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*Inventiones mathematicae*


# Existence, decomposition, and limits of certain Weierstrass points

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## Contents



**Summary.** We prove the existence, in the "expected" codimension, of curves with prescribed collections of Weierstrass points, as long as the total of the weights of the prescribed points is $\leqq$ half the genus of the curve, and in some more general circumstances.

We also study incidence relations among Weierstrass points of different types, and show that a Weierstrass point that moves in a family of the expected codimension can be obtained as the common limit of two Weierstrass points of weights adding up to $w$ in every "conceivable" way.

To do these things we use the theory developed in our [1986] to identify limits of canonical series and Weierstrass points as smooth curves degenerate to certain reducible curves. This allows us, for example, to prove the existence of certain Weierstrass points on curves of genus $g$ by induction on $g$.

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## Introduction

### *a) Main results*

Let $C$ be a smooth (complex projective) curve of genus $g$. To each point $p \in C$ we associate the ramification sequence of the canonical bundle, or *Schubert index* $\alpha(p)=(\alpha_0(0), \ldots, \alpha_{g-1}(p))$ where $0 \leqq \alpha_0(0) \leqq \ldots \leqq \alpha_{g-1}(p) \leqq g-1$ are the integers such that there exists a holomorphic differential form on $C$ vanishing to order $\alpha_i(p)+i$ at $p$ for $i=0, \ldots, g-1$. For all but finitely many $p \in C$ we have

$$\alpha_0(p)=\ldots=\alpha_{g-1}(p)=0 .$$

If $p$ does not satisfy these equalities, then $p$ is called a *Weierstrass point* of $C$, and

$$w(p):=\sum_{i=0}^{g-1} \alpha_i(p)$$

is called its (Weierstrass) *weight*. (Of course this is equivalent to the definition of Weierstrass points in terms of pole orders; see section 1 below for the well-known conversion.)

Let $\mathcal{M}_g$ be the moduli space of smooth curves of genus $g$ and let $\mathscr{C}_g=\mathcal{M}_{g,1}$ be the moduli space of pointed curves. For any *Schubert index of genus g*

$$\alpha: 0 \leqq \alpha_0 \leqq \ldots \leqq \alpha_{g-1} \leqq g-1$$

we may define a locally closed subvariety of $\mathscr{C}_g$ by

$$\mathscr{C}_\alpha=\{(C, p) \mid \alpha(p)=\alpha\} .$$

More generally, if $\alpha^1, \ldots, \alpha^s$ are such indices, we define a subset of $\mathcal{M}_{g,s}$, the moduli of $s$-pointed curves, by

$$\mathscr{C}_{\alpha^1, \ldots, \alpha^s}=\{(C, p_1, \ldots, p_s) \mid \alpha(p_i)=\alpha^i\} .$$

The question of which $\mathscr{C}_\alpha$ are nonempty was posed by Hurwitz [1893] and remains open. More generally, we would like to know:

1) What are the dimensions of the $\mathscr{C}_{\alpha^1, \ldots, \alpha^s}$ or, when they are reducible, of their components?

2) How do Weierstrass points come together? What are the incidence relations among the closures of the $\mathscr{C}_{\alpha^1, \ldots, \alpha^s}$?

As regards 1, the *weight* $w(\alpha)=\sum \alpha_i$ gives an upper bound for the codimension of any component of $\mathscr{C}_\alpha$, and similarly $\sum w(\alpha^i)$ is an upper bound for components of $\mathscr{C}_{\alpha^1, \ldots, \alpha^s}$. (Reason: In a neighborhood of any of its points $\mathscr{C}_{\alpha^1, \ldots, \alpha^s}$ is obtained by pulling back Schubert cycles $\sigma_{\alpha^i}$ from a suitable Grassmannian of $g-1$-planes, and $w(\alpha^i)=\operatorname{codim} \sigma_{\alpha^i}$.)

To simplify the discussion we will call a point $x=(C, p_1, \ldots, p_s) \in \mathcal{M}_{g,s}$ *dimensionally proper* if $\mathscr{C}_{\alpha(p_1), \ldots, \alpha(p_s)}$ has codimension $=\sum_i w(\alpha(p_i))$ in a neighborhood of $x$.

Of course the dimensionally proper points of a given $\mathscr{C}_\alpha$ form an open subset of that $\mathscr{C}_\alpha$, and in fact by Theorem 3 of our [1987], they even form an open subset of $\mathscr{M}_{g,s}$, obviously non-empty (and thus dense) because it contains the $s$-tuples of non-Weierstrass points. Thus it is a natural place to begin the study of Weierstrass points.

In this paper we provide answers to question 2) and to a considerable part of question 1) which are valid in the open set of dimensionally proper points. We indicate the results in an order different from the order of the proofs, which will be given in sections 5 and 6 below.

**Theorem 1.** *If $\alpha^1, \ldots, \alpha^s$ are Schubert indices of genus $g$ with total weight*

$$w = \sum w(\alpha^i) \leqq g/2,$$

*then $\mathscr{C}_{\alpha^1,\ldots,\alpha^s}$ contains dimensionally proper points.*

We next study the incidence relations among the closures of the $\mathscr{C}_{\alpha^1,\ldots,\alpha^s}$. Recall that the Schubert indices are partially ordered by

$$\alpha \leqq \beta \quad \text{if} \quad \alpha_i \leqq \beta_i, \qquad i = 0, \ldots, g-1;$$

this corresponds to the containment relation among the associated Schubert cycles. We have:

**Theorem 2.** *A dimensionally proper point of $\mathscr{C}_{\beta^1,\ldots,\beta^s}$ is in the closure of $\mathscr{C}_{\alpha^1,\ldots,\alpha^s}$ if and only if $\alpha^j \leqq \alpha^j$ $(j = 1, \ldots, s)$; in particular, if $\mathscr{C}_{\beta^1,\ldots,\beta^s}$ contains a dimensionally proper point and for some $t \leqq s$, $\alpha^1 \leqq \beta^1, \ldots, \alpha^t \leqq \beta^t$ then $\mathscr{C}_{\alpha^1,\ldots,\alpha^t}$ contains dimensionally proper points.*

To prove stronger existence theorems than Theorem 1, we need to impose conditions on the Schubert indices involved. The simplest reason for the emptiness of some of the $\mathscr{C}_{\alpha^1,\ldots,\alpha^s}$ is that if $\alpha = \alpha(p)$ is the Schubert index of a point $p$ on a curve, then $\alpha$ satisfies the *semigroup condition* that

$$H(\alpha) := \mathbb{N} - \{\alpha_i + i + 1 \mid i = 0, \ldots, g-1\}$$

be a subsemigroup of the natural numbers $\mathbb{N}$; $H(\alpha(p))$ is the Weierstrass semigroup of pole orders of rational functions regular away from $p$. It is easy to check that every $\alpha$ with $w(\alpha) \leqq g/2$ satisfies the semigroup condition, but that the Schubert index $(0^{g-w}1^w)$ of weight $w$ does not if $w \geqq \lfloor g/2 \rfloor + 1$. This shows that the bound $g/2$ in Theorem 1 is best possible without further hypotheses.

We are motivated by Theorem 2 to say that $\alpha$ is *primitive* if every index $\beta$ with $\beta \leqq \alpha$ satisfies the semigroup condition. This corresponds to saying that $H(\alpha)$ is a semigroup such that twice the smallest positive element of $H(\alpha)$ is $>$ the largest integer not in $H(\alpha)$; see Proposition 1.1. From Theorem 2 we obtain at once:

**Corollary.** *If $\mathscr{C}_{\alpha^1,\ldots,\alpha^s}$ contains a dimensionally proper point then each $\alpha^i$ is primitive.*

Unfortunately the converse of the Corollary is certainly false – for example, primitive indices can have weight much larger than the dimension of $\mathscr{M}_{g,1}$ (see example 5.6). Thus we are led to bound the weight as well; we prove:

**Theorem 3.** *If $\alpha$ is a primitive Schubert index of genus $g$ and weight $w(\alpha)<g-1$, then $\mathscr{C}_\alpha$ contains dimensionally proper points.*

Theorem 3 is the central existence theorem of this paper; the other results are deduced from it by means of the theory developed in our [1987] which describes, in our present context, the situation in which a group of Weierstrass points $p_i(t)\in C_t$ have a common limit $p_i(0)$ on a smooth curve $C_0$. For simplicity we describe here the case where just two points meet; the general case, in which several groups, of many points each, meet at several limit points follows the same pattern. It is treated in [1987] and will be applied below.

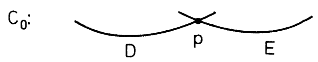
**Theorem A** [1987]. *Let $C_t$ be a family of smooth curves (for $t\in\mathbb{C}$, $|t|<\varepsilon$, say) and let $p_1(t)$, $p_2(t)$ be distinct points of $C_t$ for $t\neq 0$ with common limit $p=p_i(0)$ at $t=0$. If for $t\neq 0$ the point $p_i(t)$ is a Weierstrass point of $C_t$ with constant Schubert index $\alpha^i$, then the Schubert index $\beta$ of $p$ on $C$ is such that the class of the associated Schubert cycle $\sigma_\beta$ is contained in the class of the product $\sigma_{\alpha^1}\cdot\sigma_{\alpha^2}$ in the cohomology ring of the Grassmannian $G(g-1,\mathbb{P}^{2g-2})$.*

*Conversely, suppose that $C_0$ is a smooth curve, $p_0\in C_0$ a Weierstrass point of Schubert index $\sigma_\beta$, and $\alpha^1$ and $\alpha^2$ are Schubert indices with $[\sigma_\beta]\subseteq[\sigma_{\alpha^1}\cdot\sigma_{\alpha^2}]$ as above. If $p_0$ is a dimensionally proper Weierstrass point, then a family $C_t$, $p_1(t)$, $p_2(t)$ as above actually exists.*

Here we say that $[\sigma_\beta]\subseteq[\sigma_{\alpha^1}]\cdot[\sigma_{\alpha^2}]$ if, in the unique expression $[\sigma_{\alpha^1}]\cdot[\sigma_{\alpha^2}] = \sum m_\gamma[\sigma_\gamma]$ of the product in terms of classes of Schubert cycles, we have $m_\gamma\neq 0$ for some index $\gamma$ with

$$\gamma_0\leqq\beta_0,\ldots,\gamma_{g-1}\leqq\beta_{g-1}.$$

Our approach to these results, in sections 2–4 below, is to use the theory of limit series developed in our paper [1986] to study what happens to the Weierstrass points of a smooth curve as the smooth curve degenerates to a reducible curve of compact type. As an example of the results on such degenerations that we obtain, consider a curve $D$ of genus $g-1$, and an elliptic curve $E$, and let $C_0$ be the union of $D$ and $E$, joined transversely at a point $p$:




We will show that if $\{C_t\}$ is a family of smooth curves approaching $C_0$ in such a way that the Weierstrass points of $C_t$ do not tend to $p$, then

1) If $p$ is not a Weierstrass point of $D$ then *independently of the family* $\{C_t\}$
   a) Exactly $g^2-1$ of the Weierstrass points of $C_t$, all of weight 1, approach $E$; their limits are the $g^2-1$ points of $E$ which differ from $p$ by $g$-torsion.
   b) The remaining $(g^2-1)(g-1)$ (with multiplicities) Weierstrass points of $C_t$ approach the $(g^2-1)(g-1)$ ramification points of the complete linear series $|K_D+2p|$ on $D$.

On the other hand

2) If $p$ is a dimensionally proper Weierstrass point of weight $w>0$ on $D$, then
   a) every point of $E$ is the limit of Weierstrass points on nearby smooth curves.

b) The points of $E$ which differ from $p$ by torsion of certain orders are the limits of Weierstrass points of weight $w+1$ on certain nearby smooth curves.

### *b) Background*

Hurwitz' original question [1893] was whether $C_\alpha$ is non-empty for all $\alpha$ satisfying the semigroup condition. Almost immediately Haure [1896] incorrectly gave further restrictions and Hensel and Landsberg [1902] incorrectly showed there were none.

There have been positive results for $\alpha$ of special form: For example in the (primitive) case $\alpha=(0,\ldots,0,k)$, $k<g-1$, the existence of dimensionally proper points is proved by Pinkham [1974] and Laufer [1981]. On the other hand, the set

$$\bigcup_{\alpha_{n-2}=0,\,\alpha_{n-1}\geqq 1} \mathscr{C}_\alpha \subset \mathscr{M}_{g,1}$$

may be identified with the space of $n$-sheeted covers of $\mathbb{P}^1$ with a point of total ramification and has been studied from this point of view: It is irreducible, smooth along the locus $\alpha_{n-1}=1$, and of dimension $g-n+1$ (Arbarello [1974]; Laufer [1981], Lax [1975], Diaz [1986]). Its tangent spaces have been computed by Diaz [1984a].

There are further scattered positive results, for example when $H(\alpha)$ is symmetric (Knebl [1984]), when $g\leqq 4$ (Lax [1980]), and when $H(\alpha)$ is "negatively graded" (Rim-Vitulli [1977]), and in some cases of several Weierstrass points (Kuribayashi and Komiya [1977]; Diaz [1984b]; Vermeulen [1983]; Lugert [1981]).

On the other hand, Buchweitz [1980] finally showed that *not* every $\alpha$ satisfying the semigroup condition occurs. His first example, with $g=16$, is $\alpha=(0,\ldots,0,6,7,9,9)$. (His astonishingly simple proof is this: If $\alpha=\alpha(p)$ for some $p\in C$, then $C$ would have differentials vanishing to orders $0,\ldots,11$, 18, 20, 23, 24 at $p$, and thus would have quadratic differentials vanishing to every order between 0 and 48 except possibly 37, 39, and 45; since $49-3=46>H^0(C,K_C^2)=3g-3=45$, this is impossible.) There are a number of further negative results concerning $\mathscr{C}_{\alpha^1,\ldots,\alpha^s}$ for $s\geqq 2$ and "large" $\alpha^i$; see for example Schreyer [1983].

The reader may find a discussion, without proofs, of some of the results of this paper in our [to appear].

### *c) Open problems*

Aside from problems 1) and 2) above, which are largely open outside the dimensionally proper domain, the following seem to us particularly interesting:

3) What is the codimension of the complement of the open set of dimensionally proper points in $\mathscr{M}_{g,s}$ (or even in $\mathscr{C}_g=\mathscr{M}_{g,1}$)? It is known (Arbarello [1974]) to be $\leqq\lceil g/2\rceil$. Is it $=\lceil g/2\rceil$?
4) Are the $\mathscr{C}_{\alpha^1,\ldots,\alpha^s}$ (even, say, for $s=1$) *irreducible* if $\sum w(\alpha^i)$ is small (say $\leqq g/2$)?
5) What are the limits of Weierstrass points in families of curves degenerating to stable curves *not* of compact type?
6) Are the $\mathscr{C}_{\alpha^1,\ldots,\alpha^s}$ affine?

At any rate, one can see that they contain no complete curves. (*Proof sketch.* Of course it is enough to do the case $s=1$. Suppose $\pi:\mathscr{C}\to B$ were a complete 1-dimensional family of smooth curves of genus $g$ with section $B=S\subset\mathscr{C}$ consisting of Weierstrass points of the fibers, all with Schubert index $\alpha=(\alpha_0,\ldots,\alpha_{g-1})$. Let $\omega=\omega_{\mathscr{C}/B}$ be the relative dualizing sheaf, and let $H=\pi_*\omega$ be the Hodge bundle on $B$. By the Grothendieck-Riemann-Roch theorem, the degree of $H$ is $(\omega)^2/12$. On the other hand, there is a natural filtration of $H$ by order of vanishing along $S$, and the quotients are $(\omega|S)^{\alpha_i+i}$, whence the degree of $H$ is

$$\sum(\alpha_i+i)(-(S)^2)\geqq\frac{g(g-1)}{2}(-(S)^2).$$

Using the inequality

$$-(S)^2\geqq\frac{(\omega)^2}{4g(g-1)},$$

of Szpiro (1981) (p. 58, Prop. 2) and $(\omega)^2>0$, we get a contradiction.)

7) For which $\alpha$ is there a curve with a Weierstrass point of Schubert index $\alpha$ and no other Weierstrass point of weight $>1$? Of course for $\alpha=(0,1,2,\ldots,g-1)$, the Schubert index of a hyperelliptic Weierstrass point, the answer is "no," as it is in the "elliptic-hyperelliptic" case $\alpha=(0,0,0,1,2,\ldots,g-3)$. Are there such $\alpha$ with $w(\alpha)\leqq g/2$?

In connection with problem 1) above, it should be remarked that even for $s=1$, there is not even a plausible conjecture as to $\dim\mathscr{C}_\alpha$ for reasonably broad classes of $\alpha$. (But see Rim and Vitulli (1977) § 6, for an upper bound.)

Another class of appealing questions concerns the monodromy action on families of curves with Weierstrass points of given type. In another paper [ ] we will deal with the "first" of these, showing that the monodromy acts as the full symmetric group on the Weierstrass points of curves whose Weierstrass points all have weight 1.

## 1. Preliminaries

For the reader's convenience we collect various notation and known results.

### *a) Ramification sequences and Weierstrass points*

Let $C$ be a smooth curve. If $\mathscr{L}$ is a line bundle of degree $d$ on $C$ and $V\subset H^0(\mathscr{L})$ is an $r+1$-dimensional space of sections then we say that $L=(\mathscr{L},V)$ is a $\mathfrak{g}^r_d$ on $C$. For any $p\in C$, the sections in $V$ vanish to $r+1$ distinct orders at $p$ which we may order and write as

$$a^L(p)=(a^L_0(p),\ldots,a^L_r(p))$$

with $0\leqq a^L_0(p)<\ldots<a^L_r(p)\leqq d$.

$a^L(p)$ is called the *vanishing sequence* of $L$ at $p$. It is aften more convenient to deal with

$$\alpha_i^L(p) = a_i^L(p) - i\,;$$

then

$$\alpha^L(p) = (\alpha_0^L(p), \ldots, \alpha_r^L(p))$$

satisfies

$$0 \leqq \alpha_0^L(p) \leqq \ldots \leqq \alpha_r^L(p) \leqq d - r\,,$$

and $\alpha^L(p)$ is called the *ramification sequence* of $L$ at $p$.

For any *Schubert index*

$$\alpha : 0 \leqq \alpha_0 \leqq \ldots \leqq \alpha_r \leqq d - r$$

we set $w(\alpha) = \sum \alpha_i$, the weight of $\alpha$. A well known *Plücker formula* (see for example our [1983, section 1]) asserts that for any $g_d^r$ $L$ on a curve $C$ of genus $g$ we have

$$\sum_{p \in C} w(\alpha^L(p)) = (r+1)d + \binom{r+1}{2}(2g-2)\,.$$

Note that we are writing our Schubert indices in the *reverse* of the usual order. Otherwise the conventions we follow are those of, for example, Griffiths and Harris [1978]; thus associated to $\alpha$ as above is a *Schubert cycle* $\sigma_\alpha$ in the Grassmannian of $r$-planes in $\mathbb{P}^d$.

Let $K = (K_C, H^0(K_C))$ be the (complete) canonical series on $C$, so that if $C$ has genus $g$ then $K$ is a $g_{2g-2}^{g-1}$. The ramification sequence $\alpha(p) = \alpha^K(p)$ is called the *Schubert index* of $p$, and $p$ is a *Weierstrass point* if

$$\alpha(p) \neq (0, \ldots, 0)\,.$$

By Riemann-Roch one checks at once that

$$H(p) := H(\alpha(p)) := \mathbb{N} - \{\alpha_i(p) + i + 1 \mid i = 0, \ldots, g-1\}$$

is the set of orders of poles of rational functions on $C$ regular except at $p$; in particular, $H(p)$ is a semigroup, called the Weierstrass semigroup of $p$. The corresponding condition on $\alpha$ is the *semi-group condition*. The numbers in $H(p)$ are called *non-gaps* at $p$, the numbers $\alpha_i(p) + i + 1 = a_i^K(p) + 1$ are the *gaps*.

The *semigroup* and *primitivity* conditions of a Schubert index $\alpha$ were defined in the introduction. We have

**Proposition 1.1.** *A Schubert index $\alpha$ of genus $g$ is primitive if and only if twice the first non-gap of $H(\alpha)$ is > its last gap, or equivalently $\alpha_j = 0$ for $j \leqq (\alpha_{g-1} + g)/2 - 1$. In particular, any index of weight $\leqq g/2$ is primitive.*

*Proof.* If $\alpha_{k-1} = 0$, $\alpha_k \neq 0$ then the smallest non-gap in $H(\alpha)$ is $k+1$, and the last gap is $\alpha_{g-1} + g$, whence the equivalence. If $k \leqq (\alpha_{g-1} + g)/2 - 1$ then $2(k+1) \leqq \alpha_{g-1} + g$, and reducing the $\alpha_i$ suitably we may arrive at a $\beta \leqq \alpha$ with $\beta_{k-1} = 0$, $\beta_k \neq 0$ and

$2(k+1)=\beta_l+l+1$ for some $l$, so that $H(\beta)$ is not a semigroup. On the other hand, the condition $\alpha_j=0$ for $j\leqq(\alpha_{g-1}+g)/2-1$ clearly implies that $H(\alpha)$ is a semigroup, and is inherited by all $\beta\leqq\alpha$. The last assertion follows by elementary manipulation if we note that $\alpha_k\neq 0$ implies $w(\alpha)\geqq\alpha_g+g-k-1$. □

### b) *Curves of compact type and limit series*

*Curves* will always be complex, projective, reduced, and connected. A curve has compact type if its singularities are all ordinary nodes and its dual graph is a tree. If $C$ is a curve of compact type, then a collection

$$L=\{L_Y=(\mathscr{L}_Y, V_Y),\quad \text{a}\quad \mathfrak{g}^r_d\quad \text{on}\quad Y\}_{Y\text{ a component of }C}$$

will be called a *crude limit* $\mathfrak{g}^r_d$ *on* $C$ if it satisfies the compatibility conditions:
For every node $p=Y\cap Z$ of $C$

$$\alpha_i^{L_Y}(p)+\alpha_{r-i}^{L_Z}(p)\geqq d-r\,,\qquad i=0,\ldots,r\,.$$

If the inequalities are all equalities, then $L$ is said to be a *refined limit series*. Since the refined limit series on reducible curves play in many ways the role of ordinary linear series on smooth curves, we will sometimes use *limit series* as a synonym for refined limit series.

The reader will find background and details on these notions in our [1986].

## 2. Limit canonical series

### a) *Results and examples*

Let $C$ be a curve of compact type. By a *(crude) limit canonical series* on $C$ we mean any (crude) limit series $L$ of degree $2g-2$ and dimension $g-1$. Suppose that $Y$ is an irreducible component of $C$ and let $C_i$ $(i=1,\ldots,s)$ be the closures of the connected components of $C-Y$, meeting $Y$ in points $p_1,\ldots,p_s$:

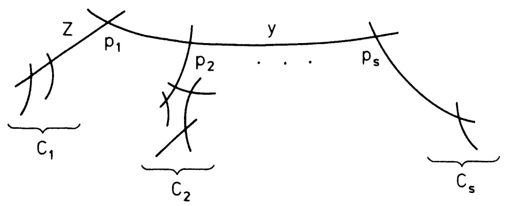



**Fig. 2.1**

We will maintain the above notation throughout this section; in particular $Y$ will always be a smooth curve of genus $g_0$, $p_1,\ldots,p_s\in Y$ will be distinct points, and

$g_1, \ldots, g_s$ will be nonnegative integers. We set

$$g = g_0 + \sum_1^s g_i.$$

To avoid trivialities we will usually suppose the $g_i$ strictly positive.

We wish to investigate the *Y-aspect* $L_Y = (\mathscr{L}_Y, V_Y)$ of an arbitrary crude limit canonical series $L$ on $C$.

Our version of Clifford's Theorem [1986, Theorem 4.1] shows that

$$\mathscr{L}_Y = K_Y\left(2 \sum_1^s g_i p_i\right),$$

and further

$$\alpha_i^{L_Y}(p_j) \begin{cases} \geqq i & \text{for } i < g_j - 1 \\ = i & \text{for } i = g_j - 1, g_j \\ \leqq i & \text{for } i > g_j. \end{cases} \tag{2.2}$$

This information goes a long way toward identifying $V_Y$:

**Proposition 2.1.** *Let $L_Y = (K_Y(2 \sum g_j p_j), V)$ be a $\mathfrak{g}_{2g-2}^{g-1}$ on Y. If*

$$\alpha_{g_j}^{L_Y}(p_j) \geqq g_j, \quad j = 1, \ldots, s$$

*then $H^0(K_Y) \subset V$ and $V$ is the sum of its subseries $V \cap H^0(K_Y(2g_j p_j))$.*

It is useful to note that the meromorphic differentials in $\bar{V} = \sum_{j=1}^s H^0 K_Y(2g_j p_j)$ $\subset H^0 K_Y\left(2 \sum_1^s g_j p_j\right)$ that appear in this result may be characterized as those whose residues at the $p_j$ are all 0: certainly they have this property since they are sums of differentials each with only one pole, and on the other hand the codimension of $\bar{V}$ in $H^0 K_Y\left(2 \sum_1^s g_j p_j\right)$ is immediately seen to be $s-1$.

We will see below that if $C$ is sufficiently generic, then the aspect $L_Y$ of a limit canonical series will actually have a $(g_j - 1)$-fold base point at $p_j$ for each $j$. Such series are unique:

**Theorem 2.2.** *There is a unique $\mathfrak{g}_{2g-2}^{g-1}$ on Y having a $(g_j - 1)$-fold base point at $p_j$ and satisfying $\alpha_{g_j}(p_j) \geqq g_j$ for each j. It is given by $L^+ = (\mathscr{L}_Y, V_Y^+)$*

$$\mathscr{L}_Y = K_Y\left(2 \sum_1^s g_j p_j\right)$$

$$V_Y^+ = \text{image}\left\{\sum_1^s H^0 K_Y((g_j + 1) p_j) \to H^0 \mathscr{L}_Y\right\}.$$

*On the other hand there is a positive dimensional family of $\mathfrak{g}_{2g-2}^{g-1}$'s on Y satisfying any weakening of the ramification conditions*

$$\alpha_i^{L^+}(p_j) \geqq \begin{cases} g_j - 1 & i \leqq g_j - 1 \\ g_j & i \geqq g_j. \end{cases} \tag{2.2}$$

*Definition.* The series $L^+$ defined in Theorem 2.2 will be called the *aresidual series* on $Y$ (with respect to the data $(\{g_j\}, \{p_j\})$; its differentials are characterized as having poles of orders $\leqq g_j+1$ and no residues at the $p_j$.

It is trivial to check that the aresidual series on the various components of $C$ fit together into a crude limit canonical series on $C$, which we will call the aresidual series on $C$, and that the aresidual series on $C$ is refined if and only if the inequalities corresponding to (2.2) for all components $Y \subset C$ are equalities. This turns out to be the generic case:

*Definition.* The datum $(Y, \{p_i\}, \{g_i\})$ is *aresidually generic* if the inequalities (2.2) on the ramification of the aresidual series are all equalities (or equivalently – by the Riemann-Roch and Clifford Theorems – if $\alpha_{g-1}^{L^+}(p_j)=g_j$ for each $j$). $C$ is called *aresidually generic* if the aresidual series on $C$ is a refined limit series.

**Proposition 2.3.** *Given nonnegative integers $g_0, \ldots, g_s$, the set of*

$$(Y, p_j) \in \mathcal{M}_{g_0,s}$$

*such that $(Y, \{p_j\}, \{g_j\})$ is aresidually generic, is open and dense. Further, if $g_0=0$, $Y=\mathbb{P}^1$, then every $(V, \{p_j\}, \{g_j\})$ is aresidually generic.*

We now come to the main result of this section:

**Theorem 2.4.** *If $C$ is aresidually generic then the aresidual series is the only (crude) limit canonical series on $C$; otherwise, $C$ has a positive dimensional family of crude limit canonical series.*

Before proving these results we give several examples, leaving the straightforward verifications to the reader.

*Example 1.* Taking $s=1$, the aresidual series on a smooth curve $Y$ with respect to a single point $p=p_1$ and any $g_1$ is the image $V$ in $H^0(K_Y(2g_1p))$ of

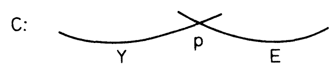
$$H^0(K_Y((g_1+1)p)).$$

If $\beta_0, \ldots, \beta_{g_0-1}$ is the ramification sequence of the canonical series of $Y$ at $p$, then the ramification sequence of $V$ at $p$ is

$$\alpha_i^V(p)=\begin{cases} g_1-1 & \text{for} \quad i<g_1 \\ g_1+\beta_{i-g_1} & \text{for} \quad g_1 \leqq i \leqq g_0+g_1-1=g-1\,. \end{cases}$$

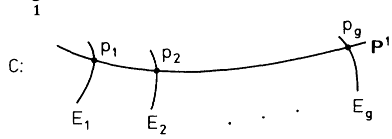
Thus independently of $g_1$ the data $Y$, $p$, $g_1$ is aresidually generic if and only if $p$ is not a Weierstrass point of $Y$.

Thus if $E$ is an elliptic curve




then the curve has a unique crude canonical limit series (which is then refined) if and only if $p$ is not a Weierstrass point of $Y$. It is not hard to see that if $p$ *is* a Weierstrass point of $Y$, then there is a positive dimensional family of refined limit series on $C$, and in some cases, to be treated below, these are all smoothable. This example is fundamental to this paper and to [ ].

*Example 2.* If $C = \mathbb{P}^1 \cup \bigcup_1^s E$




then $C$ is aresidually generic if and only if the $p_i$ are not Weierstrass points on $E_i$; in particular, this will be so when the $E_i$ are all elliptic. The aresidual series on $E_i$ is the image $|gp_i| + (g-2)p_i$ of the complete series $|gp_i|$.

*Example 3.* Taking $s=2$, $(Y, \{p_1, p_2\}, \{g_1, g_2\})$ is aresidually generic if and only if $p_2$ is not a ramification point of the complete series $K_Y((g_1+1)p_1)$, and vice versa. Thus for instance if $Y$ is an elliptic curve, our data fails to be aresidually generic if and only if $p_1 - p_2$ is torsion of order dividing $g_1$ or $g_2$.

## b) Proofs

*Proof of Proposition 2.1.* Let $H_j$ be the image of $H^0K_Y(2g_jp_j)$ in $H^0K_Y(2\sum g_jp_j)$. Because of the ramification conditions satisfied by $V$ we have

$$\begin{aligned}\dim(H_j \cap V) &= \dim V\left(-\sum_{\substack{i=1\\ i\neq j}}^{s} 2g_ip_i\right)\\ &\geqq g - \sum_{\substack{i=1\\ i\neq j}}^{s} g_i\\ &= g_0 + g_j.\end{aligned}$$

In particular, if $s=1$ we are done.

Since the $H_j$ are linearly independent modulo the image of $H^0(K_Y)$ we have

$$V/(H^0(K_Y)\cap V) = \bigoplus_1^s (H_j \cap V)/(H^0(K_Y)\cap V).$$

If $\dim H^0(K_Y) \cap V = h \leqq g_0$, then taking dimensions and using the first inequality we get

$$g - h \geqq \sum_1^s (g_0 + g_i - h) = g - h + (s-1)(g_0 - h),$$

so $H^0(K_Y) \subset V$ and equalities hold everywhere, as required. □

*Proof of Theorem 2.2.* We have

$$\dim H^0K_Y((g_j+1)p_j) = g_0 + g_j,$$

and the $H^0K_Y((g_j+1)p_j)$ are linearly independent modulo $H^0K_Y$, so $\dim V^+=g$. Furthermore, the image of $\sum_{\substack{k=1\\k\neq j}}^{s} H^0K_Y((g_k+1)p_k)$ is a subseries of $V^+$ of codimension $g_j$ whose sections vanish at $p_j$ to order $\geqq 2g_j$, so $\alpha_{g_j}^{L^+}(p_j)\geqq g_j$. Thus $L^+$ satisfies the given conditions.

Now suppose that $L'=(\mathscr{L}'_Y, V')$ is any series satisfying the given conditions. Because of the conditions $\alpha_{g_j}^{L'}(p_j)\geqq g_j$ the subseries

$$L'\left(-2\sum_1^s g_jp_j\right)$$

is a $\mathfrak{g}_{2g_0-2}^n$ for some $n\geqq g_0-1$, so it is the complete canonical series and $\mathscr{L}'_Y=K_Y\left(2\sum_1^s g_jp_j\right)$. We now get $V'=V^+$ at once from Proposition 2.1 and the basepoint condition.

Finally, to see that there is a positive dimensional family of $\mathfrak{g}_{2g-2}^{g-1}$'s satisfying any weakening of (2.2) we use the standard fact that for any $r$, $d$ every component of the family of $\mathfrak{g}_d^r$'s $L$ satisfying ramification conditions

$$\alpha_i^L(p_j)\geqq\alpha_{ij}$$

has dimension $\geqq\varrho=g-(r+1)(g-d+r)-\sum\alpha_{ij}$. This number is 0 for the conditions given in 2.2 and is strictly positive for any weaker conditions. Since such a family at least contains the aresidual series it is nonempty, and we are done. □

*Proof of Proposition 2.3*. This follows from the theory of our [1983], in which we show that if $p_1,\dots,p_s$ are general points of an arbitrary curve $Y$ of genus $g_0$, or arbitrary points of $Y=\mathbb{P}^1$, then for any $\mathfrak{g}_d^r$ $L$ on $Y$ we have

$$\sum_1^s w^L(p_j)\leqq\varrho:=g_0-(r+1)(g_0-d+r)\,. \qquad (*)$$

If $L$ is the aresidual series on $Y$ with respect to the data $\{p_j\}$, $\{g_j\}$ then

$$w^L(p_j)\geqq(g-1)g_j\,,$$

with equality if and only if

$$\alpha_i^L(p_j)=\begin{cases} g_j-1 & i\leqq g_j-1\\ g_j & i\geqq g_j\end{cases}.$$

If this inequality were strict for any $j$ then (*) would be violated. □

*Proof of Theorem 2.4*. Suppose first that the aresidual series on $C$ is not refined. Let $p$ be a node of $C$ where components $Y$ and $Z$ intersect and where one of the inequalities expressing the crude limit series compatibility conditions at $p$ is strict. Taking the aresidual series as aspects on all components of $C$ except $Y$ we see that we can construct a crude limit canonical series on $C$ by taking any aspect on $Y$ which satisfies certain ramification conditions at the nodes, and these conditions are strictly weaker than (2.2). We are now done by the last statement of Theorem 2.1.

Now suppose that $C$ is aresidually generic; we must show that the aresidual series is the only crude limit canonical series.

By Proposition 2.1 it will be enough to show that if $L$ is any limit canonical series on $C$ then for every component $Z$ of $C$ containing a node $p$ we have

$$\alpha_0^{L_Z}(p) \geqq h-1,$$

where $h$ is the genus of the connected component of $C-p$ not containing $Z$.

Adopting the notation of Fig. 2.1, and taking $p=p_1$, it is enough by the compatibility conditions for crude limit series to show that

$$\alpha_{g-1}^{L_Y}(p_1)=g_1, \qquad (*)$$

and we may assume by induction in the dual graph of $C$ that

$$\alpha_0^{L_Y}(p_j) \geqq g_j-1, \quad j=2,\ldots,s.$$

By Proposition 2.1 it follows that $L_Y(-2p_1 g_1)$ is contained in the aresidual series $L_Y^+$ series on $Y$, and since by 2.1 it has (at least) the same dimension as $L_Y^+(-2p_1 g_1)$, these coincide. Since $(Y,\{p_j\},\{g_j\})$ is aresidually generic we have

$$\alpha_{g-1}^{L_Y^+}(p_1)=g_1,$$

and (*) follows. □

## 3. Limits of Weierstrass points on aresidually generic curves

Let $C_t$ be a 1-parameter family of projective curves of genus $g$ with $C_t$ smooth for $t \neq 0$ and $C_0$ an aresidually generic curve of compact type. We will show that the limits in $C_0$ of the Weierstrass points of $C_t$ are independent of the family chosen.

**Theorem 3.1.** *The limits of the Weierstrass points of $C_t$ are smooth points of $C_0$. If $Y \subset C_0$ is a component of $C_0$ then the limits of the Weierstrass points of $C_t$ are the ramification points other than the nodes of $C_0$ on $Y$ of the aresidual series on $Y$. In particular, if the genus of $Y$ is $g_0$, then the total weight of the Weierstrass points of $C_t$ approaching points of $Y$ is*

$$g_0(g^2-1).$$

*Proof.* By Theorem 2.4 the limit on $C_0$ of the canonical series on $C_t$ is the aresidual series, which is refined because $C_0$ is aresidually generic. It follows by our [1986], Proposition 2.5, that the limits of the Weierstrass points are smooth points of $C_0$. This proves the first two statements. For the last, one computes the total weight of the ramification points of the aresidual series on $Y$ by the Plücker formula (see section 1) to be $(g_0+1)g(g-1)$ and subtracts the sum of the weights of the aresidual series at the nodes of $C$ on $Y$, which is

$$\sum_{j=1}^{s} (g-1)g_j=(g-1)(g-g_0). \quad \square$$

*Example.* With hypothesis as in the Theorem, suppose that $Y$ is an elliptic curve containing only one node $p$ of $C_0$ (an "elliptic tail"). By the theorem, precisely $g^2-1$ simple (that is, weight 1) Weierstrass points of $C_t$ approach $Y$, and their limits are the $g^2-1$ points $q \neq p$ of $Y$ with $g^2(q-p) \sim 0$ in the divisor class group of $Y$.

## 4. Limit canonical series on 2-component curves

We now consider the case of a curve $C$

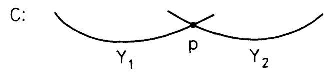



**Fig. 4.1**

with just two smooth components $Y_1$, $Y_2$ meeting transversely at a point $p$. We already know that $C$ has a unique crude limit canonical series if and only if $p$ is not a Weierstrass point on either $Y_1$ or $Y_2$, that is, if the Weierstrass weights

$$w_i := w^{K_{Y_i}}(p)$$

are both 0.

More generally, we have:

**Theorem 4.1.** *The space of crude limit canonical series on $C$ is irreducible and of dimension $w_1+w_2$. The open subset of refined limit canonical series is non-empty, and thus of the same dimension.*

*Remark.* The proof shows that the space of crude limit series is isomorphic to a product of Schubert cycles of dimensions $w_1$ and $w_2$, while the refined series are the product of open Schubert cells; thus the space of refined canonical limit series is actually smooth and rational.

*Proof.* By Proposition 2.1 and the remarks preceding it, any crude canonical limit series on $C$ has aspects of the form

$$L_{Y_1}=(K_{Y_1}(2g_2p), V_1)$$
$$L_{Y_2}=(K_{Y_2}(2g_1p), V_2)$$

and $V_i$ contains the image of $H^0(K_{Y_i})$ as the codimension $g_{3-i}$-subspace $V_i(-2g_{3-i}p)$. Thus if the Schubert index of $p$ as a Weierstrass point on $Y_1$ is $(\alpha_0^1, \ldots, \alpha_{g_1-1}^1)$ then

$$\alpha_{g_2-1}^{L_{Y_1}}(p)=g_2-1$$

and

$$\alpha^{L_{Y_1}}_{g_2+k}(p) = g_2 + \alpha^1_k, \quad k = 0, \ldots, g_1 - 1.$$

The compatibility conditions for crude limit series give

$$\left.\begin{array}{l} \alpha^{L_{Y_2}}_k(p) \geqq (g_1 - 1) - \alpha^1_{(g_1-1)-k}, \quad k = 0, \ldots, g_1 - 1 \\ \alpha^{L_{Y_2}}_{g_1}(p) \geqq g_1, \end{array}\right\} \quad (*)$$

with equalities in the refined case. But the argument is reversible, and we see that any two linear series

$$L_{Y_i} = (K_{Y_i}(2g_{2-i}p_i), V_i)$$

satisfying $V_i \supset H^0 K_{Y_i}$ and *) are the aspects of a crude limit canonical series, which is refined if and only if the inequalities (*) are all equalities.

For $i = 0, \ldots, 2g - 1$ let $f_i \subset H^0 K_{Y_1}(2g_2 p)$ be the space of sections vanishing to order $\geqq i$ at $p$. It is easy to see that $f_i$ has codimension exactly $i$, and thus that the conditions (*) and $V_i \supset H^0 K_{Y_1}$ are the conditions defining an ordinary Schubert cycle of $g$-planes in $H^0 K_{Y_1}(2g_2 p)$ whose dimension one computes to be $w_2$. Theorem 4.1 follows at once from this and the corresponding calculation on $Y_2$. □

As a special case of our fundamental smoothing result (Corollary 3.7 of [1986]) we have the first statement of the following:

**Theorem 4.2.** *If p is a dimensionally proper Weierstrass point on each of $Y_1$ and $Y_2$, then every limit canonical series on C is smoothable. On the other hand if $(Y_1, p) \in \mathscr{C}_\alpha$ is a general point and is not dimensionally proper, then the general limit canonical series on C cannot be smoothed.*

*Proof.* Since the smoothability is proved in [1986] we prove only the unsmoothability in the improper case.

Let $\pi: \tilde{\mathscr{C}} \to \tilde{B}$ be a versal deformation of the curve $C$, $B \subset \tilde{B}$ the discrimant locus, and $\mathscr{C} = \pi^{-1} B$. The space $B$ is then a finite covering of a neighborhood of the pair $((Y_1, p), (Y_2, p))$ in $\mathscr{C}_{g_1} \times \mathscr{C}_{g_2}$. Let $\alpha^i$ be the Schubert index of $p$ as a Weierstrass point on $Y_i$, with weight $w_i$. If $p$ is not dimensionally proper on $Y_1$, say, then part of $G^{g-1}_{2g-2}(\mathscr{C}/B)$ lying over $\mathscr{C}_{\alpha^1} \times \mathscr{C}_{\alpha^2}$ has dimension $\geqq \dim \mathscr{C}_{\alpha^1} + \dim \mathscr{C}_{\alpha^2} + w_1 + w_2 > \dim B$.

Since the part of $G^{g-1}_{2g-2}(\tilde{C}/\tilde{B})$ lying over the complement of the discriminant locus $B$ has dimension equal to $\dim \tilde{B} = \dim B + 1$. We see that the generic limit canonical series does not smooth. □

As an application we have:

**Corollary 4.3.** *With C as in Fig. 4.1, suppose that $(Y_1, p)$ and $(Y_2, p)$ are dimensionally proper. If p is not a Weierstrass point of $Y_2$ then the limits on $Y_1$ of Weierstrass points on smooth curves degenerating to C are precisely the ramification points of the complete series $K_{Y_1}((g_2+1)p)$. If p is a Weierstrass point of $Y_2$ then every smooth point of $Y_1$ is such a limit.*

*Proof.* By Theorem 4.2 any limit canonical series on $C$ is smoothable. From Theorem 4.1 we see that if $p$ is not a Weierstrass point of $Y_2$ then the complete series $K_{Y_1}((g_2+1)p)$ is, up to base points at $p$, the $Y_1$-aspect of any crude limit canonical series on $C$, whence the conclusion. In the opposite case, the family of $Y_1$ aspects of refined limit canonical series is a positive dimensional family of subspaces in $H^0(K_{Y_1}(2g_2p))$, while the set of subspaces having a ramification point at a given point of $Y_1$ is a hyperplane section of the corresponding Grassmannian. Since a given linear series can have only finitely many ramification points, it follows that the set of points in $Y_1$ which are ramification points of refined limit series on $C$ is dense in $Y_1$, whence the result. □

## 5. The existence of dimensionally proper Weierstrass points

We will use induction on the genus to construct dimensionally proper Weierstrass points on curves of genus $g$ from a dimensionally proper Weierstrass point on a curve $Y$ of genus $g-1$ by smoothing a refined limit series on the curve $C_0$ of Fig. 5.1.

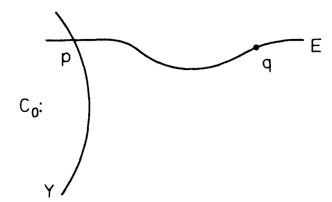



**Fig. 5.1.** $(Y,p)\in\mathscr{C}_\alpha$ dimensionally proper, $\alpha=(\alpha_0,\ldots,\alpha_{g-2})$, $E$ elliptic, $q\neq p\in E$

Using the results of the preceding sections we can describe the limit canonical series on $C_0$. We have

1) All limit canonical series on $C_0$ can be smoothed.
2) All limit canonical series have the same $Y$-aspect, namely the complete series $|K_Y(2p)|$.
3) The $E$-aspects of limit canonical series on $C_0$ are the $\mathfrak{g}^{g-1}_{2g-2}$'s contained in the complete series $\mathcal{O}_E((2g-2)p)$ which have ramification sequence

$$(g-2)-\alpha_{g-2},\ldots,(g-2)-\alpha_0,g-1$$

at $p$.

Applying Corollary 3.7 of [1986] as before, we get:

**Proposition 5.1.** *With notation above, there are dimensionally proper Weierstrass points with given Schubert index $\beta=(\beta_0,\ldots,\beta_{g-1})$ on smooth curves near $C_0$ in either of the following two cases:*

a) $\sum\beta_i=\sum\alpha_i$ *and for generic $q\in E$ there are finitely many series as in* 3) *above on $E$ with ramification index $\beta$ at $q$.*

b) $\sum \beta_i = (\sum \alpha_i) + 1$ *and for finitely many* $q \in E$ *there exist finitely many series as in* 3) *above on* $E$ *with ramification index* $\beta$ *at* $q$, *but there are no such series with ramification index* $\beta$ *at general points* $q \in E$. □

To exploit the Proposition we will look for $g^{g-1}_{2g-2}$'s contained in the complete series $\mathcal{O}_E((2g-2)p)$ which have ramification sequence

$$\alpha'_0 = (g-2) - \alpha_{g-2}, \ldots, \alpha'_{g-2} = (g-2) - \alpha_0\,, \qquad \alpha'_{g-1} = g-1$$

at $p$ and ramification

$$\beta_0, \ldots, \beta_{g-1}$$

at $q$ with

$$\beta_0 = 0 = (g-1) - \alpha'_{g-1}$$
$$\beta_i \geqq \alpha_{i-1} = (g-1) - \alpha'_{(g-1)-i} - 1\,, \qquad i \geqq 1\,,$$

and with equality holding for all $i \geqq 1$ in case a) and failing, by 1, for a unique $j$ so that

$$\beta_j = (g-1) - \alpha'_{(g-1)-j}$$

in case b). The required existence and uniqueness theorem for $g^r_d$'s on $E$ is the following, which we have stated for simplicity in terms of an arbitrary divisor $D$ instead of $(2g-2)p$, arbitrary $r$ instead of $g-1$, and vanishing sequences $a_i = \alpha'_i + i$ and $b_i = \beta_i + i$ in place of ramification sequences $\alpha'$ and $\beta$:

**Proposition 5.2.** *Let* $p, q \in E$ *be distinct points of an elliptic curve. Let* $D$ *be a divisor of degree* $d$ *on* $E$, *and let*

$$a: 0 \leqq a_0 < a_1 < \ldots < a_r \leqq d$$
$$b: 0 \leqq b_0 < b_1 < \ldots < b_r \leqq d$$

*be sequences of integers with*

$$d \geqq a_{r-i} + b_i \geqq d-1 \quad \textit{for} \quad i = 0, \ldots, r\,. \tag{*}$$

*There exists at most one* $g^r_d$ $L = (\mathcal{O}_E(D), V)$ *on* $E$ *with* $a^L(p) = a$ *and* $a^L(q) = b$; *there exists one if and only if*

$$\left.\begin{array}{ll} a_{r-i} + b_i = d & \Rightarrow a_{r-i}p + b_i q \sim D \\ \textit{and} \quad a_{r-i}p + (b_i + 1)q \sim D & \Rightarrow b_{i+1} = b_i + 1 \end{array}\right\} \tag{**}$$

*for each* $i$.

*Proof.* We first remark that if $L' = (\mathcal{O}_E(D), V')$ is any $g^r_d$ then there is at least one section on $V$ vanishing on

$$a^{L'}_{r-i}(p)p + a^{L'}_i(q)q$$

and at least a pencil vanishing on

$$a^{L'}_{r-i-1}(p)p + a^{L'}_i(q)q\,.$$

Under the hypothesis (*) there is exactly one section satisfying the first condition and exactly a pencil satisfying the second, since else we would, upon removing base points, have a $g^1_1$ or a $g^2_2$.

The necessity of condition (**) follows at once from this remark applied to a series $L$ as in the Proposition. In case $a_{r-i}+b_i=d$ the unique section vanishing on $a_{r-i}p+b_iq$ can vanish nowhere else, while in the second case the unique section vanishing on $a_{r-i}p+b_iq$ vanishes to order $b_i+1$ at $q$, so that $b_i+1$ must be among the $b_j$, and is thus $b_{i+1}$.

Now suppose that $a$ and $b$ satisfy *) and **). Let $\sigma_i \in H^0\mathcal{O}_E(D)$ be the unique section vanishing on $a_{r-i}p+b_iq$. The order of vanishing of $\sigma_i$ at $q$ is either $b_i$ or $b_i+1$. In the first case we let $V_i$ be the 1-dimensional space spanned by $\sigma_i$. In the latter case $D \sim a_{r-i}p+(b_i+1)q$, so $b_{i+1}=b_i+1$, and $\sigma_i=\sigma_{i+1}$ up to scalar multiple. In this case we let $V_i \subset H^0\mathcal{O}_E(D)$ be the space of sections vanishing on $a_{r-i-1}p+b_iq$, the sections in $V_i$ have vanishing order precisely $a_{r-i-1}$, $a_{r-i}$ at $p$ and $b_i$, $b_{i+1}$ at $q$, and $V_i \supset V_{i+1}=\langle\sigma_i\rangle=\langle\sigma_{i+1}\rangle$.

Set $V=\sum_0^r V_i$. The sections in $V$ have at least the vanishing orders $a$ at $p$ and $b$ at $q$. On the other hand each $V_i$ is either 1-dimensional or it is 2-dimensional and contains $V_{i+1}$, so $\dim V \leqq r+1$. Thus $\dim V=r+1$, and $L=(\mathcal{O}_E(D), V)$ is the required $g^r_d$.

Uniqueness follows from the fact that if $L'=(\mathcal{O}_E(D), V')$ satisfies the conditions of the proposition, then by the remark at the beginning of the proof $V'$ contains each $V_i$. □

For convenience we give the translation into the terms useful for our application:

**Corollary 5.3.** *If $0 \leqq \alpha_0 \leqq \ldots \leqq \alpha_{g-2} \leqq g-2$ is a Schubert index of genus $g-1$, and $p, q \in E$ are distinct points on an elliptic curve $E$ then there is a unique $g^{g-1}_{2g-2}$ $(\mathcal{O}_E((2g-2)p), V)$ on $E$ with ramification sequence*

$$(g-2)-\alpha_{g-2}, \ldots, (g-2)-\alpha_0, g-1$$

*at $p$ and ramification sequence*

$$0=\beta_0, \beta_1, \ldots, \beta_{g-1}$$

*at $q$ in each of the following cases*

1) *$\beta_i=\alpha_{i-1}$, $i=1,\ldots,g-1$ and the difference $p-q$ is not torsion of any order $\beta_i+i+1$ in $\operatorname{Pic}^0 E$ (in particular for generic choice of $q$).*
2) *$\beta_i=\alpha_{i-1}$ for $i=1,\ldots,g-1$ with the exception of one index $j$ for which $\beta_j=\alpha_{j-1}+1$ (in particular, if $j<g-2$ then we require $\alpha_{j-1}<\alpha_j$).*
   *The class of $p-q$ in $\operatorname{Pic}^0 E$ is killed by $\beta_j+j$; and for any $i \neq j$ such that the class of $p-q$ is killed by $\beta_i+i+1$ we have $\beta_i=\beta_{i+1}$ (and in particular $i<g-1$).*
   *In particular, this condition is satisfied if we choose $q-p$ of order exactly $\beta_j+j$ as long as either $j=g-1$ or*

$$2(\beta_j+j) > \beta_{j-1}+g.$$

From these results we deduce a numerically explicit version of 5.1:

**Theorem 5.4.** *If* $\mathscr{C}_\alpha \subset \mathscr{M}_{g-1,1}$ *contains a dimensionally proper point, then so does* $\mathscr{C}_\beta \subset \mathscr{M}_{g,1}$ *if either:*

1) $\beta_0 = 0$

$\beta_i = \alpha_{i-1} \quad (i = 1, \dots, g-1)$ *or*

2) *For some* $0 < j < g-1$

$\beta_0 = 0$

$\beta_j = \alpha_{j-1} + 1$

$\beta_i = \alpha_{i-1} \quad (i = 1, \dots, g-1, i \neq j)$

*and* $\beta$ *is a Schubert index satisfying the semigroup condition.*

*Remark.* Given that $\alpha$ must be primitive, the condition that the $\beta$ in 2) be a Schubert index satisfying the semigroup condition is equivalent to the conditions

$$\begin{cases} \alpha_j > \alpha_{j-1} \quad \text{or} \quad j = g-1 \\ \text{if} \quad \alpha_{j-1} = 0 \quad \text{then} \quad j > (\alpha_{g-2} + g - 1)/2 - 1/2, \end{cases}$$

and then $\beta$ is automatically primitive.

*Proof.* Part 1 follows at once from parts 1 of Proposition 5.1 and Corollary 5.3, and part 2 follows from parts 2 of those results together with the remark above, since $\beta$ primitive implies $2(\beta_j + j) > \beta_{g-1} + g$.

Applying this theorem we obtain Theorem 3 of the introduction:

**Corollary 5.5.** *If* $\beta$ *is a primitive Schubert index of genus* $g$, *and weight* $< g-1$, *then there exists a dimensionally proper Weierstrass point with index* $\beta$.

*Proof.* We will show that either the Schubert index

1) $\alpha_0 = \beta_1, \dots, \alpha_{g-2} = \beta_{g-1}$ or

2) $\alpha_0 = \beta_1, \dots, \alpha_{j-2} = \beta_{j-1}, \alpha_{j-1} = \beta_j - 1, \alpha_j = \beta_{j+1}, \dots, \alpha_{g-2} = \beta_{g-1}$ for some $j$ such that $\beta_j > \beta_{j-1}$

satisfies the hypothesis of the Corollary; induction, with Theorem 5.4, completes the proof.

If all $\beta_j$ are 0, then clearly 1) satisfies the hypothesis. Else let $j$ be the smallest index with $\beta_j > 0$. We claim 2) satisfies the hypothesis. Since $w(\alpha) = w(\beta) - 1$, it is enough to check that $\alpha$ is primitive.

It is easy to see that the first non-gap for $\beta$ is $j+1$. If $\beta_j = 1$, then the first nongap for the $\alpha$ of 2) is also $j+1$, and since the last gap for $\alpha$ is 1 less than that for $\beta$, we are done.

There remains the case $\beta_j \geqq 2$, in which the first nongap for $\alpha$ is $j$. Since in this case $2 \leqq \beta_j \leqq \dots \leqq \beta_{g-2}$, we have

$$g - 1 > w(\beta) \geqq \beta_{g-1} + \sum_{j}^{g-2} \beta_k \geqq \beta_{g-1} + 2(g-1-j)$$

or $2j > \beta_{g-1} + g - 1 = \alpha_{g-2} + g - 1$, which is the primitivity condition for $\alpha$. □

The techniques of Theorem 5.4 will apply to prove a result corresponding to 5.5 for some primitive indices $\alpha$ with weight $\geqq g-1$, but it cannot apply to all, or even all for which $\mathscr{C}_\alpha \neq \emptyset$, simply because the weight will be too large:

*Example 5.6.* Let $g=4k$, $\alpha=(0^{3k},(2k+1)^k)$, that is, $3k$ zeros followed by $k$ $2k+1$'s. $\alpha$ is primitive of weight $2k^2+k$, which is $>\dim \mathscr{M}_{g,1}=12k-2$ for $k \geqq 6$. For $k=1$ we have $\mathscr{C}_\alpha=\emptyset$ (the curve would be hyperelliptic), and for $k=2$ $\mathscr{C}_\alpha$ is irreducible of codimension $10=w(\alpha)$, but it appears that for larger $k$ $\mathscr{C}_\alpha$ is reducible with $\left[\frac{k-1}{2}\right]+1$ components, of codimensions $2k-4$ and $6k+1$, $6k-3, \ldots, 6k-1$ $+2\left[\frac{k-1}{2}\right]$ corresponding to trigonal curves and curves that are double covers of curves of genus $1,2,\ldots,\left[\frac{k-1}{2}\right]$ respectively.

## 6. Proofs of Theorems 1–2

Theorem 3 of the introduction, which has already been proved in the last section, is our fundamental existence theorem. Theorems 1–2 are proved by decomposing Weierstrass points already obtained. The key to this is the decomposition theory of [1987], as mentioned in the introduction.

*Proof of Theorem 1.* From the Littlewood-Richardson formula (Littlewood, 1950) one sees at once that the product of the Schubert cycles corresponding to the $\alpha^i$ is nonzero. If $\sigma_\beta$ is a Schubert cycle occuring in the product, then $w(\beta) \leqq g/2$ so $\beta$ is automatically primitive and by Theorem 3 of the introduction there exists a dimensionally proper point in $\mathscr{C}_\beta$ to which our [1987] may be applied. □

*Proof of Theorem 2.* This follows from the theory of [1987]; the point is that a $(C, p_1, \ldots, p_s) \in \mathscr{C}_{\beta^1,\ldots,\beta^s}$ which is dimensionally proper can be deformed in such a way that the $p_i$ decompose into collections of Weierstrass points in any way allowed by the Schubert conditions, as indicated in Theorem A of the introduction. If we arrange this decomposition so that $p_i$ becomes a point with index $\alpha^i$ and a number of normal Weierstrass points (ramification index $(0,\ldots,0,1)$, as we may under the given hypothesis, and then ignore these normal points, we get nearby points of $\mathscr{C}_{\alpha^1,\ldots,\alpha^t}$ as required, and by Theorem 1 a sufficiently nearby point will again be dimensionally proper.

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