Invent. math. 85, 337–371 (1986)

Inventiones mathematicae

Limit linear series: Basic theory

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Contents

Abstract

In this paper we introduce techniques for handling the degeneration of linear series on smooth curves as the curves degenerate to a certain type of reducible curves, curves of compact type. The technically much simpler special case of 1-dimensional series was developed by Beauville [2], Knudsen [21–23], Harris and Mumford [17], in the guise of "admissible covers". It has proved very useful for studying the Moduli space of curves (the above papers and Harris [16]) and the simplest sorts of Weierstrass points (Diaz [4]). With our extended tools we are able to prove, for example, that:

1) The Moduli space M_g of curves of genus g has general type for $g \geqq 24$, and has Kodaira dimension $\geqq 1$ for $g=23$, extending and simplifying the work of Harris and Mumford [17] and Harris [16].

2) Given a Weierstrass semigroup Γ of genus g and weight $w \leqq g/2$ (and in a somewhat more general case) there exists at least one component of the subvariety of M_g of curves possessing a Weierstrass point of semigroup Γ which has the "expected" dimension $3g-2-w$ (and in particular, this set is not empty).

3) The fundamental group of the space of smooth genus g curves having distinct "ordinary" Weierstrass points acts on the Weierstrass points by monodromy as the full symmetric group.

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* Both authors are grateful to the National Science Foundation, for partial support during the preparation of this work

4) If r and d are chosen so that

$$\rho := g - (r+1)(g-d+r) = 0,$$

then the general curve of genus g has a certain finite number of g_d^r's [15, 20]. We show that the family of all these, allowing the curve to vary among general curves, is irreducible, so that the monodromy of this family acts transitively. If $r=1$, we show further that the monodromy acts as the full symmetric group.

5) If r and d are chosen so that

$$\rho = -1,$$

then the subvariety of M_g consisting of curves posessing a g_d^r has exactly one irreducible component of codimension 1.

6) For any r, g, d such that $\rho \leqq 0$, the subvariety of M_g consisting of curves possessing a g_d^r has at least one irreducible component of codimension $-\rho$ so long as

$$\rho \geqq \begin{cases} -g+r+3 & (r \text{ odd}) \\ -\dfrac{r}{r+2}g+r+3 & (r \text{ even}). \end{cases}$$

In this paper we present the basic theory of "limit linear series" necessary for proving these results. The results themselves will be taken up in our forthcoming papers [8–12]. Simpler applications, not requiring the tools developed in this paper but perhaps clarified by them, have already been given in our papers [5–7].

Introduction

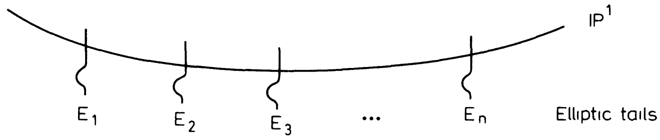
One of the most potent methods in the theory of (complex projective algebraic) curves and their linear systems since the work of Castelnuovo [3] has been degeneration to singular curves. In the work of Castelnuovo, Severi, Kleiman, Kempf, Laksov, and through Griffiths and Harris [15] the applications of this idea involved irreducible rational nodal curves:

$\mathbb{P}^1$

Some of the basic theory worked for other kinds of singular curves, in particular for irreducible curves with cusps:

$\mathbb{P}^1$

and we showed in our paper [5] that these offer substantial advantages, at least in characteristic 0. On the other hand, the existence of a nice moduli space of stable curves [24] suggests that one should consider reducible curves as well; for example, the "stable form" of the cuspidal curve above would be:



Recall that a linear series on a smooth curve Y is a pair

$$(\mathscr{L}, V)$$

where $\mathscr{L}$ is a line bundle and $V \subset H^0(Y, \mathscr{L})$ is a vectorspace of sections; if the degree of $\mathscr{L}$ is d and the dimension of V is $r+1$, then $(\mathscr{L}, V)$ is said to be a $\mathfrak{g}_d^r$ (classically a group of d points moving in a linear family of projective dimension r). The family of all $\mathfrak{g}_d^r$'s is naturally a complete projective variety, $G_d^r(Y)$, and this varies nicely in nice families of curves – see for example the forthcoming book of Arbarello et al. [1], Vol. 2.

Most problems of interest about curves are, or can be, formulated in terms of (families of) linear series. Thus, in order to use degenerations to reducible curves for studying smooth curves, it is necessary to understand what happens to linear series in the course of such a degeneration, and in particular, to understand what structure on a reducible curve plays the part of a linear series.

All the classical notions of linear series make perfectly good sense for reducible, as well as for singular irreducible curves, but they are not useful in the reducible context. To explain why, suppose that

$$\pi\colon\ X \to B$$

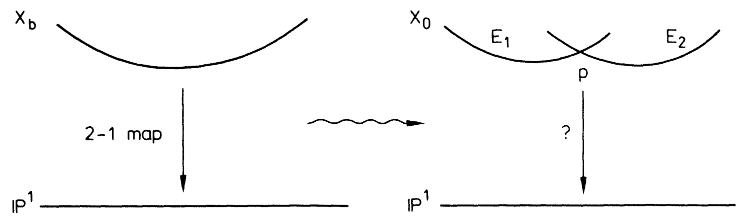
is a flat family of courves over a smooth curve (germ) B containing a point 0, with X smooth, $X_0 = \pi^{-1}(0)$ a reduced but reducible curve, and all other fibers of π smooth and irreducible. Suppose that we are given a family of $\mathfrak{g}_d^r$'s $(\mathscr{L}_b, V_b)$ for $0 \neq b \in B$ – that is, a line bundle $\mathscr{L}$ on $X - X_0$ and a subvectorbundle of rank $r+1$ of $\pi_* \mathscr{L}$ on $B - 0$. We would wish to describe a "limiting linear series" $(\mathscr{L}_0, V_0)$ on X_0, and by analyzing its geometry to say something about the geometry of the general member $(\mathscr{L}_b, V_b)$ of the family.

The difficulties are:

1. *Though limits* $(\mathscr{L}_0, V_0)$ *exist, they are not unique, and no one of them reflects the geometry of* (L_b, V_b).

Specifically, $\mathscr{L}$ may be extended to a line bundle $\tilde{\mathscr{L}}$ on all of X, and V_b will then have as limit a well-defined $(r+1)$-dimensional subspace V_0 of sections of $\mathscr{L}_0 = \tilde{\mathscr{L}}|_{X_0}$, namely the space of those admitting extensions, on a neighborhood of X_0, to sections whose restrictions to each fiber X_b lie in V_b. But if $\tilde{\mathscr{L}}$ is such an extension and D is a nontrivial divisor on X supported on X_0, then $\tilde{\mathscr{L}} \otimes_{\mathcal{O}_X} \mathcal{O}_X(D)$ is another extension, whose restriction to X_0 may for example have different degrees on each component.

To see that no one limiting $(\mathscr{L}_0, V_0)$ can reflect the geometry of $(\mathscr{L}_b, V_b)$ very satisfactorily, consider the case of the $\mathfrak{g}_2^1$'s on a curve of genus 2 as that curve degenerates to the union X_0 of two elliptic curves E_1 and E_2 meeting transversely at a point p:



In this case, the geometry that one would like to understand includes, for example, the 6 ramification points of the g^1_2 on X_b and their limits on X_0, and it is not hard to see (see for example our [8]) that in fact the 6 ramification points tend to the 6 points on X_0 which differ by 2-torsion from p on the two elliptic curves. However, if $k := \deg \mathscr{L}_0|_{E_2} \leqq 1$ then the limiting $(\mathscr{L}_0, V_0)$ has at most one section which is not identically 0 on E_2, and similarly for $k \geqq 1$ with E_1, so none of these limits can seek out all the limits of the ramification points.

A more general version of this problem is reflected in Harris and Mumford [17] where the question is raised which reducible curves X_0 are limits of smooth k-gonal curves; the answer turns out to have little to do with the presence of a g^1_k on X_0. This difficulty is exacerbated by the fact that linear series on reducible curves obey none of the usual theorems about linear series on irreducible curves. For example, there are linear series of arbitrarily high dimension but arbitrarily negative degree (the degree being the sum of the degrees of all the restrictions to the components of the curve).

2. *We know no way to tell, in interesting cases, whether a given linear series on a reducible curve occurs as the limit of linear series on smooth curves in the sense above.*

The scheme of g^r_d's $(\mathscr{L}_0, V_0)$ on X_0 is an infinite disjoint union of schemes of finite type (permuted by the group of divisor classes on X supported on X_0); but the "interesting" components have dimension typically much larger than that of $G^r_d(X_b)$ for nearby smooth X_b, so that it is difficult to tell which $(\mathscr{L}_0, V_0)$ can be "smoothed" to an $(\mathscr{L}_b, V_b)$.

In the case $r=1$ of *pencils*, these difficulties have been overcome by the theory of *admissible covers*. Essentially, the idea is to regard a g^1_d on an irreducible curve as a map to $\mathbb{P}^1$, and when the source of the map becomes reducible, to allow the $\mathbb{P}^1$ to degenerate into a reducible curve (necessarily a union of $\mathbb{P}^1$'s) too. Unfortunately, this approach has not been generalized satisfactorily to linear systems of higher dimension. This is one of the sources of the restriction to odd genus in Harris and Mumford [17], and also restricts the class of Weierstrass points that could be treated by Diaz [4] to those distinguishable by pencils.

In this paper, we will offer a notion of *limit linear series* which solves the problems above for all r, but unfortunately only a slightly restricted class of reducible curves, those of *compact type* (that is, curves which are a union of smooth curves with dual graph a tree). Roughly, a limit g^r_d on a curve of compact type X_0 is a collection of g^r_d's, one on each component of X_0, related by a compatibility condition on vanishing orders at the nodes of X_0. Given a suitable one-dimensional family X/B of curves of compact type with various marked points there will exist, after some base changes and blow-ups, in the spirit of semi-stable reduction theory

(Kempf et al. [19]), a family $G_d^r(X/B)$ whose fiber over b_0 is the family of limit $\mathfrak{g}_d^r$'s on X appropriately ramified at the marked points, so that the family $(\mathscr{L}_b, V_b)$ has a unique limit $\mathfrak{g}_d^r$ as limit, and under certain frequently occurring conditions on X_0 every limit $\mathfrak{g}_d^r$ on X_0 will be the limit of ordinary $\mathfrak{g}_d^r$'s on nearby smooth curves. Further, the geometry of nearby smooth $\mathfrak{g}_d^r$'s is rather well reflected by that of a limit $\mathfrak{g}_d^r$, as evidenced by the applications mentioned in the abstract above.

To describe briefly how a limit series arises, we return to the situation above of a generically smooth family

$$\pi: X \to B$$

of curves, with a family of linear series $(\mathscr{L}_b, V_b)$ for $b \neq 0$, and we assume in addition that X_0 is of compact type.

In this case the different limiting line bundles L_0 on X_0 are determined by the degrees of their restrictions to components of X_0, and any distribution of degrees with sum d is possible. For each component Y of X_0 we let $\mathscr{L}_Y$ be the limiting line bundle on X_0 whose degree on Y is d and whose degree on each other component is 0. We write $(\mathscr{L}_Y, V_Y)$ for the limit of the $(\mathscr{L}_b, V_b)$ corresponding to the choice $\mathscr{L}_0 = \mathscr{L}_Y$. The collection

$$L = \{(\mathscr{L}_Y, V_Y) \mid Y \text{ is a component of } X_0\}$$

is the (*crude*) *limit series* which is the limit of the $(\mathscr{L}_b, V_b)$; the individual $(\mathscr{L}_Y, V_Y)$ will be called its *aspects*. If we perform these operations after "improving" the family X/B by base changes and blow-ups so that none of the ramification points of $(\mathscr{L}_b, V_b)$ approach the nodes of X_0, then L will satisfy the compatibility conditions for a limit series – see Sects. 1–2 for precise definitions.

We now indicate the contents of this paper:

In Sect. 1, below, we give the fundamental definitions, including that of limit series.

In Sect. 2, we show how to construct the limit series which is the limit of a 1-dimensional family of linear series on a family of curves degenerating to a curve of compact type.

The goal of Sect. 3 is to give "smoothing criteria" – that is, criteria for a limit series on a curve of compact type to arise as the limit of a family of linear series on nearby smooth curves. It turns out that every limit $\mathfrak{g}_d^1$ is smoothable in this sense – a fact which is already known in the context of admissable covers [17], and for which we give a constructive analytic proof. Unfortunately this fails for $r \geqq 2$, so in the general case we must take a different approach:

We begin by constructing the scheme $G_d^r(X/B)$ of limit series associated to a suitable family of curves X/B, by giving explicit equations for it as a subscheme of a certain projective frame bundle, a space smooth over B. There are "relatively few" equations in the sense that if the special fiber X_0 is reasonably generic then $G_d^r(X/B)$ is locally a complete intersection. This allows us to give a sharp lower bound for the dimensions of components of $G_d^r(X/B)$. From this and the fact that the "boundary" of the moduli functor of curves is a union of smooth divisors, we deduce the main result of this paper, the "Smoothing Theorem": if X/B is a nice family of curves of compact type, and the dimension of $G_d^r(X/B)$ is the "expected" one, then every limit $\mathfrak{g}_d^r$ on every fiber of X/B arises as the limit of (ordinary) linear series on some nearby smooth curves. We prove a version that makes provision for prescribed ramification points, so the Theorem is of use for constructing linear series on smooth curves with special properties. For example, one can construct in this way curves with special Weierstrass points – see our paper [8]. It also allows us to compute the closures of certain divisors in moduli spaces of curves, or pointed curves. Results of this type will be crucial for the computations of monodromy groups [9, 10], and for the computation of the Kodaira dimension of the moduli space [11].

Sections 4–5 are more elementary than Sects. 2–3, and could be read after section 1.

Section 4 contains extensions to the case of limit series on curves of compact type of the fundamental "elementary" results on linear series – the theorems of Riemann and Clifford bounding r in terms of d and g, and the result of Brill-Noether-Griffiths-Harris on the dimensions of the varieties of linear series. As in the classical situation, the borderline case of Clifford's theorem, where $2g-2 \geqq d = 2r$, is particularly interesting. For example, every smooth curve of genus $g \geqq 1$ possesses a unique $\mathfrak{g}^{g-1}_{2g-2}$ – its complete canonical series. It turns out that, while a curve of compact type can have a positive dimensional family of limit $\mathfrak{g}^{g-1}_{2g-2}$'s – we call them "limit canonical series" – they all have the same underlying line bundles. This fact will be systematically exploited in our study of Weierstrass points [8].

The section closes with a discussion of the "additivity of ρ" – a manifestation of which is that a curve of compact type that is constructed by joining general curves at general points behaves as a general curve, at least from the point of view of the dimensions of its families of (limit) $\mathfrak{g}^r_d$'s. This fact will be used in the computation of the Kodaira dimensions of moduli spaces of curves and the proof of the existence of linear series with negative ρ; see the abstract above.

Finally, in Sect. 5 we compare limit series with $r=1$ with admissable covers by curves of compact type – they are not quite equivalent – and give an interpretation of a family of linear series with arbitrary dimension r degenerating to a limit series on a curve of compact type as a mapping to a family of $\mathbb{P}^r$'s with degenerate central fiber.

This paper is intended as the basis for and introduction to our series of papers [5–7] and [8–12] (although [5–7] are technically independent of it). Since it is primarily the applications which give the tools presented here their interest, we will now survey the contents of those eight papers, and their relationship to the present work.

Papers [5–7] deal essentially with the family $G^r_d(C)$ of $\mathfrak{g}^r_d$'s on a fixed general curve C. [5] contains a basic transversality result allowing one to conclude from the Schubert calculus alone the non-existence of $\mathfrak{g}^r_d$'s with certain properties on $\mathbb{P}^1$; the most significant application given there is to the embedding theorem, that on a general curve, any general $\mathfrak{g}^r_d$ with $r \geqq 3$ is very ample. We mention [6] here for its motivational value; it shows how to derive the Brill-Noether formula for the dimension of $G^r_d(C)$ in a few lines, by using reducible curves and a little conbinatorics – essentially a very special case of the transversality principle of [5] linked to a primitive notion of limit series. In [7] limit series are used in a more serious way, following Gieseker's lead, to give a proof of the Gieseker-Petri Theorem that $G^r_d(C)$ is smooth and of dimension $\rho = g-(r+1)(g-d+r)$ – the

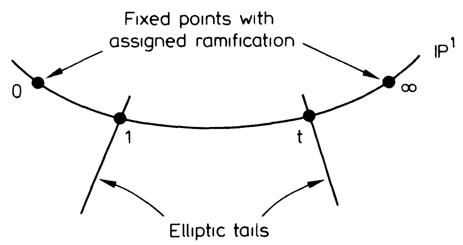
"Brill-Noether" number. [5, 6] both involve taking the limit – as a *crude* limit series in the sense of the present paper – of a family of $\mathfrak{g}^r_d$'s on a degenerating family of smooth curves.

The present paper is a watershed between the applications that do not require the Smoothing Theorem and those that do.

In [8, 9] we will apply the methods developed here to questions about Weierstrass points. In [8] we will examine the Weierstrass point locus in the moduli space of pointed curves, $M_{g,1}$, and show that for each Weierstrass semigroup Γ of weight $w \leqq g-1$, there is at least one component of the space of Weierstrass points of weight Γ having codimension w if (and only if) Γ is *primitive* – that is, the last gap is less than twice the first non-gap. In particular, this shows that every type of Weierstrass point of weight $\leqq g/2$ occurs with at least the "expected" abundance, a result which seems to have been known only in extremely restricted cases. The main method is an *induction on g* made possible by the smoothing theorem of this paper; having produced curves of genus $g-1$ with a given type of Weierstrass point, we "attach" an elliptic curve through that Weierstrass point, and then smooth a certain limit canonical series on the union to produce a family of more complex Weierstrass points. The paper also contains a discussion of the possible limit canonical series, and their uniqueness, and ends with a discussion of the ways in which two Weierstrass points can slide together on a family of smooth curves to form a more complex one, a phenomenon which turns out to be governed by simple Schubert calculus in a wide range of cases.

In [9] we make use of the Smoothing Theorem and of the identification of limit canonical series again to show that the monodromy group of the Weierstrass points on a general curve is the full symmetric group. The fundamental new technique is to *fix* a reducible curve with *many* limit canonical series, and to study the monodromy of the Weierstrass points (ramifications points of limit canonical series) as the limit canonical series *itself* varies on the fixed curve (an idea which would be the height of absurdity for irreducible curves).

As we have mentioned, $G^r_d(C)$ is smooth and of dimension ρ for $\rho \geqq 0$, and Fulton and Lazarsfeld [13] have shown that it is connected, and thus irreducible, for $\rho > 0$. In [10] we demonstrate the analogue of this result for $\rho = 0$; that is, we show that monodromy acts transitively on $G^r_d(C)$ as C varies among general curves. It seems likely that the monodromy acts in fact as the full symmetric group, and we prove this in the case $r = 1$. This involves some combinatorics ("the lattice of Schubert cycles is connected in codimension 1") and an intricate study of the ramification of the family of G^r_d's over a particularly simple 1-dimensional family of curves:



This study of ramification also plays a role in the central paper of this series, [11], in which we show that M_g is of general type for all $g \geqq 24$ (and of Kodaira dimension $\geqq 1$ for $g = 23$), improving and completing the results of Harris and Mumford [17] and Harris [16]. In the first half of Harris and Mumford [17], it is shown that to prove results such as these it is enough to exhibit effective divisors in M_g whose coefficients, when written as a linear combination of the standard divisor classes $\lambda, \delta_0, \ldots, \delta_{[g/2]}$, satisfy a certain inequality. Using our techniques, it is quite easy to show that if $\rho = g - (r+1)\ (g-d+r) = -1$ then the locus of curves possessing a $\mathfrak{g}_d^r$ contains a divisor with the desired properties; this gives an easy proof of the above result for those g such that $g+1$ is not prime.

To deal with the other values, $g = 28, 36, \ldots$ which are of course all even, we choose $r = 1$, $d = (g/2) - 1$, so that $\rho = 0$, and we show that the branch locus of the family of all $\mathfrak{g}_d^r$'s over $\bar{M}_g$ contains a divisor of the desired sort, in accordance with the program suggested in Harris [16]. This is where we must use the information on ramification developed in [10].

Finally, we remark on two further applications of our technique, which have not yet been written up: The computations described above for $\rho = -1$ suggest that the locus of curves in M_g with a $\mathfrak{g}_d^r$, where $g - (r+1)\ (g-d+r) = -1$, should be interesting. We have proved that its divisorial part is irreducible – that is, it consists of an irreducible family of curves of dimension $3g-4$ and (possibly) some components of lower dimension.

Most of the interesting curves in projective space are *not* general, and are in fact embedded by $\mathfrak{g}_d^r$'s with $\rho = g - (r+1)(g-d+r)$ quite negative. The loci of curves in M_g possessing linear series with given negative ρ is thus of interest, but very little is known. It is natural to conjecture that, at least for not-too negative ρ, these loci are non-empty and have codimension $-\rho$. We have proved that they have, at least, a component of exactly this codimension, as long as ρ satisfies

$$\rho \geqq \begin{cases} -g+r+3 & (r \text{ odd}) \\ -\dfrac{r}{r+2}\, g + r + 3 & (r \text{ even}). \end{cases}$$

These last results are very close to results of Sernesi [25] who proves that there is a component of codimension $-\rho$ of the universal family $G_d^r \to M_g$ of $\mathfrak{g}_d^r$'s whose general member defines a birational map to $\mathbf{P}^r$ as long as

$$g - r \geqq d \geqq \frac{r-1}{r}\, g + r + \frac{1}{r};$$

our result gives the existence of a component of G_d^r of codimension $-\rho$ as long as

$$d \geqq \begin{cases} \dfrac{r-1}{r+1}\, g + r + 1 + \dfrac{2}{r+1} & (r \text{ even}) \\ \dfrac{r}{r+2}\, g + r + 1 + \dfrac{2}{r+2} & (r \text{ odd}) . \end{cases}$$

0. Preliminaries

We briefly review some basic facts and definitions:

Curves in this paper will be reduced, projective curves over C with only ordinary nodes as singularities except when the contrary is explicitly stated. If two curves on a smooth surface meet in a point p, we will write $(Y, Z)_p$ for the local intersection multiplicity at p.

The *Dual Graph* of a curve X has one vertex for each irreducible component Y of X and an edge incident to that vertex for every node of X lying on Y.

We say that the curve X has *compact type* if its dual graph is a tree. This condition is well-known to be equivalent to the condition that the Jacobian $\mathrm{Pic}^0(X)$, the set of line bundles whose restriction to every component of X has degree 0, is compact.

Now let Y be a smooth curve. A $\mathfrak{g}^r_d$ on Y is a linear series L of (projective) dimension r and degree d; that is, $L = (\mathscr{L}, V)$ with $\mathscr{L}$ a degree d line bundle on Y and $V \subset H^0(Y, \mathscr{L})$ an $(r+1)$-dimensional vectorspace of sections.

If $p \in Y$ and σ is a section of $\mathscr{L}$ then we write $\mathrm{ord}_p \sigma$ for the order of vanishing of σ at p.

There are exactly $r+1$ distinct integers

$$a_0^L(p) < a_1^L(p) < \ldots < a_r^L(p)$$

which are orders of vanishing of sections in V at p. The sequence of integers $(a_0^L(p), \ldots, a_r^L(p))$ is called the *vanishing sequence* L at p, and the sequence $(\alpha_0^L(p), \ldots, \alpha_r^L(p))$ with

$$\alpha_i^L(p) := a_i^L(p) - i$$

is called the *ramification sequence* of L at p. The *weight* $w^L(p)$ of p with respect to L is defined by

$$w^L(p) := \sum_0^r \alpha_i^L(p) = \sum_0^r a_i^L(p) - \binom{r+1}{2};$$

as is well-known, it is the "expected codimension" of the pairs L, p with given ramification sequence. We say that L is *unramified* at p *if* $\alpha_0^L(p) = \ldots = \alpha_r^L(p) = 0$, else that p is a ramification point of L. *Weierstrass points* of Y are of course the ramification points of the *canonical series* $(K_Y, H^0(K_Y))$. There are only finitely many ramification points of L on Y, and we have the "*Plücker formula*"

$$\sum_{p \in Y} w^L(p) = (r+1)d + \binom{r+1}{2}(2g-2),$$

which generalizes both the Hurwitz formula ($r=1$) and the formula for the sum of the weights of (higher) Weierstrass points; see for example our paper [5] for a derivation.

1. Limit series

We now come to the central definitions:

If X is a curve of compact type, then a *crude limit* $\mathfrak{g}^r_d$ (or simply *crude limit series*) L on X is:

For each irreducible component Y of X a $\mathfrak{g}^r_d$ $L_Y=(\mathscr{L}_Y,V_Y)$ on Y, called the Y-aspect of L, satisfying the *Compatibility Condition*: If Y and Z are components of X meeting in a point p, then for $i=0,\ldots,r$ we have

$$a_i^{L_Y}(p)+a_{r-i}^{L_Z}(p)\geqq \mathrm{d}\,.$$

A *refined limit* series is a crude limit series for which all the inequalities in the definition are equalities. We shall see that the refined limit series on curves of compact type play the role of ordinary linear series on smooth curves. For this reason we will also use the term *limit series* for refined limit series. We conserve the term *refined* for emphasis.

If $p\in X$ is a smooth point contained in a component Y, and L is a crude limit series on X, we define the ramification and vanishing sequences of L at p to be those of L_Y.

The following shows that L is a limit series if it satisfies the Plücker formula given in the previous section:

Proposition 1.1. *Let X be a genus g curve of compact type. If*

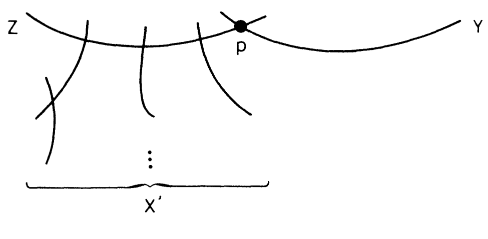
$$L=\{(\mathscr{L}_Y,V_Y)\mid Y \text{ a component of } X\}$$

is a crude limit $\mathfrak{G}^r_d$ on X, then

$$\sum_{\substack{p\text{ a smooth}\\ \text{point of }X}} w^L(p)\leqq (r+1)\,d+\binom{r+1}{2}(2g-2),$$

and equality holds if and only if L is a limit series.

Proof. If X is smooth, this is the Plücker formula as in our [5]. Else we do induction on the number of components of X, and write $X=Y\cup X'$, where X' is a curve of genus g' having fewer components and Y is smooth of genus $g-g'$, and meets a unique component Z of X'. Set $p=Y\cap X'$.



The restriction L' of L to X' is a crude limit series, and

$$w^{L_Y}(p)\geqq \sum_i \left(d-a_{r-i}^{L_Z}(p)-i\right),$$

with equality if and only if L satisfies the limit series condition at p. The desired result now follows from the equalities

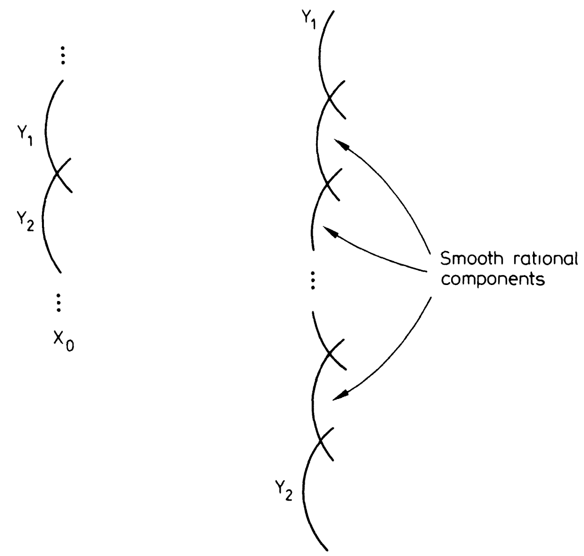
$$\sum(a_i^{L'}(p)-i)+\sum((d-a_i^{L'}(p))-i)=(d-r)(r+1)$$
$$=(r+1)d+\binom{r+1}{2}(2g'-2)$$
$$+(r+1)d+\binom{r+1}{2}(2(g-g')-2)$$
$$-\left[(r+1)d+\binom{r+1}{2}(2g-2)\right]. \quad \square$$

2. Limits of linear series

Throughout this section we will work with a family π: $X \to B$ of varieties satisfying the following conditions:

(2.0) X is a nonsingular surface, projective over B.
B is the spectrum of a discrete valuation ring $\mathcal{O}$, complete or Henselian according to taste, with parameter t.
The residue class field of $\mathcal{O}$ is algebraically closed and of characteristic 0. We write 0, η and $\bar{\eta}$ for the special, generic, and geometric generic point of B respectively.
X_0, the special fiber, is reduced and of compact type.
$X_{\bar{\eta}}$, the geometric generic fiber, is smooth and irreducible.

Let $(\mathscr{L}_{\bar{\eta}}, V_{\bar{\eta}})$ be a $\mathfrak{g}_d^r$ on $X_{\bar{\eta}}$. Our goal in this section is to show how $(\mathscr{L}_{\bar{\eta}}, V_{\bar{\eta}})$ gives rise to a limit series on a curve which is derived from X_0 by (possibly) replacing nodes of X_0 by chains of smooth rational curves, and which is thus semi-stably equivalent to X_0, and has again compact type:



Analytically, $(\mathscr{L}_{\bar{\eta}}, V_{\bar{\eta}})$ corresponds to a curve in the space of all $\mathfrak{g}_d^r$'s on fibers of $X \to B$ near 0. We first "rationalize" so that this curve becomes a section of $X \to B$: Let K be a finite extension of $k(\eta)$ such that $(\mathscr{L}_{\bar{\eta}}, V_{\bar{\eta}})$ is defined over K, and let B' be the normalization of B in Spec K. If $[K: k(\eta)] = n$ then the surface $X \times_B B'$ will have a singularity of type A_{n-1} (local equation $xy - t^n$) over each node of X_0. If $X' \to X \times_B B'$ is the minimal resolution of these singularities, obtained by repeatedly blowing up the singular points, then the family $X' \to B'$ satisfies all the hypotheses above, the special fiber X_0' is obtained from X_0 by inserting a chain of $n-1$ smooth rational curves at each node of X_0, and $\mathscr{L}_{\bar{\eta}}$ is defined over the generic point η' of B'.

Changing our notation, we now may assume that $(\mathscr{L}_{\bar{\eta}}, V_{\bar{\eta}})$ comes from a $\mathfrak{g}_d^r(\mathscr{L}_\eta, V_\eta)$ on X_η. This gives rise to a $\mathfrak{g}_d^r$ on each component Y of X_0 as follows: Since X is smooth, $\mathscr{L}_\eta$ extends to a line bundle on X, unique up to a line bundle of the form $\mathcal{O}_X(D)$, where D is supported on X_0. Since the intersection pairing on the components of X_0 is unimodular modulo the class of X_0, there is an extension $\mathscr{L}_Y$ of $\mathscr{L}_\eta$ to X such that for every component Z of X_0 other than Y,

$$\deg(\mathscr{L}_Y|_Z) = 0 .$$

Since $\mathcal{O}_X(X_0) \cong \mathcal{O}_X$, the extension $\mathscr{L}_Y$ is unique up to isomorphism, and the restriction map identifies $\mathscr{L}_Y$ with an $\mathcal{O}$-subsheaf of $\mathscr{L}_\eta$ unique up to a power of the parameter t. Of course the degree of $\mathscr{L}_Y$ on the fibers of $X \to B$ is constant, so $\deg(\mathscr{L}_Y|_Y) = d$.

For each Y, $\pi_* \mathscr{L}_Y$ is a free $\mathcal{O}$-module, and the restriction map makes $\pi_* \mathscr{L}_Y$ into an $\mathcal{O}$-lattice in the $k(\eta)$-vectorspace $\pi_* \mathscr{L}_\eta$ in a way which is unique up to a power of t. Set

$$\tilde{V}_Y = V_\eta \cap \pi_* \mathscr{L}_Y$$
$$V_Y = \tilde{V}_Y \otimes k(0),$$

so that $\tilde{V}_Y$ is a free $\mathcal{O}$-module of rank $r+1$, and

$$V_Y \cong \pi_*(\mathscr{L}_Y) \otimes k(0) \subseteq H^0(\mathscr{L}_Y|_{X_0})$$

is a vectorspace of rank $r+1$. Since $\deg(\mathscr{L}_Y|_Z) = 0$ for components Z of X_0 other than Y, and X_0 is connected, sections of $\mathscr{L}_Y|_{X_0}$ are determined by their values on Y. Thus we may (and will) identify V_Y with its image in $H^0(\mathscr{L}_Y|_Y)$, and

$$(\mathscr{L}_Y|_Y, V_Y)$$

is a $\mathfrak{g}_d^r$ on Y.

Definition. For any irreducible component Y of X_0, we call the linear series $(\mathscr{L}_Y|_Y, V_Y)$ the *Y-aspect* of the linear series $(\mathscr{L}_\eta, V_\eta)$ (with respect to the given family) and we say that the collection of aspects

$$L = \{(\mathscr{L}_Y|_Y, V_Y) \mid Y \text{ a component of } X_0\}$$

is the *limit* of $(\mathscr{L}_\eta, V_\eta)$.

Proposition 2.1. *L is a crude limit series.*

To prove this we need some results from our [7], the first in a sharper form than it appeared there.

For $\sigma \in V_\eta$ let $\tilde{\sigma}^Y = t^n \sigma$ where n is the least integer with $t^n \sigma \in \tilde{V}_Y$, and let σ^Y be the image of $\tilde{\sigma}^Y$ in V_Y. Write D_σ for the closure in X of the divisor (σ) in $X - X_0$.

Proposition 2.2. If Y and Z are components of X_0 meeting in a point p

and $\sigma \in V_\eta$ then with $D = D_\sigma$ we have:

$$\begin{aligned} \operatorname{ord}_p \sigma^Y + \operatorname{ord}_p \sigma^Z &= d + (D.Y)_p + (D.Z)_p \\ &\geqq d. \end{aligned}$$

Proof. Suppose that $\tilde{\sigma}^Y$ vanishes along Z to order a and $\tilde{\sigma}^Z$ vanishes along Y to order b. Clearly

$$\operatorname{ord}_p \sigma^Y = (Y \cdot \{\tilde{\sigma}^Y = 0\})_p = a + (D \cdot Y)_p$$

and similarly for Z, so it suffices to prove that $a + b = d$.

Let Y' and Z' be the connected components of $X_0 - p$ containing Y and Z respectively. Checking intersection numbers we see at once that $\mathscr{L}_Y(-dZ') \cong \mathscr{L}_Z$.

Now for any c, $\mathscr{L}_Y(-cZ')$ has degree 0 on components of Z' other than Z, so any section of $\mathscr{L}_Y(-cZ')$ vanishing on Z vanishes on all of Z'. It follows, since $\tilde{\sigma}^Y$ vanishes on Z to order a, that $\tilde{\sigma}^Y \in \mathscr{L}_Y(-aZ')$, and

$$\tilde{\sigma}^Z = t^{d-a} \tilde{\sigma}^Y$$

as sections of $\mathscr{L}_Z = \mathscr{L}_Y(-dZ') \cong \mathscr{L}_Y$. Similarly, $t^{d-b} \tilde{\sigma}^Z = t^{(d-a)+(d-b)} \tilde{\sigma}^Y$ is a section of $\mathscr{L}_Z(-dY') = \mathscr{L}_Y(-dX_0) \cong \mathscr{L}_Y$ that does not vanish on Y. Since $t^d \tilde{\sigma}^Y$ also has this property, we see that $(d-a) + (d-b) = d$, whence $a + b = d$ as desired. □

We use this in conjunction with the following lemmas from our [7]:

Lemma 2.3 (Adapted bases). *If Y and Z are irreducible components of X_0, and q is any point of Y,*

then there exists a basis $\{\sigma_i\}_{i=0,\ldots,r}$ of V such that

1) *$\{\tilde{\sigma}_i^Y\}$ is a basis of $\pi_* \mathscr{L}_Y$*
2) *$\{\tilde{\sigma}_i^Z\}$ is a basis of $\pi_* \mathscr{L}_Z$*
3) *The $r+1$ numbers $\{\operatorname{ord}_q \sigma_i^Y\}$ are strictly increasing with i; that is $\{\operatorname{ord}_q \sigma_i^Y\}$ is the vanishing sequence of $(\mathscr{L}_Y, V_Y)$ at q.*

We say that a basis $\{\sigma_i\}$ as above is *adapted* to $q \in Y$ and Z.

Lemma 2.4. *Suppose that $a_0 < \ldots < a_r$ are integers. If $\{b_i\}$ are distinct integers with $a_i \leqq b_i$ for each i, and σ is a permutation so that $b_{\sigma(0)} < \ldots < b_{\sigma(r)}$, then $a_i \leqq b_{\sigma(i)}$ for each i. Further, if $a_j = b_{\sigma(j)}$ for some j, then $\sigma(j) = j$, so that $a_j = b_j$.*

Proof of Proposition 2.1. Let $p = Y \cap Z$ be a node of X_0, and let $\{\sigma_i\}$ be a basis of V_η adapted to $p \in Y$ and Z. The sequence $(\operatorname{ord}_p \sigma_i^Z)$ is by Proposition 2.2 termwise $\geqq$

the decreasing sequence $(d - \mathrm{ord}_p \sigma_i^Y)$. Since the vanishing sequence of $\mathscr{L}_Z|_Z$ at p is termwise $\geqq$ a rearrangement of the first sequence, lemma 2.4 shows that the $(r-i)^{\text{th}}$ term of this vanishing sequence is $\geqq d - \mathrm{ord}_p \sigma_i^Y$ as required. □

We will show that the family X can be modified so that L becomes a limit series. For this we use condition 3) of the following characterization; condition 2) will be applied elsewhere.

Proposition 2.5. *The following are equivalent for the limit L of the series* $(\mathscr{L}_\eta, V_\eta)$:

1) *L is a limit series.*

2) *For each node* $p = Y \cap Z$ *of* X_0, *each free basis* $\tilde{\sigma}_i^Y$ *of* $\tilde{V}_Y$ *adapted to* $p \in Y$ *and Z, and each* σ *in the* $k(0)$-span *of the* $\tilde{\sigma}_i^Y$, *we have* $\mathrm{ord}_p \sigma^Y + \mathrm{ord}_p \sigma^Z = d$.

3) *No ramification point of* $(\mathscr{L}_\eta, V_\eta)$ *specializes to a node of* X_0.

Remark. If L is refined then it follows from 2) that if $\{\tilde{\sigma}_i^Y\}$ is a basis adapted to $p \in Y$ and Z then, reversing the order, $\{\tilde{\sigma}_{r-i}^Y\}$ is a basis adapted to $p \in Z$ and Y.

Proof. 2) ⇒ 1) is immediate. 1) ⇒ 2): Assume L is a limit series. Of course by virtue of Proposition 2.1 and Lemma 2.4

$$\mathrm{ord}_p \sigma_i^Y + \mathrm{ord}_p \sigma_i^Z = d$$

for all i. If $\mathrm{ord}_p \sigma^Y = \mathrm{ord}_p \sigma_i^Y$, then for suitable $0 \neq s_i$ and $s_j \in k(0)$

$$\sigma^Y = s_i \sigma_i^Y + \sum_{j>i} s_j \sigma_j^Y .$$

Since σ is in the $k(0)$ span of the $\tilde{\sigma}_i^Y$ we have

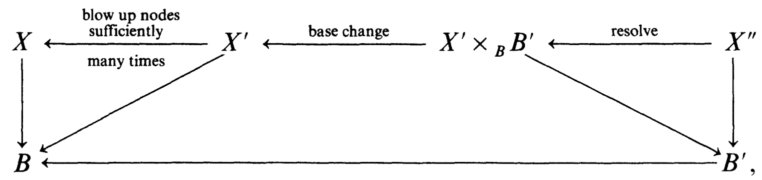
$$\sigma = s_i \tilde{\sigma}_i^Y + \sum_{j>i} s_j \tilde{\sigma}_j^Y .$$

It follows that $\sigma^Z = \sigma_i^Z$, whence the desired conclusion.

To prove the equivalence of 1) and 3), note that if a ramification point of weight w of $(\mathscr{L}_\eta, V_\eta)$ specializes to a smooth point $p \in Y$ of X_0, then $w^{L_Y}(p) \geqq w$. The result now follows at once from Proposition 1.1. □

We now come to the main result of this section:

Theorem 2.6. *Let* $(\mathscr{L}_\eta, V_\eta)$ *be a line bundle on* X_η. *After blowing up the nodes of* X_0 *sufficiently often, making a finite base change, and resolving the resulting singularities of X,*



we obtain a family $X'' \to B'$ *satisfying conditions* (2.0), *having generic fiber* $X''_\eta = X_\eta$, *special fiber* X''_0 *derived from* X_0 *by inserting chains of rational curves at the nodes of* X_0, *and such that the limit of* $(\mathscr{L}_\eta, V_\eta)$ *on* X''_0 *is a limit series.*

Proof. Let $D \in X_\eta$ be the reduced divisor of ramification points of $(\mathscr{L}_\eta, V_\eta)$, and let $\bar{D}$ be its closure in X. Since $\bar{D} \cup X_0$ is a reduced curve in X with singularities only along X_0 we may blow up points of X_0 in X, to get a surface X', such that the preimage of $D \cup X_0$ is a divisor with normal crossings – in particular such that no ramification point on X_η approaches a node in $X'_0 \cong X'$. We would be done by condition 3 of Proposition 2.5 except that the special fibre X'_0 of X' is not reduced.

Once having produced X' as above, the existence of $X'' \to B'$, obtained by base change and resolution, satisfying conditions 2.0 except for the condition that X_0 is of compact type, is the conclusion of the "semi-stable reduction theorem" [19]. On the other hand, since stable limits are unique, X''_0 is obtained from X_0 by replacing nodes by chains of rational curves and/or "blowing down" such chains, and thus X''_0 is of compact type. Since the process of obtaining X''_0 involved no blowing down, we see that the components of X_0 all survive in X'_0, whence our assertion. □

Remark. As Joe Lipman pointed out to us, in this case one can follow the stable reduction process quite easily. After blowing up X_0 many times at nodes, the family $X' \to B$ has the property that adjacent components appear in X'_0 with relatively prime multiplicities. If we now make a base change of degree the product of all the multiplicities involved, we will obtain a surface $X' \times_B B'$ whose singularities are locally of the form

$$s^{abc} - x^a y^b, \qquad (a, b) = 1,$$

where s is a parameter on B'. If we write $a'a + b'b = 1$ for suitable a', b', then

$$x' = x^{b'} s^{aa'c} / y^{a'}$$

and

$$y' = y^{a'} s^{bb'c} / x^{b'}$$

satisfy

$$x'^b = x$$

$$y'^a = y$$

$$s^c = x'y'.$$

Thus the normal singularity $s^c - x'y'$ is the normalization of $s^{abc} - x^a y^b$; as is well known its resolution has exceptional divisor a chain of $c-1$ rational curves which appear in the fiber $s = 0$ with multiplicity 1.

3. Equations and deformations

Given a reducible curve X_0 of compact type, and a limit $\mathfrak{g}^r_d$ on it, we will give in this section local equations which define the variety of limit $\mathfrak{g}^r_d$'s on reducible curves near the given one, and ordinary $\mathfrak{g}^r_d$'s on irreducible curves near the given one. We will show by dimension theory that in many cases all the limit $\mathfrak{g}^r_d$'s on X_0 can be deformed onto nearby irreducible curves.

We will do the same for $\mathfrak{g}^r_d$'s with assigned ramification, an easy generalization important for the applications, and indeed for the proof.

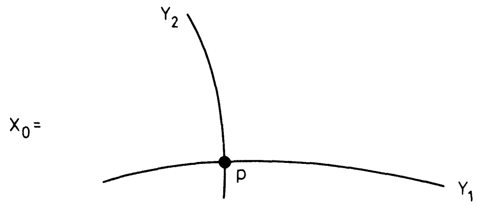
Definition. If X_0 is a curve of compact type, then we say that a limit $\mathfrak{g}^r_d$ on X_0 *can be smoothed* if there is a family X/B with central fiber X_0 as in (2.0), and a $\mathfrak{g}^r_d$ $(\mathscr{L}_\eta, V_\eta)$ on the general fiber X whose limit is the given limit $\mathfrak{g}^r_d$ on X_0.

Our original hope was to prove that every refined limit $\mathfrak{g}^r_d$ could be smoothed. This is true for $r=1$, but false in general, as the next proposition and example show:

Proposition 3.1. *Every limit $\mathfrak{g}^1_d$ can be smoothed, and the smoothing may be done so as to preserve all ramification points away from the nodes.*

Remark. By section 5a), below, Proposition 3.1 is subsumed in the corresponding assertion for admissible covers. This assertion, at least for the first statement of 3.1, is made for admissible covers on p. 61 of Harris and Mumford [17] and supported by a reference to obstruction theory; the second statement could be proved in the same way. However, a constructive analytic proof is so easy that it seems worth giving:

Proof. For simplicity, we treat only the case where X_0 has just 2 components, Y_1 and Y_2, meeting at p:



Let $(\mathscr{L}_i, V_i)$ be the Y_i-aspect of the given limit $\mathfrak{g}^1_d$, and let $\sigma^i_0, \sigma^i_1 \in V_i$ be bases whose orders of vanishing

$$a^i_j = \operatorname{ord}_p \sigma^i_j$$

satisfy

$$a^2_{1-j} = d - a^1_j \qquad (j=0,1).$$

Choose local analytic parameters x_i on disks $D_i \subset Y_i$ centered at p such that

$$\sigma^i_1/\sigma^i_0 = x_i^{a^i_1 - a^i_0}$$

on D_i, and set

$$e_i = \sigma^i_j / x_i^{a^i_j} \in H^0(D_i, \mathscr{L}_i|_{D_i}),$$

which is independent of j.

Let D be a small disk in $\mathbb{C}$ with coordinate t and let $Z \subset \mathbb{C}^2$ be the product of two small disks with coordinates z_1 and z_2, regarded as a family over D by $t = z_1 z_2$.

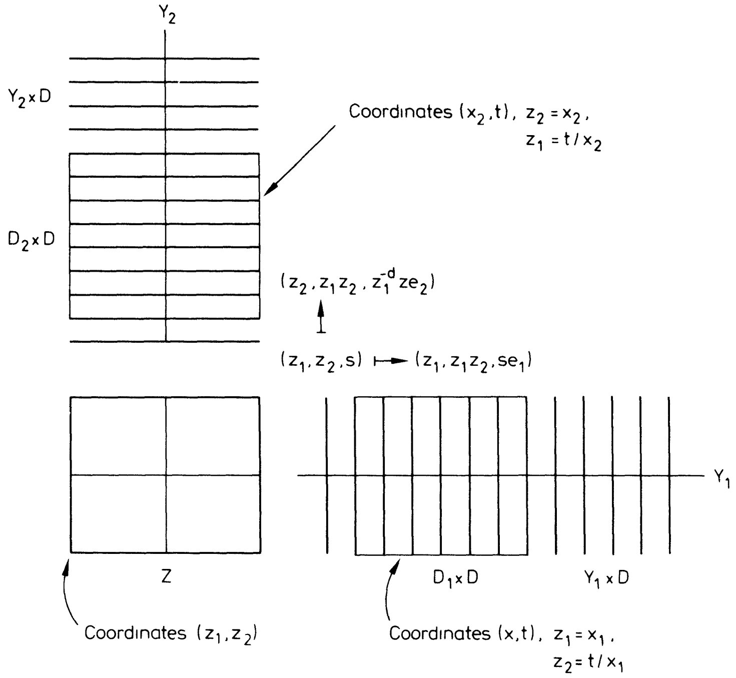
Write p_i: $Y_i \times D \to Y_i$ for the projection. Any point in the total space of the line bundle $p_i^* \mathscr{L}_i|_{D_i \times D}$ may be written in the form (x_i, t, se_i), and any point in the trivial bundle $Z \times \mathbb{C}$ may be written as (z_1, z_2, s). In these terms we glue $Y_1 \times D$, Z, and $Y_2 \times D$ together by identifying Z along open sets with $D_1 \times D$ and with $D_2 \times D$ to make a family $X \to D$, and simultaneously glue the line bundles

$$p_i^*(\mathscr{L}_i)|_{Y_i \times D - p \times D} \qquad (i=1, z)$$

and

$$Z \times \mathbb{C}$$

to make a line bundle $\mathscr{L}_{Y_1}$ on X according to the following scheme:

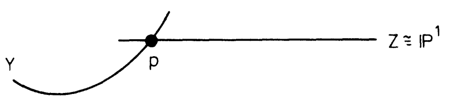


One checks immediately that the section $p_1^*\sigma_j^1$ of $p_1^*\mathscr{L}_1 = \tilde{\mathscr{L}}_{Y_1}|_{Y_1\times D}$ is represented on $D_2 \times D$ by

$$(x_2, t, t^{a_j^1-d}x_2^{d-a_j^1}e_2) = t^{a_j^1-d}p_2^*\sigma_{1-j}^2|_{D_2\times D},$$

so that $p^*\sigma_j^1$ extends to a section $\tilde{\sigma}_j^1$ of $\mathscr{L}_{Y_1}$ on all of X. Taking $\mathscr{L}_\eta = \tilde{\mathscr{L}}_{Y_1}|_{X-X_0}$ and $\tilde{\sigma}_0^1$, $\tilde{\sigma}_1^1$ as a basis for $V_\eta \subset H^0(\mathscr{L}_\eta)$, we get the desired smoothing. □

Example 3.2. Let X_0 be the curve



with Y hyperelliptic of genus $g \geqq 4$, p a ramification point of the $\mathfrak{g}_2^1$ on Y, and Z rational. We will exhibit an unsmoothable limit $\mathfrak{g}_4^2$ on X_0.

Let $(\mathscr{L}_Y, V_Y)$ be the complete $\mathfrak{g}_4^2$ on Y which is the double of the $\mathfrak{g}_2^1$. Let $\mathscr{L}_Z = \mathcal{O}_Z(4)$, and define V_Z in terms of a local coordinate t on Z centered at p as the span of the sections 1, t^2+t^3, t^4.

Each of these series has vanishing sequence 0, 2, 4 at p, so

$$L = \{(\mathscr{L}_Y, V_Y), (\mathscr{L}_Z, V_Z)\}$$

is a limit series; we claim that it cannot be smoothed.

Note that $(\mathscr{L}_Z, V_Z)$ has no base-points and defines a birational map ϕ of Z to $\mathbb{P}^2$. If now L could be smoothed in a family X/B we could regard ϕ as a limit of a (necessarily birational) mapping $\phi_{\bar\eta}$ from the geometric generic fiber $X_{\bar\eta}$, and $\phi_{\bar\eta}(X_{\bar\eta})$ would be a plane quartic of geometric genus $\geqq 4$, which is absurd. □

We turn now to the main results of this section. Fix r and d. A *ramification index* (*or sequence*) *of type* (r,d) is a sequence of integers $b=(b_0, \ldots, b_r)$ with $0 \leqq b_0 \leqq \ldots \leqq b_r \leqq d-r$. Suppose that p is a smooth point of a genus g curve X_0 of compact type, and that p is contained in a component Z of X_0. We say that a limit $\mathfrak{g}_d^r = \{(\mathscr{L}_Y, V_Y)\}$ on X_0 satisfies the ramification condition (p,b) if the ramification sequence of L at p is termwise $\geqq (b_0, b_1, \ldots, b_r)$.

If $b^1, \ldots, b^s$ are ramification indices of type (r,d) we set

$$\rho(g,r,d;b^1,\ldots,b^s) = (r+1)(d-r) - rg - \sum_{i,j} b_j^i.$$

This is the "expected dimension" of the family of limit series on X_0 satisfying ramification conditions (p_i, b^i) for fixed $p_1, \ldots, p_s \in X_0$; see Theorem 4.5.

Definition. A *smoothing family*

$$\pi\colon X \to B, \qquad p_1, \ldots, p_s\colon B \to X$$

is an s-pointed (relative) genus g curve of compact type such that:

B is irreducible.
π is flat and proper.
The fibers of π are curves of compact type.
The images of the p_i are disjoint and in the smooth locus of π.
There exists a relatively ample divisor on X whose support is disjoint from all the sections $p_i(B)$.
The components of the singular locus of π map isomorphically onto their images in B.

Our main technical result is this:

Theorem 3.3. *Let* $\pi\colon X \to B$, $p_1, \ldots, p_s\colon B \to X$ *be a smoothing family, and let* $b^1, \ldots, b^s$ *be ramification indices of type* (r,d). *There exists a scheme* $G = G_d^r(X/B; (p_1, b^1), \ldots, (p_s, b^s))$, *quasi-projective over* B, *and compatible with base change, whose points over any point* q *in* B *correspond to limit* $\mathfrak{g}_d^r$*'s on* X_q *satisfying the ramification conditions* $(p_1(q), b^1), \ldots, (p_s(q), b^s)$. *Further, if* $\rho = \rho(g,r,d;b^1,\ldots,b^s)$, *then every component of* $G_d^r(X/B;(p_1,b^1), \ldots, (p_s,b^s))$ *has dimension* $\geqq \dim B + \rho$. *If*

$$\sum_{i,j} b_j^i = (r+1)d + \binom{r+1}{2}(2g-2),$$

the maximum possible, then G is proper over B.

Remark. The essence of Theorem 3.3 is that there is a moduli space of "rigidified" limit $\mathfrak{g}_d^r$'s on smoothing families of curves, where "rigidified" means that all the ramification happens at marked (smooth) points.

Proof. We will take the scheme of limit $\mathfrak{g}_d^r$'s to be a disjoint union over such schemes on which the vanishing sequences of the aspects at the nodes of X_q are also assigned. Thus we begin by choosing, for each component Y of X_q and each node p

of X_q contained in Y, a sequence of integers $0 \leqq a_0^{Y,p} < a_1^{Y,p} < \ldots < a_r^{Y,p} \leqq d$ with the property that if Y and Z are components meeting in p then

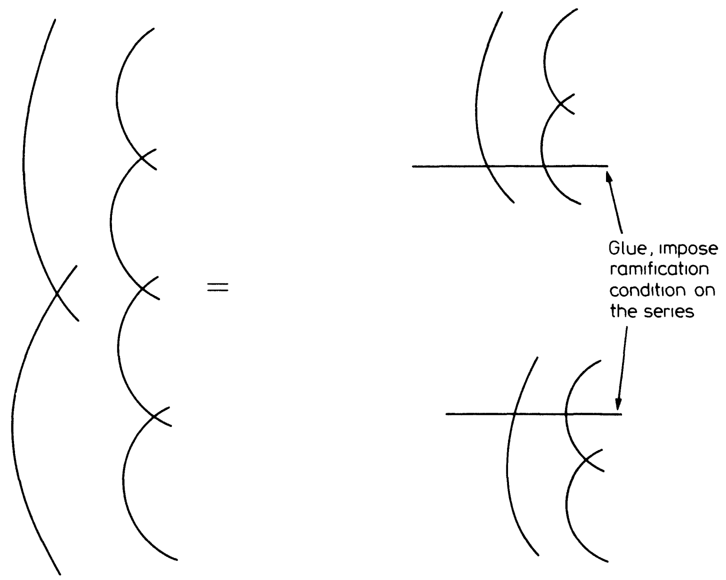
$$a_i^{Y,p} + a_{r-i}^{Z,p} = d.$$

All the remainder of our construction will be relative to this choice.

It is easy to see how to define the scheme of limit $\mathfrak{g}_d^r$'s on a curve X_q of compact type: One takes, in the product of the families of $\mathfrak{g}_d^r$'s on the components of X_q, the set of those points consisting of collections of $\mathfrak{g}_d^r$'s whose vanishing sequences at the nodes of X_q satisfy the compatibility conditions for limit series; these are obviously locally closed conditions. It is obvious that it adds no problems to consider ramification conditions at smooth points of X_q as well.

Similarly, in any smoothing family X/B where no nodes are smoothed, it is clear how to define the scheme of limit $\mathfrak{g}_d^r$'s on fibers of the family, again as a locally closed subscheme of the product of the schemes of $\mathfrak{g}_d^r$'s on the fibers of the components of X, and again one can easily include ramification conditions along sections in the smooth locus of π. These families can fail to be proper only because a family of limit $\mathfrak{g}_d^r$'s may have ramification points which approach nodes of the family, but if $\sum_{i,j} b_j^i = (r+1)d + \binom{r+1}{2}(2g-2)$ then even this cannot happen by Proposition 1.1, and the family is proper as required. A little computation shows that the dimension conditions are also satisfied – see Theorem 4.5.

When some of the nodes of fibers of π are smoothed in the generic fiber, then things become more complex. On the other hand, once one has dealt with families with smooth general fiber, with extra ramification conditions imposed along sections in the smooth locus, then it is clear how to treat arbitrary smoothing families: one regards the family as being obtained from several families, in each of which all nodes are smoothed, by glueng along sections in the smooth locus, at which some ramification conditions will have to be assigned:



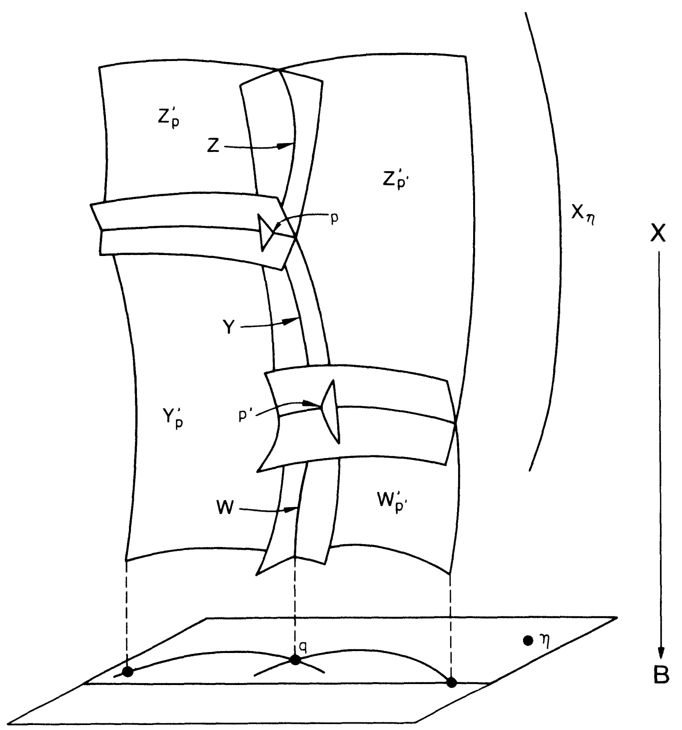
Again the dimension statements follow easily, once they are known in the case of families with geometrically smooth generic fiber.

It remains to treat the case of a smoothing family with geometrically smooth generic fiber. The problem is local on B, so we assume that B is affine.

Let D be a relatively ample divisor on X, contained in the smooth locus of π, and disjoint from the sections p_i, as in the definition of a smoothing family.

For each component Y of X_q, let $\mathrm{Pic}^Y(X/B)$ be the relative Picard scheme of locally free sheaves whose degree on Y is d and whose degree on the other components of X_q is 0. We will need certain isomorphisms $\mathrm{Pic}^Y(X/B) \to \mathrm{Pic}^Z(X/B)$, which we now define.

For each node p of X_q, let $\Delta_p' \subset X$ be the component of the singular locus of $\pi: X \to B$ that contains p, and let $\Delta_p \subset B$ be its image. The p-discriminant Δ_p is precisely the set "over which p is not smoothed", and is a Cartier divisor in B. If p is in the intersection, $p \in Y \cap Z$, of irreducible components Y and Z of X_q, then we write Y_p' and Z_p' for the closures in X of the connected components of $\pi^{-1}(\Delta_p) - \Delta_p'$ containing $Y - p$ and $Z - p$, respectively, as in the following picture:



We may assume that B is small enough so that $\mathcal{O}_B(\Delta_p) \cong \mathcal{O}_B$ and thus $\mathcal{O}_X(Y_p' + Z_p') = \mathcal{O}_X(\pi^*(\Delta_p)) = \mathcal{O}_X$. We fix such an isomorphism for each p. We also, for each Y, p, choose a section $\tau_{Y,p} \in \mathcal{O}_X(Y_p')$ which vanishes precisely on Y_p'.

Now it is easy to check that tensoring with $\mathcal{O}_Z(-dY_p')$ provides an isomorphism

$$\mathrm{Pic}^Y(X/B) \to \mathrm{Pic}^Z(X/B).$$

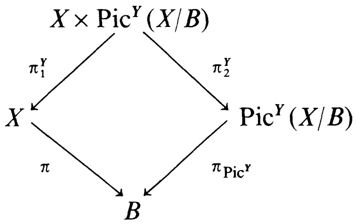
Further, the inverse has the same form, tensoring with $\mathcal{O}_X(-dZ'_p)$, since

$$\mathcal{O}_X(-dY'_p - dZ'_p) \cong \mathcal{O}_X$$

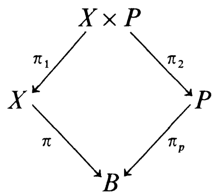
by the isomorphism chosen above. We will write P for the inverse limit of the $\mathrm{Pic}^Y(X/B)$ over this system of isomorphisms, so that $\mathrm{Pic}^Y(X/B) \cong P \cong \mathrm{Pic}^Z(X/B)$.

For each Y, let $\tilde{\mathscr{L}}_Y$ be a Poincaré sheaf on $X \times_B \mathrm{Pic}^Y(X/B)$, chosen to agree under the isomorphisms just given. Let $\tilde{\mathscr{L}}$ be the corresponding bundle on $X \times P$.

For each Y we have a square of projections and structure maps



and these induce



Replacing D with a high multiple of itself, we may assume that it meets each component Y of X_q with high degree. Let D_Y be the union of the components of D that meet Y, so that $D = \sum_Y D_Y$. Write e_Y for the degree $(D_Y \cdot Y)$ (over B) of D_Y, and $e = \sum_Y e_Y$ for the degree of D.

It follows, regarding each $\tilde{\mathscr{L}}_Y$ as a bundle on $X \times P$, that

$$\pi_{2*}\tilde{\mathscr{L}}_Y(\pi_1^* D_Y)$$

is a vector bundle on P. Let G_Y be the bundle of Grassmannians of $r+1$-planes in $\pi_{2*}\tilde{\mathscr{L}}_Y(\pi_1^* D_Y)$. For each node p of X_q and each irreducible component Y on which p lies, we let

$$F_{Y,p} \to G_Y$$

be the bundle of *projective frames*, that is, bases of $r+1$-planes considered up to scalar multiplies of their elements. We let F_Y be the product over all nodes $p \in Y$ of the $F_{Y,p}$, fibered over G_Y.

Finally, we let

$$u\colon\ F \to G$$

be the product of all the $F_Y \to G_Y$, fibered over P. Let $\tilde{V}_Y$ be the universal sub-bundle on G_Y pulled back to G, and let $\sigma_0^{Y,p}, \ldots, \sigma_r^{Y,p} \in u^* \tilde{V}_Y$ be the universal frames.

A point x of F over $q' \in B$ is thus a collection of line bundles $\mathscr{L}_Y(D_s)|_{X_{q'}}$, all determined by any one of them, an $r+1$-dimensional space of sections $(V_Y)_x$ of each of these, and for each node p of X_q lying in Y, a basis $(\sigma_0^{Y,p})_x, \ldots, (\sigma_r^{Y,p})_x$ for this space of sections, up to the equivalence $(\sigma_0^{Y,p})_x, \ldots, (\sigma_r^{Y,p})_x \sim (\lambda_0 \sigma_0^{Y,p})_x, \ldots, (\lambda_r \sigma_r^{Y,p})_x$.

We will realize the desired scheme of limit series as the image in G of an open set in the subscheme of F defined by the following three groups of equations.

(1) *Vanishing on D*. We require that the sections of $\tilde{V}_Y$ vanish on D_Y; that is, that the composite

$$\tilde{V}_Y \to \pi_{2*} \tilde{\mathscr{L}}_Y(\pi_1^* D_Y) \to \pi_{2*} \pi_1^* \mathcal{O}_{D_Y}$$

is zero.

(2) *Compatibility*. For each node $p \in X_q$ we may write $p = Y \cap Z$ for irreducible components Y and Z of X_q and we impose the conditions

$$\sigma_i^{Y,p} \, \tau_{Y,p}^{a_{r-i}^{Z,p}} = \sigma_{r-i}^{Z,p} \, \tau_{Z,p}^{a_i^{Y,p}} \qquad (i = 0, \ldots, r)$$

up to scalar multiples, as elements of the bundles

$$\pi_{2*} \tilde{\mathscr{L}}_Y(D + a_{r-i}^{Z,p} Y'_p) \cong \pi_{2*} \tilde{\mathscr{L}}_Z(D + a_i^{Y,p} Z'_p),$$

the isomorphism being the one induced by the chosen

$$\tilde{\mathscr{L}}_Z \cong \tilde{\mathscr{L}}_Y(-dZ'_p)$$

and

$$\mathcal{O}_X(Y'_p + Z'_p) \cong \mathcal{O}_X.$$

(3) *Ramification*: For each section p_i: $B \to X$ which passes through an irreducible component Y we impose the ramification condition of type b^i on the sections of $\tilde{V}_Y$ at p_i; we pull these conditions back from G to F.

We next compute a lower bound on the dimensions of components of the scheme $F' \subset F$ defined by these equations. Let n be the number of nodes of X_q, so that X_q has $n+1$ components.

Since $\pi_{2*} \tilde{\mathscr{L}}_Y(\pi_1^* D)$ is a vector-bundle of rank $d+e-g+1$, G is smooth over B, so irreducible, of dimension

$$\dim G = \dim B + g + \sum_Y (r+1)(d+e-g-r)$$

It follows that F is also irreducible and of dimension

$$\dim F = \dim G + 2n(r+1)r.$$

The vanishing conditions (1) are given locally by $(r+1)e = \sum_Y (r+1) e_Y$ equations. Each of the compatibility equations (2) is an equation on elements of a projective space bundle with fiber dimension $d+e-g$, so (2) accounts, locally, for $n(r+1)(d+e-g)$ equations in all. The ramification conditions (3) are the union of the conditions defined by the individual ramification conditions of type b^i along p_i. This condition is a pullback of a Schubert condition of codimension $\sum_j b_j^i$ in a suitable Grassmann variety so these conditions can reduce the dimension of components of the subscheme defined by (1) and (2) above by at most $\sum_{ij} b_j^i$ in all.

Simple arithmetic now shows that the dimension of every component of the subscheme F' of F defined by the conditions above is at least

$$\dim B + \rho + n(r+1)r.$$

We now examine the fibers of $F' \to G$; we wish to show that their dimension is $\leqq n(r+1)r$, at least in a certain open set $F'' \subset F'$.

Let q' be any point of B, and consider a point f in a fiber of F' over q' given by

$$f = \{V_Y \subset H^0(X_{q'}, \mathscr{L}_Y + D), \{\sigma_i^{Y,p}\}_{i=0,\ldots,r} \mid p \in Y \subset X_q, \\ p \text{ a node, } Y \text{ a component}\}.$$

If Y_1, Y_2 are components of X_q meeting in p, then there are two cases to consider:

$q' \notin \Delta_p$: This means that p has been smoothed in $X_{q'}$, $\tau_{Y_1,p}$ and $\tau_{Y_2,p}$ are both units on $X_{q'}$, and the compatibility conditions 2 serve to identify the frames $\{\sigma_i^{Y_1,p}\}$ and $\{\sigma_i^{Y_2,p}\}$ of $V_{Y_1} = V_{Y_2}$. Clearly the two projective frames associated to p then contribute at most $(r+1)r$ dimensions to the fiber of $F' \to G$ in which they lie.

$q' \in \Delta_p$: This means that there is a node $p' \in \Delta_p'$ over q' so that $X_{q'} - p' = Y_{1p}' \cup Y_{2p}'$. Since $\tau_{Y_{2,p}}$ vanishes in $X_{q'}$ precisely along $Y_{2,p}'$, we see from the compatibility conditions that

$$\sigma_i^{Y_1,p} \in \mathscr{L}_{Y_1}(D - a_i^{Y_1,p}(Y_{2,p}')_{q'})$$

so $\sigma_i^{Y_1,p}|_{(Y_{1,p}')_{q'}}$ vanishes at p' to order $\geqq a_i^{Y_1,p}$ as a section of $\mathscr{L}_{Y_1}|_{Y_{1,p}'}$.

Let F'' be the open set at F' of the f such that for every p, Y_1, p' as above, $\sigma_i^{Y_1,p}|_{(Y_{1,p}')_{q'}}$ vanishes at p' to order *exactly* $a_i^{Y_1,p}$. Since the sequence of integers $a_i^{Y_1,p}$ is strictly increasing, it easily follows that the set of frames $\sigma_i^{Y_1,p}$ for V_Y satisfying this condition has dimension at most $(r+1)r/2$, the dimension of the $(r+1)\times(r+1)$ unipotent group. Thus again the set of pairs of possible frames associated with p contributes at most $(r+1)r$ dimensions to the fiber in which f lies.

This proves that every component of the image G'' of F'' in G has dimension at least $\dim B + \rho$. It remains to identify the fiber of G'' over a point q' in B.

Let f as above be a point of F'' over $q' \in B$ and $\{\mathscr{L}_Y\} \in P$. If Y' is a component of $X_{q'}$, then Y' specializes, in a natural sense, to a certain union of components $Y_1 \cup \ldots \cup Y_m$ of X_q; indeed the Y_i are precisely those such that for every node $Y \cap Z = p \in X_q$ such that Δ_p' meets $X_{q'}$, Y_i is contained in Y_p' if and only if Y' is. If Y_i meets Y_j, in a node p, then Δ_p' automatically does not meet $X_{q'}$, so $\mathscr{L}_{Y_j}|_{X_{q'}} \cong \mathscr{L}_{Y_i}(-dY_p'|_{X_{q'}}) = \mathscr{L}_{Y_i}|_{X_{q'}}$. Thus for each component Y' of $X_{q'}$ we may define a line bundle $\mathscr{L}_{Y'}$ by the requirement $\mathscr{L}_{Y'} \cong \mathscr{L}_{Y_i}$ (for any of the Y_i in the specialization of Y'), and we will have

$$\deg \mathscr{L}_{Y'}|_{Y'} = d$$
$$\deg \mathscr{L}_{Y'}|_{Z'} = 0 \quad \text{for any component } Z' \neq Y' \text{ of } X_{q'}.$$

Next, for $i = 1, \ldots, m$ the spaces $V_{Y_i}|_{X_{q'}}$ are identified by the compatibility conditions (2). Let $V_{Y'} \subset H^0(X_{q'}, \mathscr{L}_{Y'}(D)|_{X_{q'}})$ be their common value. We may identify the image of $V_{Y'}$ in $H^0\left(Y', \mathscr{L}_{Y'}\left(\sum_1^m D_{Y_i}\right)\Big|_{Y'}\right)$ with a space of sections

$\bar{V}_{Y'} \subset H^0(Y', \mathscr{L}_{Y'}|_{Y'})$ because of the vanishing conditions (1). Finally, the compatibility conditions associated to a node $p \in X_q$ for which $\Delta'_p \cap X_{q'}$ is a node p' of $X_{q'}$ contained in Y', together with our choice of the open subset F''', show that if $Y \subset X$ is chosen so that $Y' \subset Y'_p$, then sections in $\bar{V}_{Y'}$ vanish to the $r+1$ distinct orders $a_i^{Y,p}$ at p'. This shows that the map $V_{Y'} \to \bar{V}_{Y'}$ is an isomorphism, and that the pairs

$$(\mathscr{L}_{Y'}|_{Y'}, \bar{V}_{Y'})$$

are the aspects of a refined limit $\mathfrak{g}_d^r$ on $X_{q'}$. Since this process may evidently be reversed to produce elements of F'' from any refined limit $\mathfrak{g}_d^r$ on $X_{q'}$, we have established that the image of F'' in G is the desired scheme.

We now define $G_d^r(X/B, (p_1, b^1), \ldots, (p_s, b^s))$ to be the (disjoint) union, over all choices of $a_i^{Y,p}$, of the open subsets of the varieties G'' defined above consisting of refined limit series. The dimension statement of the theorem has already been proved.

Finally, to check properness in case

$$\sum b_j^i = (r+1)\,d + \binom{r+1}{2}(2g-2)$$

it suffices, since G_d^r is quasi-projective over B, to show that each fiber is closed. This is implied by Proposition 1.1 since a limit of points in the fiber will be a crude limit series for which the equality of that proposition holds. □

Given this result it is easy to prove our main result on smoothability:

Theorem 3.4. *Let π: $X \to B$, $p_1, \ldots, p_s$: $B \to X$, D, $b^1, \ldots, b^s$, and ρ be as in Theorem 3.3. Suppose that every fiber X_q of X is reducible, and that L is a refined limit* $\mathfrak{g}_d^r$ on a fiber X_q *which is contained in some component G of*

$$G_d^r(X/B, (p_1, b^1), \ldots, (p_s, b^s))$$

with $\dim G = \rho + \dim B$. *Then L smooths in the sense that there is a 1-parameter family $\bar{X}_t$ of curves with $\bar{X}_0 = X_q$ and $\bar{X}_t$ smooth for $t \neq 0$, sections $\bar{p}_1, \ldots, \bar{p}_s$ of the family with $\bar{p}_i(0) = p_i(q)$ and a family of linear series L_t on $\bar{X}_t$, satisfying the ramification conditions $(\bar{p}_1, b^1), \ldots, (\bar{p}_s b^s)$, with limit $L_0 = L$.*

Proof. Let $\tilde{X} \to \tilde{B}$ be the versal family of curves around $X_q, p_1(q), \ldots, p_s(q), D \cap X_q$. Let f: $B \to \tilde{B}$ be the map locally inducing $X/B, p_1, \ldots, p_s, D$. Let $\tilde{G}$ be a component of $G_d^r(\tilde{X}/\tilde{B}, (p_1, b^1), \ldots, (p_s, b^s))$ such that G is a component of $f^*\tilde{G}$ and let $\tilde{L}$ be the point of $\tilde{G}$ corresponding to L. By Theorem 3.3, $\dim \tilde{G} \geq \rho + \dim \tilde{B}$. If $\tilde{G}$ does not lie entirely over the discriminant locus of $\tilde{X}/\tilde{B}$, then clearly we are done. Suppose on the contrary that its image, with that of B, is contained in a component $\tilde{B}'$ of the discriminant locus.

Now $\tilde{B}'$ is a smooth hypersurface in $\tilde{B}$, and $\dim \tilde{G} > \rho + \dim \tilde{B}'$. Thus every component of $f^*\tilde{G}$, and in particular G, has dimension $> \rho + \dim B$, a contradiction. □

There are two special cases of this result which we will use repeatedly, and we isolate them in the next corollaries. The first is the case where B is a point, which is useful for establishing results on general curves:

Corollary 3.5. *Let $(X, p_1, \ldots, p_s)$ be an s-pointed genus g curve of compact type, and let $b^1, \ldots, b^s$ be ramification indices of type (r, d). If the family of limit series $G^r_d(X; (p_1, b^1), \ldots, (p_s, b^s))$ has dimension $\rho(g, r, d; b^1, \ldots, b^s)$, then every $\mathfrak{g}^r_d$ L on X satisfying the ramification conditions $(p_1, b^1), \ldots, (p_s, b^s)$ can be smoothed to every nearby curve, maintaining the ramification conditions at prescribed nearby points.*

Proof. The corollary is of course vacuous unless $\rho = \rho(g, r, r; b^1, \ldots, b^s) \geqq 0$; and in this case the dimension statement of Theorem 3.3, applied with the semicontinuity of fiber dimensions and the irreducibility of the versal deformation space of $(X, p_1, \ldots, p_s)$, gives the result. □

Of course when examining special behavior, such as Weierstrass points, or cases when $\rho < 0$, we need a stronger form. First we introduce some terminology:

Let X be a genus g curve of compact type, L a limit g^r_d on X, and $p_j \in X$ distinct smooth points. We write $\alpha^L(p_j)$ for the ramification index

$$\alpha^L(p_j) := a_0^{L_Y}(p_j) - 0, \ \ldots, \ a_r^{L_Y}(p_j) - r$$

where Y is the component of X containing p_j and L_Y is the Y-aspect of L. We define the *adjusted Brill-Noether number* of L with respect to the p_j to be

$$\rho(X, L, \{p_j\}) = \rho(g, r, d; \{\alpha^L(p_j)\}).$$

Direct arithmetic calculation using the Plücker formula (see Sect. 0 above) gives:

Lemma 3.6. *The adjusted Brill-Noether number is additive; that is, if Y_i are the components of X as above, $\{p_{ij}\}$ is the set of those p_j lying on Y_i, and $\{q_{ij}\}$ are the points of Y_i which are nodes of X, then*

$$\rho(X, L, \{p_j\}) = \sum_i \rho(Y_i, L_{Y_i}, \{p_{ij}\} \cup \{q_{ij}\}).$$

Let Y be a smooth curve and let $\{q_j\}$ be distinct points of Y. Let

$$(*) \qquad \begin{array}{c} \tilde{Y} \longrightarrow B_Y \\ \tilde{Y} \underset{\tilde{q}_j}{\longleftarrow} B_Y \end{array}$$

be the versal deformation of $(Y, \{q_j\})$. If L_Y is a $\mathfrak{g}^r_d$ on Y, then we say that L_Y is *dimensionally proper* with respect to the q_j if an irreducible component $G_{\tilde{Y}}$ of

$$G^r_d(\tilde{Y}/B_Y; \{(\tilde{q}_j, \alpha^{L_Y}(q_j))\}$$

containing L_Y has dimension $\dim B_Y + \rho(Y, L_Y, \{q_j\})$.

More generally, if X is a curve of compact type, and $p_j \in X$ are distinct smooth points, then returning to the notation above we say that the g^r_d L on X in *dimensionally proper* with respect to $\{p_j\}$ if each aspect L_{Y_i} is dimensionally proper with respect to $\{p_{ij}\} \cup \{q_{ij}\}$. Our most useful smoothing result is:

Corollary 3.7. *If X is a curve of compact type, and L is a limit series on X which is dimensionally proper with respect to smooth points $p_1, \ldots, p_s \in X$, then L can be smoothed, maintaining the ramification conditions $\alpha^L(p_i)$ as in Theorem 3.3.*

Proof. Let $\tilde{Y}_i/B_{Y_i}$ and G_{Y_i} be the analogues for Y_i of $B_{\tilde{Y}}/B_Y$ and G_Y of *) and the discussion following, above. We may construct a family $\tilde{X}/\Pi_i B_{Y_i}$ of curves of compact type containing X by gluing the $\tilde{Y}_i$ along the sections $\tilde{q}_{ij}$, and ΠG_{Y_i} is evidently a component of $G_d^r(\tilde{X}/\Pi_i B_{Y_i};\ \{(\tilde{p}_i, \alpha^L(p_i))\})$ whose dimension is, by Lemma 3.6, equal to

$$\dim(\Pi B_i) + \rho(X, L, \{p_i\}).$$

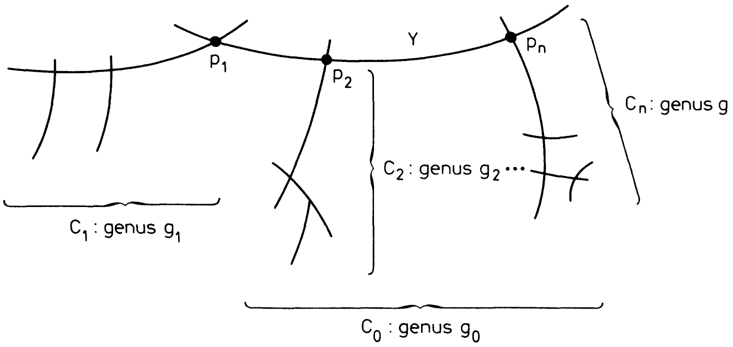
Thus Theorem 3.3 applies. □

4. Extensions of the Riemann, Clifford, and Brill-Noether theorems

The classical versions of the first two results mentioned give upper bounds on the dimension r of a $\mathfrak{g}_d^r$ on a smooth curve of genus g in terms of g and d, and classify the cases of equality. The last result gives a lower bound for the dimension of the family of all $\mathfrak{g}_d^r$'s which is sharp for general curves. In this section we give corresponding results for limit series on curves of compact type. These results help to justify our definitions; also, the result identifying the line bundles in the aspects of a $\mathfrak{g}_{2g-2}^{g-1}$ (a "limit canonical series") will play a fundamental role in our forthcoming papers on Weierstrass points, and the results on the "additivity of ρ" will be used in our treatment of the Kodaira dimension of the moduli space.

We begin with notation that will be used throughout this section:

Let C be a genus g curve of compact type, and let Y be a component of C, containing the nodes $p_1, \dots, p_n$ of C. We write K_Y for the canonical bundle of Y. For $i = 1, \dots, n$ let C_i be the closure of the connected component of $C - Y$ meeting Y at p_i, and let g_i be the genus of C_i. We further write C_0 for the closure of the connected component of $C - p_1$ containing Y, and g_0 for its genus:



$$g_0 = \text{genus } Y + \sum_2^n g_i$$

$$g = \text{genus } Y + \sum_1^n g_i = g_0 + g_1$$

Fig. 4.1

With this notation we have:

Theorem 4.1. *Let L be a crude limit $\mathfrak{g}_d^r$ on C. If $r \leqq g$ or $d \leqq 2g$ then*

$$2r \leqq d \qquad (\text{"Clifford's Theorem"})$$

while if $r \geqq g$ or $d \geqq 2g$ then

$$r + g \leqq d \qquad (\text{"Riemann's Theorem"})$$

Further, if

$$2r = d = 2g - 2$$

and $(\mathscr{L}_Y, V_Y)$ is the Y-aspect of L, then

$$\mathscr{L}_Y \cong K_Y\left(2\sum_1^n g_i p_i\right).$$

Remarks. The Canonical Case: If C is the special fiber in a family of curves whose generic fiber is smooth, then the canonical bundle on the generic curve extends to the relative dualizing sheaf on the family, whose restriction to C is the dualizing sheaf ω_C of C. It is not hard to check [26, p. 77 eq. (27)] that $\omega_C|_Y \cong K_Y(\sum p_i)$, so the Y-aspect of a limit series which is the limit of nearby canonical series will be $K_Y(2\sum g_i p_i)$; of course this follows from the theorem as well, since the limit series is a $\mathfrak{g}_{2g-2}^{g-1}$. In particular every genus g curve of compact type has at least one smoothable crude limit $\mathfrak{g}_{2g-2}^{g-1}$. In our [8] we give a criterion for this to be the unique crude limit $\mathfrak{g}_{2g-2}^{g-1}$. In any case, it is not hard to construct, aspect by aspect, a refined limit $\mathfrak{g}_{2g-2}^{g-1}$ on any such curve.

The Hyperelliptic case: As in the classical case, if C has a $\mathfrak{g}_{2r}^r$ with $r < g-1$, the C is *hyperelliptic* in the sense that it possesses a $\mathfrak{g}_2^1$, and the underlying line bundles are the r^{th} powers of those for the $\mathfrak{g}_2^1$. It is fairly easy to see what having a $\mathfrak{g}_2^1$ means for a curve C of compact type: supposing for simplicity that C has no rational components that meet only one other component, C possesses a $\mathfrak{g}_2^1$ if and only if

1) every component Y of genus $\geqq 2$ is hyperelliptic and each node of C on Y is a Weierstrass point of Y,

2) If Y is a component of C of genus 1, then the nodes of C on Y differ by 2-torsion in the group structure on C, and

3) rational components of C contain at most two nodes of C.

To prove that if C has a $\mathfrak{g}_{2r}^r$ with $r < g-1$, then C has properties 1–3, one does induction on the number of components of C, making use of Lemma 4.2, much as in the proof given below for the last part of 4.1. The only significantly different point is that one must prove the non-existence of $\mathfrak{g}_{2r}^r$'s on $\mathbb{P}^1$ having vanishing sequences of a certain form at $\geqq 3$ points, By our [5] this amounts to showing that the intersections product of certain collections of Schubert cycles in the cohomology ring of some Grassmannian is 0; but this computation seems combinatorially formidable. However, using the Plücker formula one can prove the following result which (with $d = 2r$, $\sum s_i = r-1$, $a_i = 2s_i$, $n \geqq 3$) suffices.

Lemma. *Let $L = (\mathscr{L}, V)$ be a $\mathfrak{g}_d^r$ on $\mathbb{P}^1$, let $p_1, \ldots, p_n$ be distinct points, and let $0 \leqq s_1, \ldots, s_n \leqq r$ and $a_1, \ldots, a_n$ be such that*

$$a_{s_i}^L(p_i) \geqq a_i.$$

If we let

$$e = d - \sum a_i,$$
$$k = r - \sum s_i,$$

and

$$\delta = \sum (a^L_{s_i+1}(p_i) - a_i - 1)$$

then

$$4k\delta \leqq (e+1)^2.$$

Proof Sketch. $L(-\sum a_i p_i)$ is a $\mathfrak{g}^e_2$, and thus by Plucker's formula [5] has total ramification $\leqq \frac{1}{4}(e+1)^2$: But one computes that it has ramification $\geqq k(a^L_{s_i+1}(p_i) - a_i - 1)$ at p_i. □

For the proof of Theorem 4.1 we will use the following, in which we return to the notation of Fig. 4.1:

Lemma 4.2. *Let L be a crude limit $\mathfrak{g}^r_d$ on C, let $a_0, \dots, a_r$ be the vanishing sequence of the Y-aspect $(\mathscr{L}_Y, V_Y)$ at p_1, and let $0 \leqq s \leqq r$.*

4.2a) *If $s \leqq g_1$ then $2s \leqq a_s$.*

4.2b) *If $r - g_0 \leqq s$ then $a_s \leqq d - 2r + 2s$.*

Remark. If $r \geqq g_1$, then from 4.2a), with $s = g_1$ we get $2g_1 \leqq a_{g_1}$. Since a_i are strictly increasing, we can complement 4.2a with the assertion: If $s \geqq g_1$ then $g_1 + s \leqq a_s$. Similarly, if $r - g_0 \geqq s$, then $a_s \leqq d - r - g_0 + s$. In particular, the weight $w^{L_Y}(p)$ of p with respect to $(\mathscr{L}_Y, V_Y)$ satisfies

$$\binom{r+1}{2} \leqq w^{L_Y}(p) \qquad \text{if } r \leqq g_1$$

$$\binom{g_1+1}{2} + (r-g_1)g_1 \leqq w^{L_Y}(p) \qquad \text{if } r \geqq g_1$$

$$w^{L_Y}(p) \leqq (r+1)(d-r) - \binom{r+1}{2} \qquad \text{if } r \leqq g_0$$

$$w^{L_Y}(p) \leqq (r+1)(d-r) = \binom{g_0+1}{2} - (r-g_0)g_0 \qquad \text{if } r \geqq g_0.$$

Before proving these results we introduce a very useful construction. In the case of a smooth curve Y it amounts to forming, from a linear series $(\mathscr{L}_Y, V_Y)$, the linear series $(\mathscr{L}_Y(-ap), V'_Y)$, where $V'_Y \subseteq V_Y$ is a space of sections of $\mathscr{L}_Y$ vanishing to order $\geqq a$ at a point $p \in Y$.

Construction 4.3. *Let C be a curve of compact type, and let Y be a component. Let L be a crude limit* $\mathfrak{g}^r_d$ on C, and let *$(\mathscr{L}_Y, V_Y)$ be its Y-aspect. If $p \in Y$ is a smooth point of C, a is an integer, and $V'_Y \subset V_Y$ is an $r'+1$-dimensional space of sections of $\mathscr{L}_Y$ vanishing to order $\geqq a$ at p, then we may construct a limit $\mathfrak{g}^{r'}_{d-a}$ on C,*

$$L' = \{(\mathscr{L}_Y(-ap), V'_Y)\}$$

inductively as follows:

Suppose that Z is a component of C meeting Y in a point q. Since the sections of V'_Y of $\mathscr{L}_Y$ vanish to order $\geqq a$ at p, they vanish to order $\leqq d-a$ at q. If $(\mathscr{L}_Z, V_Z)$ is the

Z-aspect of L then by the compatibility condition there is an $r'+1$-dimensional subspace $V'_Z \subset V_Z$ whose orders of vanishing at q are $\geqq a$. We may take $(\mathscr{L}_2(-aq), V'_Z)$ to be the Z-aspect of L', and continue inductively.

In the special case where V'_Y is the space of all sections in V_Y vanishing to order $\geqq a$, we will write $L(-ap)$ for L' (but note that $L(-ap)$ may not be determined uniquely by L, a, and p). If, in particular, $a = a_j^{V_Y}(p)$, the j^{th} order of vanishing of $(\mathscr{L}_Y, V_Y)$ at p, then $L(-ap)$ is a $\mathfrak{g}_{d-a}^{r-j}$ on C.

Proof. of Theorem 4.1 and Lemma 4.2:

We will prove Lemma 4.2 and the first statement, "Clifford's Theorem", of Theorem 4.1 together. By induction on the number of components of C, we may assume that "Clifford's Theorem" holds for C_0 and for C_1 separately.

Applying it to the $\mathfrak{g}_{d-a}^{r-s}$

$$L|_{C_0}(-a_s p_1)$$

on C_0 we see that if $r-s \leqq g_0$ then

$$2(r-s) \leqq d-a_s,$$

which is equivalent to 4.2b).

If now Z is the component of C meeting Y at p_1, and $b_0, \ldots, b_r$ is the vanishing sequence at p_1 of the Z-aspect of L, then by the same argument, if $r-s \leqq g_1$, then $2(r-s) \leqq d-b_s$. But $b_s \geqq d-a_{r-s}$, so this yields 4.2a).

To prove "Clifford's Theorem" for C itself, note that if $r \leqq g = g_0 + g_1$, then we can choose s satisfying $0 \leqq s \leqq r$ and $r - g_0 \leqq s \leqq g_1$, and for this s we get

$$2s \leqq a_s \leqq d - 2r + 2s, \tag{4.4}$$

so in particular $d - 2r \geqq 0$, as required.

Finally, suppose that $d \leqq 2g$. It will suffice to show that $r \leqq g$. Otherwise, taking a general point $q \in y$,

$$L(-(r-g)q)$$

would be a limit $\mathfrak{g}_g^{d-(r-g)}$ on C, whose existence would contradict what has just been shown.

Next, to prove "Riemann's Theorem", let q be a general point of C. For $0 \leqq s \leqq r$,

$$L(-sq)$$

is a limit $\mathfrak{g}_{d-s}^{r-s}$. If $r \geqq g$ we take $s = r-g$ and "Clifford's Theorem" yields $d-s \geqq 2g$, or equivalently $d \geqq g+r$, as required. The remaining case $r \leqq g$, $d \geqq 2g$ satisfies "Riemann's Theorem" trivially.

We now prove the last statement of the Theorem. Given $r-g-1 = g_0+g_1-1$, and $d=2r$ we may apply (4.4) with $s=g_1$, obtaining

$$a_{g_1} = 2g_1.$$

Thus $L|_{C_0}(-a_{g_1}p)$ is a $\mathfrak{g}_{2g_0-2}^{g_0-1}$ on C_0, whence by induction on the number of components, $L_Y(-a_{g_1}p) \cong K_Y\left(2\sum_{i>1} g_i p\right)$, so $\mathscr{L}_Y \cong K_Y\left(2\sum_{i\geqq 1} g_i\right)$ as required. $\square$

We now turn to the Brill-Noether result. In order to prove it by induction we need a slightly strengthened form, which we will often apply:

Theorem 4.5. *Let $p_1, \ldots, p_s$ be smooth points of C, and let $b^1, \ldots, b^s$ be ramification conditions,*

$$b^i\colon\ 0 \leqq b_0^i \leqq \ldots \leqq b_r^i \leqq d-r$$

for some r and d. Every component of the family

$$G_d^r(C, (p_1, b^1), \ldots, (p_s, b^s))$$

has dimension

$$\geqq \rho(g, r, d;\, b^1, \ldots, b^s) := (r-1)(d-r) - rg - \sum_{i,j} b_j^i$$

and equality holds if each component of C is a general curve of its genus and $p_1, \ldots, p_s$ are general points on the components in which they lie.

Remark. Even if $\rho(g, r, d; b^1, \ldots, b^s) \geqq 0$ there may be no components! But (using for example our [5] the problem of existence for general curves can be reduced to a question of whether the intersection of Schubert cycles in the cohomology rings of certain Grassmannians are zero.

Proof. In case C is irreducible, the inequality follows because

$$G = G_d^r(C, (p_1, b^1), \ldots, (p_s, b^s))$$

is the pullback of an intersection of Schubert varieties, of codimensions $\sum_j b_j^i$, for $i = 1, \ldots, s$, in the variety $G_d^r(C)$. On the other hand the equality for general curves follows at once from the case of a g-cuspidal rational curve, treated by "Plücker-formula" methods in our [5].

To do the general case induction on the number of components of C and the following elementary observation suffice:

Proposition 4.6. (Additivity of the Brill-Noether number.) *Let r, d, g_0, g_1 be non-negative integers and let b^i ($i=1, \ldots, s$) be ramification indices*

$$b^i\colon\ 0 \leqq b_0^i \leqq \ldots \leqq b_r^i \leqq d-r.$$

If a and $\bar{a}$ are complementary vanishing sequences

$$a\colon\ 0 \leqq a_0 < \ldots < a_r \leqq d$$
$$\bar{a}\colon\ 0 \leqq \bar{a}_0 < \ldots < \bar{a}_r \leqq d$$
$$a_i + \bar{a}_{r-i} = d \qquad i = 0, \ldots, r$$

and b, $\bar{b}$ are the corresponding ramification sequences

$$b_i = a_i - i$$
$$\bar{b}_i = \bar{a}_i - i,$$

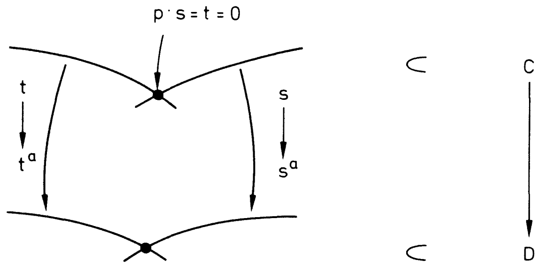
then for any k between 0 *and s*

$$\rho(g_0 + g_1, r, d; b^1, \ldots, b^s)$$
$$= \rho(g_0, r, d; b^1, \ldots, b^k, b) + \rho(g_1, r, d; b^{k+1}, \ldots, b^s, \bar{b}).$$

The proof is nothing but arithmetic. □

5. Interpretations of limit series

a) *Limit series and admissible covers.* Let C and D be reduced projective curves having only ordinary nodes as singularities. A map $\pi: C \to D$ is an *admissible cover* if the images and preimages of double points are double points, and in an analytic neighborhood of a double-point of C the map π has the form



that is, the two branches of C at p map to the two branches of D at $\pi(p)$ with equal ramification. (The version given by Harris and Mumford [17; section §4] for families of curves includes also the requirement that π has only simple ramification away from the double points, and that the branch-points other than the nodes be marked points of D. These conditions are useful if one wishes to produce a moduli space of admissible coverings, but are not relevant for our present purpose.)

We now assume that D has arithmetic genus 0 – that is, it is a union of copies of $\mathbb{P}^1$, with dual graph a tree – and that C is of compact type. For each component Y of C, let $(\mathscr{L}_Y^0, V_Y^0)$ be the pencil corresponding to the map of Y to $\pi(Y) \cong \mathbb{P}^1$.

For each component Y of C let d_Y be the degree of $\mathscr{L}_Y^0$, and for each node $p \in Y$, $p = Y \cap Z$, say, let a_p be the common index of ramification of $\pi|_Y$ and $\pi|_Z$ at p. There is a (unique) way to "add basepoints at the nodes" to π in order to obtain a limit series:

Proposition 5.1. *There is a unique limit series*

$$L = \{(\mathscr{L}_Y, V_Y) \mid Y \text{ a component of } C\}$$

such that

$$\mathscr{L}_Y = \mathscr{L}_Y^0\Big(\sum_{p \in Y \text{ a node of } C} a_{Y,p} \cdot p\Big)$$

for some family of integers $a_{Y,p}$ and V_Y is the image of V_Y^0 under the canonical inclusion

$$V_Y^0 \subset H^0(\mathscr{L}_Y^0) \to H^0(\mathscr{L}_Y).$$

L is a limit $\mathfrak{g}_d^1$ with

$$d = \sum_Y d_Y - \sum_p a_p.$$

Proof. If for given integers $a_{Y,p}$ the collection L is a limit series, then for each $p = Y \cap Z$ we have

$$a_{Y,p} + (a_p + a_{Z,p}) = d$$

by the compatibility conditions. On the other hand, since $\mathcal{L}_Y^0\left(\sum_{p\in Y} a_{Y,p}\right)$ must have degree d, we have

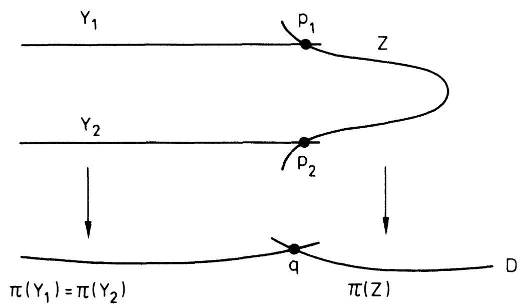
$$d_Y + \sum_{p\in Y} a_{Y,p} = d.$$

Summing the first of these equalities over the nodes, then second over the components, and subtracting gives the required value for d. A simple induction through the dual graph of C shows that the $a_{Y,p}$ are uniquely determined and, using $d_Y \geqq a_p$ for all $p \in Y$, non-negative, establishing the proposition. □

The data contained in the limit series associated to the admissible cover is in general *strictly less* than that of the admissible cover! For if Y and Z are two components of C with $\pi(Y) = \pi(Z)$ then π induces isomorphisms

$$V_Y \cong H^0(\pi(Y), \mathcal{O}_{\pi(Y)}(1)) \cong V_Z,$$

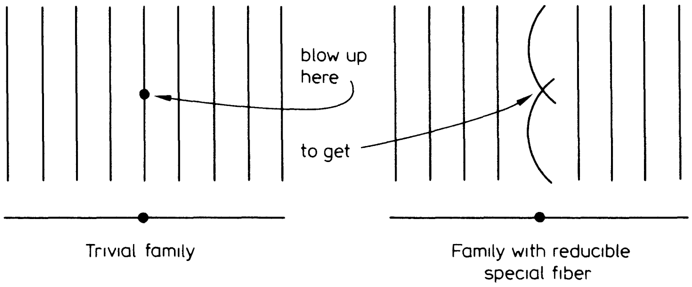
and thus V_Y and V_Z may be identified. The simplest case is perhaps the following:



where $Y_1 \cong Y_2 \cong Z \cong \mathbb{P}^1$, $\pi|_{Y_1}$ and $\pi|_{Y_2}$ are isomorphisms, and $\pi|_Z$ is a double cover, not branched over q. (Of course stable versions may also be manufactured.) Here the data of the admissible cover obviously includes, in addition to that of the limit series, the data of an isomorphism $Y_1 = Y_2$, which is consistent with our general remark since $Y_i \cong \mathbb{P}(V_{Y_i})$.

On the other hand, given a limit series $\{(\mathcal{L}_Y, V_Y)\}$ on a curve C of compact type, we may obviously construct from it an admissible cover in the sense above simply by dropping all base points of the $(\mathcal{L}_Y, V_Y)$, and regarding the collection of mappings as a mapping to a genus 0 curve with the same dual graph. In general, a given admissible cover will factor through the admissible cover associated to the limit series derived from it as in the Proposition.

b) *Limit series and mappings.* A limit $\mathfrak{g}_d^r$ on a reducible curve may obviously be interpreted (after removing base points) as a map from the curve to a union of copies of $\mathbb{P}^r$, joined at points. In case $r = 1$, this is natural and satisfactory, since a family of $\mathbb{P}^1$'s can have a union of $\mathbb{P}^1$'s, meeting at points, as a degenerate fiber:



No degeneration into a union of $\mathbb{P}^r$'s meeting at points is possible for $\mathbb{P}^r$ if $r > 1$, but we now discuss a construction which fills this gap. Let X/B be a family of curves as in (2.0), and suppose that $(\mathscr{L}_\eta, V_\eta)$ is a linear series on X_η with refined limit $\{(\mathscr{L}_Y, V_Y)\}$ on X_0. For simplicity, we suppose that X_0 has only two components, Y and Z, meeting at p. Let $a_0, \ldots, a_r$ be the vanishing sequence of V_Y at p.

We may define a rational map

$$B \times \mathbb{P}^r \to \mathbb{P}^r$$

by the formula

$$(t, X_0, \ldots, X_r) \mapsto (t^{a_0} X_0, \ldots, t^{a_r} X_r).$$

Let $\mathbb{P}^{\underline{a}} \subset B \times \mathbb{P}^r \times \mathbb{P}^r$ be the closure of the graph of this map. Because $a_0 < a_1 < \ldots < a_r$ it is defined by equations

$$\mathbb{P}^{\underline{a}} = \{(t, x, y) \in B \times \mathbb{P}^r \times \mathbb{P}^r \mid x_i y_j - t^{a_j - a_i} x_j y_i = 0,\ i < j\}.$$

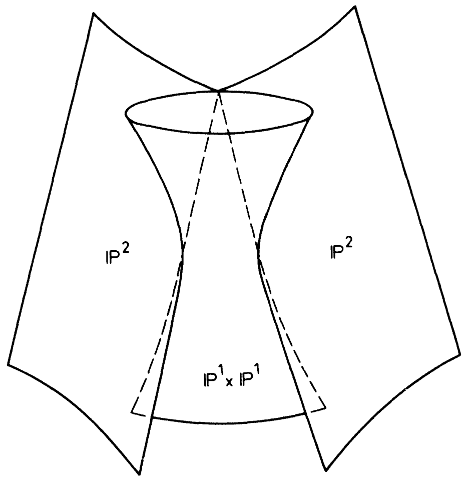
Its special fiber is thus

$$\mathbb{P}^{\underline{a}}_0 = \mathbb{P}^r \times \mathbb{P}^0 \cup \mathbb{P}^{r-1} \times \mathbb{P}^1 \cup \ldots \cup \mathbb{P}^1 \times \mathbb{P}^{r-1} \cup \mathbb{P}^0 \times \mathbb{P}^r,$$

where $\mathbb{P}^{r-i} \times \mathbb{P}^i$ is the subvariety

$$\{(0, x, y) \mid x_0 = \ldots = x_i = y_{i+1} = \ldots = y_r = 0\}.$$

For example, if $r = 2$, we have:



$\mathbb{P}^{\underline{a}}_0$ for $r = 2$, $a_0 < a_1 < a_2$

We claim that the map $X_\eta \to \mathbb{P}^r = (\mathbb{P}^{\underline{a}})_\eta$ defined by $(\mathscr{L}_\eta, V_\eta)$ extends to a map $X \to \mathbb{P}^{\underline{a}}$ over B, in such a way that Y and Z are mapped into $\mathbb{P}^r \times 0$ and $0 \times \mathbb{P}^r$, respectively, by maps corresponding to the linear series $(\mathscr{L}_Y, V_Y)$ and $(\mathscr{L}_Z, V_Z)$. Indeed, let

$$\sigma_0, \ldots, \sigma_r$$

be a basis of $\tilde{V}_Y$ adapted to $p \in Y$ and Z, so that

$$t^{a_0}\sigma_0, \ldots, t^{a_r}\sigma_r$$

is a basis of V_Z. The map $\phi: X \to B \times \mathbb{P}^r \times \mathbb{P}^r$ defined by $\phi(x) = (\pi(x), (\sigma_0(x), \ldots, \sigma_r(x)), (t^{a_0}\sigma_0(x), \ldots, t^{a_r}\sigma_r(x))$ obviously factors through $\mathbb{P}^{\underline{a}}$, and satisfies our needs.

Conversely, if we begin with a suitable map $X_0 \to \mathbb{P}^{\underline{a}}_0$ and deform it to a map $X \to \mathbb{P}^{\underline{a}}$ over B (if this is possible), we get a smoothing of a refined limit series. Unfortunately, though it is easy to write down formally a condition for this deformation to exist (at least up to first order), we have not been able to prove any smoothing results in this way, so we will not pursue the construction here.

References

1. Arbarello, E., Cornalba, M., Griffiths, P., Harris, J.: Geometry of Algebraic Curves. Berlin-Heidelberg-New York-Tokyo: Springer 1984
2. Beauville, A.: Prym varieties and the Schottky problem. Invent. Math. **41**, 149–196 (1977)
3. Castelnuovo, G.: Numero delle involutione razionali giacenti sopra una curva di dato genere. Rend. R. Acad. Lincei, Ser 4,5 (1889)
4. Diaz, S.: Exceptional Weierstrass points and the divisor on moduli space that they define. Thesis, Brown University 1982
5. Eisenbud, D., Harris, J.: Divisors on general curves and cuspidal rational curves. Invent. Math. **74**, 371–418 (1983)
6. Eisenbud, D., Harris, J.: On the Brill-Noether Theorem. In: *Algebraic Geometry – Open Problems*. Proceedings, Ravello 1982, Ed. C. Ciliberto, F. Ghione, and F. Orecchia, Lect. Notes Math. **997**, 131–137 (1983)
7. Eisenbud, D., Harris, J.: A simpler proof of the Gieseker-Petri Theorem on special divisors. Invent. Math. **74**, 269–280 (1983)
8. Eisenbud, D., Harris, J.: Existence and degeneration of Weierstrass points of low weight. Invent. Math. (to appear)
9. Eisenbud, D., Harris, J.: The monodromy of Weierstrass points. Invent. Math. (to appear)
10. Eisenbud, D., Harris, J.: The irreducibility of some families of linear series. (Preprint 1984)
11. Eisenbud, D., Harris, J.: On the Kodaira dimension of the moduli space of curves. Invent. Math. (to appear)
12. Eisenbud, D., Harris, J.: When ramification points meet. (Preprint)
13. Fulton, W., Lazarsfeld, R.: On the connectedness of degeneracy loci and special divisors. Acta Math. **146**, 271–283 (1981)
14. Gieseker, D.: Stable curves and special divisors. Invent. Math. **66**, 251–275 (1982)
15. Griffiths, P., Harris, J.: On the variety of linear systems on a general algebraic curve. Duke Math. J. **47**, 233–272 (1980)
16. Harris, J.: The Kodaira dimension of the moduli space of curves II: The even genus case. Invent. Math. **75**, 437–466 (1984)
17. Harris, J., Mumford, D.: On the Kodaira dimension of the moduli space of curves. Invent. Math. **67**, 23–86 (1982)
18. Kempf, G.: Schubert methods with an application to algebraic curves. Publ. Math. Centrum, University of Amsterdam 1971

19. Kempf, G., Knudson, F., Mumford, D., Saint-Donat, B.: Toroidal Embeddings I. Lect. Notes Math. **339**, (1973)
20. Kleiman, S.L., Laksov, D.: Another proof of the existence of special divisors. Acta Math. **132**, 163–176 (1974)
21. Knudsen, F., Mumford, D.: The projectivity of the moduli space of stable curves I: preliminaries on "det" and "Div". Math. Scand. **39**, 19–55 (1976)
22. Knudsen, F.: The projectivity of the moduli space of curves II: The stacks $M_{g,n}$. Math. Scand. **52**, 161–199 (1983)
23. Knudsen, F.: The projectivity of the moduli space of curves III: The line bundles on $M_{g,n}$ and a proof of the projectivity of $\bar{M}_{g,n}$ in characteristic 0.
24. Mumford, D.: Stability of projective varieties. Enseign. Math. **23**, 39–110 (1977)
25. Sernesi, E.: On the existence of certain families of curves. Invent. Math. **75**, 25–57 (1984)
26. Serre, J.-P.: Groupes Algébriques et Corps de Classes. Actualités scientifiques et industrielles 1264. Hermann, Paris 1959

Oblatum 29-III-1985 & 18-VII-1985