Proceedings of Symposia in Pure Mathematics
Volume **46** (1987)

On Varieties of Minimal Degree
(A Centennial Account)

DAVID EISENBUD AND JOE HARRIS

ABSTRACT. This note contains a short tour through the folklore surrounding the rational normal scrolls, a general technique for finding such scrolls containing a given projective variety, and a new proof of the Del Pezzo–Bertini theorem classifying the varieties of minimal degree, which relies on a general description of the divisors on scrolls rather than on the usual enumeration of low-dimensional special cases and which works smoothly in all characteristics.

Introduction. Throughout, we work over an algebraically closed field k of arbitrary characteristic with subschemes $X \subset \mathbf{P}^r_k$. We say that X is a *variety* if it is reduced and irreducible, and that it is *nondegenerate* if it is not contained in a hyperplane. There is an elementary lower bound for the degree of such a variety:

PROPOSITION 0. *If* $X \subset \mathbf{P}^r$ *is a nondegenerate variety, then* $\deg X \geq 1 + \operatorname{codim} X$.

(PROOF. If $\operatorname{codim} X = 1$ the result is trivial. Else we project to $\mathbf{P}^{r-1}$ from a general point of X, reducing the degree by at least 1 and the codimension by 1, and are done by induction. □)

We say that $X \subset \mathbf{P}^r$ is a *variety of minimal degree* if X is nondegenerate and $\deg X = 1 + \operatorname{codim} X$. One hundred years ago Del Pezzo (1886) gave a remarkable classification for surfaces of minimal degree, and Bertini (1907) showed how to deduce a similar classification for varieties of any dimension. Of course the case of codimension 1 is trivial, X being then a quadric hypersurface, classified by its dimension and that of its singular locus. In other cases we may phrase the result as:

THEOREM 1. *If* $X \subset \mathbf{P}^r$ *is a variety of minimal degree, then* X *is a cone over a smooth such variety. If* X *is smooth and* $\operatorname{codim} X > 1$, *then* $X \subset \mathbf{P}^r$ *is either a rational normal scroll or the Veronese surface* $\mathbf{P}^2 \subset \mathbf{P}^5$.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 14J26, 14J40; Secondary 14M20, 14M99.

(See §1 for the definition and some properties of rational normal scrolls.)

The purpose of this note is to give a short and direct proof of the Del Pezzo–Bertini theorem, valid in any characteristic. The proofs (Bertini (1907), Harris (1981), and Xambò (1981)) are all essentially similar: they treat first the cases of surfaces in general (which is also done in Nagata (1960) and Griffiths-Harris (1978)), and finally they reduce the case of arbitrary varieties to the case of surfaces, distinguishing according to whether the general 2-dimensional plane section of the given variety is a scroll or the Veronese surface. Instead, we base our discussion on the following general result (§2), which is useful in many other circumstances:

THEOREM 2. *Let $X \subset \mathbf{P}^r$ be a linearly normal variety, and $D \subset X$ a divisor. If D moves in a pencil $\{D_\lambda | \lambda \in \mathbf{P}^1\}$ of linearly equivalent divisors, then writing $\overline{D}_\lambda$ for the linear span of D_λ in $\mathbf{P}^r$, the variety*

$$S = \bigcup_\lambda \overline{D}_\lambda$$

is a rational normal scroll.

This allows us (in §3) to write an arbitrary variety X of minimal degree as a divisor on a scroll, and simple considerations of the geometry of scrolls then lead to the result.

1. Description of the varieties of minimal degree. We first explain some of the terms used in Theorems 1 and 2 above:

If $L \subset \mathbf{P}^{r+s+1}$ is a linear space of dimension s, $p_L: \mathbf{P}^{r+s+1} \to \mathbf{P}^r$ is the projection from L, and X is a variety in $\mathbf{P}^r$, then the cone over X is the closure of $p_L^{-1}X$. In equations, the cone is simply given by the same equations as X, written in the appropriate subset of the coordinates on $\mathbf{P}^{r+s+1}$. Thus a *cone in $\mathbf{P}^r$ over the Veronese surface* $\mathbf{P}^2 \hookrightarrow \mathbf{P}^5$ may be defined as a variety given, with respect to suitable coordinates $x_0, \ldots, x_r$, by the (prime) ideal of 2×2 minors of the generic symmetric matrix:

$$\begin{pmatrix} X_0 & X_1 & X_2 \\ X_1 & X_3 & X_4 \\ X_2 & X_4 & X_5 \end{pmatrix}$$

(It is easy to see that a cone over any variety of minimal degree has minimal degree; our definition of rational normal scroll is such that the cone over a rational normal scroll is another rational normal scroll.)

Note that the Veronese surface contains no lines—indeed, any curve that lies on it must have even degree, as one sees by pulling back to $\mathbf{P}^2$—and thus a cone over the Veronese surface cannot contain a linear space of codimension 1. We shall see that this property separates the varieties of minimal degree which are cones over the Veronese surface from those that are scrolls.

We now describe rational normal scrolls in the terms necessary for Theorem 1. In our proof of the theorem we reduce rapidly to the case where X is a divisor on a scroll, and we shall describe these as well.

A rational normal scroll is a cone over a smooth linearly normal variety fibered over $\mathbf{P}^1$ by linear spaces; in particular, a rational normal scroll contains a pencil of linear spaces of codimension 1 (and these are the only linearly normal varieties with this property, as will follow from Proposition 2.1, below).

To be more explicit, think of $\mathbf{P}^r$ as the space of 1-quotients of k^{r+1}, so that a d-plane in $\mathbf{P}^r$ corresponds to a $d+1$-quotient of k^{r+1}. A variety $X \subset \mathbf{P}^r$ with a map $\pi: X \to \mathbf{P}^1$ whose fibers are d-planes is thus the projectivization of a rank $d+1$ vector bundle on $\mathbf{P}^1$ which is a quotient of

$$k^{r+1} \otimes_k \mathcal{O}_{\mathbf{P}^1} = \mathcal{O}_{\mathbf{P}^1}^{r+1}.$$

Slightly more generally, let

$$\mathcal{E} = \bigoplus_0^d \mathcal{O}_{\mathbf{P}^1}(a_i)$$

be a vector bundle on $\mathbf{P}^1$, and assume

$$0 \leq a_0 \leq \cdots \leq a_d, \quad \text{with } a_d > 0,$$

so that $\mathcal{E}$ is generated by $\sum a_i + d + 1$ global sections. Write $\mathbf{P}(\mathcal{E})$, or alternately $\mathbf{P}(a_0, \ldots, a_d)$, for the projectivized vector bundle

$$\mathbf{P}(\mathcal{E}) = \operatorname{Proj} \operatorname{Sym} \mathcal{E} \xrightarrow{\pi} \mathbf{P}^1$$

(whose points over $\lambda \in \mathbf{P}^1$ are quotients $\mathcal{E}_\lambda \to k(\lambda)$), and let $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$ be the tautological line bundle. Because the a_i are ≥ 0, $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$ is generated by its global sections (see the computation below) and defines a "tautological" map

$$\mathbf{P}(\mathcal{E}) \to \mathbf{P}^{\sum a_i + d}.$$

This map is birational because $a_d > 0$. We write $S(\mathcal{E})$ or $S(a_0, \ldots, a_d)$ for the image of this map, which, as we shall see, is a variety of dimension $d+1$ and degree $\sum a_i$, so that it is a variety of minimal degree. A *rational normal scroll* is simply one of the varieties $S(\mathcal{E})$. Note that $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$ induces $\mathcal{O}_{\mathbf{P}^d}(1)$ on each fiber $F \cong \mathbf{P}^d$ of $\mathbf{P}(\mathcal{E}) \to P^1$, so F is mapped isomorphically to a d-plane in $S(\mathcal{E})$.

The most familiar examples of rational normal scrolls are probably

(o) $\mathbf{P}^d$, which is $S(0, \ldots, 0, 1)$,

(i) the rational normal curve of degree a in $\mathbf{P}^a$, which is $S(a)$,

(ii) the cone over a plane conic, $S(0,2) \subset \mathbf{P}^3$,

(iii) the nonsingular quadric in $\mathbf{P}^3$, $S(1,1)$,

(iv) the projective plane blown up at one point, embedded as a surface of degree 3 in $\mathbf{P}^4$ by the series of conics in the plane passing through the point; this is $S(1,2)$.

There is a pretty geometric description of $S(a_0, \ldots, a_d)$ from which the name "scroll" derives, and from which the equivalence of the two definitions above may be deduced:

The projection

$$\mathcal{E} = \bigoplus_0^d \mathcal{O}(a_i) \to \mathcal{O}(a_i)$$

defines a section $\mathbf{P}^1 \cong \mathbf{P}(\mathcal{O}(a_i)) \hookrightarrow \mathbf{P}(\mathcal{E})$, and

$$\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)|_{\mathbf{P}(\mathcal{O}(a_i))} = \mathcal{O}_{\mathbf{P}(\mathcal{O}(a_i))}(1) = \mathcal{O}_{\mathbf{P}^1}(a_i),$$

so this section is mapped to a rational normal curve of degree a_i in the $\mathbf{P}^{a_i} \subset \mathbf{P}^{\sum a_i+d}$ corresponding to the quotient $H^0(\mathcal{E}) \to H^0(\mathcal{O}_{\mathbf{P}^1}(a_i))$. (Of course if $a_i = 0$, the "rational normal curve of degree a_i" is a point $\subset \mathbf{P}^0$!) Thus we may construct the rational normal scroll $S(a_0, \ldots, a_d) \subset \mathbf{P}^{\sum a_i+d}$ by considering the parametrized rational normal curves

$$\mathbf{P}^1 \xrightarrow{\phi_i} C_{a_i} \subset \mathbf{P}^{a_i} \subset \mathbf{P}^{\sum a_i+d}$$

corresponding to the decomposition

$$k^{\sum a_i+d+1} = \bigoplus_0^d k^{a_i+1},$$

and letting $S(a_0, \ldots, a_d)$ be the union over $\lambda \subset \mathbf{P}^1$ of the d-planes spanned by $\phi_0(\lambda), \ldots, \phi_d(\lambda)$. In particular, we see that the cone in $\mathbf{P}^{\sum a_i+d+s}$ over $S(a_0, \ldots, a_d)$ is

$$S(\underbrace{0, \ldots, 0}_{s}, a_0, \ldots, a_d).$$

Also, $S(a_0, \ldots, a_d)$ is nonsingular iff $(a_0, \ldots, a_d) = (0, \ldots, 0, 1)$ or $a_i > 0$ for all i.

We note that this description is convenient for giving the homogeneous ideal of $S(a_0, \ldots, a_d)$. As is well known, the homogeneous ideal of a rational normal curve $S(a) \subset \mathbf{P}^a$ may be written as the ideal of 2×2 minors

$$\det{}_2 \begin{pmatrix} X_0, X_1, \ldots, X_{a-1} \\ X_1, X_2, \ldots, X_a \end{pmatrix},$$

and this expression gives the parametrization sending $(s, t) \in \mathbf{P}^1$ to the point of $\mathbf{P}^a$ where the linear forms

$$sX_0 + tX_1, \ldots, sX_{a-1} + tX_a$$

all vanish. (This is s times the first row of the given matrix plus t times the second row.) It follows at once that $S(a_0, \ldots, a_d)$ is at least set-theoretically the locus where the minors of a matrix of the form

$$\left(\begin{array}{c|c|c} X_{0,0} X_{0,1}, \ldots, X_{0,a_0-1} & X_{1,0}, \ldots, X_{1,a_1-1} & \ldots X_{d,a_d-1} \\ & & \ldots \\ X_{0,1}, X_{0,2}, \ldots, X_{0,a_0} & X_{1,1}, \ldots, X_{1,a_1} & \ldots X_{d,a_d} \end{array} \right)$$

all vanish. That these minors generate the whole homogeneous ideal follows easily as in the proof of Lemma 2.1 below.

The divisor class group of a projectivized vector bundle $\mathbf{P}(\mathcal{E})$ over $\mathbf{P}^1$ is easy to describe (Hartshorne (1977), Chapter II, exc. 7.9): Writing H for a divisor in the class determined by $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$, and F for the fiber of $\mathbf{P}(\mathcal{E}) \to \mathbf{P}^1$, the divisor class group may be written (confusing divisors and their classes systematically)

$$\mathbf{Z}H + \mathbf{Z}F.$$

Moreover, the chow ring is given by

$$\mathbf{Z}[F,H]/\left(F^2, H^{d+2}, H^{d+1}F, H^{d+1} - \left(\sum a_i\right) H^d F\right).$$

We shall only need a numerical part of this, giving the degree of a scroll:

$$\text{degree } S(a_0, \ldots, a_d) = H^{d+1} = \sum_0^d a_i.$$

The simplest way to understand this is perhaps from the geometric description given above: In $\mathbf{P}^{\sum_0^d a_i + d}$ we may take a hyperplane containing the natural copy of $\mathbf{P}^{\sum_1^d a_i + d - 1}$ and meeting $C_{a_0} \subset \mathbf{P}^{a_0}$ transversely. The hyperplane section is then the union of $S(a_1, \ldots, a_d)$ with a_0 copies of F (which is embedded as a d-plane).

It is also easy to compute the cohomology of the line bundles on $\mathbf{P}(\mathcal{E})$. In particular, the tautological map

$$\pi^*\mathcal{E} \to \mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$$

induces for any integer a a map

$$\operatorname{Sym}_a \mathcal{E} = \pi_* \operatorname{Sym}_a \pi^*\mathcal{E} \to \pi_*\mathcal{O}_{\mathbf{P}\mathcal{E}}(a)$$

and thus for every a, b a map

$$\mathcal{O}_{\mathbf{P}^1}(b) \otimes \operatorname{Sym}_a \mathcal{E} \to \mathcal{O}_{\mathbf{P}^1}(b) \otimes \pi_*\mathcal{O}_{\mathbf{P}\mathcal{E}}(a) \cong \pi_*(\pi^*\mathcal{O}_{\mathbf{P}^1}(b) \otimes \mathcal{O}_{\mathbf{P}\mathcal{E}}(a)),$$

which is an isomorphism, as one easily checks locally. Since π is surjective, π_* induces an isomorphism on global sections, and we see that an element

$$\sigma \in H^0(\pi^*\mathcal{O}_{\mathbf{P}^1}(b) \otimes \mathcal{O}_{\mathbf{P}\mathcal{E}}(a))$$

may be represented as an element of

$$\begin{aligned} H^0(\mathcal{O}_{\mathbf{P}^1}(b) \otimes \operatorname{Sym}_a \mathcal{E}) &= H^0\left(\mathcal{O}_{\mathbf{P}^1}(b) \otimes \sum_{|I|=a} \mathcal{O}_{\mathbf{P}^1}\left(\sum_{i\in I} a_i\right)\right) \\ &= \sum_{|I|=a} H^0\left(\mathcal{O}_{\mathbf{P}}\left(b + \sum_{i \in I} a_i\right)\right), \end{aligned}$$

where the notation $\sum_{|I|=a}$ indicates summation over all collections I consisting of a elements (with repetitions) from $\{0, \ldots, d\}$.

From this we may derive a useful representation of divisors in $\mathbf{P}\mathcal{E}$, generalizing the idea of "bihomogeneous forms" in the case of $\mathbf{P}(\mathcal{O}_{\mathbf{P}^1}^{d+1}) = \mathbf{P}^1 \times \mathbf{P}^d$. If we let

$$x_i \in H^0(\pi^*\mathcal{O}_{\mathbf{P}^1}(-a_i) \otimes \mathcal{O}_{\mathbf{P}\mathcal{E}}(1)) = H^0\mathcal{E}(-a_i)$$

be an element corresponding to a generator of the ith summand

$$\mathcal{O}_{\mathbf{P}^1}(a_i - a_i) = \mathcal{O}_{\mathbf{P}^1} \subset \mathcal{E}(-a_i),$$

and write

$$x^I := \prod_{i\in I} x_i \in H^0\Big[(\mathrm{Sym}_a \mathcal{E})\Big(-\sum_{i\in I} a_i\Big)\Big]$$
$$= H^0\Big(\pi^*\mathcal{O}_{\mathbf{P}^1}\Big(-\sum_{i\in I} a_i\Big)\otimes \mathcal{O}_{\mathbf{P}\mathcal{E}}(a)\Big)$$

for the product, then we may represent σ conveniently as a "polynomial":

$$\sigma = \sum_{|I|=a} \alpha_I(s,t)x^I,$$

where s, t are homogeneous coordinates on $\mathbf{P}^1$ and where $\alpha_I(s,t)$ is a homogeneous form of degree

$$\deg \alpha_I(s,t) = b + \sum_{i\in|I|} a_i.$$

This representation is convenient because the "variables"

$$x_i \in H^0(\pi^*\mathcal{O}_{\mathbf{P}^1}(-a_i)\otimes\mathcal{O}_{\mathbf{P}\mathcal{E}}(1))$$

restrict to a basis of the linear forms on each fiber of $\mathbf{P}\mathcal{E} \to \mathbf{P}^1$, and the divisor D of σ meets the $\mathbf{P}^d \cong F_{(u,v)}$ over $(u,v)\in\mathbf{P}^1$ in the hypersurface with equation $\sum_{|I|=a}\alpha_I(u,v)x^I$.

In practice, we wish to use this idea on a Weil divisor X of a scroll $S(\mathcal{E})$. Since $S(\mathcal{E})$ is normal and $\mathbf{P}\mathcal{E} \to S\mathcal{E}$ is birational, we may do this by defining $\tilde{X} \subset \mathbf{P}\mathcal{E}$ to be the "strict transform" of X—that is, for an irreducible subvariety X of codimension 1, $\tilde{X}$ is the closure of the image in $\mathbf{P}\mathcal{E}$ of the complement, in X, of the fundamental locus of the inverse rational map, $S(\mathcal{E}) \to \mathbf{P}\mathcal{E}$. Then $\tilde{X}$ occupies a well-defined divisor class on $\mathbf{P}\mathcal{E}$, and we may apply the above technique to it.

2. Rational normal scrolls in the wild. The proof of Theorem 2 rests on a technique of constructing scrolls from their determinantal equations, as follows:

We say that a map of k-vector spaces

$$\phi: U\otimes V \to W$$

is *nondegenerate* if $\phi(u\otimes v)\neq 0$ whenever $u, v\neq 0$, or equivalently if each map $\phi_u: u\otimes V \to W$ is a monomorphism. The typical example, for our purposes, comes from a (reduced, irreducible) variety X and a pair of line bundles $\mathcal{L}$, $\mathcal{M}$; if $U = H^0(\mathcal{L})$, $V = H^{(}\mathcal{M})$, and $W = H^0(\mathcal{L}\otimes\mathcal{M})$, then the multiplication map is obviously nondegenerate in the above sense. In our application, X will be embedded linearly normally in $\mathbf{P}^r$ by $\mathcal{L}\otimes\mathcal{M}$, so we may identify $H^0(\mathcal{L}\otimes\mathcal{M})$ with $H^0(\mathcal{O}_{\mathbf{P}^r}(1))$.

In general, given any map

$$k^\gamma \otimes k^\delta \to H^0\mathcal{O}_{\mathbf{P}^r}(1),$$

we define an associated map of sheaves

$$A_\phi: \mathcal{O}^\delta_{\mathbf{P}^r}(-1) \to \mathcal{O}^\gamma_{\mathbf{P}^r}$$

by twisting the obvious map

$$k^{\delta} \otimes \mathcal{O}_{\mathbf{P}^r} \to k^{\gamma^*} \otimes \mathcal{O}_{\mathbf{P}^r}(1)$$

by $\mathcal{O}_{\mathbf{P}^r}(-1)$. Taking $\gamma = 2$, we have

LEMMA 2.1. *If $\phi: k^2 \otimes k^{\delta} \to H^0\mathcal{O}_{\mathbf{P}^r}(1)$ is a nondegenerate pairing, then the ideal of 2×2 minors $\det_2 A_{\phi}$ is prime, and $V(\det_2 A_{\phi})$ is a rational normal scroll of degree δ.*

PROOF. If the image of ϕ is a proper subspace of $H^0\mathcal{O}_{\mathbf{P}^r}(1)$, then $V(\det_2 A_{\phi})$ is a cone. Since the cone over a scroll is a scroll, we may by reducing r assume that ϕ is an epimorphism, so that the rank of A_{ϕ} never drops to 0 on $\mathbf{P}^r$. It follows that $\mathcal{L} = \operatorname{Coker} A_{\phi}$ is a line bundle on $S = V(\det_2 A_{\phi})$, generated by the image of $V = k^{2^*}$. The linear series $(\mathcal{L}, V)$ defines a map $\pi: S \to \mathbf{P}^1$. If $(s,t) \in \mathbf{P}^1$, then the fiber F of π over (s,t) is the scheme defined by the vanishing of the composite map

$$\mathcal{O}^{\delta}_{\mathbf{P}^r}(-1) \to \mathcal{O}^2_{\mathbf{P}^r} \xrightarrow{(s,t)} \mathcal{O}_{\mathbf{P}^r};$$

and this scheme is, by our nondegeneracy hypothesis, given by the vanishing of δ linearly independent linear forms, so F is a plane of codimension δ. By the general formula for the maximum codimension of (any component of) a determinantal variety we have $\operatorname{codim} S \le \delta - 1$, so the map $S \to \mathbf{P}^1$ is onto, and since the fibers are smooth and irreducible, and the map is proper, S is smooth and irreducible of codimension $\delta - 1$.

Since $\det_2 A_{\phi}$ thus has height $\delta - 1$ in the homogeneous coordinate ring of $\mathbf{P}^r$, it is perfect, and in particular unmixed (Arbarello et al. (1984), Chapter II, 4.1; note that the characteristic 0 hypothesis there is irrelevant). Thus $\det_2 A_{\phi}$ is the entire homogeneous ideal of S, and since $\det_2 A_{\phi}$ is perfect, S is arithmetically Cohen-Macaulay, so in particular S is linearly normal.

The fibers of π, being linear spaces in $\mathbf{P}^r$, correspond to quotients of k^{r+1}, and this defines a vector bundle on $\mathbf{P}^1$ of rank $r - \delta + 1$ such that $S \to \mathbf{P}^1$ is the associated projective space bundle; thus S is a rational normal scroll as claimed. □

REMARK. Using the same ideas, one sees that the height of $\det_2(A_{\phi})$ is $\delta - 1$ iff the rank of ϕ_u never drops by more than 1; then $X = V(\det_2 A_{\phi})$ is a "crown", that is, the union of a scroll of codimension $\delta - 1$ and some linear spaces of codimension $\delta - 1$ which intersect the scroll along linear spaces of codimension δ (fibers of π)—see Xambò (1981).

With this result in hand, it is easy to complete the proof of Theorem 2:

PROOF OF THEOREM 2. Let $k^2 \cong V \subset H^0\mathcal{O}_X(D)$ be the vector space of sections corresponding to the pencil D_{λ}, and let H be the hyperplane section of X. The natural multiplication map

$$V \otimes H^0\mathcal{O}_X(H - D) \to H^0\mathcal{O}_X(H) = H^0\mathcal{O}_{\mathbf{P}^r}(1)$$

is nondegenerate, and thus gives rise to a scroll S containing all the D_{λ}, and thus X. The linear space $\overline{D}_{\lambda}$ is the intersection of all the hyperplanes containing

D_λ, which correspond to elements of $H^0\mathcal{O}_X(H-D)$, so $\overline{D}_\lambda$ is the fiber over λ of $S \to \mathbf{P}^1$, as desired. □

EXAMPLES. (i) Let C be a hyperelliptic (or elliptic) curve, $C \subset \mathbf{P}^r$ an embedding by a complete series of degree d. C is a divisor on the variety S which is the union of the secants corresponding to the $\mathfrak{g}_2^1$ on C (or, if C is elliptic, any $\mathfrak{g}_2^1$ on C). This variety is a rational normal scroll $S(\mathcal{E})$ and $\tilde{C} \sim 2H + (d-2r+2)F$ on $\mathbf{P}(\mathcal{E})$. More generally, a linearly normal curve $C \subset \mathbf{P}^r$ which possesses a $\mathfrak{g}_d^1$ lies on a scroll of dimension $\leq d$; if $C \subset \mathbf{P}^r$ is the canonical embedding, then this scroll is of dimension $\leq d-1$, so in particular the canonical image of any trigonal curve is a divisor on a 2-dimensional scroll $S(\mathcal{E})$, and $\tilde{C} \sim 3H + (4-g)F$ on $\mathbf{P}(\mathcal{E})$. See Schreyer (1986) for a study of canonical curves using this idea.

(ii) A $K3$ surface, embedded linearly normally in any projective space, is a divisor on a 3-dimensional scroll if it contains an elliptic cubic (which then moves in a nontrivial linear series). See for example Saint-Donat (1974).

3. The classification theorem. Before giving our proof of the Del Pezzo–Bertini Theorem, we record three elementary observations about projections:

(1) If X is a variety of minimal degree, then X is linearly normal. (*Proof*: If X were the isomorphic projection of a nondegenerate variety X' in $\mathbf{P}^{r+1}$, then X' would have degree less than that allowed by Proposition 0.)

(2) If $X \subset \mathbf{P}^r$ is a variety of minimal degree and $p \in X$, then the projection $\pi_p X \subset \mathbf{P}^{r-1}$ is a variety of minimal degree, the map $X - p \to \pi_p X$ is separable, and if p is singular then X is a cone with vertex p. (*Proof*: Indeed, $\pi_p X$ is obviously nondegenerate. If X is a cone with vertex p, the result is obvious. Else $\dim \pi_p X = \dim X$ but $\deg \pi_p X \leq \deg X - 1$. The inequality must actually be an equality by Proposition 0, which shows in particular that p is a nonsingular point, and $\pi_p\colon X - p \to \pi_p X$ is birational.)

(3) If $p \in X \subset \mathbf{P}^r$ is any point on any variety, E_X the exceptional fiber of the blow-up of p in X, and $E_{\mathbf{P}^r} \cong \mathbf{P}^{r-1}$ the exceptional fiber of the blow-up of $\mathbf{P}^r$ at p, then E_X is naturally embedded in $E_{\mathbf{P}^r}$, which is mapped isomorphically to $\mathbf{P}^{r-1}$ by the map induced by π_p. Thus $E_X \subset \pi_p(X) \subset \mathbf{P}^{r-1}$. In particular, if p is a nonsingular point on X, so that E_X is a linear subspace of $\mathbf{P}^{r-1}$, then the "image of p" under $\pi_p\colon X \to \pi_p(X) \subset \mathbf{P}^{r-1}$ is a linear subspace of $\mathbf{P}^{r-1}$ which is a divisor on $\pi_p(X)$. More naively, this is the image of the tangent plane to X at p.

In view of observation (3) it will be useful to begin with the following result, which "recognizes" scrolls:

PROPOSITION 3.1. *If $X \subset \mathbf{P}^r$ is a variety of minimal degree, and X contains a linear subspace of $\mathbf{P}^r$ as a subspace of codimension 1, then X is a scroll.*

PROOF. By Proposition 2.1 it suffices to show that X contains a pencil of linear divisors, though the given subspace itself may not move.

Let $F \subset X$ be the given linear subspace. We may assume (by projecting, if necessary) that X is smooth along F. Let $H \subset \mathbf{P}^r$ be a general hyperplane containing F, and let $S = H \cap X - F$. Let π_F be projection from F. We distinguish two cases:

Case 1. $\dim \pi_F(X) \geq 2$. By Bertini's Theorem and observation (2) above, S is then a reduced and irreducible variety, of degree and dimension one less than that of X. Thus by Proposition 0, S is degenerate in H, so $F = H \cdot X - S$ moves in (at least) a pencil of linear spaces, and we are done.

Case 2. $\dim \pi_F(X) = 1$. By observation (1), $\pi_F(X)$ is a curve of minimal degree, say of degree s in $\mathbf{P}^s$. Projecting $\pi_F(X)$ from $s-1$ general points on it gives a birational map to $\mathbf{P}^1$, so $\pi_F(X) \cong \mathbf{P}^1$. Further, the cone on S with vertex p is a union of s planes, the spans of F with the points of a general hyperplane section of $\pi_F(X)$, so S has s components. But $s = r - \dim F - 1 = \operatorname{codim} X = \deg X - 1 = \deg S$, so S is the union of s planes, and these are linearly equivalent to each other since the points of $\pi_F(X)$ are. Thus a component of S is a linear space moving in a pencil as desired. □

PROOF OF THEOREM 1. Let $X \subset \mathbf{P}^r$ be a variety of minimal degree. We may assume that the codimension c of X is ≥ 2 and that X is not a cone. By Proposition 3.1 we may as well also assume that X contains no linear space of codimension 1, so that in particular the dimension d of X is ≥ 2, and we must prove that under these hypotheses X is the Veronese surface $\mathbf{P}^2 \subset \mathbf{P}^5$. In fact, it suffices to prove that $X \cong \mathbf{P}^2$; for the embedding of $\mathbf{P}^2$ by the complete series of curves of degree d gives a surface of degree d^2 and codimension

$$\binom{2+d}{2} - 3,$$

which is $< d^2 - 1$ for $d \geq 3$.

Let $p \in X$ be any point. By observation (3) and Proposition 3.1, $\pi_p(X)$ is a scroll, so the cone $S \subset \mathbf{P}^r$ with vertex p over $\pi_p(X)$ (or over X) is a scroll, say $S = S(\mathcal{E})$, with $\mathcal{E} = \bigoplus_0^d \mathcal{O}_{\mathbf{P}^1}(a_i)$ and $0 \leq a_0 \leq \cdots \leq a_d$. X is a divisor on S.

Consider the strict transform $\tilde{X} \subset \mathbf{P}(\mathcal{E})$ of X under the desingularization $\mathbf{P}(\mathcal{E}) \to S(\mathcal{E}) = S$, and let its divisor class be $aH - bF$. We will prove under the hypotheses above that $a = 2$ and X is a surface. (Along the way we will see numerically that $b = 4$, $(a_0, a_1, a_2) = (0, 1, 2)$, so $c = 3$ and $X \subset \mathbf{P}^5$ as befits the Veronese, but we will not use this directly.)

First, because the degree $c+1$ of X is 1 more than that of S, and on the other hand is $H^{d-1} \cdot (aH - bF)$, we get $b = (a-1)c - 1$.

To bound a, first note that X must meet every fiber of $\mathbf{P}\mathcal{E} \to \mathbf{P}^1$, so $aH - bF|_F = aH|_F > 0$, and $a \geq 1$. If a were 1, then $\tilde{X}$ would meet each fiber F in a linear space of dimension $d-1$. Since each fiber F is mapped isomorphically to a d-plane in $\mathbf{P}^r$ under $\mathbf{P}(\mathcal{E}) \to S(\mathcal{E})$, X would contain linear spaces of dimension $d-1$, contrary to our hypothesis. Thus $a \geq 2$.

As in §2, $\tilde{X}$ may be represented by an equation $g = 0$ with g of the form:

$$g = \sum_{|I|=a} \alpha_I(s,t) x^I,$$

with

$$\deg \alpha_I = \left(\sum_{i\in I} a_i\right) - b = \sum_{i\in I} a_i - (a-1)c + 1.$$

If the variable x_0 did not occur in g, then $\tilde{X}$ would meet each fiber F in a cone over the preimage of p, and X itself would be a cone contrary to hypothesis. But for x_0 to occur we must have

$$0 \le \deg \alpha_{0,d,\ldots,d} = a_0 + (a-1)a_d - (a-1)c + 1.$$

Since S is a cone we have $a_0 = 0$, and we derive

$$(*) \qquad a_d \ge c - 1/(a-1).$$

If x_d occurred in every nonzero term of g, then for every fiber F, $\tilde{X} \cap F$ would contain the $(d-1)$-plane $x_d = 0$, and again X would contain a $(d-1)$-plane, contradicting our hypotheses. Thus

$$(**) \qquad 0 \le \deg \alpha_{d-1,d-1,\ldots,d-1} = a a_{d-1} - (a-1)c + 1.$$

Now if $a \ge 3$, then $a_d = c$ by $(*)$; but $c = \deg X - 1 = \deg S = \sum_{i=0}^{d} a_i$, so this implies $a_{d-1} = 0$, and $(**)$ gives a contradiction. Thus $a = 2$ as claimed, and $a_d \le c - 1$. Condition $(*)$ now gives $a_d = c - 1$, so $a_{d-1} = 1$ and $a_0 = \cdots = a_{d-2} = 0$. Applying $(**)$ again we get $a_d = 1$ or $a_d = 2$.

In the first case $(a_0, \ldots, a_d) = (0, \ldots, 0, 1, 1)$, so S is a cone over $\mathbf{P}^1 \times \mathbf{P}^1 \subset \mathbf{P}^3$. A suitable hypersurface section of S will consist of the union of two planes F_1 and F_2, the cones over the rulings of $\mathbf{P}^1 \times \mathbf{P}^1$. Since each of these rulings sweeps out all of $\mathbf{P}^1 \times \mathbf{P}^1$, X must meet each of F_1 and F_2 in codimension 1. Because $c = 2$ we have $\deg X = 3$, so either $X \cap F_1$ or $X \cap F_2$ must be a linear space, contradicting our assumption on X.

We thus see that $a = 2$, $c = 3$, and $(a_0, \ldots, a_d) = (0, \ldots, 0, 1, 2)$. Under these circumstances the sum of the terms of g involving $x_0, \ldots, x_{d-2}$ may be written

$$\left(\sum_0^{d-2} \alpha_{i,d} x_i\right) x_d,$$

with $\alpha_{i,d}$ constant. Thus if $d \ge 3$ the locus $g = 0$ in each fiber F is a cone with vertex the $(d-3)$-dimensional linear space given by

$$x_d = x_{d-1} = \sum_0^{d-2} \alpha_{i,d} x_i = 0.$$

Of course S is itself a cone with $(d-2)$-dimensional vertex L, say. The $(d-2)$-dimensional subspaces of the fibers F given by $x_d = x_{d-1} = 0$ are all mapped isomorphically to L under $\mathbf{P}(\mathcal{E}) \to S$, and the restrictions of the coordinates $x_0, \ldots, x_{d-2}$ are all identified, and become coordinates on L. Thus X meets the image of each fiber in a cone with vertex given in L by $\sum_0^{d-2} \alpha_{i,d} x_i = 0$, so X is a cone, contradicting our assumption. This shows $d = 2$.

We have now shown that $a = 2$ and X is a surface. In this case, for every fiber $F \cong \mathbf{P}^2$ of $\mathbf{P}(\mathcal{E})$, $F \cap \tilde{X}$ is a conic, necessarily nonsingular since else X would contain a line. Thus $\tilde{X}$ is a rational ruled surface. But the preimage in X of p is a line, so $\tilde{X}$ is the blow-up of X at p, and is not a minimal surface. This is only possible if $\tilde{X} \cong \mathbf{P}(\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(1))$ and $X \cong \mathbf{P}^2$, as required. □

REFERENCES

1. E. Arbarello, M. Cornalba, P. A. Griffiths, and J. Harris, *Geometry of algebraic curves*, vol. I, Springer-Verlag, New York, 1984.
2. E. Bertini, *Introduzione alla geometria proiettiva degli iperspazi*, Enrico Spoerri, Pisa, 1907.
3. Del Pezzo, *Sulle superficie di ordine n immerse nello spazio di n + 1 dimensioni*, Rend. Circ. Mat. Palermo **1** 1886.
4. P. Griffiths and J. Harris, *Principles of algebraic geometry*, Wiley, New York, 1978.
5. J. Harris, *A bound on the geometric genus of projective varieties*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **8** (1981), 35–68.
6. R. Hartshorne, *Algebraic geometry*, Springer-Verlag, New York, 1977.
7. M. Nagata, *On rational surfaces*. I: *Irreducible curves of arithmetic genus* 0 *or* 1, Mem. Coll. Sci. Univ. Kyoto, Ser. A, **32** (1960), 351–370.
8. B. Saint-Donat, *Projective models of K-3 surfaces*, Amer. J. Math. **96** (1974), 602–639.
9. F. O. Schreyer, *Syzygies of curves with special pencils*, Thesis, Brandeis University (1983); Math. Ann. **275** (1986), 105–137.
10. S. Xambò, *On projective varieties of minimal degree*, Collect. Math. **32** (1981), 149–163.

BRANDEIS UNIVERSITY

BROWN UNIVERSITY