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LIMIT LINEAR SERIES, THE IRRATIONALITY OF M_g, AND OTHER APPLICATIONS

BY DAVID EISENBUD AND JOE HARRIS[1]

ABSTRACT. We describe degenerations and smoothings of linear series on some reducible algebraic curves. Applications include a proof that the moduli space of curves of genus g has general type for all $g \geq 24$, a proof that the monodromy action is transitive on the set of linear series of dimension r and degree d on a general curve of genus g when $\rho := g-(r+1)(g-d+r) = 0$, a proof that there exist Weierstrass points with every semigroup of a certain class—in particular, on curves of genus g, all those semigroups with weight $w \leq g/2$ occur and a proof that the monodromy group acts as the full symmetric group on the $g^3 - g$ Weierstrass points of the general curve.

Curves will here be reduced, connected, and complex algebraic.

The study of general curves (Brill-Noether theory, etc.) and of moduli of curves depends on the degeneration of smooth curves to singular ones. Originally, the singular curves used were irreducible curves with nodes ([**G-H**] is a recent avatar) or, more recently, cusps [**E-H1**], but from the work of Mumford and others on the moduli space of stable curves it is apparent that reducible curves should be considered as well.

Unfortunately the degeneration of a linear series on a curve which degenerates to a reducible curve has not been well understood except in the particularly simple case of pencils; there the "limit" of the linear series, after removing base points, corresponds to an admissible covering, in the sense of Beauville, Knudsen and Harris-Mumford [**B**, **K**, **H-M**], of a curve of genus 0. The potential of a general theory is indicated, for example, by work of Gieseker [**G**].

In this announcement we describe the limits of linear series on some reducible curves and give some applications.

We call a curve *tree-like* if its irreducible components meet only two at a time, in ordinary nodes, in such a way that its dual graph (a vertex for each component, an edge for each intersection between distinct components) has no loops.

We say that a curve is of *compact type* if its (generalized) Jacobian is compact, or, equivalently, if it is tree-like and its irreducible components are all nonsingular.

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DEFINITION. A *limit* g^r_d on a tree-like curve Y is a collection of g^r_d's, one on each irreducible component Z of Y,

L_Z a line bundle of degree d on Z,

$V_Z \subset H^0(Z, L_Z)$ an $r+1$-dimensional subspace

such that whenever two components of Y meet in a point, say $p = Z_1 \cap Z_2$, there is for each $\sigma \in V_{Z_1}$ a $\tau \in V_{Z_2}$ such that $\operatorname{ord}_p \sigma + \operatorname{ord}_p \tau = d$.

The following result is implicit in [**E-H3**]:

THEOREM 1. *Let $\mathcal{O}$ be a discrete valuation ring, and let $X \to \operatorname{Spec} \mathcal{O}$ be a family of curves with irreducible geometric general fiber $X_{\overline{\eta}}$ and reduced, special fiber of compact type. Given a line bundle $\mathcal{L}$ and a g^r_d $k(\overline{\eta})^{r+1} \cong V \subset H^0(X_{\overline{\eta}}, \mathcal{L})$ on $X_{\overline{\eta}}$, there is a family $\pi'\colon X' \to \operatorname{Spec} \mathcal{O}'$ obtained from X by base change, blow-ups of points in the central fiber, and normalizations, with reduced, special fiber Y of compact type such that:*

(1) *For each irreducible component $Z \subset Y$ there is an extension $\mathcal{L}_Z$ of $\mathcal{L}$ to X with*

$$\deg(\mathcal{L}_{Z|Z}) = d,$$
$$\deg(\mathcal{L}_{Z|Z'}) = 0 \text{ for irreducible components } Z' \neq Z.$$

(2) *The images*

$$V_Z = \operatorname{im}(V \hookrightarrow \pi'_*(\mathcal{L}_Z) \xrightarrow{\text{restriction}} H^0(Z, \mathcal{L}_{Z|Z}))$$

form a limit g^r_d on Y.

See [**E-H2,3**] for applications of this result to Brill-Noether theory.

We will say that a limit g^r_d on a tree-like curve Y is *smoothable* if it can be obtained from a family with geometrically irreducible general fiber as in Theorem 1. Every limit g^1_d is smoothable, as is shown in [**H-M**]; an explicit analytic smoothing can actually be constructed with little effort. Unfortunately there are nonsmoothable g^r_d's with $r \geq 2$. But these only occur on rather atypical curves, as our next result shows:

THEOREM 2. *Let $X \to B$ be a family of tree-like curves of arithmetic genus g over an irreducible base B, and let $G^r_d(X/B)$ be the corresponding family of limit g^r_d's. Set $\rho = g-(r+1)(g-d+r)$. (ρ may be negative!) If $\dim G^r_d(X/B) \leq \dim B + \rho$, then every limit g^r_d on every curve of the family is smoothable.*

Curves satisfying the hypothesis of Theorem 2 (with B a point) may be found in [**E-H2,3**]. It is also satisfied (for every r, d) by the union of three general curves of genus g_1, g_2 with $g_1 + g_2 = g$, joined at general points of each, and by many other simple curves and families of curves.

Theorem 2 is proved by giving explicitly the "right number" of local equations for the family of g^r_d's (or rather, for a certain associated frame-bundle) in the neighborhood of a given limit g^r_d. This approach was suggested by conversations with Ziv Ran, to whom we are grateful.

We now indicate three applications beyond those of [**E-H2,3**]:

First, we may complete and simplify the ideas in the second half of [**H-M**] and [**H**], where it is shown that the moduli space M_g of curves of genus g has general type for g odd and ≥ 25 or even and ≥ 40:

APPLICATION 1 [**E-H5**]. M_g has general type for all $g \geq 24$.

For the proof of this we make use of the ideas and methods of the first 3 sections of [**H-M**] as described in the introduction to [**H**]; these methods require the choice and computation of a divisor in M_g with certain properties.

We distinguish 2 (overlapping) cases:

(i) If $g+1$ is not prime, then for suitable r and d we have

$$\rho = g-(r+1)(g-d+r) = -1,$$

and the closure of the set of smooth curves possessing a g_d^r forms a suitable divisor in M_g if $g \geq 24$. This covers in particular the cases g odd and $g =$ 24, 26.

(ii) If g is even, say $g = 2k-2$, and $g \geq 28$, we use the closure of the ramification divisor of the map from the moduli space of curves C of genus g with chosen pencil $\mathbf{C}^2 \cong V \subset H^0(C, \mathcal{L})$ of degree k to M_g, in accordance with the program expressed in the introduction to [**H-M**]. To circumvent the problem mentioned in the introduction [**H**] we interpret ramification as being signalled by the presence of a nonzero section of $K_C \otimes \mathcal{L}^{-2}$, where K_C is the canonical class of C.

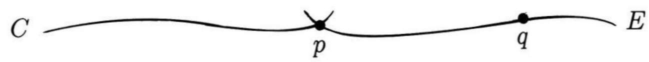
As a second application, we can complete, in a certain sense, the result of Fulton and Lazarsfeld [**F-L**] who prove (using the result of Gieseker proved in [**G**] and [**E-H3**]) that if C is a general curve, then the variety $G_d^r(C)$ of g_d^r's on C is irreducible as long as $\rho := g-(r+1)(g-d+r) > 0$. For $\rho = 0$ and C general, $G_d^r(C)$ is a reduced set of points. We prove:

APPLICATION 2 [**E-H4**]. Assume $\rho = g-(r+1)(g-d+r) = 0$. The fundamental group of the moduli space of curves C with $G_d^r(C)$ reduced and finite acts transitively by monodromy on each such $G_d^r(C)$. Equivalently, there is a family of such curves $X \to B$ such that the associated family $G_d^r(X/B)$ is irreducible.

The key to the proof of this is the fact that on the curve used in [**E-H3**] the different g_d^r's can be labelled, in the $\rho = 0$ case, by certain chains of Schubert cycles in a Grassmann variety. Further, if two of these chains differ in only one element, then a family of curves can be constructed (by allowing two "elliptic tails" to hang at varying points from one rational component of a curve as in [**E-H3**]) whose monodromy interchanges the corresponding g_d^r's. Since the simplicial complex of chains of Schubert cycles is connected in codimension 1 (even Cohen-Macaulay—see for example [**D-E-P**]), this suffices to prove transitivity.

APPLICATION 3. Certain semigroups occur as the Weierstrass semigroups of smooth curves. In particular, if $\Gamma = \{0, a_1, a_2, \dots\} \subset \mathbf{N}$ is a subsemigroup without common divisor of the natural numbers, then Γ occurs as the Weierstrass semigroup of a curve of genus $g = |\mathbf{N}-\Gamma|$ if $a_1 > w$ or, more particularly, $w \leq g/2$, where $w = \sum_{i=1}^{g+1}(g+i-a_i)$ is the weight of Γ. Moreover, there is at least one component of the subvariety of Weierstrass points with semigroup Γ, in M_g^1, with codimension$= w$.

This is proved inductively by smoothing "limit canonical series" on curves of the form



where C is a curve of genus $g-1$ with a suitable Weierstrass point p of a certain type, moving in a family whose dimension is the weight of p, E is an elliptic curve, $q-p$ is torsion of a suitable order, and the limit series is chosen to have ramification at q corresponding to a Weierstrass point of the desired type.

APPLICATION 4. The monodromy group acts on the g^3-g Weierstrass points of a general curve as the symmetric group on g^3-g letters.

This is proved by specializing to a reducible curve with a positive dimensional family of "limit canonical series", and examining the monodromy of this family.

REMARK. It seems possible to give a related, but substantially more complicated, description of "limit g^r_d's" on arbitrary stable curves. It may be possible to use this fact to study other types of Weierstrass points of low weight.

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DEPARTMENT OF MATHEMATICS, BRANDEIS UNIVERSITY, WALTHAM, MASSACHUSETTS 02254

DEPARTMENT OF MATHEMATICS, BROWN UNIVERSITY, PROVIDENCE, RHODE ISLAND 02912