their relationship.

Proposition 3.1: R has either 1 or 2 isomorphism classes of nonfree indecomposable MCM modules. In case there are 2, they have the same rank, and are first syzygies of one another. If char $k \neq 2$, then there is just 1 if either $\dim V$ is odd or $\dim V$ is even but

$$(-1)^{(\dim V)/2} \det Q$$

is not a square in k. If char $k = 2$, then there is just 1 iff either Q is defective or Q is non-defective but

$$\operatorname{arf} Q \neq 0.$$

Further, for any MCM R-module M without free summands,

$$M^* \cong \Omega^1(M) \quad \text{if} \quad \dim V \equiv 0 \quad (4)$$
$$M^* \cong M \quad \text{otherwise.}$$

Similar ideas allow us to determine the rank of the MCM modules. We already know when R has only 1 nonfree indecomposable MCM module and when 2, and we know that when there are two they have the same rank. Thus we need only compute the rank of the sum, A, of the 1 or 2 indecomposable, nonfree MCM R-modules.

Recall from Theorem 1.2 that $C_0(Q)$ may be written as a tensor product of a central-simple k-algebra and an algebra which is either a $k \times k$, or a field extension. The central simple factor is of course itself the product of a central division algebra and a full matrix ring. For our purposes here we will split things differently: We write $C_0(Q) = D(Q) \otimes C_0'(Q)$, where $D(Q)$ is a (not necessarily central) division algebra and $C_0'(Q)$ is a full matrix ring over k or a product of such. Note that $D(Q)$ may be identified with the endomorphism ring of (either of) the simple $C_0(Q)$-module(s), and thus, by Theorem 2.1, with the degree 0 endomorphism ring of (one of) the nonfree indecomposable R-module(s).

Recall that the dimension of a division algebra over its center is the square of an integer called the <u>degree</u> of the algebra.

<u>Proposition 3.2:</u> Let D be the division algebra which is the degree 0 part of the endomorphism ring of a nonfree indecomposable MCM module. Set

$$m = \begin{cases} \left[\dfrac{\dim V}{2}\right] - 1 & \text{if char } k \neq 2 \text{ or } Q \text{ is non-defective} \\ \left[\dfrac{\dim V + \dim V_{ins} - 2}{2}\right] - 1 & \text{if } Q \text{ is defective.} \end{cases}$$

If A is the sum of the (1 or 2) distinct indecomposable nonfree MCM modules over R, then rank A is a power of 2 and

$$\text{rank } A = 2^m \deg D.$$

In particular,

$$2^m \,\big|\, \text{rank } A \,\big|\, 2^{\dim V - 2} .$$

<u>Proof of Proposition 3.1:</u> By Theorem 2.1, the indecomposable nonfree MCM R-modules of some rank n correspond to the irreducible $C_0(Q)$-modules of dimension $2n$, so the statements about the number and rank of the MCM R-modules follow directly from the theory of modules over semi-simple rings and Theorem 1.2.

If $C_0(Q)$ is simple, so that there is just 1 indecomposable nonfree MCM R-module, then the other statements of the Proposition are trivial, so we suppose that $C_0(Q)$ is not simple, and its center is then $k \times k$.

To show that the two indecomposable nonfree MCM R-modules are first syzygies of one another, we must, by Theorem 2.1, show that if A_0 is an irreducible $C_0(Q)$-module, then

$$A_1 := C_1(Q) \otimes_{C_0(Q)} A_0 \ncong A_0.$$

To prove this we will first choose an element of $V \subset C_1(Q)$ which is a unit and does not centralize $k \times k = \text{Center } C_0(Q)$. To see that this is possible, we note that $k \times k = k[c]$ where $c \in C_0(Q)$. If the characteristic is not 2 then c may be taken to be Πe_i, where e_i is an orthogonal base of V. If the characteristic is 2, then c may be taken to be $\Sigma e_i f_i$, where $\{e_i, f_i\}$ is a symplectic base of V. In the first case any e_i will do for v; in the second case any $e_i + f_i$ will do.

Conjugation by v now defines a non-trivial automorphism of $k \times k$, which must therefore interchange the two factors. Since $C_1(Q) = vC_o(Q)$, we may think of $C_1(Q) \otimes_{C_o(Q)} A_o$ as $v \otimes A_o$, with the action given by $w \cdot (v \otimes a) = v(v^{-1}wv) \otimes a = v \otimes (v^{-1}wv)a$, that is, as A_o with the action given via conjugation by v; since conjugation by v interchanges the two simple factors of $C_o(Q)$, it interchanges the two modules.

To prove the last statement of the Proposition, we may assume that $C_o(Q)$ is not simple and M is indecomposable.

If $e_1, e_2 \in k \times k \subset C_o(Q)$ are the two central primitive idempotents, then we may distinguish between the simple $C_o(Q)$-modules by saying which of the e_i annihilates them. Suppose e_1 annihilates $T(M)$. By what was proved above, $T(\Omega^1 M) \not\cong T(M)$, so e_2 annihilates $T(\Omega^1 M)$. Thus $M^* \cong \Omega^1 M$ iff the main involution of $C_o(Q)$ fixes e_2 or, equivalently, acts trivially on the center of $C_o(Q)$. But if $C_o(Q)$ ist not simple, this is true iff rank $Q \equiv 0(4)$ by the results of section 1, concluding the proof.

<u>Proof of Proposition 3.2:</u> Since $C_o(Q)$ has dimension $2^{\dim V - 1}$, it will be a ring of $2^s \times 2^s$ matrices over $D(Q)$ for some s, and the degree of $D(Q)$ will also be a power of 2, say 2^d, as will the dimension of the center (of $D(Q)$ or of $C_o(Q)$), say 2^c. Thus

$$\dim V - 1 = 2s + 2d + c,$$

and the dimension of the sum of the 1 or 2 distinct $C_o(Q)$-modules is

$$2^{s+2d+c}.$$

By the last statement of Theorem 2.1, the rank of A is

$$2^{s+2d+c-1},$$

and putting this together with the fact that the dimension of the center of $C_o(A)$ is

$2^c = 1$ if char $k \neq 2$ or Q non-defective, dim V odd

$2^c = 2$ if char $k \neq 2$ or Q non-defective, dim V even

$2^c = 2^{\dim V_{ins}-1}$ if char = 2, Q defective,

the first 2 statements follow.

Remark:

Although we were only concerned with quadrics, parts of the result may be extended to hypersurfaces of multiplicity two:

Let $f \in k[[V^*]]$ be an element of order two, not necessarily homogeneous, and denote f_2 its leading form in $gr_{(V^*)}P = k[V^*]$.

Let $R = P/f$ be as usual the hypersurface ring defined by f and $\overline{R} = gr_{(V^*)}R$ its tangent cone. If now M is a MCM over R, given by a reduced matrix-factorization (ϕ,ψ), say, the leading forms $(\overline{\phi},\overline{\psi})$ define a matrix-factorization of f_2 and hence a MCM $\overline{M}$ over $\overline{R}$. Now, if f has multiplicity two, its leading term f_2 is a quadric, denoted Q_f.

It is not hard to see that M and $\overline{M}$ have the same rank. In particular, if Q_f is regular, we may apply Proposition 3.2 to show that over R as over $\overline{R}$ the rank of a MCM without free summands is divisible by an appropriate power of 2 depending only on $C_o(Q_f)$.

4. Explicit examples

Again in this section we maintain the notations $S = k[V^*]$, $R = S/Q$, V_{ins} = nullspace $(\ ,\ )_Q$ (in char. 2) of sections 2, 3, and we continue to assume that Q is nonregular, as in section 3.

We will also adopt the notation of Proposition 3.2: A will denote the sum of the (1 or 2) distinct indecomposable nonfree MCM R-modules, and

$$m = \begin{cases} \left[\dfrac{\dim V}{2}\right] - 1 & \text{if char } k \neq 2 \text{ or } Q \text{ is non-defective} \\ \left[\dfrac{\dim V + \dim V_{ins} - 2}{2}\right] - 1 & \text{if } Q \text{ is defective.} \end{cases}$$

Let us first consider the case of a quadratic form of maximal Witt index. It is given by either

$$Q \equiv \sum_i x_i y_i \quad \text{in case dim V is even, or}$$

$$Q \equiv \sum_i x_i y_i + \alpha z^2 \quad \text{in case dim V is odd.}$$

The indecomposable modules over the Clifford-algebra correspond then - as is well known - to the two half-spin representations - for V even-dimensional - or the spin representation - for V odd-dimensional - of the orthogonal group of Q.

Not surprisingly, therefore, the construction of the corresponding MCM's, as given in Buchweitz-Greuel-Schreyer [1986] for example is the "same" as the classical construction of these group representations, a fact now elucidated by Theorem 2.1.

Proposition 3.1 answers rather satisfactorily the question of the number of MCM modules over R. By contrast, Proposition 3.2 refers the problem to another, computing the degree of a division algebra, which is not easy without some technique.

In this section we make some remarks on the situation over various interesting fields. We then treat the case $k = \mathbb{R}$ in somewhat more detail; as we shall show, the mod 8 periodicity associated with the orthogonal groups appears here.

In general we know that

$$2^m \mid \text{rank } A \mid 2^{\dim V - 2},$$

and it follows from Proposition 3.2 that in many cases only the lower estimate is achieved:

Corollary 4.1: If k is finite, algebraically closed, or a rational function field in 1 variable over an algebraically closed field, then

$$\text{rank } A = 2^m.$$

Proof: Classical results assert that every division algebra is commutative over such fields.

More delicate is the situation of global fields such as the rational numbers. Since the Clifford algebra is isomorphic to a product of quaternion algebras we may apply Theorem 1.3 and get:

Corollary 4.2: If k is a local or global field, then

$$\text{rank } A = 2^m \quad \text{or} \quad 2^{m+1},$$

depending on the Hasse-Witt invariant of Q.

Nevertheless it is not hard to construct examples of k and Q for which rank A takes on any of the values allowed by the last statement of Theorem 2. The key observation is that the Clifford algebra of the generic quadratic form is a division algebra - see Theorem 1.4.

Corollary 4.3: Let $k = \mathbb{Q}(s_1, \ldots, s_n)$ where the s_i are transcendental over $\mathbb{Q}$. If Q is of the form $Q \sim \sum_0^{n_1} s_i x_i^2 + \sum_1^{n_2} y_i z_i$, so that $\dim V = 1 + n_1 + 2n_2$, then

$$\text{rank } A = 2^{m+n_1};$$

in particular, if $n_2 = 0$ then

$$\text{rank } A = 2^{\dim V - 2}.$$

We turn now to forms over the reals (more generally one could easily treat the forms over any field of characteristic $\neq 2$ which are equivalent to $\sum_i \pm x_i^2$).

Choosing an appropriate basis, we may write

$$Q \sim \sum_{i=1}^{n_1} x_i^2 - \sum_{i=n_1+1}^{n} x_i^2 .$$

Recall that the signature $\sigma(Q)$ of Q is defined to be the number of positive minus the number of negative signs, i. e.

$$\sigma(Q) = 2n_1 - n.$$

Writing as usual $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$ for the reals, complexes, and quaternions, the following table, which follows closely Table 1 of Atiyah, Bott, Shapiro [1964], summarizes what is true:

$\sigma(Q) \bmod 8$	$C_o(Q)$ is a full matrix ring over	If rank $Q = r$: number of indecomposable non-free MCM modules	rank of indecomposable non-free MCM modules
1	$\mathbb{R}$	1	$2^{\frac{r-3}{2}}$
2	$\mathbb{C}$	1	$2^{\frac{r-2}{2}}$
3	$\mathbb{H}$	1	$2^{\frac{r-1}{2}}$
4	$\mathbb{H} \times \mathbb{H}$	2	$2^{\frac{r-2}{2}}$
5	$\mathbb{H}$	1	$2^{\frac{r-1}{2}}$
6	$\mathbb{C}$	1	$2^{\frac{r-2}{2}}$
7	$\mathbb{R}$	1	$2^{\frac{r-3}{2}}$
0	$\mathbb{R} \times \mathbb{R}$	2	$2^{\frac{r-4}{2}}$

Notes on a proof: By virtue of Proposition 3.2, it suffices to establish the correctness of the second column of the table. This is contained in section 4 and Proposition 5.4 of the paper by Atiyah-Bott-Shapiro just cited.

Remark: As Q and $-Q$ have obviously the same MCM's, it is clear that the table only depends on $|\sigma(Q)|$ mod 8.

It is not difficult to explicitly compute the linear matrix factorizations of smallest possible size for a given real quadric Q. By the results of section 2, such a factorization corresponds to an indecomposable $\mathbb{Z}/2\mathbb{Z}$-graded module M over the Clifford algebra $C(Q)$. Choose a basis $e_1,\dots,e_r$ of V, and bases of the even and odd part of M. With respect to the chosen bases let $(a_{ij}^{(k)}) \in Gl(n;k)$ be the matrix describing the multiplication map $M_o \xrightarrow{e_k} M_1$, and $(b_{ij}^{(k)}) \in Gl(n;k)$ the matrix for $M_1 \xrightarrow{e_k} M_o$. In terms of these, Q has the matrix factorization $Q\cdot 1 = \varphi\psi$ with $\varphi = (\varphi_{ij})$, $\psi = (\psi_{ij})$, where

$$\varphi_{ij} = \sum_{k=1}^{r} a_{ij}^{(k)} x_k \ , \ \psi_{ij} = \sum_{k=1}^{r} b_{ij}^{(k)} x_k$$

We use this method to compute the matrix factorizations of real quadratic forms. By Knörrer's periodicity theorem [1986] it suffices to factorize the forms $Q_n = \sum_{i=1}^{n} x_i^2$. For $n \leq 8$, we obtain the following result:

$$Q_n \cdot 1 = \varphi_n \circ \varphi_n^t \ , \quad \text{where}$$

$$\varphi_1 = (x_1),\ \varphi_2 = \begin{pmatrix} x_1 & x_2 \\ x_2 & -x_1 \end{pmatrix}$$

$$\varphi_4 = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ x_2 & -x_1 & -x_4 & x_3 \\ x_3 & x_4 & -x_1 & -x_2 \\ x_4 & -x_3 & x_2 & -x_1 \end{bmatrix}$$

$$\varphi_8 = \left[\begin{array}{cccc|cccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \\ x_2 & -x_1 & -x_4 & x_3 & -x_6 & x_5 & x_8 & -x_7 \\ x_3 & x_4 & -x_1 & -x_2 & -x_7 & -x_8 & x_5 & x_6 \\ x_4 & -x_3 & x_2 & -x_1 & -x_8 & x_7 & -x_6 & x_5 \\ \hline x_5 & x_6 & x_7 & x_8 & -x_1 & -x_2 & -x_3 & -x_4 \\ x_6 & -x_5 & x_8 & -x_7 & x_2 & -x_1 & x_4 & -x_3 \\ x_7 & -x_8 & -x_5 & x_6 & x_3 & -x_4 & -x_1 & x_2 \\ x_8 & x_7 & -x_6 & -x_5 & x_4 & x_3 & -x_2 & -x_1 \end{array}\right]$$

φ_3 is obtained from φ_4 by setting $x_3 = 0$, and φ_i $(i = 5,6,7)$ from φ_8 by setting $x_j = 0$ for $i < j \leq 8$.

Notice that the entries of these matrices are 0 or $\pm x_i$. Combinatorists call such matrices orthogonal designs. A survey on the theory of orthogonal designs is given in the book of Geramita and Seberry [1979]. It seems hence plausible that the theory of MCM's has also applications to combinatorial questions.

We will use our result on matrix factorizations of real quadrics to give one more proof of the famous theorem of Hurwitz [1898]:

Let $z_1,\ldots,z_n$ be real bilinear forms in the variables $x_1,\ldots,x_n$ and $y_1,\ldots,y_n$ such that

$$z_1^2 + \ldots + z_n^2 = (x_1^2 + \ldots + x_n^2)(y_1^2 + \ldots + y_n^2),$$

then $n = 1,2,4$ or 8.

To relate this theorem with matrix factorizations we consider a finite-dimensional k-vectorspace A together with a bilinear map

$$\begin{aligned} A \times A &\longrightarrow A \\ (a,b) &\mapsto a \cdot b \end{aligned}$$

We call such a pair a <u>k-algebra</u>. The k-algebra A is said to be a

<u>composition algebra</u> if there exists a quadratic form $Q : A \to k$ such that $(\, , \,)_Q$ is nondegenerate and such that

$$Q(a \cdot b) = Q(a) \cdot Q(b) \quad \text{for all } a,b \in A .$$

Recall that if $(\, , \,)_Q$ is nondegenerate then any k-linear map $\varphi : A \to A$ admits a unique k-linear map $\varphi^{ad} : A \to A$, the adjoint of φ, such that

$$(\varphi(a),b)_Q = (a,\varphi^{ad}(b))_Q \quad \text{for all } a,b \in A.$$

<u>Proposition 4.4</u>: Let A be a finite dimensional k-vectorspace, and $Q : A \to k$ a quadratic form for which $(\, , \,)_Q$ is nondegenerate. Consider the following conditions:

1) There exists a k-algebra structure on A such that

$$Q(a \circ b) = Q(a) \circ Q(b) .$$

2) There exists a k-linear map $\varphi : A \to \mathrm{End}_k(A)$ such that

$$Q(a) \cdot id_A = \varphi(a)^{ad} \ \varphi(a) \quad \text{for all } a \in A.$$

Then 1) implies 2), and if char $k \neq 2$, then 2) implies 1).

First let us derive the Hurwitz theorem from the Proposition and our classification of matrix factorizations of real quadratic forms.

It is clear and well known that a composition of square sums as in the Hurwitz theorem exists exactly for those n for which there exists a composition algebra A over $\mathbb{R}$ of $\mathbb{R}$-dimension n with norm $Q = \sum_{i=1}^{n} x_i^2$. Now Proposition 4.4 implies that composition algebras of dimension n exist exactly for those n for which there exists a matrix factorization $Q \circ 1 = \varphi\varphi^t$, where the size of $\varphi = n = \text{rank } Q$.

The last column of the table on page 90 tells us the rank of an indecomposable nonfree MCM over S/Q (depending on the signature of Q). The minimal size of a matrix factorization of Q is just twice this number.

Checking the table we see that we must have $n \leq 8$, simply because the minimal size of the matrices in a matrix factorization of Q is $\geq 2^{\frac{n-2}{2}}$, and $2^{\frac{n-2}{2}} > n$ for $n > 8$.
Checking the numbers $n \leq 8$, we see that only $n = 1,2,4$ and 8 are possible. That, in fact, these numbers allow a decomposition of square sums as in the Hurwitz theorem follows from Proposition 4.4 and our explicit factorization of $Q = \sum_{i=1}^{n} x_i^2$, $n \leq 8$, where we have seen that $Q \cdot 1 = \varphi\varphi^t$.

Proof of 4.4: 1) $\Rightarrow$ 2): We define $\varphi : A \to \mathrm{End}_k(A)$ by $\varphi(a)(b) = a \circ b$ for all $a,b \in A$. Then for all $a,b_1,b_2 \in A$ we have
$(b_1,(\varphi(a)^{ad} \circ \varphi(a))(b_2))_Q = (\varphi(a)(b_1), \varphi(a)(b_2))_Q = (a \circ b_1, a \cdot b_2)_Q =$
$= Q(a \cdot b_1 + a \circ b_2) - Q(a \circ b_1) - Q(a \circ b_2) = Q(a)(Q(b_1+b_2) - Q(b_1) - Q(b_2)) =$
$= Q(a) \circ (b_1,b_2)_Q = (b_1, Q(a) \circ b_2)_Q$. Since $(\ ,\ )_Q$ is nondegenerate, this implies that $Q(a)\mathrm{id}_A = \varphi(a)^{ad} \circ \varphi(a)$ for all $a \in A$.

2) $\Rightarrow$ 1): We define $a \circ b = \varphi(a)(b)$ for all $a,b \in A$, then
$2Q(a \circ b) = (a \circ b, a \circ b)_Q = (\varphi(a)(b), \varphi(a)(b))_Q = (b, (\varphi(a)^{ad} \circ \varphi(a))(b))_Q =$
$= (b, Q(a) \cdot b)_Q = Q(a) \circ (b,b)_Q = 2Q(a) \circ Q(b)$. Since $\mathrm{char}\, k \neq 2$, we may divide by 2.

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Appendix: The Comparison Theorem.
by Ragnar-Olaf Buchweitz

Introduction

The aim of this appendix is to put theorem 2.1. of the foregoing paper in its proper conceptual context. In addition, it provides examples for which the theory of maximal Cohen-Macaulay modules over (not necessarily commutative) Gorenstein rings, as developed in [Bu]; has a concrete geometric content.

From our point of view, the theorem, which describes maximal Cohen-Macaulay modules over a quadric in terms of representations of the Clifford-algebra of the associated quadratic form, is just a special application of the generalized Bernstein-Gelfand-Gelfand-correspondence which compares (complexes of) modules over certain graded k-algebras with (complexes of) modules over their associated Yoneda-Ext-algebras. This correspondence was originally established in [BGG] for the case of (graded) polynomial algebras, to obtain a general "monadic" description of complexes of coherent sheaves on the corresponding projective space.

Essentially the same proof extends this correspondence to the class of homogeneous coordinate rings R of complete intersections defined by quadrics.

The distinguishing features of this class of rings R are:

- their Yoneda-Ext-algebra is again noetherian (on both sides). By a theorem of Bøgvad-Halperin [B-H], complete intersections are the only commutative, noetherian rings for which this holds.
- The double Yoneda-Ext-algebra $\mathrm{Ext}^*_{\mathrm{Ext}^*_R(k,k)}(k,k)$ of R coincides with R. Quadratic complete intersections are the only ones with this property as well. (cf. [B-R]).

For this special class of rings, the generalization of the results in [BGG] has been obtained independentely by Kapranov, - to appear in Funkt. Anal. - , as S. I. Gelfand informed us.

Although the above remarks indicate that complete intersections defined by quadrics constitute a natural domain to which the Bernstein-Gelfand-Gelfand-correspondence extends, one may nevertheless drop either restriction of being "commutative" or "noetherian" and still obtain similar results for other classes of graded algebras. This will be explained somewhere else.

To see how the correspondence may be used to study coherent sheaves on projective schemes or maximal Cohen-Macaulay modules, let R denote the (commutative) homogeneous coordinate ring of a projective complete intersection X defined by quadrics, $E^{\cdot} = Ext_R^{*}(k,k)$ the Ext-algebra over R of the ground field k, - considered as an R-module via the augmentation - , graded by homological degree and with the multiplicative structure defined by the Yoneda-product of extensions.

The main result is the existence of a natural - and explicit - exact equvalence between the triangulated categories $D^b(R.)$ and $D^b(E^{\cdot})$, each denoting the derived category of complexes with bounded cohomology of graded, finitely generated (right) modules over the algebra in question.

Under this equivalence, the full triangulated subcategories $D^b_{perf}(\ .\ )$, consisting of all objects in $D^b(\ .\ )$ isomorphic to a finite, free (="perfect") complex, and $D^b_{art}(\ .\ )$, given by all objects with artinian cohomology, are transformed into each other. Now Serre's description of coherent sheaves of modules on projective schemes tells us that $D^b(R.)/D^b_{art}(R.)$ is canonically equivalent to $D^b(X)$, the derived category of such sheaves of modules on the projective complete intersection X.
On the other hand, the main result in [Bu] states that $D^b(R.)/D^b_{perf}(R.)$ is naturally equivalent to $\underline{MCM}(R.)$, the category of graded maximal Cohen-Macaulay R.-modules modulo projective modules.

To emphasize even more the duality between R. and $E^{\cdot}$, remark that $E^{\cdot}$ is <u>strongly</u> <u>Gorenstein</u>, ([Bu] and proposition 1), so that also $D^b(E^{\cdot})/D^b_{perf}(E^{\cdot})$ might be identified with $\underline{MCM}(E^{\cdot})$, the corresponding category of graded maximal Cohen-Macaulay-modules over $E^{\cdot}$ modulo projectives (in the sense of (loc. cit.)).

Denoting furthermore <u>Proj</u> $E^{\cdot}$ the quotient of the abelian category of all finitely generated, graded right $E^{\cdot}$-modules by its Serre-subcategory of artinian modules, one has that $D^b(\underline{Proj}\ E^{\cdot})$, its derived category, is naturally equivalent to $D^b(E^{\cdot})/D^b_{art}(E^{\cdot})$.

Hence, the generalized Bernstein-Gelfand-Gelfand correspondence can now be stated as follows:

Theorem 1: With the notations as above, there is a natural exact equivalence, the Bernstein-Gelfand-Gelfand-correspondence, between $D^b(R.)$ and $D^b(E^\cdot)$.
It induces exact equivalences between triangulated categories

$$D^b(\underline{\text{Proj}}\ R.) \xrightarrow{\sim} \underline{\text{MCM}}(E^\cdot) \quad \text{and}$$
$$\underline{\text{MCM}}(R.) \xrightarrow{\sim} D^b(\underline{\text{Proj}}\ E^\cdot)\ .$$

Furthermore, these equivalences are functorial in $(R.,E^\cdot)$. □

To use this theorem efficiently, it is necessary to obtain structural results on $E^\cdot$, and as an example we deduce theorem 2.1. of the foregoing article which motivated this appendix. All that is needed is:

Theorem 2: Assume R. is the homogeneous coordinate ring of a single quadric Q . Then

(i) Proj $E^\cdot$ is naturally equivalent to mod-$C^\cdot(Q)$, the category of $\mathbb{Z}/2\mathbb{Z}$-graded, finitely generated right modules over the (full) Clifford-algebra $C^\cdot(Q)$ of Q.

(ii) If Q is not zero, mod-$C^\cdot(Q)$ is equivalent to mod-$C^o(Q)$, the category of finitely generated right modules over the even Clifford-algebra $C^o(Q)$ of Q.

(iii) If $C^o(Q)$ is semi-simple, $D^b(\text{mod-}C^o(Q))$ is abelian and equivalent to grmod-$C^o(Q)$, the category of $\mathbb{Z}$-graded, finitely generated right $C^o(Q)$-modules, $C^o(Q)$ being concentrated in degree zero. □

Here of course only (i) needs a proof, (ii) and (iii) being essentially well-known. (For (ii), see [A-B-S] and for (iii) remark that over a semi-simple ring every complex of modules splits, so that passing to cohomology is an equivalence of categories.)

Combining these two theorems with the description of quadrics which define a semi-simple even Clifford-algebra (theorem 1.1. of the foregoing article), we get the desired result on maximal Cohen-Macaulay modules on a regular quadric:

Corollary 1: Assume that R. is the homogeneous coordinate ring of a regular quadric Q . Then $\underline{\text{MCM}}(R.)$ is an abelian category and as such naturally equivalent to grmod-$C^o(Q)$. □

Considering the algebraic K-groups of the categories involved, one can extend R.G. Swan's result, [Sw], for smooth quadrics to regular ones over a field.

Another application is the following description of those maximal Cohen-Macaulay modules on a quadric which can occur as syzygy-modules of artinian R.-modules.

Corollary 2: With the same assumptions as above, an equi-generated maximal Cohen-Macaulay R.-module occurs as the syzygy-module of some artinian R.-module if and only if the corresponding module over the even Clifford-algebra is free. In particular, an indecomposable maximal Cohen-Macaulay module occurs in this way iff $C^{o}(Q)$ is a division-ring.

1. The Correspondence

As in the introduction, let $R.$ be the homogeneous coordinate ring of a complete intersection defined by quadrics, $E^{\cdot} = \mathrm{Ext}^{*}_{R.}(k,k)$ its Yoneda-Ext-algebra.

Let $A.$ denote either of these two algebras, (where, contrary to conventions in other sources, raising or lowering of indices will not change signs, i. e. $A_i = E^i$ in case $A.$ represents $E^{\cdot}$!)

Set $\varepsilon = +1$ if $A. = R.$ and $\varepsilon = -1$ if $A. = E^{\cdot}$. If $M. = \bigoplus_i M_i$ is any $\mathbb{Z}$-graded right $A.$-module, for any integer ν we denote $M.(\nu)$ the shifted module with grading $M_i(\nu) = M_{i+\nu}$ and $A.$-module structure determined by

$$m(\nu)a = \varepsilon^{\nu j} ma(\nu) \in M_{i+j}(\nu) \ ,$$

for $m \in M_{i+\nu}$, $a \in A_j$.

To fix notations on module categories, let Mod-A. be the category of graded, right A.-modules with degree-preserving, A.-linear maps as morphisms, mod-A. its full subcategory of right noetherian modules.

$D^b(A.)$ will denote the full subcategory of the derived category(1) of Mod-A. , whose objects are those complexes of (arbitrary, graded right) A.-modules which have only finitely many non-vanishing cohomology modules, each of which is furthermore in mod-A. .

$D^b_{perf}(A.)$ refers to the full subcategory of complexes isomorphic to finite complexes of graded free A.-modules of finite rank. (Such complexes are also called "perfect".)

$D^b_{art}(A.)$ denotes the full subcategory of $D^b(A.)$ whose objects are all those complexes with right artinian cohomology modules. Both $D^b_{perf}(A.)$ and $D^b_{art}(A.)$ are closed under translation and mapping-cones, whence they inherit a triangulated structure from $D^b(A.)$.

As already remarked, art-A. , the category of right artinian and noetherian A.-modules, is a Serre-subcategory of mod-A. , hence the quotient Proj A. = mod-A./art-A. is again abelian. As in the commu-

(1) For definitions, notations and results on derived or triangulated categories, which are not explicitely given, we refer to [Ver], [BBD] or [Ha]. Remark that in [Ha], $D^b(A.)$ would be denoted $D^b_{\text{mod-}A.}(\text{Mod-}A.)$.

tative case, where this is essentially Serre's description of the coherent sheaves on a projective scheme, the quotient $D^b(A.)/D^b_{art}(A.)$ exists as a triangulated category and is naturally equivalent to D^b(Proj A.) , the derived category with bounded cohomology of Proj A. .

Also, as we will see in a moment, in our case A. is left and right noetherian and of finite injective dimension as either left or right module over itself, hence strongly Gorenstein in the sense of [Bu], and so $D^b(A.)/D^b_{perf}(A.)$ is naturally equivalent to MCM(A.) , the category of graded maximal Cohen-Macaulay A.-modules modulo projectives, as a triangulated category. Let us only remark that the equivalence is induced from the functor which to a maximal Cohen-Macaulay module over A. associates the complex with exactly this module as its only non-zero component, placed in degree zero.

We will write T or -[1] as usual for the translation-functor in any triangulated category (to be) encountered. We may combine it with the shift-functor - with which it commutes - and set

$$X[i,j) = T^i X(j)$$

for all $i,j \in \mathbb{Z}$, X in $D^b(A.)$.

In particular, any two complexes X,Y in $D^b(A.)$ define a bigraded Ext-group:

$$\mathrm{Ext}^i_{A.}(X,Y)_j = \mathrm{Hom}_{D^b(A.)}(X,Y[i,j)) \; ; i,j \in \mathbb{Z};$$

and, given a third complex Z in $D^b(A.)$, the Yoneda-product

$$\mathrm{Ext}^p_{A.}(Z,Y)_q \times \mathrm{Ext}^r_{A.}(X,Z)_s \to \mathrm{Ext}^{p+r}_{A.}(X,Y)_{q+s}$$

is defined for all integers p,q,r and s - see [Ver; II.1.2; 2.3]. This product is associative and functorial in the usual way. Thus for any object X in $D^b(A.)$ we get a bigraded algebra $\bigoplus_{p,q} \mathrm{Ext}^p_{A.}(X,X)_q$.

In particular, considering k as an A.-module via the augmentation $A. \to A_o = k$ and as a complex with k as its only non-zero component placed in degree zero, we recover the usual (bigraded) Yoneda-Ext-algebra of k over A.

We want to emphasize here that $\mathrm{Ext}^p_{A.}(k,k)_q$ as a vectorspace is independent of the sign $\varepsilon = \pm 1$ used in the definition of the shifts but that the multiplicative structure on the whole Ext-algebra depends on this choice!

Leaving abstract homological algebra, let us come back to the algebras in question.

Let $S.$ be a polynomial ring generated by variables of degree 1, and let $R.$ be a quotient of $S.$ by a regular sequence of d quadrics. Set $n = \dim R.$. The properties of $R.$ and $E.$, its Ext-algebra, are summarized in:

Proposition 1: Let $A.$ be either of the augmented, positively graded k-algebras $R.$ and $E^{\cdot}$.

(i) As an associative algebra, $A.$ is generated by its elements in degree one and defined by quadratic relations among these generators. It is left and right noetherian.

(ii) $A.$ is of (the same) finite injective dimension as either left or right module over itself. More precisely, $\operatorname{injdim}_{R.} R. = n = \dim R.$, $\operatorname{injdim}_{E.} E. = d = \operatorname{codim}_{S.} R.$.

(iii) The minimal, graded free resolution of k as an $A.$-module is linear and hence $\operatorname{Ext}^i_{A.}(k,k)_{-j} = 0$ unless $i = j \geq 0$.

(iv) $E^i = \operatorname{Ext}^i_{R.}(k,k)_{-i}$ and $R_i = \operatorname{Ext}^i_{E.}(k,k)_{-i}$ for all i and the multiplicative structure on $E^{\cdot}$ or $R.$ coincides with the Yoneda-product.

(v) $\operatorname{Ext}^i_{R.}(k,R.)_j = 0$ unless $i = \dim R.$, $j = \operatorname{codim}_{S.} R. - \dim R.$,
$\operatorname{Ext}^i_{E.}(k,E^{\cdot})_j = 0$ unless $i = \operatorname{codim} R.$, $j = \dim R. - \operatorname{codim}_{S.} R.$.

Proof: As $R.$ is a complete intersection defined by quadrics, (i) is obvious for $R.$. Furthermore, $R.$ is Gorenstein, whence (ii) in this case. That k has a linear resolution as an $R.$-module follows most easily from the explicit knowledge of that resolution as originally determined by J. Tate; see also [Eis]. Part (v) may be proved by induction on d, the codimension of $R.$ in $S.$: if $d=0$, i.e. $R.=S.$, one has $\operatorname{Ext}^i_{S.}(k,S.)_j = 0$ unless $i = \dim S. = -j$. If the result is assumed for $\tilde{R}.$ of codimension $\tilde{d} \geq 0$ in $S.$, and Q is a non-zero-divisor of degree 2 in $\tilde{R}.$, the exact sequence

$$0 \to \tilde{R}.(-2) \xrightarrow{Q} \tilde{R}. \to \tilde{R}./Q \to 0$$

shows $\operatorname{Ext}^i_{\tilde{R}.}(k,\tilde{R}./Q)_j = \operatorname{Ext}^{i+1}_{\tilde{R}.}(k,\tilde{R}.)_{j-2}$.

As $\mathrm{Ext}^i_{\tilde{R}.}(k,M.)_j = \mathrm{Ext}^i_{\tilde{R}./Q}(k,M.)_j$ for any $\tilde{R}.$-module M. annihilated by Q , (v) is still true for $\tilde{R}./Q$.

In (iv), the statement on $E^{\cdot}$ is tautological. What remains is to see that $R. = \mathrm{Ext}^{\cdot}_{\mathrm{Ext}^*_{R.}(k,k)}(k,k)$.

But this follows already from (iii) in case A. = R. by the work of S. Priddy, [Pr], and C. Löfwall, [Löf]. These papers also show that $E^{\cdot}$ is necessarily generated by E^1 , the defining relations being quadric, and that k over $E^{\cdot}$ again has a linear resolution.

That $E^{\cdot}$ is noetherian and of finite injective dimension over itself follows from the fact that $E^{\cdot}$ is indeed a graded, cocommutative Hopf-algebra (as it is the Ext-algebra of a commutative ring), whose graded Lie-algebra of primitive elements is finite-dimensional and (therefore) nilpotent (as R. is a complete intersection). This is explained, for example, in [Av], cf. also [B-R]. That this structural result on $E^{\cdot}$ implies the ring-theoretic properties claimed is then a consequence of the Poincaré-Birkhoff-Witt-theorem for graded Lie-algebras.

It remains to prove (v) for $E^{\cdot}$. This again can be deduced from the Poincaré-Birkhoff-Witt-theorem, given the explicit knowledge of the Lie-algebra of primitives in $E^{\cdot}$. □

Another proof - and a description of the Lie-algebra - will be given below.

The essential properties for the existence of the Bernstein-Gelfand-Gelfand-correspondence are (iii) - with its consequence (iv) - and (v).

Following S. Priddy and C. Löfwall, (loc. cit.), an algebra A. satisfying (iii) is called a Koszul-algebra . As this implies - by [Löf] - that $B^{\cdot} = \mathrm{Ext}^{\cdot}_{A.}(k,k)$ also satisfies (iii), one may rather call $(A.,B^{\cdot})$ a Koszul-pair. It is for such pairs that natural functors $D(A.) \rightleftarrows D(B^{\cdot})$ as in [BGG] can be set up.

A necessary condition for these functors to be inverse equivalences is (v), that is $\mathrm{Ext}^*_{A.}(k,A.)$ and $\mathrm{Ext}^*_{B^{\cdot}}(k,B^{\cdot})$ have to be one-dimensional k-vectorspaces. But for A. commutative, noetherian, this is equivalent to A. being Gorenstein, whence a pair $(A.,B^{\cdot})$ may be called (numerically) Gorenstein, (even without A. commutative).

Hence, the Bernstein-Gelfand-Gelfand correspondence should be considered a property of such <u>Gorenstein-Koszul-pairs</u>.

Now we come to the actual correspondence:

Assume $A.$ is a positively graded k-algebra, (with all A_i finite-dimensional k-vectorspaces), such that $k = A_o$ as a right $A.$-module admits a <u>linear</u> resolution $\mathbb{P}(A.,k) = (P^{\cdot}_{\cdot}, d^{\cdot})$ by finitely generated, graded free $A.$-modules.

Let the shifts on right graded $A.$-modules be defined by using a "commutation-factor" $\varepsilon = \pm 1$ as in the beginning. If $P^{-i}_{\cdot}$ denotes the i-th term in the resolution, (so that $d^i : P^{i-1}_{\cdot} \to P^i_{\cdot}$ increases the complex-degree), it can be identified - non-canonically - as

$$P^{-i}_{\cdot} = T_i \otimes_k A.(-i) \ ,$$

where $T_i = \mathrm{Tor}_i^{A.}(k,k)_i = H_o^{-i}(\mathbb{P}(A., k)(i) \otimes_{A.} k)$. (Of course, $P^{-i} = 0$ for $i < 0$.)

More generally, the graded pieces of $\mathbb{P}(A.,k)$ are given by

$$P^{-i}_{j+i} = T_i \otimes_k A_j = \mathrm{Tor}_i^{A.}(k(i),A_j) \ ,$$

where here and in the sequel the homogeneous components A_j of $A.$ are considered as $A.$-bimodules in degree zero; i. e. A_j is identified with $(A_{\geq j}/A_{\geq j+1})(j)$

Consider now the class $\Theta_j \in \mathrm{Ext}^1_{A.^{op}}(A_j, A_{j+1}(-1))$, represented by the extension of <u>left</u> $A.$-modules

$$0 \to A_{j+1}(-1) \to (A_{\geq j}/A_{\geq j+2})(j) \to A_j \to 0 \ .$$

Then the restriction of the differential $d^{\cdot}$ in $\mathbb{P}(A.,k)$ to $P^{-i}_{j+i} = \mathrm{Tor}_i^{A.}(k(i),A_j)$ can be recovered from the <u>left</u> action of $\mathrm{Ext}^1_{A.^{op}}(A_j, A_{j+1}(-1))$ on this Tor-group:

$$\mathrm{Ext}^1_{A.^{op}}(A_j, A_{j+1}(-1)) \times \mathrm{Tor}_i^{A.}(k(i),A_j) \to$$

$$\to \mathrm{Tor}_{i-1}^{A.}(k(i),\ A_{j+1}(-1)) \simeq \mathrm{Tor}_{i-1}^{A.}(k(i-1),A_{j+1}) = P^{-i+1}_{j+i} \ ,$$

in other words, $d^{\cdot}|P^{-i}_{j+i}$ is one of the connecting homomorphisms in the exact sequence obtained by applying $\mathrm{Tor}_*^{A.}(k(i),-)$ to the above short exact sequence, (cf. [ALG X.130; Prop. 7(b)]).

Secondly, we have the natural action

$$\mathrm{Ext}^1_{A.}(k,k(-1)) \times \mathrm{Tor}_i^{A.}(k(i),A_j) \to \mathrm{Tor}_{i-1}^{A.}(k(i-1),A_j)$$

which anti-commutes with Θ_j by [ALG X.129, Prop. 6 (15)] but defines on $\bigoplus_i \mathrm{Tor}_i^{A.}(k(i),A_j) = (\bigoplus_i \mathrm{Tor}_i^{A.}(k,k)_i) \otimes_k A_j$ the structure of a graded left module over $B^{\cdot} = \bigoplus_i \mathrm{Ext}_{A.}^i(k,k)_{-i}$ with its Yoneda-product. Furthermore, $\bigoplus_i \mathrm{Tor}_i^{A.}(k,k)_i \simeq B^{\cdot *}$, the graded k-dual of $B^{\cdot}$, as a left $B^{\cdot}$-module. Hence, forgetting the differential, $\mathbb{P}(A.,k)$ may be considered as the left $B^{\cdot}$-, right A.-module $(B^{\cdot})^* \otimes_k A.$.

Having said all this, let M. be any graded, right A.-module. Then $\bigoplus_i \mathrm{Hom}_{A.}^{\cdot}(\mathbb{P}(A.,k),\ M.[i,-i])$ is a complex of graded k-vector-spaces with j-th term:

$$\begin{aligned}\bigoplus_i \mathrm{Hom}_{A.}^j(\mathbb{P}(A.,k),\ M.[i,-i]) &= \bigoplus_i \mathrm{Hom}_{A.}(T_{j+i} \otimes_k A.(-j),\ M.)\\ &= \bigoplus_i M_j \otimes_k T_{j+i}^*\\ &= M_j \otimes_k \Big(\bigoplus_i \mathrm{Ext}_{A.}^{i+j}(k,k)_{-i-j}\Big) .\end{aligned}$$

But defining now the shifts for B.-modules with the opposite commutation-factor $-\varepsilon$, $B^{\cdot}(j) = \bigoplus_i \mathrm{Ext}_{A.}^{i+j}(k,k)_{-i-j}$ is endowed with the "correct" structure, and this complex becomes even a complex of graded free right B.-modules which we denote

$$\beta_{\cdot}^{\cdot}(M) = M. \otimes_k B^{\cdot}(.) ,$$

with j-th term $\qquad \beta_j^{\cdot}(M) = M_j \otimes_k B^{\cdot}(j)$,

(this time the complex-degree as lower index), the differential $\delta^M = \bigoplus_i \mathrm{Hom}_{A.}^{\cdot}(d^{\cdot},M[i,-i])\colon\ M. \otimes_k B^{\cdot}(.) \to M.+1 \otimes_k B^{\cdot}(.+1)$ still raising the complex-degree (but preserving $B^{\cdot}$-degree).

Remark 1: Obviously, $\mathrm{Ext}_{A.}^1(k,k(-1)) = \mathrm{Hom}_k(\mathrm{Tor}_1^{A.}(k,k)_1,k)$ is the k-dual of A_1. From the explicit construction of the linear resolution of k over a Koszul-algebra - as given in [Löf] - one gets the following description of δ_j^M . Denote $t : k \to A_1 \otimes_k A_1^* = \mathrm{Hom}_k(A_1,A_1)$ the map sending $1 \in k$ to the identity on A_1 . Denote $\mu_M : M. \otimes_k A_1 \to M._{+1}$ the right action of the 1-forms in A. on M. , $\mu_B : B^1 \otimes_k B^{\cdot} \to B^{\cdot}(1)$ the left multiplication of B^1 on $B^{\cdot}$. Then $\delta_j^M : M_j \otimes_k B^{\cdot}(j) \to M_{j+1} \otimes_k B^{\cdot}(j+1)$ is just the composition

$$\delta_j^M = (\mu_M \otimes_k \mu_B) \cdot (1_{M_j} \otimes_k t \otimes_k 1_{B^{\cdot}(j)}) ,$$

or, in terms of dual bases $\{x_i\} \subseteq A_1;\ \{x_i^*\} \subseteq B^1 = A_1^*$,

$$\delta_j^M(m \otimes b) = \sum_i mx_i \otimes x_i^* b .$$

(This is the form of the differential as given in [BGG].) □

The above construction is obviously functorial on Mod-A. and can hence be extended to complexes of such modules by applying it to every term, obtaining a double complex, and finally passing to the total complex. As one easily sees, homotopies of complexes are then carried into such.

Remark that the cohomology of $\beta(M)$ is given by

$$\begin{aligned} H_j^i(M) &= H^j(\mathrm{Hom}_{A.}^{\cdot}(\mathbb{P}(A.,k), M.[i,-i)) \\ &= \mathrm{Ext}_{A.}^{i+j}(k,M)_{-i} , \end{aligned}$$

so that $H_j^{\cdot}(M) = \bigoplus_i \mathrm{Ext}_{A.}^{i+j}(k,M)_{-i}$ as a graded right $B^{\cdot}$-module with respect to the natural Yoneda-product of $B^{\cdot} = \mathrm{Ext}_{A.}^{\cdot}(k,k)_{-.}$ on this module.

In particular, we see that $H_j^{\cdot}(k) = 0$ unless $j = 0$ in which case $H_0^{\cdot}(k) = B^{\cdot}$. Also, for $A. = R.$ or $A. = E^{\cdot}$, $H_j^{\cdot}(A.) = 0$ except for $j = \mathrm{injdim}_B . B^{\cdot}$ and

$$H^i_{\mathrm{injdim}\, B^{\cdot}}(A^{\cdot}) = 0 \quad \text{unless} \quad i = \mathrm{injdim}\, A. - \mathrm{injdim}\, B^{\cdot} ,$$

in which case the cohomology is just k , by statement (v) of proposition 1. This shows already that always this functor transforms $D^b_{art}(A.)$ into $D^b_{perf}(B^{\cdot})$ and that it transforms $D^b_{perf}(A.)$ into $D^b_{art}(B^{\cdot})$, as soon as $\mathrm{Ext}^*_{A.}(k,A.)$ is one-dimensional over k, (artinian would be enough).

Exchanging the roles of A. and $B^{\cdot}$, let α denote the corresponding functor on Mod-$B^{\cdot}$, which is possible as, by [Löf], k has also a linear resolution as a $B^{\cdot}$-module. Given a right graded A.-module M. , the double complex obtained by applying α term by term to $\beta(M.)$ is just the triply-graded k-vectorspace

$$\alpha\beta(M.) = M. \otimes_k B^*(.) \otimes_k A_-(*) ,$$

$\{.,*,-\}$ denoting the three degrees. The two differentials are $d_1 = \delta^M \otimes 1_A$ and $d_2 = 1_M \otimes \delta_B$. Hence, considering the spectral sequence obtained by first passing to cohomology with respect to d_2 yields the E_1-terms

$$E_1 \equiv M. \otimes_k H(B^{\cdot} \otimes_k A.) = M. \otimes_k \mathrm{Ext}_{B^{\cdot}}(k,B^{\cdot}) ,$$

conveniently graded.

We did not indicate the specific degrees as they do not matter. The point is, that <u>if</u> $\mathrm{Ext}^{*}_{B^{\cdot}}(k,B^{\cdot})$ <u>is one-dimensional</u>, this spectral sequence obviously collapses and yields the E_∞-term M. as a graded A.-module. In other words, as soon as the spectral sequence converges, the associated total complex of $\alpha\beta(M.)$ has a single cohomology module, namely M. , and hence, in the derived category of A. , represents M. again.

Now to ensure the convergence, it is certainly enough that $\beta(M)$ represents an object in $D^b(B.)$, that is, there are only finitely many j such that

$$H^{\cdot}_j(\beta(M)) = \bigoplus_i \mathrm{Ext}^{i+j}_{A.}(k,M.)_{-i} \neq 0$$

and each $H^{\cdot}_j$ is finitely generated over $B^{\cdot}$. But it is known - see [B-R] or [Av, 5.7] and dualize - that for A. = R. and M. a finitely generated R.-module, these two conditions are satisfied.

It follows then that α transforms $D^b(E^{\cdot})$ into $D^b(R.)$, for example by using once again the Poincaré-Birkhoff-Witt theorem for the Hopf-algebra $E^{\cdot}$ and induction on d , the codimension of R. in S. , the case d = 0 being precisely the result in [BGG]. Hence (α,β) is a pair of inverse exact equivalences between $D^b(A.)$ and $D^b(B^{\cdot})$.

As far as functoriality is concerned - which is alluded to above - let R! be another complete intersection defined by quadrics, lying between S. and R. : S. → R! → R. . Set $d' = \mathrm{codim}_{S.} R!$ and $'E^{\cdot} = \mathrm{Ext}^{\cdot}_{R!}(k,k)$. Then any graded right R!-module N. - or more generally an object in $D^b(R!)$ - gives rise to a spectral sequence

$$\mathrm{Ext}^i_{R.}(k,\mathrm{Ext}^j_{R!}(R.,N)) \Rightarrow \mathrm{Ext}^{i+j}_{R!}(k,N.) ,$$

which is in fact a spectral sequence of $E^{\cdot} = \mathrm{Ext}^{\cdot}_{R.}(k,k)$-modules. A priori, the abutment is an $'E^{\cdot}$-module, but it becomes an $E^{\cdot}$-module via the algebra morphism $E^{\cdot} = \mathrm{Ext}^{\cdot}_{R}(k,k) \to {}'E^{\cdot} = \mathrm{Ext}^{\cdot}_{R'}(k,k)$, which in turn is a boundary map in the above spectral sequence for N. = k.

Conversely, any $E^{\cdot}$-module $L^{\cdot}$ - or a complex of such in $D^b(E^{\cdot})$ - defines a spectral sequence of $R! = \mathrm{Ext}^{\cdot}_{'E^{\cdot}}(k,k)$-modules

$$\mathrm{Ext}^i_{'E^\cdot}(k, \mathrm{Ext}^j_{E^\cdot}('E^\cdot, L^\cdot)) \Rightarrow \mathrm{Ext}^{i+j}_{E^\cdot}(k, L^\cdot) \ .$$

Writing these spectral sequences in terms of double-complexes in the usual way and interpreting them as statements on exact functors between derived categories, this yields the following diagram of exact functors between triangulated categories

$$\begin{array}{ccccc} & D^b(R^!) & \underset{'\varepsilon}{\overset{\rho'}{\rightleftarrows}} & D^b('E^\cdot) & \\ R\,\mathrm{Hom}_{R^!}(R.,-) & \downarrow \uparrow \diagup & \text{"forget"} & \diagdown \downarrow \uparrow & R\,\mathrm{Hom}_{E^\cdot}('E^\cdot,-) \\ & D^b(R.) & \underset{\varepsilon}{\overset{\rho}{\rightleftarrows}} & D^b(E^\cdot) & \end{array}$$

whose two squares commute, and where the horizontal pairs $(\rho', '\varepsilon)$ and (ρ, ε) are the equivalences of the Bernstein-Gelfand-Gelfand-correspondence described above.

Taking $R^! = S.$, in which case $'E^\cdot = \mathrm{Ext}^\cdot_{S.}(k,k)$ is an exterior algebra $\Lambda^\cdot$, one obtains another proof that the equivalences transform $D^b_{perf}(R.)$ into $D^b_{art}(E^\cdot)$, hence induce equivalences between MCM(R.) and D^b(Proj $E^\cdot$): By Hilbert's syzygy theorem, $D^b(S.) = D^b_{perf}(S.)$ and the image under $R\,\mathrm{Hom}_{S.}(R.,-)$ has $D^b_{perf}(R.)$ as its thick hull (cf. [Bu]). Conversely, $D^b(\Lambda^\cdot) = D^b_{art}(\Lambda^\cdot)$, as $\Lambda^\cdot$ is an artinian k-algebra, and hence it generates $D^b_{art}(E^\cdot)$ under the forgetful functor $D^b(\Lambda^\cdot) \to D^b(E^\cdot)$.

This finishes the sketch of the proof of theorem 1 .

Remark 2: (a) As R. is Gorenstein, one could as well describe ρ in terms of Tor's - which is essentially done in theorem 2.1. of the foregoing article for maximal Cohen-Macaulay modules. For this, consider the two spectral sequences with same limit $\mathbb{E}^\cdot$ for an R.-module M.:

$$\mathrm{Ext}^i_{R.}(\mathrm{Tor}^{R.}_j(k,M.),R.) \Rightarrow \mathbb{E}^{i+j}$$

and

$$\mathrm{Ext}^i_{R.}(k, \mathrm{Ext}^j_{R.}(M.,R.)) \Rightarrow \mathbb{E}^{i+j} \ .$$

Now the first of these degenerates as R. is Gorenstein, showing that $\mathrm{Tor}^{R.}_j(k,M.)^*$, the k-dual, is isomorphic to $\mathbb{E}^{\dim R.+j}$. If furthermore M. is MCM , also the second spectral sequence degenerates, so that $\mathrm{Ext}^i_{R.}(k, M.^*) = \mathrm{Hom}_k(\mathrm{Tor}^{R.}_{i-\dim R}(k,M.),k)$, where $M.^*$ is now the R.-dual of M. . Hence the functors $R\,\mathrm{Hom}_{R.}(k, R\,\mathrm{Hom}_{R.}(M.,R.))$ and

$\mathrm{Hom}_k(k \overset{\mathbb{L}}{\otimes}_{R.} M.[-\dim R.],k)$ are equivalent - and compatible with the action of $E^{\cdot} = \mathrm{Ext}^{\cdot}_{R.}(k,k)$ on the right.

(b) Both $D^b(R.)$ and $D^b(E^{\cdot})$ are by definition "derived" from abelian categories, hence carry natural t-structures - see [BBD] for the definition - whose "hearts" are mod-R. and mod-$E^{\cdot}$ respectively. As the derived categories are equivalent, each of them carries hence two such structures, which are different:

The cohomology-theory attached to the ordinary t-structure on $D^b(R.)$, for example, is just the usual cohomology of complexes, whereas the "perverse" cohomology given by the equivalence (and with "heart" equivalent to mod-$E^{\cdot}$) is given by ${}^pH(X) = H^{\cdot}_p(\underline{\varepsilon}(X)) = \bigoplus_i \mathrm{Ext}^{i+p}_{E^{\cdot}}(k,\underline{\varepsilon}(X))_{-i}$, where X is any object in $D^b(R.)$. In particular, a complex X corresponds to a single $E^{\cdot}$-module under $(\underline{\rho},\underline{\varepsilon})$ iff ${}^pH(X) = 0$ for $p \neq 0$, that is, iff X is isomorphic to a <u>linear</u> complex of free modules in $D^b(R.)$, (the i-th module being generated in degree i). As a consequence, the intersection of the two "hearts" - on either side - is given by those <u>modules which have linear resolutions</u>.

2. The maximal Cohen-Macaulay modules

We want to investigate Proj $(E^\cdot)$ to obtain more insight into the structure of MCM (R.) and to prove theorem 2 and its corollaries.

So far we did not use any representation of R. or $E^\cdot$. This will be done now. By assumption, R. is the homomorphic image of $S. = k[V]$, $\deg V = 1$, by an ideal generated by quadrics $W \subseteq S_2^k V$, where both V and W are finite-dimensional k-vectorspaces. Set $d = \dim_k W = \operatorname{codim}_{S.} R.$ and $n = \dim R.$, so that $\dim V = d+n$. Dualizing the inclusion $W \to S_2^k V$ and composing it with the universal quadratic map $\chi : V^* \to (S_2V)^*$, given by $\chi(\lambda)(vw) = \lambda(v)\lambda(w)$ one obtains a k-quadratic map

$$q : V^* \to W^* , \text{ which satisfies the usual conditions:}$$

(i) $q(\alpha\lambda) = \alpha^2 q(\lambda)$ for $\alpha \in k$,

and

(ii) $(\lambda,\mu)_q = q(\lambda+\mu)-q(\lambda)-q(\mu)$ is a bilinear map from $V^* \times V^*$ into W^*.

The triple $L^\cdot = (V^*,W^*,q)$ is a simple kind of a graded k-Lie-algebra - as introduced by Milnor-Moore and amended by G. Sjödin and L. Avramov to capture the case char k = 2 , see [Av]. Here $L^1 = V^*$, $L^2 = W^*$ and $L^i = 0$, for $i \neq 1,2$, the bracket being given by $(\ , \ )_q$.

Now $E^\cdot$ is precisely its graded universal enveloping algebra $U^\cdot(L)$, that is, $E^\cdot$ admits a presentation $T^k_\cdot(V^*) \otimes_k S^k_\cdot(W^*)/I$, where

- $T^k_\cdot(V^*)$ is the tensor-algebra on V^*, graded by $\deg V^* = 1$,
- $S^k_\cdot(W^*)$ is the symmetric algebra on W , with $\deg W^* = 2$,
- the multiplicative structure is given by

$$(x \otimes y).(x' \otimes y') = x \otimes x' \otimes yy'$$

for $x,x' \in T^k_\cdot(V^*)$; $y,y' \in S^k_\cdot(W^*)$, so that $S^k_\cdot(W^*)$ becomes a central subalgebra,

and finally,

- I is the two-sided ideal generated by all elements

$$\lambda \otimes \lambda - q(\lambda) \quad \text{for } \lambda \in V^*.$$

The Lie-algebra $L^\cdot$, which following L. Avramov, [Av], is called the homotopy Lie-algebra of R., is nilpotent in an obvious sense,

$W^* = L^{\geq 2}$ being an abelian ideal in the centre of $L^\cdot$, $V^* = L^\cdot/L^{\geq 2}$ an abelian quotient.

Another interpretation of $L^\cdot$ is obtained from the André-Quillen cohomology of the cotangent-complex. It follows from [Qu] that $L^\cdot = T^\cdot_{k/R.}(k)$, the tangent or André-Quillen cohomology of k over R. , that is, $L^\cdot = H^\cdot(\mathrm{Hom}_k(\mathbb{L}_{k/R.},k))$, where $\mathbb{L}_{k/R.}$ is the cotangent-complex of k over R. . This is easily seen by direct computation and from (loc. cit.) it follows - in any characteristic, as R. is a complete intersection - that $\mathrm{Ext}^\cdot_{R.}(k,k)$ is precisely the graded enveloping algebra of this graded Lie-algebra, whence the claimed result on the structure of $E^\cdot$.
As to the functoriality of this construction, let us remark only that a k-linear map $\lambda : W_1 \to W$ between two k-vectorspaces W_1 and W defines obvious morphisms of graded Lie-algebras,

$$L^\cdot(\lambda^*) : L^\cdot = (V^*,W^*,q_W) \to L_1^\cdot = (V^*,W_1^*,\lambda^*\cdot q_W),$$

as well as of their graded universal enveloping algebras,

$$U^\cdot(\lambda^*) : E^\cdot = U^\cdot(L^\cdot) \to E_1^\cdot = U^\cdot(L_1).$$

If $i : W_1 \to W$ is a subspace of W, it also defines a complete intersection $R_1 = S./_{(W_1)}$, and $U^\cdot(i^*) : E^\cdot = U^\cdot(L) \to E_1^\cdot = U^\cdot(L_1)$ is the morphism of Ext-Algebras $\mathrm{Ext}^\cdot_R(k,k) \to \mathrm{Ext}^\cdot_{R_1}(k,k)$ referred to earlier.

Two particular cases are worth mentioning:

If $W_1 = 0$, (hence $R_1 = S.$), $L_1^\cdot = V^*$ is the graded abelian Lie-algebra concentrated in degree one and its graded universal enveloping algebra is just $\Lambda^\cdot = \Lambda^\cdot(V^*)$, the exterior k-algebra generated by V^* in degree one. The corresponding morphism $E^\cdot \to \Lambda^\cdot$ is then nothing but the canonical projection $E^\cdot \to E^\cdot \otimes_{S^k(W^*)} k \simeq \Lambda^\cdot$.

If $W_1 = k.Q \subseteq W$ is the subspace generated by a single quadric Q in $W \subseteq S_2^k(V)$, $R_Q = S./_{(Q)}$ is a quadric hypersurface ring with Ext-algebra

$$E_1^\cdot = E_Q^\cdot = T^k_\cdot(V^*)[\sigma]/_{(\lambda \otimes \lambda - \lambda^2(Q).\sigma)}$$

where σ is a central variable of degree two, and λ^2 denotes the linear form on $S_2^k(V)$ given by $\lambda^2(vw) = \lambda(v)\lambda(w)$, i.e. $\lambda^2 = \chi(\lambda)$.

Q can of course also be considered as a linear form on W^* and the corresponding morphism $E^\cdot \to E_Q^\cdot$ is just the projection

$E^{\cdot} \to E^{\cdot} \otimes_{S^k(W^*)} k[\sigma]$, $k[\sigma]$ considered a graded $S^k(W^*)$-module via the linear form $Q : W^* \to k.\sigma$.

$E^{\cdot}_Q$ is nothing but the <u>homogenized Clifford-algebra</u> of the quadratic form defined by Q, in other words, sending σ to 1 defines a morphism of $\mathbb{Z}/_{2\mathbb{Z}}$-graded algebras - as $\deg \sigma = 2$ - :

$$E^{\cdot}_Q \xrightarrow{\sigma=1} C^{\cdot}(Q) = E^{\cdot} \otimes_{S^k=W^*)} k[\sigma]/_{(\sigma-1)} = T^k_{\cdot}(V^*)/_{(\lambda\otimes\lambda-\lambda^2(Q))} .$$

These morphisms allow us to visualize $E^{\cdot}$ geometrically. An equivalent form of the Poincaré-Birkhoff-Witt theorem for such simple-minded graded Lie-algebras $L^{\cdot} = (V^*,W^*,q_W)$ can be stated as saying that $E^{\cdot}$ is a <u>central</u>, <u>flat</u> $S^k_{\cdot}(W^*)$-algebra with <u>fibre</u> $\Lambda^{\cdot} = E^{\cdot} \otimes_{S^k(W^*)} k$ over the "<u>origin</u>" $\{S^k_+(W^*)\}$ in the affine space W^*.
Sheafifying $E^{\cdot}$ to obtain a sheaf of $\mathbb{Z}/_{2\mathbb{Z}}$-graded algebras $\mathcal{E}^{\cdot}$ on $\mathbb{P}(W^*)$, the underlying projective space of hyperplanes in W^*, the <u>fibre</u> over a k-rational point $W^* \to k$ is the Clifford-algebra $C^{\cdot}(Q)$ of the corresponding quadric in $W=W^{**}$ defining this point.

In other words, $E^{\cdot}$ may be considered as the total space of a flat deformation deforming the exterior algebra $\Lambda^{\cdot}$ into the various Clifford-algebras $C^{\cdot}(Q)$, $Q \in W$.
Recall now that <u>Proj</u> $(E^{\cdot})$ is the category of graded, finitely generated, right $E^{\cdot}$-modules modulo its Serre-subcategory of artinian modules. As it was said above, $E^{\cdot}$ is a central, flat $S^k_{\cdot}(W^*)$-algebra, hence free as a module over this polynomial ring and its rank equals $\dim_k \Lambda. = 2^{\dim V}$. As W^* is concentrated in degree two, any graded $E^{\cdot}$-module $N^{\cdot}$ defines two graded $S^k_{\cdot}(W^*)$-modules,

- $N^{\cdot}_+$ given by $N^i_+ = N^{2i}$ and
- $N^{\cdot}_-$ given by $N^i_- = N^{2i+1}$ for all i.

As an $E^{\cdot}$-module is artinian iff its positive and negative part are artinian $S^k_{\cdot}(W^*)$-modules, the forgetful-functor defines a functor $(+,-)$: <u>Proj</u> $E^{\cdot} \to \mathrm{Coh}(\mathbb{P}(W^*)) \times \mathrm{Coh}(\mathbb{P}(W^*))$, associating to (the class of) a graded $E^{\cdot}$-module $N^{\cdot}$ the sheaves $\tilde{N}^{\cdot}_+$ and $\tilde{N}^{\cdot}_-$ on the projective space $\mathbb{P}(W^*)$.
Denoting as usual by $\mathcal{O}(1)$ the Hopf-bundle on $\mathbb{P}(W^*)$ - which is given as a graded module by $S^k_{\cdot}(W^*)(2)$ as $\deg W^* = 2$ - , the extra-structure on $N^{\cdot}$, that is the action of V^*, can be recovered as follows:

A pair of coherent sheaves $(\mathcal{N}_+, \mathcal{N}_-)$ on $\mathbb{P}(W^*)$ comes from a graded $E^{\cdot}$-module $N^{\cdot}$ iff there are $\mathcal{O}_{\mathbb{P}(W^*)}$-linear morphisms

$$\varphi : \mathcal{N}_+ \underset{k}{\otimes} V^* \to \mathcal{N}_- \quad \text{and}$$
$$\psi : \mathcal{N}_- \underset{k}{\otimes} V^* \to \mathcal{N}_+(1)$$

such that the following diagrams are commutative:

$$\begin{array}{ccccc}
\mathcal{N}_+ \underset{k}{\otimes} V^* & & \xrightarrow{\quad 1 \otimes q \quad} & & \mathcal{N}_+ \underset{k}{\otimes} W^* \\
\downarrow{\scriptstyle 1\otimes\tilde{\chi}} & & & & \downarrow{\scriptstyle \text{can.}} \\
\mathcal{N}_+ \underset{k}{\otimes} V^* \underset{k}{\otimes} V^* & \xrightarrow[\varphi\otimes 1_{V^*}]{} & \mathcal{N}_- \underset{k}{\otimes} V^* & \xrightarrow[\psi]{} & \mathcal{N}_+(1)
\end{array}$$

and

$$\begin{array}{ccccc}
\mathcal{N}_- \underset{k}{\otimes} V^* & & \xrightarrow{\quad 1 \otimes q \quad} & & \mathcal{N}_- \underset{k}{\otimes} W^* \\
\downarrow{\scriptstyle 1\otimes\tilde{\chi}} & & & & \downarrow{\scriptstyle \text{can.}} \\
\mathcal{N}_- \underset{k}{\otimes} V^* \underset{k}{\otimes} V^* & \xrightarrow[\psi\otimes 1_{V^*}]{} & \mathcal{N}_+(1) \underset{k}{\otimes} V^* & \xrightarrow[\varphi(1)]{} & \mathcal{N}_-(1)\ .
\end{array}$$

Here $\tilde{\chi}$ denotes the composition of the universal quadratic map $\chi : V^* \to S_2(V)^*$ with the natural inclusion $S_2(V)^* \to V^* \underset{k}{\otimes} V^* = (V \underset{k}{\otimes} V)^*$, given explicitely by $\tilde{\chi}(\lambda) = \lambda \otimes \lambda$ for $\lambda \in V^*$. The map labeled "canonical" is just the tensorproduct of the identity on $\mathcal{N}_{+/-}$ with the evaluation of global sections $\mathcal{O} \underset{k}{\otimes} W^* = \mathcal{O} \underset{k}{\otimes} H^0(\mathbb{P}(W^*), \mathcal{O}(1)) \to \mathcal{O}(1)$. Hence:

Proposition 2 (i): The category Proj $E^{\cdot}$ is isomorphic to the category of quadruples $(\mathcal{N}_+, \mathcal{N}_-, \varphi, \psi)$, the $\mathcal{O}$-linear morphisms φ and ψ satisfying the compatibility conditions above. (The description of the morphisms is left to the reader.).
(ii) Under the Bernstein-Gelfand-Gelfand-correspondence, Proj $E^{\cdot}$ corresponds to the subcategory of MCM(R.) given by those modules which admit a linear resolution over R. .

Remark 3: The natural t-structure on D^b(Proj $E^{\cdot}$) - see remark 2, (b) - , if interpreted on the equivalent category MCM(R.), has hence the linear modules (generated in degree zero) as its "heart".

Proof (i) is obvious from the above. To prove (ii), let $\tilde{N}$ be an object in Proj $E^{\cdot}$ and represent it by a graded $E^{\cdot}$-module $N^{\cdot}$. By remark 2, (b), the image $\underline{\rho}(N^{\cdot})$ in $D^b(R.)$ is isomorphic to a linear complex of free R.-modules. Using [Bu], a maximal Cohen-Macaulay module representing $\underline{\rho}(N^{\cdot})$ in $\underline{MCM}(R.)$ is obtained - up to translation - as a syzygy-module sitting sufficiently far back in the free resolution of $\underline{\rho}(N^{\cdot})$, whence this module - and its translates - have a linear resolution. (A proof using Tate-cohomology, [Bu], can be obtained as follows: Let M. be a MCM representing $\underline{\rho}(N^{\cdot})$. Then, for sufficiently large i and all j, $\underline{\mathrm{Ext}}^i_{R.}(k,M)_{-j} = \mathrm{Ext}^i_{R.}(k,\rho(N.))_{-j}$ and $\underline{\mathrm{Ext}}^i_{R.}(k,M.)$ is dual to $\mathrm{Tor}^{R.}_{i-\dim R.}(k,M^*_.)$, which shows that $M^*_.$ - and then also M. - has a linear resolution.)

As $\underline{\varepsilon}(M.)$ represents $N^{\cdot}$, it follows that M. is generated in degree zero (and the i-th term in its resolution by elements of degree i). This establishes the claim. □

In case that W is one-dimensional, hence generated by a single non-zero quadric Q, $\mathbb{P}(W^*)$ is just a point and the pair (N_+,N_-) defines a $\mathbb{Z}/_{2\mathbb{Z}}$-graded module over the full Clifford-algebra $C^{\cdot}(Q)$. Hence in this case, $\underline{\mathrm{Proj}}\ E^{\cdot} = \text{mod-}C^{\cdot}(Q)$ as stated in theorem 2.

It remains to prove corollary 2. If M. is an <u>equi-generated</u> graded MCM over R_Q for some quadric Q, that is, generated by its elements of some fixed degree l, M.(l) is generated in degree zero and has a linear resolution by Proposition 2.3. of the foregoing article.
We may replace M. by M.(l) and can hence assume that M. is already generated in degree zero. Then $\underline{\varepsilon}(M.)$ is (isomorphic in $D^b(E^{\cdot})$ to) a single module $N^{\cdot}$. If now M. is a syzygy-module of some <u>artinian</u> R.-module A., this signifies the existence of a distinguished triangle

$$M.[r-1] \to X^{\cdot}_{\cdot} \to A. \xrightarrow{(1)}$$

in $D^b(R.)$, (the notations as in [BBD]), where $X^{\cdot}_{\cdot}$ is a <u>perfect</u> complex - namely the beginning of a free resolution of A. which exhibits M. as its r-th syzygy, (of course $r \geq \dim R.$, if M. is MCM, but that does not matter).

Applying $\underline{\varepsilon}$ to this triangle and using theorem 1, we obtain a triangle

$$N^{\cdot}[r-1] = \underline{\varepsilon}(M.[r-1]) \to \underline{\varepsilon}(X^{\cdot}_{\cdot}) \to \underline{\varepsilon}(A.) \xrightarrow{(1)}$$

in $D^b(E^{\cdot})$, where now $\underline{\varepsilon}(A.)$ is perfect and $\underline{\varepsilon}(X^{\cdot}_{\cdot})$ is a complex with artinian cohomology. Projecting into $D^b(\underline{\text{Proj}}\ E^\cdot)$, $\underline{\varepsilon}(X_\cdot^\cdot)$ becomes the zero-object, hence the images of $\underline{\varepsilon}(A.)$ and $N^\cdot[r]$ become isomorphic. This means that the image of $\underline{\varepsilon}(A.)$ has a single cohomology object, namely the image $\tilde{N}^\cdot$ of $N^\cdot$ in (complex-)degree $-r$.

Now assume $C^O(Q)$ semi-simple, so that the image of $\underline{\varepsilon}(A.)$ in $D^b(\text{mod-}C^O(Q)) \cong D^b(\underline{\text{Proj}}\ E^\cdot)$ is (isomorphic to) a finite complex of free $C^O(Q)$-modules, still having a single cohomology module. But $C^O(Q)$ semi-simple implies that this complex splits, whence its only cohomology module is free. (By [Kap; Thm 180] it follows a priori only that the cohomology module is projective having a free complement. But $C^O(Q)$ semi-simple implies that one can cancel.) This proves corollary 2.

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