PM 60: Geometry of Today
Giornate di Geometria, Roma 1984

Recent Progress in the Study of Weierstrass Points

David Eisenbud and Joe Harris (*)

In this note we will discuss some questions concerning the behavior of Weierstrass points on curves, and some recent progress towards their answers. While most of the results have to do a priori with the behavior of Weierstras points on families of curves, the existence of certain configurations of Weierstrass points on fixed curves (cf. Corollary 5 below) will follow as corollaries. Proofs, stronger versions of some of the results given, results for reducible curves of compact type, and a somewhat more complete list of related work may be found in our [198?a].

To begin with, let us fix notation. If p is a point on a smooth complete curve/compact Riemann surface C, we may write the orders of zeroes at p of regular differentials on C as the g-tuple

$$\{\mathrm{ord}_p\omega : \omega \in H^0(C,K_C)\} = \{\alpha_0, 1+\alpha_1, 2+\alpha_2, \ldots, g-1-\alpha_{g-1}\}$$

with $\alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_{g-1}$. We call the sequence $\alpha = \alpha(C,p) = (\alpha_0,\alpha_1,\ldots,\alpha_{g-1})$ the <u>Schubert index</u> of the point $p \in C$, or of the pair (C,p); we say that p is a Weierstrass point if $\alpha \neq (0,\ldots,0)$ and in general we define the <u>weight</u> of p as a Weierstrass point to be the sum $|\alpha| = \sum \alpha_i$. (Traditionally, such points are characterized in terms of the <u>Weierstrass semigroup</u> $H(p)$ of integers n such that there exists a rational function f on C with polar divisor $(f)_\infty = np$; the Schubert index is equivalent data since by Riemann-Roch the <u>gap sequence</u> $\mathbb{N} - H(p)$ is just $1+\alpha_0, 2+\alpha_1, \ldots, g+\alpha_{g-1}$.)

Let M_g be the moduli space of curves, i.e., the space of isomorphism classes of smooth complete curves; and let C_g be the moduli space of pointed curves -- that

---

(*) Brandeis University, Waltham, MA 02254/USA
Brown University, Providence, RI 02912/USA

is, the set of isomorphism classes of pairs (C,p) with $C \in M_g$ and $p \in C$. $\mathcal{C}_g$ of course maps to $\mathcal{C}_g$ with fiber over a point $C \in M_g$ isomorphic to C, as long as C has no automorphisms. We have a stratification of $\mathcal{C}_g$ by Weierstrass points: for each index α, we may set

$$\mathcal{C}_\alpha = \{(C,p): \alpha(C,p) = \alpha\}$$

to express $\mathcal{C}_g$ as a disjoint union of locally closed subvarieties. It is also convenient to consider the associated closed subvarieties of $\mathcal{C}_g$:

$$\mathcal{C}_{\bar{\alpha}} = \{(C,p): \alpha(C,p) \geq \alpha\}$$

(here $\beta \geq \alpha$ means $\beta_i \geq \alpha_i$ for all i). Note that while the closure $\overline{\mathcal{C}_\alpha}$ of $\mathcal{C}_\alpha$ in $\mathcal{C}_g$ is contained in $\mathcal{C}_{\bar{\alpha}}$, they need not in general be equal. (If they were then in particular the hyperelliptic locus $\mathcal{C}_{0,1,2,\ldots,g-1}$ would lie in the closure of $\mathcal{C}_\alpha$ for all α; for $g > 10$ it does not lie in the closure of the locus $\mathcal{C}_{0,0,0,1,2,\ldots,g-3}$ which consists of pairs (C,p) such that C is a double cover of an elliptic curve E ramified at p.)

We can now propose a list of questions regarding Weierstrass points.

1) The most basic question, posed by Hurwitz [1893], is simply existence: for which α is $\mathcal{C}_\alpha \neq \emptyset$? This is a problem with a rather sordid history: Haure [1896] gave necessary conditions that α had to satisfy in order for $\mathcal{C}_\alpha$ to be nonempty (apart, of course, from the condition that $H_\alpha = \mathbb{N} - \{1+\alpha_0,\ldots,g+\alpha_{g-1}\}$ be a semigroup) but these conditions were later seen not to be correct. Then Hensel and Landsberg [1902] proved that in fact $\mathcal{C}_\alpha \neq \emptyset$ for any α for which H_α was a semigroup. Unfortunately, their proof was false. The question of whether all such α in fact occur was finally correctly answered by Buchweitz, who in [1980] exhibited certain such α with $\mathcal{C}_\alpha = \emptyset$. The question of which α do occur in general remains mysterious.

2) What may be the codimension of C_α? Here we have a basic upper bound: if $C_\alpha \neq \emptyset$, <u>the codimension of</u> C_α <u>in</u> C_g <u>is at most the weight</u> $|\alpha|$. This follows from the fact that under the (locally defined) map

$$C_g \xrightarrow{\phi} G(g,\infty)$$

sending a pair (C,p) to the g-dimensional space of power series expansions at p of regular differentials on C in infinite Grassmannian, the variety $\overline{C_\alpha}$ is the inverse image

$$C_{\overline{\alpha}} = \phi^{-1}(\sigma_{\alpha_{g-1},\alpha_{g-2},\ldots,\alpha_1,\alpha_0})$$

of the Schubert cycle $\sigma_\alpha = \sigma_{\alpha_{g-1},\ldots,\alpha_0}$ (defined with respect to the flag determined by order of zero). (The C_α themselves are of course just the inverse images of the Schubert cells). Thus

$$(*) \quad \operatorname{codim}(C_\alpha, C_g) \leq \operatorname{codim}(\sigma_\alpha, G(g,\infty)) = |\alpha| \; .$$

By way of terminology, we will call a point $(C,p) \in C_\alpha$ a <u>dimensionally proper</u> Weierstrass point if, in a neighborhood of (C,p) in C_g, we have $\operatorname{codim}(C_\alpha, C_g) = |\alpha|$. Clearly we do not always have equality in (*): for one thing, because H_p must be a semigroup, not all the Schubert conditions may be independent (for example, the existence of $g-1$ independent differentials on C vanishing to order at least 2 at p implies that (C,p) is hyperelliptic, which implies the existence of $g-i$ independent differentials on C vanishing to order at least $2i$ at p). Indeed, the situation in general seems almost hopeless: one can find Schubert indices $\alpha_0,\ldots,\alpha_{g-1}$ for which C_α has any given number of irreducible components of different dimensions.

What does seem to be the case, at least on an experimental basis, is that these aberrations tend to occur among Weierstrass points of higher weight; in fact, every Weierstrass point of weight $w \leq g/2$ found so far has been dimensionally proper. A precise statement of this phenomena is currently lacking; but Theorem 1 below contains some information.

A lower bound for the codimension of the components of C_α, when it is non-empty, is also known, and is due essentially to Deligne (see Rim-Vitulli [1977]); it may be interpreted in terms of the semigroup H associated to α as the cardinality of the set

$$\{n \in \mathbb{N} \mid \exists h > 0, h \in H, n + h \notin H\}.$$

3) What are the geometric relations among the C_α? We have already encountered the question of when $C_{\bar{\alpha}} = \bar{C}_\alpha$, i.e. when a general Weierstrass point of index $\beta \geq \alpha$ is the limit of Weierstrass points of index α. We can ask more generally how the loci $C_{\bar{\alpha}}$ intersect each other: Also (and the question is distinct!) if various Weierstrass points $(C_\lambda, p_i(\lambda))$ come together, what does the limit look like? Specifically, suppose we have a family of curves C_λ and distinct points $p_i(\lambda) \in C_\lambda$ for $\lambda \neq 0$ with Schubert indices $\alpha(C\ ,\ p_i(\lambda)) = \alpha_i$; say $\lim_{\lambda \to 0} p_i(\lambda) = p$ for each i and assume moreover that these are all the Weierstrass points on the general fiber C_λ with limit p. What, we ask, can we say about $\alpha(C_0, p)$?

In regard to all of these questions, the problem in general seems very far from solution. There is however, now evidence that the behavior of Weierstrass points of relatively low weight is, by contrast, very tractable. Specifically, we have the following theorems.

Theorem 1. *For any genus* g *and any Schubert index* α *of weight* $|\alpha| \leq g/2$, *there exists a dimensionally proper Weierstrass point* (C,p) *of genus* g *having Schubert index* α. *In particular, every semigroup of weight at most* g/2 *occurs; and* C_α *has at least one component of the expected dimension* $2g - 2 - |\alpha|$.

As mentioned above, on the basis of available evidence we may make the

Conjecture. Every Weierstrass point of weight $g/2$ or less is dimensionally proper, i.e. if $|\alpha| \leq g/2$ then $\dim C_\alpha = 3g - 2 - |\alpha|$.

Our second result is that in a neighborhood of a dimensionally proper Weierstrass point (C,p), everything is as well-behaved as it could be.

Theorem 2. The locus of dimensionally proper Weierstrass points is open in C_g.

Further, if (C,p) is dimensionally proper, then in small neighborhoods of (C,p) in C_g we have

$$\operatorname{codim}(C_\alpha, C_g) = |\alpha| \quad \text{for all} \quad \alpha \leq \alpha(C,p);$$

$$C_{\bar{\alpha}} = \bar{C}_\alpha \quad \text{for all} \quad \alpha;$$

in particular, any neighborhood of (C,p) will contain Weierstrass points of all indices $\alpha \leq \alpha(C,p)$.

Lastly, in the situation of question 3 above -- in which a collection of Weierstrass points $(C_\lambda, p_i(\lambda))$ with indices α^i come together to form a single Weierstrass point (C_0,p) with index α -- we have in general

Theorem 3. i) $|\alpha| = \sum |\alpha^i|$; and

ii) the index α is such that the Schubert cycle σ_α appears with non-zero coefficient in the expression of the product $\Pi\, \alpha^i \in H^{2|\alpha|}(G(g,\infty))$ as a linear combination of Schubert cycles.

Thus, for example, if a Weierstrass point with Schubert index $(0,\ldots,0,1)$ collides with one of index $(0,\ldots,0,2)$, they may form a point of index $(0,\ldots,3)$ or $(0,\ldots,1,2)$, but not one of index $(0,\ldots,0,1,1,1)$.

Remarkably, in the case of dimensionally proper Weierstrass points, we actually have a converse to Theorem 3, namely, Theorem 4. *Let* (C_0,p) *be a dimensionally proper Weierstrass point with Schubert index* $\alpha = \alpha(C_0,p)$, *and let* $\alpha^1,\ldots,\alpha^k$ *be any collection of indices such that* σ_α *appears with non-zero coefficient in the product* $\Pi\,\sigma_{\alpha^i}$. *Then there exists a family of curves* C_λ *tending to* C_0, *and a collection of sections* $p_i(\lambda) \in C_\lambda$, $i = 1,\ldots,k$, *all tending to* p, *with Schubert indices* $\alpha(C_\lambda, p_i(\lambda)) = \alpha^i$.

Combining this with Theorem 1 above, we arrive at the

Corollary 5. *Let* $\alpha^1,\ldots,\alpha^k$ *be any Schubert indices of total weight* $\sum|\alpha^i| \le g/2$. *Then there exists a curve* C *and points* $p_1,\ldots,p_k \in C$ *with Schubert indices* $\alpha(C,p_i) = \alpha^i$.

To finish, we will mention some of the work that has been done toward describing the finer structure of some of the C_α. Working with the coarser stratification of C_g given by

$$C^{(n)} = (C,p):\begin{cases}\alpha_i(C,p) = 0, & i < n-1\\ \alpha_{n-1}(C,p) > 0\end{cases}$$

$$= (C,p):\left\{\begin{matrix}\text{an } n\text{-sheeted map } \pi: C \to \mathbb{P}^1\\ \text{totally ramified at } p\end{matrix}\right\}$$

Arbarello [1974] shows that $C^{(n)}$ is irreducible, Lax [1975] that $C^{(n)}$ is smooth at points (C,p) such that $\alpha_{n-1}(C,p) = 1$; and Diaz [198?] determines its tangent space (Diaz computes tangent spaces to other unions of strata as well).

Another question concerns the geometry of the map $C_g \to M_g$; specifically, one can ask whether it is birational onto its image, or in other words whether a curve with a Weierstrass point of index α in general has only one. Obviously this is not the case with hyperelliptic Weierstrass points, and there are other examples; but one might still expect this to hold for indices of low weight (excepting, of course, the stratum $C_{0,\ldots,0,1}$ of normal Weierstrass points). Diaz [1981] has verified this in a number of cases.

Finally, one can look at the locus $\mathcal{W}$ in C_g of all Weierstrass points -- equivalently $\mathcal{W} = \bigcup_{\alpha\neq 0} C_\alpha = \bar{C}_{0,\ldots,0,1}$ -- and look at the map $\mathcal{W} \to \mathcal{M}_g$, which is a branched cover of degree $g^3 - g$; in particular, one can ask for the monodromy group (Galois group) of this cover. The answer, recently obtained in our [198?b], is that for all genera g, the Galois/monodromy group of the Weierstrass points on a generic/general curve is the symmetric group on $g^3 - g$ letters. (This has previously been proven by Harris for $g = 3$ and Canuto for $g = 4$.)

The authors are grateful to the NSF for partial support during the preparation of this work.

References

Arbarello, E.: Weierstrass points and moduli of curves. Compositio Math., [1974] 325-342.

Buchweitz, R.-O.: On Zariski's criterion for equisingularity and non-smoothable monomial curves [1980] (preprint).

Diaz, S.: Tangent spaces in moduli via deformations with applications to Weierstrass points. Duke J. Math. [198?] (to appear).

Eisenbud, D., and Harris.: Existence, decomposition and limits of certain Weierstrass points [198?a] (to appear).

Eisenbud, D., and Harris, J.: The Monodromy of Weierstrass points [198?b] (in preparation).

Haure, M.: Recherches sur les points de Weierstrass d'une coube plane algèbrique, Ann. de l'Ecole Normale, [1896].

Hensel, K., and Landsberg, G.: Theorie der algebraischer Funktionen einer Variablen, Leipzig [1902].

Hurwitz, A.: Über algebraischer Gebilde mit eindeutigen Transformationen in sich. Math. Ann. 41, 403-442 [1893].

Lax, R.: Weierstrass points on the universal curve. Math. Ann. 216 [1975] 35-42.

Rim, D. S. and Vitulli, M.: Weierstrass points and monomial curves. J. Alg. [1977] 454-476.