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Linear Free Resolutions and Minimal Multiplicity

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INTRODUCTION

Let $S = k[x_1,..., x_n]$ be a polynomial ring over a field and let $M = \bigoplus_{\nu \geqslant -\infty} M_\nu$ be a finitely generated graded module; in the most interesting case M is an ideal of S. For a given natural number p, there is a great interest in the question: *Can M be generated by (homogeneous) elements of degree $\leqslant p$?* No simple answer, say in terms of the local cohomology of M, is known; but somewhat surprisingly the stronger question: *Can the jth syzygy of M be generated by elements of degree $\leqslant p + j$ for all $j = 0, 1,..., n$?* does admit a simple response. The following is neither new nor difficult to prove, though it seems not well known:

PROPOSITION. *With the notation above, the following are equivalent:*

(a) *The jth syzygy of M is generated by elements of degree $\leqslant p + j$ for all $j = 0, 1,...$.*

(b) *M is p-regular in the sense that the local cohomology $H^j_m(M)$ vanishes in degree $> p - j$ for $j = 0, 1,...$.*

(c) *The "truncation" $M_{\geqslant p} = \bigoplus_{\nu \geqslant p} = \bigoplus_{\nu \geqslant p} M_\nu$ of M at degree p admits a* linear free resolution: *that is, $M_{\geqslant p}$ is generated by forms of degree p, and all the elements of the matrices of the maps in the minimal free resolution of $M_{\geqslant p}$ are linear forms.*

(See Section 1 for a proof and comments on the origins of this result.)

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One of the two thrusts of this paper is to study modules which, like $M_{\geqslant p}$ in part (c) above, admit linear resolutions, and in particular study the homogeneous factor-rings $R = S/I$ of S, above, such that I is generated by forms of a given degree p and such that I admits a linear free resolution; we will call such homogeneous rings R "p-linear." (For other approaches to the study of linear resolutions, see Herzog, Simis, and Vasconcelos [16, Sect. 13], where a connection is made with d-sequences, proper sequences, and relative regular sequences, as well as the related work of Steurich [23].)

We write $\mathscr{M}$ for the maximal homogeneous ideal $(X_1,..., X_n)$ of S. If J is any ideal, then, for large p, the above proposition says that $J_{\geqslant p} = J \cap (X_1,..., X_n)^p$ admits a linear resolution. (So, for example, every projective scheme is defined by such an ideal!) But there is also another, more special, source of interesting examples. It is known that if $R = S/I$ is either Cohen–Macaulay, or geometrically integral (that is, I is prime and k is algebraically closed in R/I), then there is an inequality connecting the *multiplicity* or *degree* $\delta(R)$ of R (the *degree* of the corresponding projective scheme) and the *codimension* $\operatorname{codim} R = \dim_k R_1 - \dim R$ (or simply $n - \dim R$ if I contains no linear forms):

$$\delta(R) \geqslant 1 + \operatorname{codim} R.$$

The case of equality ("minimal" degree or multiplicity, or "maximal" codimension) is very special; in this case R is automatically Cohen–Macaulay (even if the original hypothesis was "geometrically integral"), I is generated by forms of degree $\leqslant 2$, and, assuming for simplicity that I contains no linear forms, I has a linear free resolution. This paper originated in, and has as its second thrust, an attempt to understand these phenomena and some related ones (Goto [11]), and to extend this understanding to the non-Cohen–Macaulay case.

The contents of the paper are as follows: In Section 1 we examine linear free resolutions in general, and characterize homogeneous rings with p-linear resolutions, and modules with linear resolution, both cohomologically in terms of "regularity" and by the behavior of the factor-ring or module, obtained from reduction modulo a general linear form in S. The characterization in terms of local cohomology was certainly known to Mumford and others (it is implicit, for example, in Mumford [20]) and part of it was used by Wahl [24], who credits Mumford, but seems not to have appeared explicitly in its full strength in the literature. We then deduce a number of the obvious numerical invariants of rings with p-linear resolution: the Hilbert function, Tor-series, and betti numbers. This extends work of Sally [21], Schenzel [22], Steurich [23], and others, who worked mostly in the Cohen–Macaulay case. We also explain a part of the relationship between p-linear rings and extremal Buchsbaum rings.

In an Appendix to Section 1, we show how to construct explicitly the "linear part" of the resolution of any module M from a knowledge of the two lowest degree parts of M and the linear transformations between them induced by multiplication by linear forms. This generalizes the ideas of Buchsbaum and Eisenbud [5] and Eisenbud, Riemenschneider, Schreyer [7] to this general, not necessarily Cohen–Macaulay, case. The technique has already been applied, as well, in the work of Akin, Buchsbaum, and Weyman [1].

In Section 2 we turn to the question of bounds on the multiplicity δ of a homogeneous ring $R = S/I$, in terms of the (embedding) codimension and local cohomology of R. The relations we get generalize the relation

$$\delta \geqslant 1 + \operatorname{codim} R$$

that holds in the Cohen–Macaulay case, and are essentially this relation plus "correction" terms coming from the cohomology. It is known that the relation above holds for graded domains over an algebraically closed field, and it seems very interesting to ask whether the correction terms that we introduce, at least the subtler ones of Theorem 2.1, vanish for the homogeneous coordinate rings of projective varieties. This is true at least for smooth varieties in characteristic 0 (even for the coarse version of the correction terms), by the Kodaira vanishing theorem. We deduce the results from a result which, roughly speaking, bounds the number of generators of a graded module in terms of its rank and local cohomology.

We say of a ring which achieves equality in our (coarse) estimate for the multiplicity, that it has *minimal multiplicity*; of a module which achieves the upper bound on the number of generators we say that it is *top heavy*. In Section 3 we study these two notions. We are able to characterize the rings of minimal multiplicity cohomologically, and we show that they always have 2-linear resolutions, generalizing the Cohen–Macaulay case. We also classify completely the top-heavy modules; modulo their Artinian parts, they are just direct sums of certain shifts of syzygies of the residue class field. More generally, we give a condition for a module to be determined by certain of its local cohomology modules.

In Section 4, we examine domains with 2-linear resolutions. We show among other things that if $R = S/I$ is such a domain, or, more generally, if R is a homogeneous domain of dimension d and if there are parameters $x_1, \dots, x_d \in R$ such that $\mathscr{M}_R^2 \subset (x_1, \dots, x_d) R$, then the integral closure of R is obtained by extending the groundfield only, and is Cohen–Macaulay with 2-linear resolution (and minimal multiplicity, therefore). This generalizes certain features of the classification theorem of Castelnuovo–Bertini of homogeneous domains R over an algebraically closed field with multiplicity

$\delta \leqslant 1 + \operatorname{codim} R$. (We review this theorem in a somewhat generalized form due to Xambò [25].)

We have already mentioned that large truncations of any module have linear resolutions. For a homogeneous ring $R = S/I$, this amounts to saying that for all large p, the ring

$$R' = S/(I \cap \mathscr{M}^p)$$

has p-linear resolution. Of course $I \cap \mathscr{M}^p$ is never prime for large p, so it seems particularly interesting to ask about domains with p-linear resolution. Section 5 presents two (overlapping) contexts in which these appear. In part (a) the curves in P^3 whose homogeneous coordinate rings have p-linear resolutions are described as curves of certain degrees and genera satisfying the "maximal rank" condition, conjecturally satisfied by all "sufficiently general" curves. The simplest (reduced and irreducible) non-Cohen–Macaulay example is that of a smooth elliptic quintic; the homogeneous coordinate ring of any such curve has 3-linear resolution

$$0 \to S(-5) \to S^5(-4) \to S^5(-3) \to S \to R \to 0$$

(since R is codimension 2, not 3, it is not Gorenstein).

In part (b) of the section we give a construction based on the "Bertini" Theorem of Kleiman [18] (or, in some cases, "Bourbaki's Theorem") to construct examples starting with S-modules that are the exterior powers of the S-module associated to the tangent bundle on $\mathbb{P}^{n-1}$ (the syzygies of the residue class field).

Perhaps the most interesting open problem concerning linear resolutions is concerned with finding sharp bounds on the Castelnuovo regularity of "nice" modules or ideals (sharp bounds are known for arbitrary modules and ideals; see the forthcoming thesis of Bayer [3] for details). For example, consider a prime ideal I of $S = k[X_1, \ldots, X_n]$. For simplicity we may suppose $I \subset \mathscr{M}^2$. What is the smallest p for which $I \cap \mathscr{M}^p$ has a linear resolution? If I is the homogeneous ideal of a reduced irreducible *curve* of degree δ, then Gruson, Lazarsfeld, and Peskin [12] have recently shown that $p \leqslant \delta - n + 3$, improving the result of Castelnuovo that $p \leqslant \delta - 1$ for smooth curves. (By contrast, the bound for the homogeneous ideal of a 1-dimensional subscheme has the form

$$p \leqslant \binom{\delta}{2} + 1 - p_a,$$

where p_a is the arithmetic genus.) In the light of this and some further examples, it seems natural to guess:

Conjecture. If I is a prime ideal of S and R/I has multiplicity δ, then

$$p \leqslant \delta - \operatorname{codim} R + 2.$$

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0. Preliminaries

We collect here a number of more or less standard definitions, results, and notations. The experienced reader is advised to skip this section. References for many of the unproved assertions can be found in the book of Altman and Kleiman [2], the section on local cohomology in Herzog and Kunz [15] and the notes of Eisenbud [6].

A *homogeneous ring* over (a field) k is a graded ring $R = R_0 \oplus R_1 \oplus \cdots$, with $R_0 = k$, generated by R_1 as a k-algebra. We write $\mathscr{M}$ for the maximal ideal $R_1 \oplus R_2 \oplus \cdots$ of R, and note that, with respect to the theory of graded modules, R behaves as though it were local.

Now let R be a homogeneous ring over k and let M and N be graded R-modules.

If p is an integer, then we write M_p for the pth graded component of M, so that $M = \oplus_{-\infty}^{\infty} M_p$. If q is an integer, we write $M(q)$ for the qth shift of M, defined by the formula

$$M(q)_p = M_{p+q}.$$

We say that M is *free* if $M \cong \oplus_i R(q_i)$ for suitable q_i. When we write isomorphisms and exact sequences of graded modules, we will always arrange that the maps are graded of degree 0. However, we write $\operatorname{Hom}(M, N)$ for the set of homogeneous maps of all degrees; it is again a graded module, graded by degrees of maps. Since graded free resolutions exist, this construction extends to a grading on the modules $\operatorname{Ext}_R^q(M, N)$. Similar remarks hold for $\otimes$ and Tor.

Now suppose M is finitely generated.

If V is a k-vectorspace, we will write $|V|$ for $\dim_k V$. For $p \gg 0$ the Hilbert function $|M_p|$ is a polynomial in p. The degree of this polynomial, increased by 1, is $\dim M$, the (Krull-) dimension of M. The *degree* (or *multiplicity*) of M is by definition $(\dim M)!$ times the leading coefficient of this polynomial.

If

$$\cdots \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0 \to M \to 0$$

is a minimal (graded) free resolution of M, then we write $\mathrm{Syz}_q M$ for Coker $(F_{q+1} \xrightarrow{\phi_{q+1}} F_q)$, the qth syzygy of M. We write $M^* = \mathrm{Hom}(M, R)$ for the graded dual of M and $\mathrm{Tr}\, M = \mathrm{Coker}(F_0^* \xrightarrow{\phi_1^*} F_1^*)$, the *transpose* of M in the sense of Auslander.

By the depth of M we will mean the length of a maximal M-regular sequence in $\mathscr{M}$.

The local cohomology modules $H^q_{\mathscr{M}}(M)$ are naturally graded modules; we set

$$h^q_{\mathscr{M}}(M)_p = |H^q_{\mathscr{M}}(M)_p|.$$

If $R = k[X_1, ..., X_n]$ is the polynomial ring in n variables, with degree $X_i = 1$, then R satisfies "local duality" in the graded form

$$\mathrm{Ext}^q_R(M, R(-n)) \cong \mathrm{Hom}_R(H^{n-q}_{\mathscr{M}}(M), k).$$

If $\tilde{M}$ denotes the sheaf associated to M on the scheme $X = \mathrm{Proj}\, R$, then we have an exact sequence

$$0 \to H^0_{\mathscr{M}}(M) \to M \to \bigoplus_{-\infty}^{\infty} H^0(X, \tilde{M}(p)) \to H^1_{\mathscr{M}}(M) \to 0$$

and isomorphisms

$$H^q_{\mathscr{M}}(M) \cong \bigoplus_{-\infty}^{\infty} H^{q-1}(X, \tilde{M}(p)) \qquad \text{for} \quad q \geqslant 2.$$

Finally, if $\mathbf{x} = x_1, ..., x_r$ is a sequence of homogeneous elements of R, then the Koszul complex $\mathbb{K}(\mathbf{x})$ is naturally graded: if $\deg x_i = p_i$ then $\mathbb{K}(\mathbf{x})$ has the form

$$\mathbb{K}(\mathbf{x}): \cdots \to \bigoplus_{i<j} R(-p_i - p_j) \to \bigoplus_i R(-p_i) \to R.$$

We write $\mathbb{K}(\mathbf{x}, M)$ for $\mathrm{Hom}_R(\mathbb{K}(\mathbf{x}), M)$. We denote the homology by $H^q(\mathbf{x}, M)$, and we write $h^q(\mathbf{x}, M)_p$ for $|H^q(\mathbf{x}, M)_p|$. Thus $H^q(\mathbf{x}, M)$ is the homology at the indicated point, of a complex of the form

$$0 \to M \to \bigoplus_i M(p_i) \to \cdots \to \underbrace{\bigoplus_{i_1 < \cdots < i_q} M(p_{i_1} + \cdots + p_{i_q})}_{q\text{th step}} \to \cdots.$$

There is a spectral sequence, derived for example from the double complex underlying an injective resolution of the complex $\mathbb{K}(\mathbf{x}, M)$, of the form

$$H^p_{\mathscr{M}}(\mathbf{x}, H^q(M)) \Rightarrow H^{p+q}(\mathbf{x}, M).$$

1. Rings and Modules with Linear Resolution

Let $S = k[x_1,..., x_n]$ be a homogeneous polynomial ring, with k a field, and let M be a finitely generated graded S-module. We will say that M has *p-linear resolution* (over S) if the minimal graded free resolution of M has the form

$$\cdots \to S^{\beta_2}(-p-2) \to S^{\beta_1}(-p-1) \to S^{\beta_0}(-p) \to M \to 0$$

for suitable integers β_i; that is, M has *p-linear resolution* if $M_r = 0$ for $r < p$, M is generated by M_p, and M has a resolution where all the maps are represented by matrices of linear forms.

It turns out that $M_{\geqslant p}$ always has p-linear resolution if p is large enough:

Proposition 1.1. *Let M be a finitely generated graded $S = k[x_1,..., x_n]$-module. The module $M_{\geqslant p}$ has p-linear resolution if and only if*

$$p \geqslant \max\{t \mid \mathrm{Tor}_S^q(M, k)_{t+q} \neq 0 \text{ for some } q\}.$$

Proof. By definition, $M_{\geqslant p}$ has p-linear resolution if and only if

$$\mathrm{Tor}_q(M_{\geqslant p}, k)_r = H_q(\mathbb{K}(x_1,..., x_n) \otimes M_{\geqslant p})_r = 0$$

for $r \neq p + q$.

From the graded structure it is clear that for $r < p + q$,

$$H_q(\mathbb{K}(x_1,,..., x_n) \otimes M_{\geqslant p})_r = 0,$$

while for $r > p + q$

$$\begin{aligned} H_q(\mathbb{K}(x_1,...,, x_n) \otimes M_{\geqslant p})_r &= H_q(\mathbb{K}(x_1,..., x_n) \otimes M)_r \\ &= \mathrm{Tor}_S^q(M, k)_r, \end{aligned}$$

and the desired result follows.

In view of this simple result there is great interest in knowing the smallest p for which a module $M_{\geqslant p}$ has a p-linear resolution.

It turns out that the condition for $M_{\geqslant p}$ to have a p-linear resolution can be conveniently stated as a cohomological condition called p-regularity which was first used by Castelnuovo in the study of space curves. This connection was remarked by Mumford, and is used, for example in the paper of Wahl [24]. The condition, adapted from the case of coherent sheaves on $\mathbb{P}^{n-1}$, is

Definition. M is *p-regular* if, for all q,

$$H^q_{\mathscr{M}}(M)_r = 0 \qquad \text{for} \quad r = p - q + 1$$

and $H^0_{\mathscr{M}}(M)_r = 0$ for all $r \geqslant p + 1$.

In the case of greatest interest to us, M is an ideal I contained in $\mathscr{M}^2$; abusing the terminology, we say that $R = S/I$ *has p-linear resolution* if I has, and that R *is p-regular* if I is. This notion really depends only on R, since S is simply the polynomial ring generated by R_1. Since $H^q_{\mathscr{M}}(S)_r = 0$ unless $q = n$ and $r \leqslant -n$, the long exact sequence coming from the short exact sequence $0 \to I \to S \to R \to 0$ shows that: *R is p-regular if and only if*

$$H^q_{\mathscr{M}}(R)_{p-q} = 0 \qquad \text{for all } q.$$

It is well known, and follows easily from the exact sequence associated to a non-zerodivisor on $M/H^0_{\mathscr{M}}(M)$ (or $R/H^0_{\mathscr{M}}(R)$) that p-regularity implies p'-regularity for all $p' \geqslant p$; that is, in the case of a factor ring, $R = S/I$, for example, that if R is p-regular, then

$$H^q_{\mathscr{M}}(R)_r = 0$$

for all $r \geqslant p - q$.

The fundamental connection between regularity and p-linearity is:

THEOREM 1.2. *Let $S = k[x_1, ..., x_n]$.*

(1) *If M is a finitely generated S-module, then $M_{\geqslant p}$ has p-linear resolution if and only if M is p-regular.*

(2) *If $R = S/I$ is a factor ring of S, then $R' = S/(I \cap \mathscr{M}^p)$ has p-linear resolution if and only if R is p-regular. In particular, R has p-linear resolution if and only if R is p-regular and $I \subset \mathscr{M}^p$.*

Proof. In view of the definitions and the fact that $I \cap \mathscr{M}^p = I_{\geqslant p}$, part (2) follows at once from part (1).

To prove part (1), we may suppose that $p = 0$ and $M = M_{\geqslant 0}$. Suppose, first, that M has a linear resolution of the form:

$$\mathbb{F}\colon \cdots \to S^{\beta_2}(-2) \to S^{\beta_1}(-1) \to S^{\beta_0} \to M \to 0.$$

Since

$$\operatorname{Ext}^i_S(M, S) = H^i(\operatorname{Hom}_S(\mathbb{F}, S)),$$

we see at once that $\operatorname{Ext}^i_S(M, S)_t = 0$ for $t < -i$. By duality,

$$\operatorname{Ext}^i_S(M, S(-n))^v = H^{n-i}_{\mathscr{M}}(M),$$

whence, setting $q = n - i$

$$H^q_{\mathscr{M}}(M)_r = 0 \qquad \text{for} \quad r > p - q,$$

as required.

Next, suppose that M is 0-regular. Since the residue class field k has the Koszul complex $\mathbb{K}(x_1,..., x_n)$ as a linear resolution, the following observation allows us to reduce to the case $H^0_{\mathcal{M}}(M) = 0$:

LEMMA 1.3. *If M is a graded S-module such that $M_r = 0$ for $r < 0$ and $H^0_{\mathcal{M}}(M)_r = 0$ for $r > 0$, then $H^0_{\mathcal{M}}(M)$ is a direct summand of M and is a direct sum of copies of k, all in degree* 0.

Proof. If M'_0 is a vectorspace complement for $H^0_{\mathcal{M}}(M)_0$ in M_0, and M' is the submodule $M' = M'_0 + M_1 + M_2 + \cdots \subset M$, then $M = H^0_{\mathcal{M}}(M) \oplus M'$. The rest is immediate.

We may now assume that $H^0_{\mathcal{M}}(M) = 0$, so that M has positive depth. Since we may harmlessly assume that k is infinite, we may suppose that some $x \in S_1$ is a non-zerodivisor on M. From the cohomology exact sequence associated to

$$0 \to M(-1) \xrightarrow{x} M \to M/xM \to 0$$

we see that M/xM is 0-regular, so by induction on the dimension of M, we may suppose that M/xM has linear resolution as an S/xS-module. But if $\mathbb{F}$ is the minimal free resolution of M over S, then $\mathbb{F} \otimes S/xS$ is the minimal free resolution of M/xM over S/xS, so the result follows.

Remarks. Familiar examples of modules with linear resolution are the syzygies of a polynomial ring modulo a power of its homogeneous maximal ideal. Others related to determinantal ideals can be found in Buchsbaum and Eisenbud [4] and Akin, Buchsbaum, and Weyman [1].

We see that if M is a module over $k[x_1,..., x_n]$, and $M_r = 0$ for $r < 0$, then the trivial extension $k[X_1,..., X_n] \propto M(-1)$ has 2-linear resolution (as a ring) if and only if M has linear resolution. Further, if R has p-linear resolution over S, then $S \propto R(-1)$ has $p+1$-linear resolution. This gives a first family of examples of rings with p-linear resolutions. Further examples will be considered later.

The next result allows one to do inductive arguments on rings and modules with linear resolutions by factoring out general linear forms, almost as if everything were Cohen–Macaulay. It was partly inspired by the criterion for linear resolutions of Herzog, Simis, and Vasconcelos [16] (see the remark at the end of this section).

THEOREM 1.4. *Let $S = k[x_1,..., x_n]$.*

(a) *If M is a finitely generated graded S-module, and $x \in S_1$ is a non-zerodivisor on $M/H^0_{\mathcal{M}}(M)$, then M has linear resolution if and only if M/xM has a linear resolution and $(0:_M x) \subset M_0$. In this case, if $\mathbb{F}$ is the minimal* *free resolution of M over S and $\mathbb{G}$ is the minimal free resolution of M over S/xS, and if $y_1,\dots,y_{n-1}$ generate $(S/xS)_1$, then*

$$\mathbb{F}\otimes S/xS \cong \mathbb{G}\oplus(\mathbb{K}(y_1,\dots,y_{n-1};S)(-1)[-1])^{h^0_{\mathcal{M}}(M)}.$$

(Here the $[-1]$ indicates that the Koszul complex is shifted to the left.)

(b) *If R is a factor-ring of S with $R_1=S_1$, and $x\in R_1$ is a non-zerodivisor on $R/H^0_{\mathcal{M}}(R)$, then R has p-linear resolution if and only if R/xR has p-linear resolution and $(0:_R x)\subseteq R_{p-1}$.*

In this case, if $\mathbb{F}$ is a minimal free resolution of R as an S-module, $\mathbb{G}$ is a minimal free resolution of R/xR as an S/xS-module, and $y_1,\dots,y_{n-1}$ generate $(S/xS)_1$, then

$$\mathbb{F}\otimes S/xS\cong G\oplus(\mathbb{K}(y_1,\dots,y_{n-1})(-p)[-1])^{(h^0_{\mathcal{M}}(R))_{p-1}}.$$

Note. If $M=H^0_{\mathcal{M}}(M)$ or $R=H^0_{\mathcal{M}}(R)$, the condition "x is a non-zerodivisor on $M/H^0_{\mathcal{M}}(M)$" or "on $R/H^0_{\mathcal{M}}(R)$" is to be interpreted as "$x\neq 0$." The proof simplifies somewhat in this case.

Proof. (a) If M has linear resolution, the result is immediate from Lemma 1.3. If M/xM has linear resolution, then at least $M_r=0$ for $r<0$. Further, the hypothesis $(0:_M x)\subset M_0$ implies $(0:_M x)=H^0_{\mathcal{M}}(M)$, and thus, as in Lemma 1.3, $M=M'\oplus H^0_{\mathcal{M}}(M)$, with x a non-zerodivisor on M'. Again, the result is immediate.

(b) First, suppose that R has p-linear resolution, and write $R=S/I$, so that $R/xR=(S/xS)/(I/xS\cap I)$. Of course $I(p)$ is an S-module with linear resolution, so $(I/xI)(p)$ is an S/xS-module with linear resolution.

We will show that

$$(0:_R x)=(H^0_{\mathcal{M}}(R))_{p-1}\cong(xS\cap I/xI)(1). \qquad (*)$$

In view of the exact sequence

$$0\to(xS\cap I)/xI\to I/xI\to I/(xS\cap I)\to 0$$

this will imply that

$$(xS\cap I)/xI=H^0_{\mathcal{M}}(I/xI).$$

Of course if $\mathbb{F}'$ is the minimal free resolution of I, then $\mathbb{F}'\otimes S/xS$ is the minimal S/x – free resolution of I/xI; applying Lemma 1.3, we will be done.

To establish (*), note first that

$$(xS\cap I)/xI=\operatorname{Tor}_1^S(S/xS,R).$$

From the exact sequence $0 \to H^0_{\mathcal{M}}(R) \to R \to R/H^0_{\mathcal{M}}(R) \to 0$ we get

$$\operatorname{Tor}_1^S(S/xS, R) \cong \operatorname{Tor}_1^S(S/xS, H^0_{\mathcal{M}}(R)),$$

and from $0 \to S(-1) \to^x S \to S/xS \to 0$ we get

$$\operatorname{Tor}_1^S(S/xS, H^0_{\mathcal{M}}(R)) \cong (0:_{H^0_{\mathcal{M}}(R)} x)(-1).$$

Since $(H^0_{\mathcal{M}}(R))_r = 0$ for $r > p-1$ and $(H^0_{\mathcal{M}}(R))_r \subset R_r = S_r$ for $r \leqslant p-1$ we see that $(0:_{H^0_{\mathcal{M}}(R)} x) = (H^0_{\mathcal{M}}(R))_{p-1}$, and the proof is complete.

Next suppose that R/xR has p-linear resolution (necessarily as a factor ring of S/xS) and that $(0:_R x) \subset R_{p-1}$. Using the above identification, we see that $(xS \cap I)/xI = H^0_{\mathcal{M}}(I/xI)$, and is concentrated in degree p, so that $(I/xI)(p)$ has linear resolution. Since x is a non-zerodivisor on I, this implies that $I(p)$ has linear resolution, so R has p-linear resolution as required.

COROLLARY 1.5. *Let R be a homogeneous ring of dimension d with p-linear resolution over an infinite field k. If $x_1,\dots,x_d \in R_1$ is a general system of parameters, and $x_1,\dots,x_d, y_1,\dots,y_c$ is a vectorspace basis of R_1, then*

$$R/(x_1,\dots,x_d) = k[y_1,\dots,y_c]/(y_1,\dots,y_c)^p.$$

Proof. By Theorem 1.3, $R/(x_1,\dots,x_d)$ has p-linear resolution. By Theorem 1.1,

$$(R/(x_1,\dots,x_d))_r = H^0_{\mathcal{M}}(R/(x_1,\dots,x_d))_r = 0 \qquad \text{for} \quad r \geqslant p.$$

On the other hand, we may write $R/(x_1,\dots,x_d) = k[y_1,\dots,y_c]/J$ for some ideal $J \subseteq (y_1,\dots,y_c)^p$, so $J = (y_1,\dots,y_c)^p$ as required.

We digress for a moment to point out that p-regularity is alone enough to guarantee part of this.

PROPOSITION 1.6. *Let R be a homogeneous ring of dimension d over an infinite field, and let $x_1,\dots,x_d \in R_1$ be a general system of parameters. If R is p-regular, then $\mathcal{M}^p \subseteq (x_1,\dots,x_d)R$.*

Proof. One sees immediately that $R/H^0_{\mathcal{M}}(R)$, and thus $R/(H^0_{\mathcal{M}}(R), x_1R)$, are p-regular with R. It follows from the exact sequence

$$0 \to H^0_{\mathcal{M}}(R)/x_1R \cap H^0_{\mathcal{M}}(R) \to R/x_1R \to R/(H^0_{\mathcal{M}}(R), x_1R) \to 0$$

that R/x_1R is also p-regular with R. Induction on $\dim R$ now finishes the argument.

From Corollary 1.5 we see that Cohen–Macaulay rings with p-linear resolutions are just liftings of $k[y_1,\dots,y_c]/(y_1,\dots,y_c)^p$ modulo regular sequences (of course this can be proved directly quite easily!). Such

Cohen–Macaulay rings have been rather extensively studied, see, for example, Wahl [24] and Sally [21] for the case $p=2$, Schenzel [22] and Eisenbud, Riemenschneider, and Schreyer [7], where a universal family of resolutions is made explicit, for the general case. In particular, the following results are rather easy:

PROPOSITION 1.7. *If R is a homogeneous Cohen–Macaulay ring with p-linear resolution over the polynomial ring S, and if R has dimension d and codimension c $(=|R_1|-d)$, then*:

(a) *The Hilbert function of R is*

$$H_{c,d,p}(n) := |R_n| = \begin{cases} \binom{c+d+n-1}{c+d-1} & \text{for } n<p \\ \sum_{i=0}^{d-1} \binom{n-p+i}{i}\binom{c+d+p-2-i}{p-1} & \text{for } n \geqslant p. \end{cases}$$

(b) *In particular, the multiplicity of R is $\binom{c+p-1}{c}$.*

(c) *The Betti numbers of R as an S-module—that is, the ranks of the free modules in a minimal free resolution of R—are given by*

$$\beta_q(R) = \beta_q(c,p) := \begin{cases} 1 & \text{for } q=0 \\ \binom{c+p-1}{q+p-1}\binom{q+p-2}{p-1} & \text{for } q>0. \end{cases}$$

(d) *In particular, if we write $R=S/I$, then the number of generators of I is $\beta_1(R)=\binom{c+p-1}{c-1}$.*

Using Theorem 1.4 and Corollary 1.5 we can compare these formulae with corresponding formulae for rings with p-linear resolutions. The difference is conveniently expressed in terms of the following invariants. If R is defined over an infinite field k, we choose elements $x_i \in R_1$ $(i=1,\dots,d-1)$ such that x_{i+1} is a non-zerodivisor on

$$(R/(x_1,\dots,x_i))/H^0_{\mathcal{M}}(R/(x_1,\dots,x_i)),$$

and we set

$$s_i(R) := \left| \frac{(x_1,\dots,x_{i-1}):x_i}{(x_1,\dots,x_{i-1})} \right| \qquad (i=0,\dots,(\dim R)-1).$$

If R is defined over a finite field k, and if K is an infinite extension of k, we set $s_i(R) := s_i(R \otimes_k K)$; this is independent of K.

If R has p-linear resolution, we can compute the $s_i(R)$ differently:

COROLLARY 1.8. *If R is a homogeneous ring with p-linear resolution over an infinite field, and $x_i \in R_1$ are elements such that x_{i+1} is a non-zerodivisor on $(R/(x_1,..., x_i))/H^0(R/x_1,..., x_i)$, then*

$$\frac{(x_1,..., x_i): x_{i+1}}{(x_1,,..., x_i)} = H^0_{\mathfrak{m}}(R/(x_1,..., x_i))_{p-1}$$
$$= \text{socle } R/(x_1,..., x_i).$$

Proof. By Theorem 1.3 it suffices to treat the case $i = 0$, and then the result is easy.

PROPOSITION 1.9. *If R is a homogeneous ring with p-linear resolution over the polynomial ring $S = k[x_1,..., x_{c+d}]$, and if R has dimension d, codimension c, and $s_i(R) = s_i$, then:*

(a) *The Hilbert function of R is*

$$H_R(n) =: |R_n| = H_{c,d,p}(n) - \sum_0^{d-1} s_i \binom{n-p+i}{i}.$$

(b) *In particular the multiplicity of R is*

$$\binom{c+p-1}{c} - s_{d-1}.$$

(c) *The Betti numbers of R are given by*

$$\beta_q(R) = \beta_q(c, p) + \sum_{i=0}^{d-1} s_i \binom{n-i-1}{q-1}.$$

(d) *In particular, if $R = S/I$, then the minimal number of generators of I is*

$$\beta_1(R) = \binom{c+p-1}{c-1} + \sum_{i=0}^{d-1} s_i.$$

(e) *R is a Golod ring and the 2-variable Poincaré series*

$$P_R(s, t) := \sum |\mathrm{Tor}_m^R(k, k)_n| \, s^m t^n$$

is given by

$$P_R(s, t) = \frac{(1+st)^{c+d}}{1 - \sum_{i \geqslant 1} \beta_i(R) \, s^{i+1} t^{p+i-1}}.$$

Proof. Extending k if necessary, we may assume it is infinite, and we may then choose an element $x \in R_1$ which is a non-zerodivisor on $R/H^0_{\mathscr{M}}(R)$. By Theorem 1.3, we have $(0\colon x) \subset R_{p-1}$, and we get an exact sequence

$$0 \to (0\colon x)(-1) \to R(-1) \xrightarrow{x} R \to R/xR \to 0.$$

Since R/xR has p-linear resolution, we may do induction; parts (a) and (b) of the proposition reduce to standard identities on binomial coefficients, and parts (c) and (d) follow from the explicit information on the resolution in Theorem 1.3.

For part (e), note that the second statement follows from the first by standard techniques; essentially, if R is Golod, the $\mathrm{Tor}^R_*(k, k)_*$ is, as a doubly graded vectorspace, the tensor algebra on the doubly graded vectorspace $H_{*+1}(x_1,..., x_{c+d}; R)_*$; see for example the book of Gulliksen and Levin [13]. To establish that R is Golod, we must prove that the Massey products on

$$H_*(x_1,..., x_{c+d}; R)_*$$

are 0. But, having shown inductively that Massey products of orders $\leqslant m-1$ are 0, a Massey product of order $m \geqslant 2$ will be a map of the form

$$\phi\colon \bigotimes_{i=1}^{m} H_{q_i}(\mathbf{x}; R)_{r_i} \to H_{(\sum q_i)+m-2}(\mathbf{x}; R)_{\sum r_i},$$

where $q_i \geqslant 1$. Since $q > 0$, $H_q(\mathbf{x}; R)_r$ can only be nonzero if $r = q + p - 1$. We see that if the source of ϕ is nonzero, then the target has the form

$$H_{(\sum q_i)+m-2}(\mathbf{x}; R)_{(\sum q_i)+m(p-1)},$$

and since $m(p-1) \neq (m-2) + p - 1$ for $p \geqslant 2$, $m \geqslant 2$, we see that the target is 0.

Remark. A more general, and perhaps more elegant, but less directly applicable version of some of the above results may be given as follows (we leave the proofs to the reader):

Let M be a graded S-module with linear free resolution. Set

$$U_0(M) = \dim_k H^0_{\mathscr{M}}(M),$$

and, for $x_1,..., x_n$ a sufficiently general sequence of linear forms, and $i \geqslant 1$,

$$U_i(M) = \dim_k H^0_{\mathscr{M}}(M/(x_1,..., x_i)\, M) - \dim_k H^0_{\mathscr{M}}(M/(x_1,..., x_{i-1})\, M).$$

Then the Betti numbers β_i of M are related to the $U_i = U_i(M)$ by a Pascal diagram

$$
\begin{array}{ccccccc}
 & & & U_0 & & & \\
 & & & & U_1 & & \\
 & /\!/ & \cdot^{\cdot^{\cdot}} & & & \ddots & \\
/\!/ & & & & & & U_n \\
\beta_n & \beta_{n-1} & & & \beta_0 & & 0
\end{array}
$$

(that is, each element in the interior of the diagram is the sum of the two elements above it.)

Further, we may write the Hilbert function of M as

$$H_M(v) = \sum_{i=0}^{n} U_i \binom{v+i-1}{i-1} \qquad (v \geqslant 0).$$

(Here $\binom{a}{b}$ is to be interpreted as 0 if $a < b$; further $\binom{a}{b} = 1$ for every a, and $\binom{a}{-1} = 1$ for $a = -1$, and 0 otherwise.)

Corollary 1.10. *Let $S = k[X_1, \ldots, X_n]$ and let $R = S/I$ be a homogeneous factor ring with p-linear resolution of the form*:

$$
\begin{array}{c}
0 \to S^{\beta_u}(-p-u+1) \xrightarrow{\phi_u} \cdots \to S^{\beta_1}(-1) \xrightarrow{\phi_1} S \to R \to 0. \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \|_{\beta_0} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad S
\end{array}
$$

If height $I \geqslant 2$, then

$$p = 1 + \sum_j (-1)^j j\beta_j.$$

Proof. A combinatorial deduction from Proposition 1.9 is left to the interested reader. We indicate a different approach: Set $r_i = \operatorname{rank} \phi_i$, and let d_i be the degree of the $r_i \times r_i$ minors of ϕ_i, so that $d_1 = p$, and $d_i = r_i$ for $i > 1$. From the "first Structure Theorem" of Buchsbaum and Eisenbud [4] it follows at once that $\sum_{i=1}^{u} (-1)^i d_i = 0$, or

$$p = -\sum_{i=2}^{u} (-1)^i r_i.$$

On the other hand,

$$r_i = \sum_{j \geqslant i} (-1)^{j-i} \beta_j$$

substituting this into the expression above, and simplifying, we get the desired result.

COROLLARY 1.11. *If $S = k[x_1,..., x_n]$ is a polynomial ring and if $R = S/I$ is a homogeneous ring of codimension c with p-linear resolution, then the following are equivalent*:

(a) *R is Cohen–Macaulay*;

(b) *The minimal number of generators of I is $\leqslant \binom{c+p-1}{c-1}$*;

(c) *$\beta_q(R) \leqslant \beta_q(c, p)$ for some $q \leqslant c+1$*;

(d) *$s_{d-1} = 0$*;

(e) *The multiplicity of R is $\geqslant \binom{c+p-1}{c}$.*

If these conditions are satisfied, then all the inequalities are equalities.

Proof. (a) $\Rightarrow$ (b) from Proposition 1.7.

(b) $\Rightarrow$ (c) from Proposition 1.8, parts (c) and (d).

(c) $\Rightarrow$ (d) from Proposition 1.8, part (c).

(d) $\Rightarrow$ (e) from Proposition 1.8, part (b).

(e) $\Rightarrow$ (a) Assuming, as we may, that k is infinite, and choosing a general system of parameters $x_1,..., x_d \in R_1$, we have $|R/(x_1,..., x_d)| = \binom{c+p-1}{c}$ by Corollary 1.4. Since, quite generally, $\operatorname{mult}(R) \leqslant |R/(x_1,..., x_d)|$, with equality only if R is Cohen–Macaulay, we see that (e) $\Rightarrow$ (a).

In the Cohen–Macaulay case all the inequalities become equalities by Proposition 1.8.

Remark. In the 2-linear case it is obvious that the natural map socle $R \to$ socle R/xR (for general $x \in R_1$) is an inclusion, so we have $s_0 \leqslant s_1 \leqslant \cdots \leqslant s_{d-1}$, and it is clear that $s_{d-1} = 0$ implies $s_i = 0$ for all i, as required by the corollary. In general, however, the s_i may even be strictly decreasing: for example, if f is a form of degree $p-k$ in $S = k[x_1,..., x_n]$ and $I = (x_1,..., x_n)^k f$, then $R = S/I$ has p-linear resolution, and

$$s_i = \binom{n-i+k-1}{k-1} \qquad \text{for} \quad i = 0,..., d-1 = n-2.$$

It is also possible to bound the Betti numbers of R from above, by using Proposition 1.8, part (d) and the following generalization of it:

PROPOSITION 1.12. *If R is a homogeneous ring with p-linear resolutions and $|R_1| = n$, then*

$$\beta_q(R) \leqslant \beta_q(n,p) = \binom{n+p-1}{q+p-1}\binom{q+p-2}{p-1}.$$

Proof. Write $R = S/I$ with $S = k[X_1,..., X_n]$. The proposition follows by applying the following lemma to the inclusion $I \to (X_1,..., X_n)^p$.

LEMMA 1.13. *If $M' \subset M$ are two S-modules with linear resolutions, then M/M' also has a linear resolution and*

$$\beta_i(M) = \beta_i(M') + \beta_i(M/M').$$

Proof. For each i we have an exact sequence

$$\operatorname{Tor}_i^S(M', k) \to \operatorname{Tor}_i^S(M, k) \to \operatorname{Tor}_i^S(M/M', k) \to \operatorname{Tor}_{i-1}(M', k),$$

so we see that $\operatorname{Tor}_i^S(M/M', k)_r = 0$ for $r > i$. Of course, for any module N with $N_r = 0$ for $r < 0$ we have $\operatorname{Tor}_i^S(N, k)_r = 0$ for $r < i$, so we see that M/M' has linear resolution and the sequence

$$0 \to \operatorname{Tor}_i^S(M', k)_i \to \operatorname{Tor}_i^S(M, k)_i \to \operatorname{Tor}_i^S(M/M', k)_i \to 0$$

is exact, whence the conclusion.

The following proposition and its corollary exhibit the link between our theory and the theory of extremal Buchsbaum rings that was one of our starting points (see also Steurich [23]).

PROPOSITION 1.14. *Suppose that $R = S/I$ has p-linear resolution. Then the canonical map*

$$[\operatorname{Ext}_S^i(S/\mathcal{M}, R)]_{p-(i+1)} \to [H^i_{\mathcal{M}}(R)]_{p-(i+1)}$$

is surjective for every $i \in \mathbb{Z}$.

Proof. Let

$$0 \to R \to J^0 \to \cdots \to J^i \xrightarrow{f^i} J^{i+1} \to \cdots \qquad (\#)$$

be a graded minimal injective resolution of the S-module R. Apply the functor $H^0_{\mathcal{M}}(\cdot)$ to it and get a complex

$$0 \to H^0_{\mathcal{M}}(J^0) \to \cdots \to H^0_{\mathcal{M}}(J^i) \to H^0_{\mathcal{M}}(J^{i+1}) \to \cdots$$

of graded artinian injective S-modules. The local cohomology modules $H^i_{\mathcal{M}}(R)$ appear as cohomology modules of this complex. We put $I^i = H^0_{\mathcal{M}}(J^i)$

and $E = E_s(S/\mathscr{M})$, the injective hull of the graded S-module $S/\mathscr{M}$. Then the structure theorem of injective modules allows us to express

$$I^i = \bigoplus_{j=1}^{n_i} E(-a_{ij})$$

with integers n_i and a_{ij} ($n_i \geqslant 0$).

Claim. $a_{ij} \leqslant p-(i+1)$ for every $i \geqslant 0$ and $1 \leqslant j \leqslant n_i$.

In fact assume that this fails to hold and choose i as small as possible among such counterexamples. We may assume $a_{i1} > p-(i+1)$. Let $e \neq 0$ be an element of $[E(-a_{i1})]_{a_{i1}}$ ($\subset I^i$). Then $e \in (0: \mathscr{M})$ (notice that $E = S^*$, the graded k-dual where $k = S_0$) and so $f^i(e) = 0$ since the resolution ($\#$) is minimal. Therefore e determines an element of $H^i_{\mathscr{M}}(R)$ with degree a_{i1}, which must be 0 because $[H^i_{\mathscr{M}}(R)]_q = (0)$ for $q > p-(i+1)$. Hence $e = f^{i-1}(e')$ for some e' ($\neq 0$) $\in (I^{i-1})_{a_{i1}} = \oplus_{j=1}^{n_{i-1}} E_{a_{i1}-a_{i-1,j}}$. Let $1 \leqslant j \leqslant n_{i-1}$ and assume that $a_{i1} \leqslant a_{i-1,j}$. Then

$$p-(i+1) < a_{i1} \leqslant a_{i-1,j} \leqslant p-i$$

by our choice of i and so we have that $a_{i1} = a_{i-1,j}$. Because

$$e' \in \bigoplus_{\substack{1 \leqslant j \leqslant n_{i-1} \\ \text{such that } a_{i1} \leqslant a_{i-1,j}}} E_{a_{i1}-a_{i-1,j}},$$

this argument tells us that $e' \in (0:_{I^{i-1}} \mathscr{M})$. Therefore $e = f^{i-1}(e') = 0$ again by the minimality of ($\#$)—this is the required contradiction.

By this claim we get that $[I^i]_{p-(i+1)}$ is contained in the socle $(0:_{I^i} \mathscr{M})$ of I^i. Thus the canonical map

$$[\mathrm{Ext}^i_S(S/\mathscr{M}, R)]_{p-(i+1)} \to [H^i_{\mathscr{M}}(R)]_{p-(i+1)}$$

is surjective for every $i \in \mathbb{Z}$.

COROLLARY 1.15. *Suppose that $I_q = (0)$ for $q \leqslant p-1$ and that the field S_0 is infinite. Then the following conditions are equivalent*:

(1) *$R = S/I$ is a Buchsbaum ring and $\mathscr{M}_R^p = (f_1, f_2, \ldots, f_d)\mathscr{M}_R^{p-1}$ for some $f_1, f_2, \ldots, f_d \in S_1$.*

(2) *$H^i_{\mathscr{M}}(R) = [H^i_{\mathscr{M}}(R)]_{p-(i+1)}$ for $i \neq d$ ($=\dim R$) and $[H^d_{\mathscr{M}}(R)]_q = (0)$ for $q > p-(d+1)$.*

(3) *R is a Buchsbaum ring and has p-linear resolution. In this case $\mathscr{M}_R^p = (f_1, f_2, \ldots, f_d)\mathscr{M}_R^{p-1}$ for every linear system $f_1, f_2, \ldots, f_d$ of parameters for R.*

Proof. (1) ⇒ (2). If $d=0$ we have nothing to prove (recall that $R_q=(0)$ for $q \geqslant p$ by our assumption). Assume $d>0$. We put $U=H^0_{\mathscr{M}}(R)$ and $\bar{R}=R/f_1R$. Then as R is Buchsbaum one has exact sequences

$$0 \to H^i_{\mathscr{M}}(R) \to H^i_{\mathscr{M}}(\bar{R}) \to [H^{i+1}_{\mathscr{M}}(R)](-1) \to 0 \qquad (0 \leqslant i \leqslant d-2), \qquad (\#)$$

$$0 \to H^{d-1}_{\mathscr{M}}(R) \to H^{d-1}_{\mathscr{M}}(\bar{R}) \to [H^d_{\mathscr{M}}(R)](-1) \xrightarrow{f_1} H^d_{\mathscr{M}}(R) \to 0 \qquad (\#\#)$$

of graded S-modules, and the latter part of condition (2) follows by induction on d from the exact sequence (##). The former part also follows by induction on d from the exact sequence (#), if $d \geqslant 2$. Hence we may assume that $d=1$ and it suffices to show that $U=U_{p-1}$ in this case. Notice that $U_q=(0)$ for $q \geqslant p$ as $U \cap f_1R=(0)$. Let $z \in U_q$ with $q<p-1$ and express $z=Z \bmod I$ with $Z \in S_q$. Then as $\mathscr{M}U=(0)$ we get $f_1Z \in I$, which yields that $f_1Z=0$ because $I_k=(0)$ for $k<p$. Thus $z=0$ and so $U_k=(0)$ for $k<p-1$. Therefore we find that $U=U_{p-1}$ as required.

(2) ⇒ (3). The latter part of condition (3) follows from Theorem 1.2. The fact that R is Buchsbaum follows from the above proposition and the surjectivity criterion.

(3) ⇒ (1). This follows from the proof of Corollary 1.5.

The last assertion also follows from the proof of Corollary 1.5.

The construction in Section 5 gives many examples of rings satisfying the conditions in the above corollary.

Finally, we note that Herzog, Simis, and Vascoucelos [16] prove that the associated graded ring of a local ring R has p-linear resolution (for some $p \geqslant 2$) if and only if there is a system of generators $x_1,\ldots,x_n$ of the maximal ideal such that $H_q(x_1,\ldots,x_r;R)$ is a vectorspace for all $q>0$ and $1 \leqslant r \leqslant n$.

Appendix to Section 1. Explicit Construction of Linear Complexes

We present a relatively explicit construction of the minimal free resolution of a module M with linear free resolution over a polynomial ring $S=k[X_1,\ldots,X_n]$. The construction generalizes exactly that introduced in Buchsbaum and Eisenbud [5] for the case where M is the ideal $(X_1,\ldots,X_n)^p$. It was also obtained independently by Herzog, Simis, and Vascoucelos [16] as a biproduct of their work on approximation complexes. In general, it yields "the linear part" of a minimal free resolution of any module.

Let F be the vectorspace of linear forms of S, so that we may write $S=\mathrm{Sym}(F)=\sum_{i \geqslant 0} S_i(F)$, where $S_i(F)$ denotes the ith symmetric power of F.

The multiplication map $F \otimes_k M_0 \to M_1$, together with the diagonalization $\bigwedge^q F \to \bigwedge^{q-1} F \otimes_k F$ gives rise to a natural map of vectorspaces:

$$\phi_k : \bigwedge^q F \otimes M_0 \to \bigwedge^{q-1} F \otimes M_1.$$

We write $\tilde{F}$ for the free S-module $S \otimes_k F$.

DEFINITION.

$$\begin{aligned} L^{q+1}(M) &= (S \otimes \ker \phi_q)(-q) \\ &= \ker \left(\bigwedge^q \tilde{F}(-q) \otimes M_0 \to \bigwedge^q \tilde{F}(-q) \otimes M_1 \right). \end{aligned}$$

Note that $L^{q+1}(M)$ is a free module whose generators all have degree k.

We now make the $L^q(M)$ into a linear complex, as follows. Let d: $\bigwedge^q \tilde{F} \to \bigwedge^{q-1} \tilde{F}$ be the differential of the Koszul complex of the elements $x_1,..., x_n$, coming from the natural map

$$F(-1) = S(-1) \otimes_k F = S(-1) \otimes_k S_1 \xrightarrow{\text{multiply}} S.$$

The diagram

$$\begin{array}{ccc} \bigwedge^q \tilde{F}(-q) \otimes_k M_0 & \xrightarrow{d \otimes M_0} & \bigwedge^{q-1} \tilde{F}(-q+1) \otimes_k M_0 \\ {\scriptstyle S(-q) \otimes \phi_q} \downarrow & & \downarrow {\scriptstyle S(-q+1) \otimes \phi_{q-1}} \\ \bigwedge^{q-1} \tilde{F}(-q) \otimes M_1 & \xrightarrow{d \otimes M_1} & \bigwedge^{q-2} \tilde{F}(-q+1) \otimes_k M_1 \end{array}$$

commutes, and thus induces a differential

$$d_M : L^q(M) \to L^{q-1}(M).$$

THEOREM 1.5. *Let*

$$\mathbb{F} : 0 \to F_d \to \cdots \to F_0 \to M \to 0$$

be the minimal free resolution of any graded module M with $M_r = 0$ for $r < 0$ over $S = k[X_1,..., X_n]$. Any comparison map of the complex

$$\mathbb{L}(M) : 0 \to L^n(M) \xrightarrow{d_M} \cdots \xrightarrow{d_M} L^0(M) \to M \to 0$$

to $\mathbb{F}$, covering the identity map $M \to M$, induces an isomorphism of $L^k(M)$ with the largest free summand of F_k generated by elements of degree $-k$. In *particular, $\mathbb{L}(M)$ is a free resolution of M whenever M has a linear resolution over S.*

Proof. $\mathbb{L}(M)$ is in a natural way a subcomplex of the "universal resolution"

$$\left(M \otimes_k \bigwedge F\right): \cdots \to M \otimes_k \bigwedge^k F \to M \otimes_k \bigwedge^{k-1} F \to \cdots \to M \otimes_k S \to M \to 0$$

described by Scheja and Storch [S–S] (see also [E–R–S [7]]; Avramov has remarked that this is just the "bar construction"), and by construction the induced map

$$\mathbb{L}(M) \otimes_S k \to M \otimes_k \bigwedge \tilde{F} \otimes_S k$$

maps

$$L^q(M) \otimes_S k = H_q(\mathbb{L}(M) \otimes_S k)$$

isomorphically to

$$\left[H_q\left(M \otimes_k \bigwedge \tilde{F} \otimes_S k\right)\right]_q = \operatorname{Tor}_q^S(M, k)_q.$$

Thus any comparison map $\mathbb{L}(M) \to \mathbb{F}$ induces isomorphisms

$$L^q(M) \otimes_S k \simeq \operatorname{Tor}_q^S(M, k) \cong (F_q \otimes_S k)_q$$

as required.

2. A Bound on the Degree of a Homogeneous Ring

Theorem 2.1. *Let R be a homogeneous ring of dimension d with homogeneous maximal ideal $\mathcal{M}$ over a field k. If $\{x_1, \ldots, x_{d-1}\}$ is a set of $d-1$ sufficiently general linear forms, then the degree $\delta = \delta(R)$ satisfies*

$$\begin{aligned}\delta &\geqslant 1 + \operatorname{codim} R - \sum_{p+q=d-1} h^p(\mathbf{x}; H^q_{\mathcal{M}}(R))_{2-d} \\ &\geqslant 1 + \operatorname{codim} R - \sum_{q=0}^{d-1} \binom{d-1}{q} h^q_{\mathcal{M}}(R)_{1-q}.\end{aligned}$$

The second inequality is trivial since $H^p(x; H^q_{\mathcal{M}}R)_{2-d}$ is a subquotient of the sum of $\binom{d-1}{q}$ copies of $H^q_{\mathcal{M}}(R)_{1-q}$. We note further that none of the three quantities in question change if R is replaced by $R/H^0_{\mathcal{M}}(R)$.

We will deduce this result, via the Noether Normalization Theorem, from the following general result on graded modules:

PROPOSITION 2.2. *Let M be a finitely generated graded $k[x_1,..., x_n]$-module. If $M_i = 0$ for $i < 0$, then*

$$|M_0| \leqslant \sum_{p+q=n} h^p(\mathbf{x}; H^q_{\mathscr{M}}(M))_{-n} \leqslant \sum_{q=0}^{n} \binom{n}{q} h^q_{\mathscr{M}}(M)_{-q}.$$

Proof of Proposition 2.2. From the spectral sequence

$$H^p(\mathbf{x}; H^q_{\mathscr{M}}(M)) \Rightarrow H^{p+q}(\mathbf{x}; M)$$

we see that $M_0 = (M/\mathbf{x}M)_0 = H^n(\mathbf{x}; M)_{-n}$ is filtered by subquotients of the $H^p(\mathbf{x}; H^q_{\mathscr{M}}(M))_{-n}$ for $p+q=n$, and this gives the first inequality. The second inequality is, as above, trivial.

Proof of Theorem 2.1. Choose general linear forms $y_1,..., y_d \in R$, so that R is finitely generated as a $T = k[y_1,..., y_d]$ – module.

In this situation δ is the rank of R as a T-module. The element $1 \in R$ generates a free T-module of rank 1 (which is even a direct summand). Let $\sum_{i=1}^{\delta-1} T(-e_i) \subset R/T \cdot 1$ be a free T-submodule of rank $\delta - 1$ generated by forms of degree $e_i > 1$, and consider the module

$$M = \left[R\Big/\left(T \cdot 1 \oplus \sum_{i=1}^{\delta-1} T(-e_i)\right)\right](1).$$

We see at once that $M_i = 0$ for $i < 0$, and $|M_0| = \operatorname{codim} R$. Since δ is the rank of R, M is a torsion T-module, so for sufficiently general linear forms $x_1,..., x_{d-1}$ in T, M will be a finitely generated $S = k[x_1,..., x_{d-1}]$-module. Applying Proposition 2.2 we obtain

$$\operatorname{codim} R = |M_0| \leqslant \sum_{p+q=d-1} h^p(\mathbf{x}; H^q_{\mathscr{M}}(M))_{1-d}.$$

We must now compare $H^q_{\mathscr{M}}(M)$ with $H^q_{\mathscr{M}}(R)$. From the exact sequence

$$0 \to T \oplus \sum T(-e_i) \to R \to M(-1) \to 0$$

we obtain isomorphisms

$$H^q_{\mathscr{M}}(R) = H^q_{\mathscr{M}}(M)(-1) \qquad \text{for} \quad q \leqslant d-2,$$

whence

$$h^p(\mathbf{x}; H^q_{\mathscr{M}}(M))_{1-d} = h^p(\mathbf{x}; H^q_{\mathscr{M}}(R))_{2-d} \qquad \text{for} \quad q \leqslant d-2.$$

Further, we obtain a left-exact sequence

$$0 \to H^{d-1}_{\mathcal{M}}(R) \to H^{d-1}_{\mathcal{M}}(M)(-1) \to H^{d}_{\mathcal{M}}\left(T \oplus \sum T(-e_i)\right),$$

so that

$$h^0(\mathbf{x}; H^{d-1}_{\mathcal{M}}(M))_{d-1} \leqslant h^0(\mathbf{x}; H^{d-1}_{\mathcal{M}}(R))_{2-d} + h^0(\mathbf{x}; H^{d}_{\mathcal{M}}(T))_{2-d}$$
$$+ \sum h^0(\mathbf{x}; H^{d}_{\mathcal{M}}(T(-e_i)))_{2-d}.$$

But $H^d_{\mathcal{M}}(T) = \mathrm{Hom}_k(T, k)(d)$, so

$$H^0(\mathbf{x}; H^d_{\mathcal{M}}(T(-e)))_r = \mathrm{Hom}_k(T/\mathbf{x}T, k)(d-e)_r$$
$$= \mathrm{Hom}_k((T/\mathbf{x}T)_{e-d-r}, k).$$

Since $T/\mathbf{x}T$ is a polynomial ring in 1 variable, we see that

$$h^0(\mathbf{x}; H^d_{\mathcal{M}}(S)(-e))_r = \begin{cases} 1, & e \geqslant d+r \\ 0, & e < d+r. \end{cases}$$

Combining this with the previous inequality and the fact that $e_i \geqslant 2$ for $i = 1, \ldots, \delta - 1$ we get

$$h^0(\mathbf{x}; H^{d-1}_{\mathcal{M}}(M))_{1-d} \leqslant h^0(\mathbf{x}; H^{d-1}_{\mathcal{M}}(R))_{2-d} + \delta - 1,$$

whence

$$\operatorname{codim} R \leqslant \sum_{p+q=d-1} h^p(\mathbf{x}; H^q_{\mathcal{M}}(R))_{2-d} + \delta - 1,$$

the desired relation.

Remark. If R is Cohen–Macaulay, then Theorem 2.1 (in either form) becomes the well-known bound

$$\delta \geqslant 1 + \operatorname{codim} R.$$

(*Proof.* If $x_1, \ldots, x_d$ is a linear regular sequence, then $\delta = |R/\mathbf{x}R| \geqslant 1 + \operatorname{codim} R$.) In fact it is not hard, using the same ideas as in the proof of the theorem, to show that if R is Cohen–Macaulay, then

$$\delta = 1 + \operatorname{codim} R + h^0(\mathbf{x}, H^d_{\mathcal{M}}(R))_{2-d},$$

and something similar for the Rees ring of a local Cohen–Macaulay ring has been noted by Sally.

If now R is a (not necessarily Cohen–Macaulay) reduced ring which is connected in codimension 1, and k is an algebraically closed field, then again

$$\delta \geqslant 1 + \operatorname{codim} R.$$

The irreducible case is standard, and the general case was noted by Xambò [25]. (Proof in the irreducible case: R is the homogeneous coordinate ring of a projective variety: project this from a general point on it and do induction. Alternately, Lemma 4.4 implies that if V is a connected, reduced subscheme of $\mathbb{P}^n$ of dimension $\geqslant 1$, not contained in any hyperplane, then a hyperplane section of V which does not contain any component of V is itself not contained in a linear space of dimension $n-2$. Knowing this, the codimension of the general hyperplane section of V is equal to the codimension of V, and the result follows by induction using Bertini's theorem.)

It would be interesting to know whether this follows from Theorem 2.1. In particular, we may formulate:

Problem. If R is a homogeneous domain of dimension d over an algebraically closed field, is $h^p(x_1,..., x_{d-1}; H^q_{\mathcal{M}}(R))_{2-d} = 0$ for $p + q = d - 1$ and $x_1,..., x_{d-1}$ sufficiently general linear forms?

The answer is (almost trivially) yes for $d \leqslant 3$. For $d = 4$ we have verified the cases $p > 0$, leaving only the equations: does $H^3_{\mathcal{M}}(R)_{-2}$ contain elements annihilated by three general linear forms?

3. Rings with Minimal Degree and Top-Heavy Modules

We say that a homogeneous ring R of dimension d has *minimal degree* (or minimal multiplicity) if the inequalities in Theorem 2.1 are both equalities; that is, if

$$\delta(R) = 1 + \operatorname{codim} R - \sum_{q=0}^{d-1} \binom{d-1}{q} h^q_{\mathcal{M}}(R)_{1-q}.$$

Note that R has minimal multiplicity if and only if $R/H^0_{\mathcal{M}}(R)$ does.

Similarly, we will say that a graded $k[x_1,..., x_n]$-module M with $M_i = 0$ for $i < 0$ is *top heavy* if the equalities hold in Proposition 2.2; that is, if

$$|M_0| = \sum \binom{n}{q} h^q_{\mathcal{M}}(M)_{-q}.$$

Again, M is top heavy if and only if $M/H^0_{\mathcal{M}}(M)$ is.

We remark that minimal multiplicity and top heavyness are connected by more than the formal similarity in the defining formula. For example, if M

is a $k[x_1,..., x_n]$-module, and $M_r = 0$ for $r < 0$ we may make M into a $k[x_1,..., x_{n+1}]$-module by letting $x_{n+1}M = 0$, and then M is top heavy as a $k[x_1,..., x_n]$-module if and only if the trivial extension $k[x_1,..., x_{n+1}] \propto M(-1)$ is a ring of minimal multiplicity. Of course we could also have let x_{n+1} act in some other way. This explains why there are "more" rings of minimal multiplicity than top-heavy modules;

THEOREM 3.1. (a) *If R is a homogeneous ring of dimension d, then R has minimal multiplicity if and only if $R/H^0_{\mathscr{M}}(R)$ has linear resolution and $H^q_{\mathscr{M}}(R)$ is concentrated in degree $1-q$ for $0 < q < d-1$.*

(b) *Let M be a module over $S = k[x_1,..., x_n]$ with $M_r = 0$ for $r < 0$. If we set $h^q = h^q_{\mathscr{M}}(M)_{-q}$, then M is top heavy if and only if*

$$M/H^0_{\mathscr{M}}(M) \cong \bigoplus_{q>0} (\mathrm{Syz}_q k(q))^{h^q}.$$

Remark. One can also show that M is top heavy if and only if $M/H^0_{\mathscr{M}}(M)$ has linear resolution and $H^q_{\mathscr{M}}(M)$ is concentrated in degree $-q$ for $1 < q < n$.

The proof of part (b) will involve a somewhat more general splitting criterion:

THEOREM 3.2. *Let M be a module over a local ring $(S, \mathscr{M})$ or a graded module over a homogeneous ring $(S, \mathscr{M})$ and let $k = S/\mathscr{M}$. Let $\rho < \infty$ be the projective dimension of M. If the canonical maps*

$$\mathrm{Ext}^q(M, S) \to \mathrm{Ext}^q(M, k)$$

are inclusions for $0 < q < \rho$, then M is the direct sum of a free module and the module

$$\sum_{0<q} \mathrm{Tr}\, \mathrm{Syz}_{q-1}\, \mathrm{Ext}^q_S(M, S).$$

Proof of Theorem 3.1. (a) We may harmlessly suppose $H^0_{\mathscr{M}}(R) = 0$ and that the ground-field is infinite, so we may choose a non-zerodivisor $x \in R_1$. From the exact sequence $0 \to R(-1) \to^x R \to R/xR \to 0$ we deduce sequences

$$\cdots \xrightarrow{x} (H^q_{\mathscr{M}}(R))_r \to (H^q_{\mathscr{M}}(R/xR))_r \to H^{q+1}_{\mathscr{M}}(R)_{r-1} \xrightarrow{x} \cdots.$$

Thus $h^q_{\mathscr{M}}(R/xR)_x \leqslant h^q_{\mathscr{M}}(R)_r + h^{q+1}_{\mathscr{M}}(R)_{r-1}$. In particular we see that

$$\sum_0^{d-2} \binom{d-2}{q} (h^q_{\mathscr{M}}(R/xR)_{1-q}) \leqslant \sum_0^{d-1} \binom{d-1}{q} (h^q_{\mathscr{M}}(R))_{1-q}$$

with equality if and only if the appropriate maps vanish, while $\delta(R)=\delta(R/xR)$ in any case, so R has minimal multiplicity if and only if R/xR does and the maps

$$(H^q_{\mathcal{M}}(R))_{r-1} \xrightarrow{x} H^q_{\mathcal{M}}(R)_r$$

are 0 for $q \leqslant d-2$ and $r=1-q$ and for $q \leqslant d-1$ and $r=2-q$.

If R satisfies the cohomological conditions of the theorem, then it is easy to check that R/xR satisfies the corresponding conditions, and that the above maps vanish; by induction, R/xR has minimal multiplicity, so R does too.

Now suppose that R has minimal multiplicity, so that R/xR does too, and the above maps vanish. By induction we may suppose that the cohomological conditions are satisfied by R/xR, and we must check them for R. If R is Cohen–Macaulay, then a standard argument shows that R has 2-linear resolution, and we are done. Thus we may assume that R is not Cohen–Macaulay, so in particular $d \geqslant 2$.

Consider the sequences

$$(H^{q-1}_{\mathcal{M}}(R/xR))_r \to (H^q_{\mathcal{M}}(R))_{r-1} \xrightarrow{x} (H^q_{\mathcal{M}}(R))_r \to (H^q_{\mathcal{M}}(R/xR))_r.$$

By the inductive hypothesis the middle map is onto for all $r > 1-q$, and if $q>1$ it is a monomorphism for $r > 2-q$; since $H^q_{\mathcal{M}}(R)_r = 0$ for large r, this implies $H^q_{\mathcal{M}}(R)_r = 0$ for $q>1$ and $r>1-q$. Note that even if $q=1$, the middle map is an epimorphism for $r > 1-q=0$. Since we have assumed $d \geqslant 2$, we know the middle map is 0 for $r=2-q=1$; thus $H^1_{\mathcal{M}}(R)_r = 0$ for all $r > 1-q$.

We must now show that $H^q_{\mathcal{M}}(R)_r = 0$ for $1 \leqslant q \leqslant d-2$ and $r < 1-q$.

But $H^{q-1}_{\mathcal{M}}(R/xR)=0$ for $r < 1-(q-1) = 2-q$ by induction, so the middle map in the above sequence is a monomorphism for $r<2-q$. Since it is 0 for $r=1-q$, we see that the $H^q_{\mathcal{M}}(R)_{r-1}=0$ for $r<2-q$, that is, $r-1<1-q$, as required.

Proof of Theorem 3.1(b). An easy computation shows that the module $\bigoplus_{q>0} (\operatorname{Syz}_q k(q))^{h^q}$ is top-heavy and satisfies $h^q = h^q_{\mathcal{M}}(M)_{-q}$. Thus it remains to prove the converse.

Let M be a top-heavy module. Since M is top heavy if and only if $M/H^0_{\mathcal{M}}(M)$ is, we may assume $H^0_{\mathcal{M}}(M)=0$, and we may also suppose that M has no free direct summands. Imitating the argument in part (a), we see that M has linear resolution and $H^q_{\mathcal{M}}(M)$ is concentrated in degree $-q$ for $0<q<n$. (Another approach to this is to prove first, by an induction, that M is generated in degree 0, and then to use part 3.1(a) on the ring $k[x_1,..., x_n, x_{n+1}] \propto M$, where x_{n+1} acts trivially on M.)

By local duality, $\operatorname{Ext}_S^q(M, S)$ is a vectorspace, concentrated in degree $-q$, for each q. Since the resolution of M has the form

$$\mathbb{F}: \to S^{\beta_q}(-q) \to \cdots \to S^{\beta_1}(-1) \to S^{\beta_0} \to M,$$

we see that the cycles representing non-zero elements of $\operatorname{Ext}_S^q(M, S)$ in $(S^{\beta_q}(-q))^* = S^{\beta_q}(q)$ must be outside of $\mu S^{\beta_q}(q)$; thus the canonical map $\operatorname{Ext}_S^q(M, S) \to \operatorname{Ext}_S^q(M, k)$ is a monomorphism for $0 < q < n - 1$. We can now apply Theorem 3.2 and conclude that

$$\begin{aligned} M &\cong \sum_{q>0} \operatorname{Tr} \operatorname{Syz}_{q-1} \operatorname{Ext}_S^q(M, S) \\ &\cong \sum_{n>q>0} \operatorname{Tr}(\operatorname{Syz}_{q-1}(k(q)^{h^{n-q}})) \oplus \operatorname{Tr} \operatorname{Syz}_{n-1} \operatorname{Ext}^n(M, S), \end{aligned}$$

where the second isomorhism comes from local duality.

The self-duality of the Koszul complex which is the minimal free resolution of k over S yields $\operatorname{Tr} \operatorname{Syz}_{q-1} k(q) \cong \operatorname{Syz}_{n-q} k(n-q)$. Also, because $H^0_{\mathscr{M}}(M) = 0$, the projective dimension of M is $\leqslant n-1$, so $\operatorname{Ext}^n(M, S) = 0$. Putting this together we get

$$\begin{aligned} M &\cong \sum_{n>q>0} \operatorname{Syz}_{n-q} k(n-q)^{h^{n-q}} \\ &= \sum_{0<q} \operatorname{Syz}_q k(q)^{h^q}, \end{aligned}$$

as required.

Before beginning the proof of Theorem 3.2, we state an easy lemma which extend the familiar result that a comparison map between minimal resolutions of the same module which induces an isomorphism on the module is itself an isomorphism. The lemma works equally well in the local or homogeneous case, but the local case implies the homogeneous one (by localization!) so we state it in that case only:

LEMMA 3.3. *Let*

$$\begin{aligned} &\mathbb{A}: \cdots \to A_1 \to A_0 \to 0, \\ &\mathbb{B}: \cdots \to B_1 \to B_0 \to 0 \end{aligned}$$

be free complexes over a local ring with residue field k. Suppose that $\mathbb{A}$ *and* $\mathbb{B}$ *are* minimal *in the sense that the differentials of* $\mathbb{A} \otimes k$ *and* $\mathbb{B} \otimes k$ *are* 0, *and that* $\phi: \mathbb{A} \to \mathbb{B}$ *is a map of complexes inducing isomophisms* $H_q\mathbb{A} \to H_q\mathbb{B}$ *for* $q \leqslant m$, *and an epimorphism* $H_{m+1}\mathbb{A} \to H_{m+1}\mathbb{B}$. *Then* $\phi_q: A_q \to B_q$ *is an isomorphism for* $q \leqslant m$, *and an epimorphism for* $q = m + 1$.

Proof. The conditions on ϕ imply that the mapping cylinder

$$\mathbb{M}(\phi)\colon \cdots \to A_1 \oplus B_2 \xrightarrow{d_2} A_0 \oplus B_1 \xrightarrow{d_1} B_0 \to 0$$

satisfies

$$H_q(\mathbb{M}(\phi)) = 0 \qquad \text{for} \quad k \leqslant m+1.$$

Since $\mathbb{M}(\phi)$ is a free complex, it follows that the sequence

$$\mathbb{M}(\phi)'\colon A_{m+1} \oplus B_{m+2} \to A_m \oplus B_{m+1} \to \cdots,$$

having no homology, is split exact. Thus $\mathbb{M}(\phi)' \otimes k$ is exact; since the map $d_{q+1} \otimes k$ has the form

$$d_{q+1} \otimes k = \begin{pmatrix} 0 & \phi_q \otimes k \\ 0 & 0 \end{pmatrix},$$

it follows that $\phi_q \otimes k$ is an isomorphism for $q \leqslant m$, and an epimorphism for $q = m+1$ whence the result.

Proof of Lemma 3.3. Let $\mathbb{F}$ be a minimal free resolution of M, and let $\mathbb{G}^q$ be a minimal free resolution of $\mathrm{Ext}^q_S(M, S)$, numbered as follows:

$$\mathbb{G}^q\colon \cdots \to G^q_{q-1} \to G^q_q \to \mathrm{Ext}^q_S(M, S) \to 0.$$

We will construct a map of complexes $\psi\colon \mathbb{F}^* \to \sum_{q>1} \mathbb{G}^q$ which induces isomorphisms from the homology of $\mathbb{F}^*$ at F^*_r to the homology of $\sum_{q>1} \mathbb{G}^q$ at $\sum_{q>1} G^q_r$. Of course $\sum_{q>1} \mathbb{G}^q$ has no homology at $\sum G^q_0$, so by Lemma 3.3, ψ_r is an isomorphism for $r > 0$ and an epimorphism for $r = 0$. If we write $K = \ker \psi_0$, then K is a free summand of F_0, and, dualizing we get $M = K^* \oplus \sum_{q>0} \mathrm{Tr}\,\mathrm{Syz}_{q-1} \mathrm{Ext}^q(M, S)$, as required.

We will take ψ to be the sum of maps of complexes

$$\begin{tikzcd}
\mathbb{G}^q & G^q_{q-1} \arrow[r] & G^q_q \arrow[r] & 0 & \cdots \\
\mathbb{F}^* \arrow[u, "\psi^q"'] & F^*_{q-1} \arrow[r] \arrow[u] & F^*_q \arrow[r] \arrow[u] & F^*_{q+1} \arrow[u] & \cdots
\end{tikzcd}$$

inducing isomorphisms

$$\mathrm{Ext}^q_S(M, S) = H^q(\mathbb{F}^*) \to H^q(\mathbb{G}^q) = \mathit{Ext}^q_S(M, S).$$

Set

$$K_q = \ker F^*_q \to F^*_{q+1},$$

$$B_q = \mathrm{im}\, F^*_{q-1} \to F^*_q,$$

so that $\mathrm{Ext}^q(M, S) = K_q/B_q$. To define the map ψ^q it is sufficient (and necessary) to split the inclusion map $K_q/B_q \subset F_q^*/B_q$; we may then take ψ^q to be a map of complexes covering the map $F_q^* \to F_q^*/B_q \to K_q/B_q = \mathrm{Ext}_S^q(M, S)$ (and, if ψ^q exists, then the composite map $F_q^* \to G_q^q \to \mathrm{Ext}_S^q(M, S) \cong K_q/B_q$ induces such a splitting).

For $q = \rho$ we have $K = F_q^*$, so there is no problem. For $0 < q < \rho$, we have assumed that $\mathrm{Ext}_S^q(M, S) \hookrightarrow \mathrm{Ext}_S^q(M, k) = F_q^* \otimes k$, and since $F_q^* \otimes k$ is a vectorspace, the inclusion splits; any splitting σ defines a splitting

$$F_q^*/B_q \to F_q^* \otimes k \to \mathrm{Ext}_S^q(M, S) = K_q/B_q,$$

since $B_q \subset \mathscr{M} F_q^*$.

4. Integral Domains with 2-Linear Resolutions or Minimal Multiplicity

We have seen that, for any homogeneous ring R of dimension d we have the following implications:

(1) R is Cohen–Macaulay and $e(R) \leqslant 1 + \operatorname{codim} R$ (in this case $e(R) = 1 + \operatorname{codim} R$).

$\Rightarrow$ (2) R has a 2-linear resolution.

$\Rightarrow$ (3) (If R_0 is infinite:) There exist d elements $x_1, \ldots, x_d \in R_1$ such that $R_2 \subset (x_1, \ldots, x_d) R_1$.

$\Rightarrow$ (4) $e(R) \leqslant 1 + \operatorname{codim} R$.

These implications are in general strict. However:

Theorem 4.1. *If R is a homogeneous domain of dimension d such that there exist elements $x_1, \ldots, x_d \in R_1$ with $R_2 \subset (x_1, \ldots, x_d) R_1$, then the integral closure $\bar{R}$ of R is obtained by adjoining elements of degree 0 to R, and $\bar{R}$ satisfies all the conditions 1–4 above.*

Example. Let $k \subseteq K$ be a non-trivial finite extension of fields, and let $R = k + (X_1, \ldots, X_d) K[X_1, \ldots, X_d]$. We have $\bar{R} = K[X_1, \ldots, X_d]$. Also, $H^i_{\mathscr{M}}(R) = 0$ for $i \neq i, d$, while

$$H^1_{\mathscr{M}}(R) = K/k, \quad \text{concentrated in degree 1}$$

and

$$\begin{aligned} H^d_{\mathscr{M}}(R) &= H^d_{\mathscr{M}}(K[X_1, \ldots, X_d]) \\ &= \mathrm{Hom}_{\text{graded } K\text{-vectorspaces}}(\bar{R}, K(-d)), \end{aligned}$$

so R has 2-linear free resolution, but R is not Cohen–Macaulay. It is easy to calculate that

$$\delta(R) = [K:k]$$

and

$$\operatorname{codim}(R) = ([K:k]-1)\,d.$$

Thus R satisfies conditions 2–4, but not 1.

Recall that a homogeneous domain R is *geometrically integral* if, writing K for the algebraic closure of R_0, $K \otimes_{R_0} R$ is a domain, or, equivalently, if R_0 is algebraically closed in the (homogeneous) field of quotients of R. Of course Theorem 4.1 implies that a geometrically integral homogeneous domain satisfying condition 3, above, satisfies all the conditions.

However, more is true. To make the statements convenient, we now restrict ourselves to the case when R_0 is algebraically closed. In this case it is well known in algebraic geometry that if R is a homogeneous domain, then $\delta(R) \geqslant 1 + \operatorname{codim} R$, and a complete classification of the homogeneous domains with $\delta(R) = 1 + \operatorname{codim} R$ was made classically by Castelnuovo for $\dim R \leqslant 3$ and extended to all dimensions by Bertini. This classification was recently extended by Xambò to the case when R is reduced and connected in codimension 1, and this seems to be the natural limit.

From the classification, which we will state in a moment, the following result may be deduced in a case-by-case manner. We will give a relatively direct, algebraic proof, using only Bertini's theorem on the integrality of general hyperplane sections (proved in characteristic p by Matsusaka [19]):

THEOREM 4.2. *Let R be a homogeneous ring, reduced and connected in codimension* 1, *and suppose that R_0 is algebraically closed. If $\delta(R) \leqslant 1 +$* $\operatorname{codim} R$, *then R is Cohen–Macaulay, so that R satisfies all conditions* 1–4 *above.*

With Theorem 4.1, we see that, under the hypotheses of Theorem 4.2, if P is a minimal prime of R, then R/P is normal, and we will prove this directly along the way. Of course this also follows from the classification theorem, which, for completeness, we now state.

First, we describe a class of normal irreducible varieties called *rational normal scrolls*, whose homogeneous coordinate rings satisfy conditions 1–4:

Let $\delta_1,\ldots,\delta_d$ be integers $\geqslant 0$. The rational normal scroll $S(\delta_1,\ldots,\delta_d)$ is a variety of degree $\delta = \sum \delta_i$, dimension d, and codimension $\delta - 1$ in $\mathbb{P}^{\delta+d-1}$. It may be defined in the modern manner as the image of the projectivized vector bundle over $\mathbb{P}^1$

$$\mathbb{P} = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(\delta_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(\delta_d))$$

under the map defined by the complete linear series $H^0(\mathcal{O}_{\mathbb{P}}(1)) \cong H^0(\mathcal{O}_{\mathbb{P}^1}(\delta_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(\delta_d))$; picturesquely, by taking independent subspaces

$$\mathbb{P}^{\delta_1}, \ldots, \mathbb{P}^{\delta_d} \subset \mathbb{P}^{\delta+d-1},$$

choosing parametrized rational normal curves

$$\rho_i : \mathbb{P}^1 \to \mathbb{P}^{\delta_i}, \qquad \operatorname{degree} \rho_i(\mathbb{P}^1) = d_i,$$

and taking the scroll to be the union of the spans of sets of corresponding points on the curves $\rho_i(\mathbb{P}^1)$, thus:

$$S(\delta_1, \ldots, \delta_n) = \bigcup_{t \in \mathbb{P}^1} \overline{\rho_1(t), \ldots, \rho_d(t)};$$

or algebraically as the variety whose homogeneous ideal is given in $\mathbb{P}^{\delta+d-1}$, with respect to variables

$$X_{10}, \ldots, X_{1\delta_1}, X_{20}, \ldots, X_{2\delta_2}, \ldots, X_{d0}, \ldots, X_{d\delta_d}$$

by the $\binom{\delta}{2}$ 2×2 minors of the $2 \times \delta$ matrix

$$\left(\begin{array}{cccc|c|c|c} X_{10}\ X_{11}\ X_{12} & \cdots & X_{1\delta_1-1} & & X_{20} \cdots & \cdots & X_{d0} \cdots X_{d\delta_d-1} \\ X_{11}\ X_{12}\ X_{13} & \cdots & X_{1\delta_1} & & X_{21} \cdots & & X_{d1} \cdots X_{d\delta_d} \end{array}\right).$$

From any of these definitions one sees for example that the rational normal curve of degree δ is $S(\delta)$ and the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^{\delta-1}$ is

$$S(\underbrace{1, 1, \ldots, 1}_{\delta}).$$

Any scroll is fibered by linear subspaces of codimension 1.

THEOREM 4.3 (DelPezzo–Bertini–Xambò). *If the homogeneous coordinate ring R of a d-dimensional variety $X \subset \mathbb{P}^n$ satisfies the conditions of Theorem 4.2 then either:*

(1) *X is a quadric hypersurface; $R = k[X_0, \ldots, X_n]/(Q)$ for some quadratic polynomial Q,*

(2) *X is a cone over the Veronese surface in $\mathbb{P}^5$; R is isomorphic to a polynomial ring over $k[X_{00}, X_{01}, X_{02}, X_{11}, X_{12}, X_{22}]$ modulo the ideal of 2×2 minors of the generic symmetric matrix:*

$$\begin{pmatrix} X_{00} & X_{01} & X_{02} \\ X_{01} & X_{11} & X_{12} \\ X_{02} & X_{12} & X_{22} \end{pmatrix},$$

or

(3) $\mathbb{P}^n$ *contains linear subspaces* $L_1,\dots,L_r$ *and there are d-dimensional scrolls* $X_i \subset L_i$ *such that* $X = \bigcup X_i$ *and for each* $j = 1,\dots, r$ *we have*

$$X_j \cap (X_1 \cup \cdots \cup X_{j-1}) = L_j \cap \overline{(L_1 \cup \cdots \cup L_{j-1})},$$

which is a linear subspace of dimension $d-1$.

We will not prove Theorem 4.3 here; we refer to the paper of Xambò [25] for a recent treatment.

We now turn to the proof of Theorem 4.2. The key element is the following:

LEMMA 4.4. *Let* R *be a homogeneous ring of dimension* $\geqslant 2$ *with* R_0 *algebraically closed in the total quotient field of* R. *If* R *is reduced and* $\operatorname{Proj} R$ *is connected, then*

$$H^1_{\mathscr{M}}(R)_r = 0 \qquad \text{for all} \quad r \leqslant 0.$$

Remark. The restriction $\dim R \geqslant 2$ is necessary but not serious; the only 1-dimensional ring satisfying the rest of the hypothesis is $R = R_0[X]$.

Proof. We first note that if R is a homogeneous ring of depth $\geqslant 1$ and dimension $\geqslant 2$ and if, for some $r \leqslant 0$ we have $H^1_{\mathscr{M}}(R)_r = 0$, then $H^1_{\mathscr{M}}(R)_s = 0$ for all $s \leqslant r$; indeed, if $x \in R_1$ is a non-zerodivisor, then the sequence $0 \to R(-1) \to^x R \to R/xR \to 0$ gives rise to the sequences

$$H^0_{\mathscr{M}}(R/xR)_s \to H^1_{\mathscr{M}}(R)_{s-1} \to H^1_{\mathscr{M}}(R)_s.$$

Since $\dim R/xR \geqslant 1$, we have $H^0_{\mathscr{M}}(R/xR)_s = 0$ for all $s \leqslant 0$, whence the desired result follows by descarding induction, starting with $s = r$.

It remains to show that, under the given hypotheses, $H^1_{\mathscr{M}}(R)_0 = 0$. Let Q be the graded total quotient ring of R, and set $\tilde{R} = \{x \in R \mid \mathscr{M}^n x \in R$ for $n \gg 0\}$, the "$\mathscr{M}$-transform" of R. It is easy to see that $H^1_{\mathscr{M}}(R) = H^0_{\mathscr{M}}(Q/R) = \tilde{R}/R$, so we must show that $\tilde{R}_0 = R_0$.

Since $H^1_{\mathscr{M}}(R)$ is artinian, we see that $\tilde{R}_{\geqslant 0} := \sum_{r \geqslant 0} \tilde{R}_r$ is a finitely generated R_0-algebra with $\operatorname{Proj} \tilde{R}_{\geqslant 0} = \operatorname{Proj} R$. If $\tilde{R}_0 \neq R_0$, then since R_0 is algebraically closed and $\tilde{R}_0$ is a finitely generated R_0-algebra, there would be a non-trivial idempotent in $\tilde{R}_0$. Since $\operatorname{Proj} \tilde{R}_{\geqslant 0}$ is connected, such an idempotent could be chosen to be annihilated by a power of $\mathscr{M}$, so that $H^0_{\mathscr{M}}(Q) \neq 0$, contradicting the hypothesis that R is reduced and positive dimensional. Thus $\tilde{R}_0 = R_0$ and $H^1_{\mathscr{M}}(R)_0 = 0$ as claimed.

Proof of Theorem 4.2. We will show by induction on $\dim R$ that R is Cohen–Macaulay. Since any reduced purely 1-dimensional ring is Cohen–

Macaulay, we may suppose $\dim R \geqslant 2$. By Bertini's theorem we may choose $x \in R_1$ to be a non-zerodivisor such that

$$\bar{R} = R/((x) + H^0_{\mathscr{M}}(R/xR))$$

is reduced with components corresponding to those of R. Of course, $\bar{R}$ is still connected in codimension 1, so we may suppose by induction that $\bar{R}$ is Cohen–Macaulay, and it suffices to show that $H^0_{\mathscr{M}}(R/xR) = 0$. Of course $H^0_{\mathscr{M}}(R/xR)_r = 0$ for $r \leqslant 0$ because $\dim R/xR \geqslant 1$ and R_0 is a field. From the exact sequence

$$0 \to R(-1) \xrightarrow{x} R \to R/xR \to 0$$

we deduce sequences:

$$0 \to H^0_{\mathscr{M}}(R/xR)_r \to H^1_{\mathscr{M}}(R)_{r-1} \to H^1_{\mathscr{M}}(R)_r \to H^1_{\mathscr{M}}(R/xR)_r. \qquad (*)$$

By induction, $\bar{R}$ admits a 2-linear resolution, and $H^1_{\mathscr{M}}(\bar{R}) = H^1_{\mathscr{M}}(R/xR)$, so $H^1_{\mathscr{M}}(R/xR)_r = 0$ for $r \geqslant 1$ by Theorem 1.1. From the sequence above it thus follows that if, for $r \geqslant 1$, we have $H^1_{\mathscr{M}}(R)_{r-1} = 0$, then

$$H^0_{\mathscr{M}}(R/xR)_r = 0 = H^1_{\mathscr{M}}(R)_r.$$

By the lemma, $H^1_{\mathscr{M}}(R)_0 = 0$, and it follows that $H^1_{\mathscr{M}}(R/xR))_r = H^1_{\mathscr{M}}(R) = 0$ for all $r \geqslant 1$, as required.

To prove Theorem 4.1, we need an easy corollary of Theorem 4.2.

COROLLARY 4.5. *Let R be a homogeneous domain with R_0 algebraically closed. If $\delta(R) \leqslant 1 + \operatorname{codim} R$, then R is normal.*

Proof. From Theorem 4.2, R is Cohen–Macaulay, and it suffices to prove that the singularities of R are in codimension $\geqslant 2$. Cutting with general hyperplanes, we may reduce to the case $\dim R = 2$, so that the integral closure $\bar{R}$ is again Cohen–Macaulay. We wish to prove $R = \bar{R}$. Note that $\bar{R}_0 = R_0$, but $\bar{R}$ may a priori not be generated by forms of degree 1. Set $\mathscr{M} = R_+$, the maximal homogeneous ideal. We have

$$\begin{aligned}\delta(\bar{R}_{\mathscr{M}}) = \delta(R_{\mathscr{M}}) = \delta(R) &\leqslant 1 + \operatorname{codim} R \\ &= 1 + \operatorname{codim} R_{\mathscr{M}} \\ &\leqslant 1 + \operatorname{codim} \bar{R}_{\mathscr{M}},\end{aligned}$$

so, since $\bar{R}_{\mathscr{M}}$ is Cohen–Macaulay, $\delta(\bar{R}) = 1 + \operatorname{codim} \bar{R}_{\mathscr{M}}$ and codim $\bar{R}_{\mathscr{M}} = \operatorname{codim} R$. It follows that $\bar{R}_+$ is generated by R_1, and we are done.

Proof of Theorem 4.1. Let K be the algebraic closure of R_0 in the field of fractions of R, so that $K = \bar{R}_0$, and set $R' = KR$. Of course R' is a homogeneous domain satisfying $R'_2 \subset (x_1, ..., x_d) R'_1$, so

$$\delta(R') \leqslant 1 + \operatorname{codim} R',$$

where the codimension of R' is as a K-algebra. But R' is a geometrically integral K-algebra, so if $\bar{K}$ is the algebraic closure of K, we see from Theorem 4.2 and Corollary 4.5 that $\bar{K} \otimes_K R'$ is Cohen–Macaulay and integrally closed. It follows that R' is Cohen–Macaulay and is the integral closure of R, as required.

5. Integral Domains with p-linear Resolutions

In this section we present two approaches to the construction of homogeneous integral domains with p-linear resolutions. As usual we take $S = k[x_1, ..., x_n]$, and we assume that k is infinite.

(a) *Curves in* $\mathbb{P}^3$

Let $n = 4$, and let $R = S/I$ be non-degenerate and purely 2-dimensional—that is, R is the homogeneous coordinate ring of a non-degenerate purely 1-dimensional subscheme C of $\mathbb{P}^3$. Set

$$s = \dim_k \operatorname{socle} H^1_{\mathscr{M}}(R).$$

The Hilbert polynomial $H_R(v)$ may be written in the form

$$H_R(v) = \delta v + (1 - p_a) \qquad \text{for} \quad v \gg 0,$$

where $\delta = \delta(R)$ is the *degree*, and p_a the "arithmetic genus" of C.

From Theorem 1.2 and Proposition 1.9, using the fact that R has positive depth, we immediately derive:

COROLLARY 5.1. (1) *R has p-linear resolution if and only if*

(a) *I contains no forms of degree $<p$,*

(b) $H^1_{\mathscr{M}}(R)_{p-1} = H^1(\mathbb{P}^3, I(p-1)) = 0$ *and* $H^2_{\mathscr{M}}(R)_{p-2} = H^1(C, \mathscr{O}_C(p-2)) = 0$.

(2) *If R has a p-linear resolution, then the resolution has the form*

$$0 \to S^s(-p-2) \to S^{2s+p}(-p-1) \to S^{s+p+1}(-p) \to S \to R \to 0.$$

Further, C has degree

$$\delta = \binom{p+1}{2} - s$$

and arithmetic genus

$$p_a = \frac{(2p+3)(p-1)(p-2)}{6} - s(p-1),$$

and satisfies the maximal rank condition *that the natural maps*

$$H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(v)) \to H^0(C, \mathcal{O}_C(v))$$

have maximal rank for all v.

Conversely, any non-degenerate purely 1-dimensional $C \subset \mathbb{P}^3$ with δ and p_a given by the formulae above, satisfying the maximal rank condition, has p-linear resolution.

Some examples:

p	s	δ	p_a	Description
2	0	3	0	The twisted cubic; or a line meeting a conic in one point; or 3 lines forming a connected non planar curve.
	1	2	-1	2 skew lines $(H^1{}_{\mathcal{M}}(R) = k)$.
3	0	6	3	Any linearly normal curve of genus 3 and degree 6.
	1	5	1	Any smooth eliptic quintic, or rational quintic with one double-point $(H^1{}_{\mathcal{M}}(R) = k(-1))$.
4	2	8	5	The embedding of a general curve of genus 5 by a general line bundle of degree 8 $(H^1{}_{\mathcal{M}}(R) = k^2(-2))$.

It is conjectured that a general curve of genus g embedding by a general linear series of any dimension $\geqslant 3$ and any degree $\delta \geqslant (3/4)g + 3$ satisfies the maximal rank condition. If this is so, then we get, in this way, examples of p-linear curves with every $p_a \geqslant 0$ and $\delta \geqslant (3/4)p_a + 3$ satisfying the conditions of the corollary. For comparison with the examples treated below, we note that for many values of p and s, with $p > 4$, a general embedding of a general curve of genus and degree as in the corollary will have an $H^1{}_{\mathcal{M}}(R)$ which is not a vectorspace; indeed, even if $\mathcal{O}_C(1)$ is special, it follows from the theorem of Gieseker [9] that $\mathcal{O}_C(2)$ will not be special, and will therefore have dimension

$$2\delta - p_a + 1,$$

which is in many cases $>h^0(\mathcal{O}_{\mathbb{P}^3}(2))=10$, so that $H^1_{\mathcal{M}}(R)_2 \neq 0$. Since by Theorem 1.4 the socle of $H^1_{\mathcal{M}}(R)$ is concentrated in degree $p-2>2$, this shows that $H^1_{\mathcal{M}}(R)$ is not annihilated by $\mathcal{M}$ in this case.

(b) *Codimension* 2 *Primes from Vectorbundles*

We are grateful to R. Hartshorne and A. P. Rao for discussions which helped clarify the material of this section.

Consider a graded S-module E of rank r with the following properties:

(1) E is generated by E_0,

(2) E is reflexive,

(3) E corresponds to a vector-bundle on $\mathbb{P}^{n-1}$; that is, for every non-maximal (homogeneous) prime $Q \subset S$, E_Q is free of rank r.

It is well-known that E has many quotients which are prime ideals:

LEMMA 5.2. *If $e_1,\ldots,e_{r-1} \in S$ are sufficiently general elements of degree* 1, *then there is a reduced homogeneous ideal $I \subset S$ of height* 2, *representing a smooth subscheme of $\mathbb{P}^{n-1}$, and an integer p such that*

$$E\Big/\left(\sum_1^{r-1} Se_i\right) \cong I(p).$$

If the characteristic of k is 0, *then as many as desird of the $r-1$ elements e_i may be chosen to be of degree* 0 *instead of degree* 1. *Further, if $R=S/I$, then $H^i_{\mathcal{M}}(R)=H^{i+1}_{\mathcal{M}}(E)(-p)$ for $i<n-2$. If $H^2_{\mathcal{M}}(E)_{-p}=0$, then I is prime. If depth $E \geqslant 3$, then for sufficiently general e_i as above, $R=S/I$ will be a normal domain with* depth $R=(\text{depth } E)-1$.

Proof. If the e_i are part of a minimal generating set for E locally at every homogeneous prime of height $\leqslant 1$, then $E/\sum_1^{r-1} Se_i$ is isomorphic after a shift in grading to the ideal defining the locus where $e_1,\ldots,e_{r-1}$ fail to be locally part of a basis of E. The first part of the result now follows, from the "Bertini Theorem" of Kleiman [18, Sect. 6]. The case in which all the e_i have degree 1 is treated algebraically by Evans and Griffith [8]. The identification of $H^i_{\mathcal{M}}(R)$ follows from the exact sequence

$$0 \to S^{r_1} \oplus S^{r_2}(-1) \to E \to S(p) \to R(p) \to 0.$$

If $H^1_{\mathcal{M}}(R)_0 = H^2_{\mathcal{M}}(E)_{-p}=0$, then the projective variety associated to I is connected, and since it is smooth, it must then be irreducible— that is, I is a prime. If depth $E \geqslant 3$, it similarly follows that depth $S/I \geqslant 2$, so S/I is normal. [We are grateful to A. P. Rao for having explained the characteristic 0 case to us. He also pointed out that in characteristic p, if E is the

Frobenius pull-back of the Null-correlation bundle, and if $e_i \in E_0$, then the ring $R = S/I$ defined in the Lemma is not reduced.]

If E has a linear resolution, then so will $I(p)$:

PROPOSITION 5.3. *Suppose that E is a homogeneous S-module satisfying* 1–3, *as above, and having a linear free resolution of length u and Betti numbers $\beta_j = \beta_j(E)$. If*

$$e_1, \ldots, e_{r_1} \in E_0,$$
$$e'_1, \ldots, e'_{r_2} \in E_1,$$

with $r_1 + r_2 = r - 1$, and such that

$$E \Big/ \Big(\sum_1^{r_1} Se_i\Big) + \Big(\sum_1^{r_2} Se'_i\Big) = I(p)$$

for some ideal $I \subset S$ of height 2, *then $R = S/I$ has p-linear resolution with*

(1) $p = r_1 + 2r_2 + 1 + \sum_{j=1}^{u} (-1)^j j\beta_j$;

(2) $H^i_{\mathscr{M}}(R) = H^{i+1}_{\mathscr{M}}(E)(-p)$ *for* $i = 0, \ldots, n-3$;

(3) *The Betti numbers of R are*

$$\beta_0(R) = 1,$$
$$\beta_1(R) = \beta_1 - r_1,$$
$$\beta_2(R) = \beta_2 + r_2,$$
$$\beta_j(R) = \beta_j, \qquad j > 2.$$

Proof. The minimal free resolution of $I(p)$ is obtained from that of E by adding a free module $S^{r_2}(-1)$ to S^{β_2}, corresponding to the new relations $e'_i = 0$, and factoring out a free module S^{r_1} from S^{β_1}, corresponding to the loss of generators e_i. This gives the formulas in (3). Part (1) then follows at once from Proposition 1.10. Part (2) is immediate from the exact sequence

$$0 \to S^{r_1} \oplus S^{r_2}(-1) \to E \to S(p) \to R(p) \to 0.$$

EXAMPLES. Let $T = \operatorname{coker}(S \xrightarrow{\phi} S^n(1))$, where ϕ is the map with matrix

$$\phi = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

Set $T_i = \bigwedge^{n-i} T(i-n)$ for $i = 1,..., n$. As usual, a free presentation for $\bigwedge^{n-i} T$ may be derived from that for T; it is

$$\bigwedge^{n-i} T = \operatorname{coker} \bigwedge^{n-i-1} (S^n(1)) \otimes S \xrightarrow{1 \wedge \phi} \bigwedge^{n-i} (S^n(1)),$$

so that we see that the $\bigwedge^{n-i} T$ are the syzygies of $k(n)$ in the twisted Koszul complex

$$\mathbb{K}(x_1,..., x_n)(n): 0 \to S \to S^n(1) \to \bigwedge^2 S^n(2) \to \cdots \to S(n) \to k(n) \to 0.$$

In particular, we get depth $T_i = i$, and $H^i_{\mathcal{M}}(T_i) = k(i)$ (for $i \neq n$) while $H^j_{\mathcal{M}}(T_i) = 0$ for $j \neq i, n$.

For $i \geqslant 2$ the module T_i satisfies conditions 1–3 above, so if $h_2,..., h_{n-1}$ are positive integers we may apply the lemma and proposition to a module of the form

$$\sum_{i=2}^{n} T_i^{k_i}$$

which has rank $r = \sum_i h_i \binom{n-i}{h-i}$ and Betti numbers $\beta_j = \sum_j h_i \binom{n}{n-i-j+1}$. The resulting ring R is reduced of dimension $n-2$ and has p-linear resolution, for p as in the proposition. If we take $h_2 = 0$, it is normal. Further, $H^i_{\mathcal{M}}(R) = (k(i+1-p))^{h_{i+1}}$ for $i = 1,..., n-2$, so R is Buchsbaum and is a domain if $p \neq 2$. (The case $p = 2$ is illustrated by the two skew lines in $\mathbb{P}^3$.) For simplicity, consider the case $n = 4$, so that E has resolution

$$0 \to S^{h_2}(-2) \to S^{h_3+4h_2}(-1) \to S^{h_4+4h_3+6h_2} \to E \to 0,$$

and $r = h_4 + 3h_3 + 3h_2$. If we assume char $k = 0$, then by adjusting r_1 and r_2 with $r_1 + r_2 = r - 1$ we may achieve any value of p with

$$h_3 + 2h_2 \leqslant p \leqslant h_4 + 4h_3 + 5h_2 - 1.$$

The resulting ring $R = S/I$ has

$$H^1_{\mathcal{M}}(R) = k(2-p)^{h_2}.$$

Thus, in the language of part (a), we have constructed smooth irreducible p linear curves C with every value of s and p such that

$$p \geqslant \max(3, 2s)$$

with the additional property that

$$H^1_{\mathcal{M}}(R) = H^1(\mathbb{P}^3, I_C) = (k(2-p))^s.$$

Remark. The construction above yields another proof of the existence theorem for Buchsbaum rings due to the second author (Goto [10]).

APPENDIX: HOMOGENEOUS ARTINIAN RINGS WHOSE RESOLUTIONS ARE ALMOST p-LINEAR

In this section we will characterize those homogeneous Artinian factor-rings $A = S/I$, with $S = k[x_1,..., x_n]$, k being a field, whose minimal S-free resolution has the form

$$0 \to \sum_{j \geqslant 0} S^{b_j}(-p-n+1-j) \to S^{b_{n-1}}(-p-n+2) \to \cdots$$
$$\to S^{b_2}(-p-1) \to S^{b_1}(-p) \to S \to A \to 0;$$

that is, Artinian rings whose resolutions are "p-linear except at the last term." We will say that a ring A with resolution as above is "almost" p-linear.

We first note that almost p-linear rings can be characterized internally in a simple way. Recall that the *socle* of A is the graded vectorspace

$$\mathrm{Soc}(A) = (0 :_A \mathscr{M}),$$

where $\mathscr{M} = (x_1,..., x_n)$ as usual.

The following result grew out of conversations with Jürgen Herzog and A. Iarrobino:

THEOREM A1. *Let $S = k[X_1,..., X_n]$, and let $A = S/I$ be a homogeneous Artinian ring. For each i, let $H(i) = \dim_k S_i = \binom{n-1+i}{i}$ and let $\mu_i =$ $\dim(\mathrm{Soc}\, A)_i$. If all the generators of I have degree $\geqslant p$, then $\sum_i \mu_i H(i-p+1) \geqslant H(p-1)$; and A is almost p-linear if and only if equality holds.*

Proof. Consider the module $A^\vee := \mathrm{Hom}_k(A, k)$. Since $A^\vee/\mathscr{M}A^\vee =$ $(\mathrm{Soc}\, A)^\vee$, $A^\vee$ has a minimal free presentation of the form

$$\sum_i S^{c_i}(i) \to \sum_i S^{\mu_i}(i) \to A^\vee \to 0$$

for suitable integers c_i. Note that since the generators of I have degrees $\geqslant p$, we have $\mu_i = 0$ for $i < p-1$, $\dim(A^\vee)_{-p+1} \leqslant \sum \mu_i H(i-p+1)$, as required. Further, we see that if equality holds, then $c_i = 0$ for $i \geqslant p-1$.

We now use the local duality theorem, which says that

$$A^\vee = \mathrm{Ext}^n(A, S(-n)),$$

so the minimal free resolution of A has the form

$$0 \to \sum S^{\mu_i}(-i-n) \to \sum S^{c_i}(-i-n) \to \cdots$$
$$\cdots \to \sum S(-p_{2j}) \to \sum S(-p_{ij}) \to S \to A \to 0,$$

where we have supposed that the generators of I have degrees $p_{1j} \geqslant p$. Since the resolution is minimal, the numbers $p_i = \min_j p_{ij}$ must be strictly increasing with i, so that $p_{ij} \geqslant p+i-1$, and in particular $c_i = 0$ for $i < p-1$. This shows that if the inequality of the theorem is an equality, then $c_i = 0$ for $i \neq p-1$. Since the dual of the resolution of A is again a minimal free resolution, the principal used above implies that the numbers $\bar{p}_i = \max_j p_{ij}$ are also strictly increasing with i, so that given the equality of the theorem we get

$$p+i-1 = ((p-1)+n)-(n-i) \geqslant \bar{p}_i \geqslant p_{ij} \geqslant p_i \geqslant p+i-1,$$

and we see that A is almost p-linear.

Conversely, if A is almost p-linear, then the local duality used above shows that $A^\vee$ has a presentation of the form above with $c_i = 0$ for $i \neq p-1$; it follows that $\dim(A^\vee_{-p+1}) = \sum \mu_i H(i-p+1)$. But since all the generators of I lie in degrees $\geqslant p$, we have $\dim(A^\vee_{-p+1}) = \dim(A_{p-1}) = H(p-1)$, and we are done.

Iarrobino [17] has proven the following very strong existence theorem for almost p-linear factor-rings of $S = k[x_1, \ldots, x_n]$:

PROPOSITION A2. *If* $\{\mu_i\}_{i \geqslant p-1}$ *are positive integers satisfying* $\sum_i \mu_i H(i-p+1) = H(p-1)$, *then there is an almost p-linear factor-ring* $A = S/I$ *such that* $\dim(\operatorname{Soc} A)_i = \mu_i$ *for all i. Furthermore, if r is an integer with* $H(p) \geqslant r \geqslant H(p) - (1/n)\,H(p-1)$, *then for an open dense set of r-dimensional subspaces* V *of* S_p, *the ring* $A = S/VS$ *is almost p-linear.*

Remarks. Almost p-linear algebras are a special case of what Iarrobino calls compressed algebras, and the first statement of the proposition is the corresponding case of his existence theorem for compressed Artin algebras. The second statement follows from the first; if we take

$$\mu_p = H(p) - r,$$
$$\mu_{p-1} = H(p-1) - n\mu_p,$$
$$\mu_i = 0 \qquad \text{for} \quad i \neq p-1, p,$$

the equality holds in Theorem A1, so by the first part of the proposition, an r-dimensional vectorspace $V \subset S_p$ with the property that $A = S/VS$ is almost

p-linear exists, and we must have $VS_1 = S_{p+1}$. On the other hand, if $V \subset S_p$ is any subspace, with $VS_1 = S_{p+1}$, then the socle of $A = S/VS$ is concentrated in degrees $p-1$ and p, so the argument of Theorem A1 shows that A is almost p-linear. Since the condition $VS_1 = S_{p+1}$ is an open condition of subspaces of S_p of fixed dimension, we are done.

Remark. If $n \geqslant 3$, then any almost p-linear ring $A = k[X_1,..., X_n]/I$ is *strictly Cohen–Macaulay* in the sense of Herzog [14]; that is, if A' is a local (not necessarily graded) Cohen–Macaulay ring of dimension d, and $x_1,..., x_d \in A$ is a system of parameters $\notin \mathscr{M}_A^2$, such that $A'/(x_1,..., x_d) \cong A$, then the associated graded ring $\operatorname{gr} A$ is Cohen–Macaulay. Indeed, Herzog proves that S/I is strictly Cohen–Macaulay whenever I is generated by forms of the same degree and $\operatorname{Hom}(I, S/I)_j = 0$ for $j \leqslant -2$, a condition which is easily seen to be satisfied if I has a linear presentation.

We now turn to the construction of the minimal free resolutions of almost p-linear rings.

Let A be a homogeneous Artinian ring and let $\phi_i \in A^\vee$, $i = 1,..., \mu$, be a minimal set of homogeneous generators of $A^\vee$. Composing with the $S \to A$, we may regard ϕ_i as a functional on S; setting $d_i = -\deg \phi_i$, we lose no information in regarding ϕ_i as an element of $S_{d_i}^\vee$, which we will henceforward do.

It follows from the proof of Theorem A1 that A is almost p-linear if and only if

$$S_{p-1}^\vee = A_{p-1}^\vee = \bigoplus S_{d_i-p+i}\phi_i.$$

The map ϕ_i induces maps $S_a \to S_{d_i-a}^\vee$, which we will also call ϕ_i, by the rule $\phi_i(f)(g) = \phi_i(fg)$ for $f \in S_a$, $g \in S_{d_i-a}$. Thus we may reformulate the above as:

PROPOSITION A3. *Let $A = S/I$ be a homogeneous Artinian ring such that I is generated by forms of degree $\geqslant p$. If $\phi_i \in A^\vee$, $i = 1,..., \mu$, is a minimal set of homogeneous generators for $A^\vee$, then A is almost p-linear if and only if the map*

$$S_{p-1} \xrightarrow{(\phi_1,...,\phi_\mu)} \sum_1^\mu S_{d_i-p+1}^\vee$$

is an isomorphism. Furthermore, we have

$$I_p = \ker S_p \xrightarrow{(\phi_1,...,\phi_\mu)} \sum_1^\mu S_{d_i-p}^\vee.$$

Also, if $\phi_i : S_{d_i} \to k$, are any functionals such that the induced map

$$S_{p-1} \xrightarrow{(\phi_1,\dots,\phi_\mu)} \sum S^\vee_{d_i-p+1}$$

is an isomorphism, then, setting

$$V = \ker S_p \xrightarrow{(\phi_1,\dots,\phi_\mu)} \sum S^\vee_{d_i-p},$$

the ring $A = S/VS$ is almost p-linear, and the functionals ϕ_i induce functionals on A which form a minimal set of generators for $A^\vee$.

The second part of the proposition may be deduced from what has already been said or from the construction of the free resolution of S/VS which we will now give. We will make use of the notation of section 1 and the special case of Theorem A1 in which M is a power of the ideal $\mathscr{M} = (x_1,\dots, x_n)$ of S. Thus if we regard S as $\mathrm{Sym}_k F$, with F a vectorspace of rank n, and set

$$L_p^{q+1} = \ker \left(\bigwedge^q F \otimes S_p \to \bigwedge^{q-1} F \otimes S_{p+1} \right),$$

then $L_p^{q+1}(p)) = L_p^q \otimes_k S(-q)$.

If $A = S/I$ is almost p-linear, then it is easy to see that, except for the last step, the resolution of A will be a subcomplex of the resolution $\mathbb{L}(\mathscr{M}^p)$ of $S/\mathscr{M}^p$. We will define maps $\psi_i : L_p^q \to L^{n-q+1}_{d_i-p+1}$, depending on the functions $\phi_i : S_{d_i} \to k$, and set

$$K^i = \ker L^i(\mathscr{M}^p)(-p) \to \sum_1^\mu (L^{n-i+1}(\mathscr{M}^{d_i-p+1}))(1-n) \qquad \text{for} \quad i = 0,\dots, n-1$$

so that the following diagram commutes, and so that if $S_p \to^{(\phi_1,\dots,\phi_\mu)} \sum S^\vee_{d_i-p+1}$ is an isomorphism, then all the vertical maps are epimorphisms, and the last is an isomorphism, as indicated:

$$\begin{tikzcd}[column sep=small]
 & & & 0 \arrow[d] & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & & K^{n-1} \arrow[r] & \cdots \arrow[r] & K^2 \arrow[r] \arrow[d] & K^1 \arrow[r] \arrow[d] & S \arrow[d, equal] \\
 & & L^n(\mathscr{M}^p)(-p) \arrow[r] \arrow[d, equal, "(\phi_i \otimes S(-p))"] & L^{n-1}(\mathscr{M}^p)(-p) \arrow[from=u] \arrow[r] \arrow[d, "(\phi_i \otimes S(-p))"] & \cdots \arrow[r] & L^2(\mathscr{M}^p)(-p) \arrow[r] \arrow[d, "(\phi_i \otimes S(-p))"] & L^1(\mathscr{M}^p)(-p) \arrow[r] \arrow[d, "(\phi_i \otimes S(-p))"] & S \\
0 \arrow[r] & \sum_i S(-n-d_i) \arrow[r] \arrow[uurr, dashed, rounded corners, to path={|- node[pos=0.75, above]{\lambda} (\tikztotarget)}] & \sum_i L^1(\mathscr{M}^{d_i-p+1})^*(1-n) \arrow[r] & \sum_i L^2(\mathscr{M}^{d_i-p+1})^*(1-n) \arrow[r] \arrow[d] & \cdots \arrow[r] & \sum_i L^{n-1}(\mathscr{M}^{d_i-p+1})^*(1-n) \arrow[r] \arrow[d] & \sum_i L^n(\mathscr{M}^{d_i-p+1})^*(1-n) \arrow[r] \arrow[d] & 0 \\
 & & & 0 & \cdots & 0 & 0 &
\end{tikzcd}$$

Here the maps $L^1(\mathscr{M}^p)(-p) \to S$, and the map $\sum S(-n-d_i) \to \sum L^1(\mathscr{M}^{d_i-p+1})^*\,(1-n)$, come from the natural composite

$$L^1(\mathscr{M}^a) \to \mathscr{M}^a(a) \to S(a)$$

(note that the twists arise because the generators of $\mathscr{M}^a$ are in degree a, not in degree 0). If we define the map λ, indicated in the diagram by a dotted arrow, by requiring commutativity, then an easy diagram chase shows that $A = S/VS$ has minimal free resolution

$$0 \to \sum_1^{\mu} S(-n-d_i) \xrightarrow{\lambda} K^{n-1} \to \cdots \to K^1 \to S \to A \to 0.$$

Since K^i is generated by elements of degree $p+i-1$, the resolution is almost linear, as required.

It remains to define maps ψ_i with the desired properties. Since the complex

$$\bigwedge^{q+2} F \otimes S_{p-2} \to \bigwedge^{q+1} F \otimes S_{p-1} \to \bigwedge^{q} F \otimes S_p \to \bigwedge^{q-1} F \otimes S_{p+1}$$

is exact for every p, q, we may define the ψ_i by taking $\xi : \bigwedge^a F \to \bigwedge^{n-a} F^*$ to be the usual duality isomorphisms defined by a generator $\xi \in \bigwedge^n F$ and requiring that the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & L_p^{q+1} \arrow[r] \arrow[d, dashed, "\psi_i"] & \bigwedge^{q} F \otimes S_p \arrow[r] \arrow[d, "\xi \otimes \phi_i"] & \bigwedge^{q-1} F \otimes S_{p+1} \arrow[d, "\xi \otimes \phi_i"] \\
0 \arrow[r] & L_{d_i-p+1}^{n-q}{}^* \arrow[r] & \bigwedge^{n-q} F^* \otimes S_{d_i-p}^* \arrow[r] & \bigwedge^{n-q+1} F^* \otimes S_{d_i-p+1}^* .
\end{tikzcd}$$

The (easy) verification that the right-hand square of this diagram commutes, and that the previous diagrams commute, is left to the reader. To see that the maps $\psi_i \otimes S(-p)$ are epimorphisms, it is of course enough to show that the ψ_i are. Extending the sequences above to the left and summing we get a commutative diagram with exact rows

$$\begin{tikzcd}
\bigwedge^{q+1} F \otimes S_{p-1} \arrow[r] \arrow[d] & L_p^{q+1} \arrow[r] \arrow[d, "(\psi_1,\ldots,\psi_\mu)"] & 0 \\
\sum_1^{\mu} \bigwedge^{n-q-1} F^* \otimes S_{d_i-p+1}^* \arrow[r] & \sum_1^{\mu} L_{d_i-p+1}^{n-q}{}^* \arrow[r] & 0
\end{tikzcd}$$

By our hypothesis, the left-hand vertical arrow is an isomorphism, so $(\psi_1,\ldots,\psi_\mu)$ is an epimorphism. In case $q+1=n$ the horizontal maps become isomorphisms, so $(\psi_1,\ldots,\psi_\mu)$ is an isomorphism, as required.

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