On the Brill-Noether Theorem

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The purpose of this note is to give a short, self-contained proof of the Brill-Noether theorem:

Theorem (1): Let C be a general curve of genus g , and suppose that C possesses a linear system of degree d and dimension r . Then

$$\rho = g - (r+1)(g-d+r) \geq 0 .$$

This was originally proved in [G-H], and more recently in [E-H]; the converse was established earlier in [K-L I], [K-L II] and [K].

As with all existing proofs, the approach here will be to study the behavior of linear series on a family of curves degenerating to a singular and/or reducible curve. We introduce our family here:

(2) Notational conventions: For the remainder of this paper, $\mathcal{O}$ will be a discrete valuation ring with parameter t , $T = \operatorname{Spec} \mathcal{O}$ its spectrum, and 0 and η the closed and generic points of T respectively. $\pi : X \longrightarrow T$ will be a flat, projective family with total space X smooth, and central fiber $X_0 = \pi^{-1}(0)$ the reduced curve pictured in fig. 1.

Our object is to prove theorem (1) specifically for the geometric general fiber $X_{\bar{\eta}} = X \times_T \operatorname{Spec} \overline{k(\eta)}$ of X ; since families X exist for all genera g (see [W]), and since the non-existence of linear series of given degree and dimension is an open condition among smooth curves, this will suffice to prove Theorem (1). We first observe that any line bundle L on $X_{\bar{\eta}}$ is defined over some finite extension of $k(\eta)$. But if we make any finite base change $T' \longrightarrow T$ and minimally

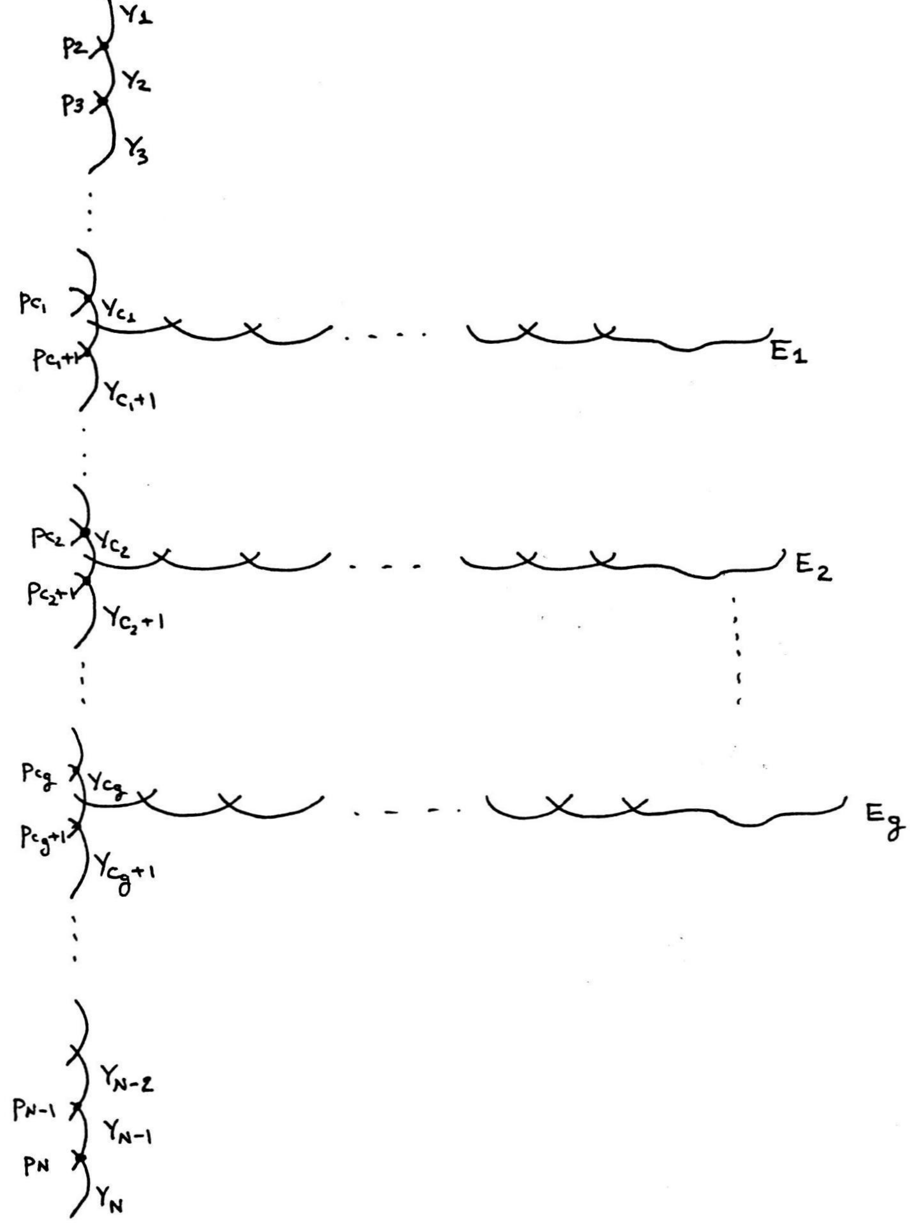

Fig. 1

Components are smooth, and intersect transversally as shown. The E_i are elliptic; all others are rational.

resolve the singularities of $X' = X \times_T T'$, we find that $X' \longrightarrow T'$ is again a family of the same form as X ; thus we may assume L is defined over $k(\eta)$. Moreover, since the total space of X is smooth, any line bundle on X_η extends to one on X . Thus, Theorem (1) will follow once we establish

Theorem (3): Let $X \longrightarrow T$ be as in (2), and let L be any line bundle on X ; let d be the relative degree of L and $r+1 = \mathrm{rank}(\pi_* L)$. Then $\rho = g - (r+1)(g-d+r) \geq 0$.

To prove (3) we consider the limiting behavior of a linear series on X as follows: since the intersection pairing among components of X_0 is unimodular, for each component Y of X_0 there exists a unique line bundle L_Y on X agreeing with L on X_η and such that L_Y has degree d on Y , 0 on all other components of X_0 . We define V_Y to be the linear series

$$V_Y = (\pi_* L_Y) \otimes k(0) \subset H^0(X_0, L_Y) \subset H^0(Y, L_Y) ,$$

the last inclusion coming from the fact that any section of L_Y vanishing on Y vanishes on X_0 . Since the V_Y are all limits of the same linear series $(\pi_* L_Y) \otimes k(\eta)$, it is reasonable to expect that they satisfy some compatibility conditions; and indeed, once we establish those conditions Theorem 3) will follow immediately. These conditions may be expressed as follows: for any point $p \in Y$, we define the vanishing sequence $a_0(V_Y, p) < \ldots < a_r(V_Y, p)$ of V_Y at p to be the $(r+1)$ distinct orders of vanishing of sections $\sigma \in V_Y$ at p ; in particular, for each $\ell = 2, \ldots, N$ we let $a_0^\ell < \ldots < a_r^\ell$ be the vanishing sequence of the series V_{Y_ℓ} at the point p_ℓ (cf. fig. 1). Our basic condition is then

(4) (i) For all ℓ and i ,

$$a_i^{\ell+1} \geq a_i^{\ell} \text{ ; and}$$

(ii) If $\ell = c_j$ for some j , then for all but at most one value of i ,

$$a_i^{\ell+1} > a_i^{\ell} .$$

We note that Theorem (3) follows immediately from (4): trivially, we have, for any ℓ , $i \leq a_i^{\ell} \leq d-r+i$, so that altogether

$$\begin{aligned}(r+1)(d-r) &\geq \sum_i a_i^N - a_i^2 \\ &= \sum_{i,\ell} a_i^{\ell+1} - a_i^{\ell} \\ &\geq rg\end{aligned}$$

and hence $\rho = (r+1)(d-r) - rg \geq 0$.

We begin the proof of (4) with two lemmas. Both refer to a pair of components Y, Z of X_0 , meeting at a point p with p' another point of Y , as in Fig. 2. In this situation, $X_0 - \{p\}$ has two connected components; we will

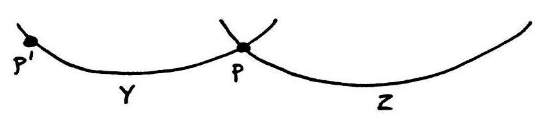

Fig 2

denote by E the divisor on X consisting of the sum of the curves in X_0 in the connected component containing Z . In particular, we have then

$$L_Z \cong L_Y(-dE) ;$$

we accordingly regard L_Z as a subsheaf of L_Y and $\pi_* L_Z$ as a submodule of $\pi_* L_Y$. Finally, for any element $\sigma \in \pi_* L_Y$ we will write $\mathrm{ord}_{p,Y}(\sigma)$ for the order of vanishing of the corresponding section of L_Y along Y . With these conventions, then, we have

<u>Lemma 5</u>. There exists a basis $\sigma_0, \ldots, \sigma_r$ of $\pi_* L_Y$ such that

i) for suitable integers $\alpha_i \geq 0$ the set $t^{\alpha_i}\sigma_i$ is a basis for $\pi_* L_Z$;

and ii) the orders $\mathrm{ord}_{p',Y}(\sigma_i)$ are all distinct.

<u>Proof:</u> The matrix expressing the inclusion of free $\mathcal{O}$-modules $\pi_* L_Z \longrightarrow \pi_* L_Y$ may be diagonalized over $\mathcal{O}$ by applying Gaussian elimination to its rows and columns; this procedure yields a basis $\sigma_0, \ldots, \sigma_r$ of $\pi_* L_Y$ satisfying (i). Now, if $g \in \mathcal{O}$ and $\alpha_i \geq \alpha_j$, then i) will still hold if we replace σ_i by $\sigma_i + g\sigma_j$; these transformations suffice for passing to a basis satisfying (ii) as well.

<u>Lemma 6</u>. If $\sigma \in \pi_* L_Y - t \cdot \pi_* L_Y$ and $\tau = t^\alpha \cdot \sigma \in \pi_* L_Z - t\pi_* L_Z$, then we have

$$\mathrm{ord}_{p',Y}(\sigma) \leq d - \mathrm{ord}_{p,Y}(\sigma) \leq \alpha \leq \mathrm{ord}_{p,Z}(\tau)$$

<u>Proof:</u> The first inequality is trivial (but is the key to (7)(iii) below). For the second inequality, observe that since $t^\alpha\sigma \in \pi_* L_Z$, the divisor

$$\alpha X_0 + (\sigma) = (t^\alpha \sigma) \geq dE ;$$

thus $(\sigma) \geq (d-\alpha)E$ and correspondingly $\mathrm{ord}_{p,Y}(\sigma) \geq d-\alpha$. Likewise, for the last inequality we see that since $t^{-\alpha}\tau \in \pi_* L_Y$,

$$-\alpha X_0 + (\tau) = (t^{-\alpha}\tau) \geq -dE$$

so $(\tau) \geq \alpha(X_0 - E)$ and hence $\mathrm{ord}_{p,Z}(\tau) \geq \alpha$.

Combining lemmas 5 and 6, we have

Lemma 7. With Y, Z, p and p' as in Fig. 2.,

i) $a_i(V_Y,p) + a_{r-i}(V_Z,p) \geq d$

ii) $a_i(V_Z,p) \geq a_i(V_Y,p')$; and

iii) $a_i(V_Z,p) = a_i(V_Y,p')$ for more than one value of i only if there are two or more independent sections of V_Y vanishing only at p and p'.

(In fact, we conclude from Lemmas 5 and 6 that $a_i(V_Z,p) \geq a_{\rho(i)}(V_Y,p')$ for some permutation ρ of $\{0,\ldots,r\}$, and hence that $a_i(V_Z,p) \geq a_i(V_Y,p')$; and similarly for parts (i) and (iii)).

Part (i) of (4) follows immediately from (7)(ii), applied to $Y = Y_\ell$, $Z = Y_{\ell+1}$, $p = p_{\ell+1}$ and $p' = p_\ell$. Part (ii) of (4), and thereby Theorem (3), will follow similarly from 7(iii) once we establish

Lemma 8. If $\ell = c_m$, there is at most one section $\sigma \in V_{Y_\ell}$ non-zero on $Y_\ell - \{p_\ell, p_{\ell+1}\}$.

Proof: Label the components of X_0 between $Y = Y_\ell$ and $E = E_m$ as in Fig. 3:

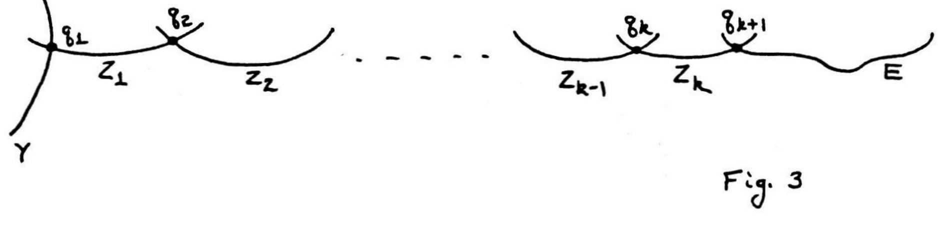

Fig. 3

Suppose there are two independent sections $\sigma,\tau \in V$ vanishing only at p_ℓ and $p_{\ell+1}$. The pencil they span will be totally ramified at p_ℓ and $p_{\ell+1}$ and hence unramified elsewhere; in particular, there will exist sections $\sigma_o,\tau_o \in V$ vanishing to orders exactly 0 and 1 at q_1. Applying (7)(i) once and (7)(ii) k times, then, we have

$$a_0(V_Y,q_1) = 0 \text{ , } a_1(V_Y,q_1) = 1$$

$$\Rightarrow a_r(V_{Z_1},q_1) = d, \; a_{r-1}(V_{Z_1},q_1) = d-1$$

$$\Rightarrow a_r(V_{Z_2},q_2) = d, \; a_{r-1}(V_{Z_2},q_2) = d-1$$

$$\vdots$$

$$\Rightarrow a_r(V_{Z_k},q_k) = d, \; a_{r-1}(V_{Z_k},q_k) = d-1$$

$$\Rightarrow a_r(V_E,q_{k+1}) = d \text{ , } a_{r=1}(V_E,q_{k+1}) = d-1 \text{ .}$$

But this is absurd; a pencil of degree d on an elliptic curve can't have d-1 base points.