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Divisors on general curves and cuspidal rational curves

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Introduction

In this paper we introduce a new method in the theory of divisors on curves in characteristic 0, and use it to prove two new results on the maps of general curves into projective spaces associated with general linear series of given degree and dimension, as well as to give a new proof of a sharpened form of the Brill-Noether result proved by Griffiths and Harris [1980].

As an application we have, for example:

Theorem. *A general curve C of genus $g \neq 1, 3$ admits an embedding of degree d in some projective space if and only if*

$$d \geqq \tfrac{3}{4} g + 3.$$

Furthermore, if the condition is satisfied, then C can be embedded in $\mathbb{P}^3$ so that C has no flexes and only stalls of multiplicity 1.

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Here a *flex* $x \in C$ is a point such that every plane containing the tangent line to C at x meets C with multiplicity $\geqq 3$ at x, and a *stall* $x \in C$ of multiplicity μ is a point such that the osculating plane to C at x at meets C with multiplicity $3+\mu$ at x. It follows easily from the general Plücker formulas (see Sect. 1, below) that a curve as in the Theorem must have $4d+12g-12$ ordinary stalls.

Here a statement about a general curve means that the corresponding statement holds for all the curves in an open dense set of the moduli space of curves (of any given genus), and a statement about a general linear series of dimension r on a curve C means that for any $d>0$ the corresponding result holds for all the linear series in an open dense of the variety $G_d^r(C)$ which parametrizes linear series of degree d and dimension r. The Brill-Noether result is that, for a general curve of genus g, $G_d^r(C)$ is reduced and has dimension $g-(r+1)(g-d+r)$. Our first main result is:

Theorem 1 (Embedding). *Let C be a general curve. If $\mathscr{L}$ is a general linear series of dimension $r \geqq 3$, then $\mathscr{L}$ has no base points and the associated map $\phi_{\mathscr{L}}: C \to \mathbb{P}^r$ is an embedding.*

This, and the Brill-Noether result are proved by a reduction to the case of certain singular rational curves; the reduction step is similar to the now standard reduction of the Brill-Noether result to the case of general nodal curves, though once in the singular case, the embedding result is far harder than the Brill-Noether result.

To state our second main result, recall that, to any r-dimensional linear series L on a curve C, and smooth point $x \in C$, is associated a *ramification sequence* $0 \leqq \alpha_0 \leqq \ldots \leqq \alpha_r < \infty$; α_0 is the multiplicity of x as a base point of L, and, for $i \geqq 1$, $\alpha_i + i$ is the multiplicity of x as a basepoint of the linear series of divisors in L whose multiplicity at x is $> \alpha_{i-1} + (i-1)$. Thus, for example, ϕ_L is regular at x iff $\alpha_0 = 0$, and ϕ_L is locally an embedding near x iff $\alpha_0 = \alpha_1 = 0$. We will say that x is an *ordinary* ramification point (with respect to L) if the ramification sequence is $(0, 0, \ldots, 0, 0)$ or $(0, \ldots, 0, 1)$.

Theorem 2 (Ordinary ramification). *If C is a general curve and $\mathscr{L}$ a general linear series on C, of any degree and dimension, then $\mathscr{L}$ has only ordinary ramification.*

This theorem is again proved by reduction to the singular rational case; but here, though the corresponding result for cuspidal rational curves is easy, the reduction is novel, and involves, for example, the Picard group of the punctured spectrum of certain singular surfaces.

It follows, in particular, from these results that if C is general and r and d are chosen so that $G_d^r(C)$ is finite, which by Brill-Noether means

$$d = \frac{r}{r+1} g + r,$$

then the two main results hold for every linear series of degree d and dimension r. The best known case is that of the canonical series, with $d = 2g-2$, $r = g$

-1; Theorem 1 is well-known and trivial in this case, Theorem 2 says that a general curve has only "normal" Weierstrass points, which is also well-known, but considerably deeper (the book of Griffiths and Harris [1978] contains a plausibility argument; we are unable to locate a proof in the literature.)

The new method that we use is to reduce the proof of results such as those above to problems on cuspidal rational curves, that is, rational curves whose singularities are all ordinary cusps (local equation: $x^3+y^2=0$) instead of problems on nodal rational curves as has been done since Castelnuovo [1889] and Severi [1921]. This seemingly small difference is significant for the theory because rational curves with g ordinary cusps behave, from the point of view of linear series, far more like general curves of genus g than do rational curves with g nodes.

A simple indication of the difference may be seen at once: Consider the open set of G_2^1 parametrizing regular degree 2 maps of a curve C to $\mathbb{P}^1$. If C is a rational curve with g nodes, obtained by identifying the g pairs of points $(p_1,q_1)\dots(p_g,q_g)$ on $\mathbb{P}^1$, the maps as above may be represented as projections of a conic in $\mathbb{P}^2$ from a point x that lies on all the chords $\overline{p_iq_i}$; the case $g=3$ is illustrated below:

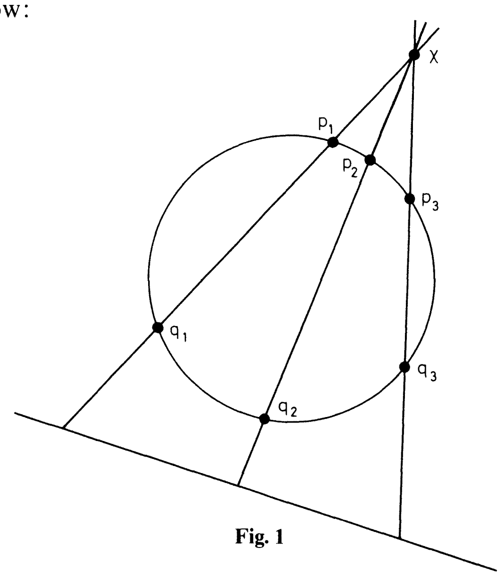


Fig. 1

If, on the other hand, C is a rational curve with g cusps, corresponding to points $p_1,\dots,p_g$ in $\mathbb{P}^1$, then the maps may be represented as projections of the conic from a point x that lies on all the tangent lines to the conic at the points p_i; the case $g=2$ is illustrated:

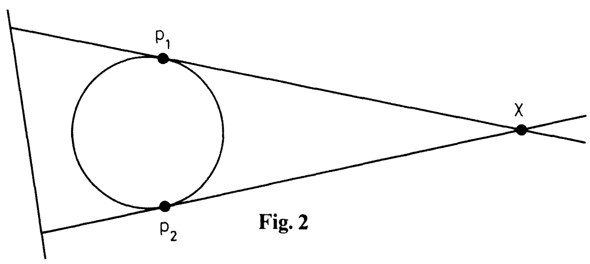


Fig. 2

Now for a general curve, Brill-Noether asserts in particular that G_2^1 is empty if and only if $g \geqq 3$. For a general g-nodal curve this is true, three or more general chords do not meet in a point; but it fails for some special g-nodal curves, as in our illustration. By contrast, *no* three tangents to a conic meet in a point:

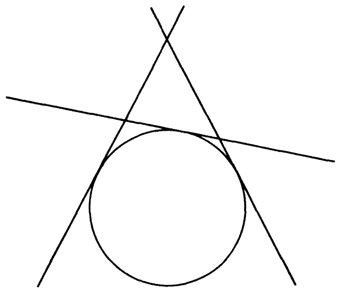


Fig. 3

so *every* 3-cuspidal rational curve satisfies at least this small part of Brill-Noether.

This phenomenon proves to be rather general, and is the basis of this paper. We prove in particular that general g-cuspidal curves satisfy the theorems above; the smooth case is a consequence. We also use our results on cuspidal curves to give explicitly the dimension of various families of embedded cuspidal rational curves. These computations are a cornerstone of the proof by Eisenbud-Van de Ven [1982] that the space of smooth rational curves with given degree and normal bundle is irreducible, and were the original motivation for our work on cuspidal curves.

Our results on cuspidal rational curves rest on two theorems concerning the osculating flags to a nondegenerate curve C in $\mathbb{P}^d$ and the Schubert varieties of planes of a fixed dimension, say k, associated to them. The case of greatest interest is that in which C is the rational normal curve. In this case we prove Theorem 2.3, generalizing the remarks about tangent lines to a conic above, that Schubert varieties defined with respect to osculating flags at a set of distinct points of C are dimensionally transverse. This means roughly that distinct osculating flags to a rational normal curve are in rather general position with respect to one another, even in a certain "non-linear" sense. Dually, this theorem asserts that the space of linear series of given degree d and dimension r on $\mathbb{P}^1$, with prescribed ramification sequences at a given set of points, always has the "expected" dimension, or equivalently, that the space of $r+1$-dimensional vector spaces V of forms of degree d in 2 variables satisfying a condition of the form

$$\{V \text{ contains a } k_{ij} \text{ dimensional subspace of forms divisible by } (L_i)^j \text{ for } i=1,\ldots,g, j\geqq 0\}$$

for a given set of linear forms L_i and suitable numbers k_{ij}, has always the "expected" dimension, independently of the set of distinct linear forms L_i.

In this last form the result was first proved by *A*. Iarrobino (unpublished, about 1973), and rediscovered by us; the proof we give, which is identical to that of Iarrobino, passes by way of a certain "Plücker formula" giving the degree of a "Wronskian" determinant.

Our second main result in this direction (Theorem 8.1) analyzes the degeneration of the intersection of the Schubert varieties corresponding to osculating flags at a pair of distinct points on an arbitrary analytic arc in $\mathbb{P}^d$ as the two points move toward each other and coalesce. In its simplest non-trivial special case, the result asserts that if C is a non-degenerate arc in $\mathbb{P}^3$ and if $T_{C,p}$ and T_{C,p_1} are the tangent lines to C at points p and p_1 in C, then the limit as $p_1 \to p$ of the variety of lines in $\mathbb{P}^3$ which meet both $T_{C,p}$ and T_{C,p_1} is the union of the variety of lines through p and the variety of lines lying in the osculating 2-plane to C at p.

This special case (and a more general special case needed for the embedding theorem) is susceptible to a reasonably direct geometric proof (Proposition 7.1) but in the general case we give an algebraic proof based on explicit knowledge of the equations for unions of Schubert varieties in the Plucker embedding, which we use in a form codified in the "Hodge algebra" structure (or "Straightening law") for the homogeneous coordinate ring of the Grassman variety, exposed in Deconcini et al. [1982] and Eisenbud [1979].

We now pose some open problems in the direction of the results treated below:

1) If C is an arbitrary cuspidal rational curve, is the general member of $G^r_d(C)$ very ample for $r \geqq 3$ as is the case for general cuspidal curves?

2) If C is a general cuspidal rational curve, is $G^r_d(C, \operatorname{Sing} C)$, the set of linear series in $G^r_d(C)$ without base points in $\operatorname{Sing} C$, smooth? Equivalently, is the set of $d-r-1$-planes in $\mathbb{P}^d$ which meet g general tangent lines to the rational normal curve, but do not meet the rational normal curve in the points of tangency, smooth? (This goes in the direction of the Gieseker-Petri theorem that $G^r_d(C)$ is smooth if C is a general curve; see Gieseker [1982] or Eisenbud-Harris [1983].)

3) Corollary 5.3 below may be rephrased as follows: if C is a general curve of genus g, then the locus in G^r_d of complete linear series with an inflection point of weight greater than 1 has codimension at least 1; and the locus of such series with an inflection point of weight greater than $g+1$ has codimension at least $g+1$ (i.e., is empty). Interpolating, we may ask whether it is the case that the locus of series with an inflection point of weight $w+1$ or more will always have codimension at least w in G^r_d. In particular, does the locus in G^r_d of linear series which fail to immerse C in $\mathbb{P}^r$ have codimension $r-1$?

4) Another question, along the lines of 3), which suggests itself is this: is it the case that if C is general then the locus in G^r_d of linear series which fail to embed C (i.e., which are not very ample) has codimension $r-2$?

It should be noted that the statements of 3) and 4) are all true for a general cuspidal curve; the reductions to this case, however, present some difficulty.

5) Finally, a major open problem in curve theory is the *maximal rank conjecture:* that if C is a general curve of genus g, and $|\mathscr{L}|$ a general point of G_d^r, the maps

$$\rho_m\colon \operatorname{Sym}^m H^0(C,\mathscr{L}) \to H^0(C,\mathscr{L}^m)$$

have maximal rank for all m. In the present circumstance, we ask: does the maximal rank conjecture hold for a general cuspidal curve? An affirmative answer would immediately imply the same for a general curve of genus g.

The organization of this paper is as follows. Section 1 contains a brief account of the necessary Plücker formula, and Sect. 2 gives the application to linear series on $\mathbb{P}^1$ (Iarrobino's theorem and its dual version on osculating flags). In Sect. 3 we apply these results to questions about families of rational curves with cusps and inflectionary behavior of various sorts (in particular: how many cusps can a rational curve in $\mathbb{P}^r$ have?)

In Sect. 4 we explain in some detail the connection of the results on $\mathbb{P}^1$ with results on $G_d^r(C)$, where C is a cuspidal rational curve; more generally, we give some results connecting linear series on a curve with "universal singularities" - that is, singularities for which the conductor is the maximal ideal of the singular curve at each singular point - with certain Schubert conditions on linear series on the normalized curve. We also prove that if C is a cuspidal rational curve, then $G_d^r(C)$ is dense in its natural compactification; this is the main step in the reduction of the smooth case to the cuspidal case in the embedding theorem and the Brill-Noether theorem.

In Sect. 5, we state our main results on smooth curves (Theorem 5.1) and show how these may all be deduced from the corresponding statements for general cuspidal curves. This completes in particular the proofs of the dimension-theoretic part of the Brill-Noether theorem, and of our ordinary ramification theorem. We give in an appendix an explicit construction of a family of smooth curves of genus g degenerating to a g-cuspidal rational curve.

In Sect. 6 we complete the proof of our embedding theorem by establishing the corresponding statement for g-cuspidal curves.

Section 7 is devoted to illustrative material on degeneration of Schubert intersections; it contains the direct geometric proof of a special case of Theorem 8.1, referred to above, and a relatively complete treatment of the possible limits of the variety of lines meeting two given lines in $\mathbb{P}^3$ as the given lines coalesce.

Section 8 contains the proof of our main degeneration result.

Finally, in Sect. 9, we complete the proof of Theorem 5.1 by showing that if C is a cuspidal rational curve with cusps in general position, then $G_d^r(C)$ is reduced. We also give an example to show that the general position hypothesis is necessary: we show that there is only one line meeting a certain set of 4 tangent lines to the twisted cubic, instead of 2 lines, as for 4 general tangent lines.

Discussions with many mathematicians have helped us clarify the material of this paper. In particular, we are grateful to Michael Artin, Maria-Grazia Ascenzi, David Buchsbaum, Tony Iarrobino, and Steven Kleiman.

0. Preliminaries

We now review some notation, terminology, and standard results.

First, we will always work in characteristic 0, over an algebraically closed groundfield which we call $\mathbb{C}$. (g-cuspidal rational curves are all hyperelliptic in characteristic 2!). Nevertheless, some of our results are characteristic free, and we have occasionally noted the fact.

Second, *curve* throughout means *reduced, irreducible 1-dimensional projective variety*. An *ordinary cusp* is a point on a curve which is analytically isomorphic to the singular point on the plane curve with equation $y^2=x^3$. An *ordinary node* is similarly represented by $xy=0$. A *g-cuspidal rational curve* is a curve whose normalization is the projective line $\mathbb{P}^1$ and whose singularities are precisely g ordinary cusps. Such a curve C can be constructed, with cusps at points corresponding to $p_1, \ldots, p_g \in \mathbb{P}^1$, say, by taking the underlying set to be $\mathbb{P}^1$ and the stalk of the structre sheaf at p_i to be $\mathcal{O}_{C,p_i}=\mathbb{C}+m^2_{\mathbb{P}^1,p_i}$; see for example Serre [1959] Ch. IV, §4. A g-cuspidal rational curve is completely determined by the positions of the cusps; g-cuspidal curves thus have a $g-3$-dimensional moduli space, consisting of the g-tuples of distinct points on $\mathbb{P}^1$, modulo $\operatorname{Aut}\mathbb{P}^1$.

Finally, we establish notation for Grassmann and Schubert varieties; see, for example, Griffiths and Harris [1978] for more details. We write $\operatorname{Gr}(k,\mathbb{P}^n)$ for the Grassmannian of projective k-planes in $\mathbb{P}^n$. Its dimension is $(k+1)(n-k)$.

A *flag* in $\mathbb{P}^n$ is a sequence of linear subspaces

$$\mathscr{F}^0 \subset \mathscr{F}^1 \subset \ldots \subset \mathscr{F}^{n-1} \qquad (\subset \mathscr{F}^n = \mathbb{P}^n)$$

in $\mathbb{P}^n$ with $\dim \mathscr{F}^i = i$.

If $\mathscr{F}$ is a flag in $\mathbb{P}^n$, and $\underline{a}$ is a $k+1$-tuple of integers.

$$\underline{a}=(a_0, a_1, \ldots, a_k)$$

with $n-k \geqq a_0 \geqq \ldots \geqq a_k \geqq 0$ then we define the Schubert variety $\sigma_{\underline{a}}(\mathscr{F}) \subset \operatorname{Gr}(k,\mathbb{P}^n)$ by

$$\sigma_{\underline{a}}(\mathscr{F})=\{L \mid \dim(L \cap \mathscr{F}^{n-k-a_i+i}) \geqq i \text{ for all } i.\}$$

The codimension of $\sigma_{\underline{a}}(\mathscr{F})$ is $\sum a_i$. We sometimes abbreviate $\sigma_{(a,0,0,\ldots)}(\mathscr{F})$ to $\sigma_a(\mathscr{F})$ or $\sigma(\mathscr{F}^{n-k-a})$, and call it a *special Schubert variety*. We use the standard duality isomorphism $\operatorname{Gr}(k,\mathbb{P}^n) \cong \operatorname{Gr}(n-k-1,\mathbb{P}^{n\vee})$. This carries a k-space $L \subset \mathbb{P}^n$ to $L^\perp$, the space of linear forms on $\mathbb{P}^n$ vanishing on L, in $\mathbb{P}^{n\vee}$. If we write $\mathscr{F}^\perp=\{\mathscr{F}^{n-1\perp} \subset \ldots \subset \mathscr{F}^{0\perp}\}$ for the flag dual to $\mathscr{F}$, then the isomorphism carries $\sigma_{\underline{a}}(\mathscr{F})$ to $\sigma_{\underline{b}}(\mathscr{F}^\perp)$, where b_i is the number of elements of the sequence $\underline{a}$ which are $\geqq i+1$.

If V is a vector space, then we write $\mathbb{P}(V^\vee)$ for the set of hyperplanes in $V^\vee$, and we identify this with the set of lines in V.

We will sometimes consider the Grassmann variety $\operatorname{Gr}(k+1, V)$ of $k+1$-dimensional (vector-) subspaces of V. Of course we have, canonically,

$$\mathrm{Gr}(k+1, V) \cong \mathrm{Gr}(k, \mathbb{P}(V^\vee)).$$

We will always number the subspaces in flags, and the Schubert varieties, as if we were in the projective case; thus, for example, $\mathscr{F}^r \subset V$ will be an $r+1$-dimensional subspace, and, if V is $(n+1)$-dimensional, then

$$\begin{aligned}\sigma_{a,0,\ldots}(\mathscr{F}) = \sigma_a(\mathscr{F}) = \sigma(\mathscr{F}^{n-k-a}) \\ = \{V' \in \mathrm{Gr}(k+1, V) \mid \dim V' \cap \mathscr{F}^{n-k-a} \geqq 1\}.\end{aligned}$$

1. A Plücker formula

The result of this section is a formula for the total ramification of a linear series which is at the heart of our method. We include the short proof because, although the result is classical we do not know a satisfactory reference (see however Weyl [1943] and Piene [1977]) and because we will need part of the argument again in Sect. 8.

Let C be a smooth, complete, complex curve of genus g. Consider a line bundle $\mathscr{L}$ of degree d and an r-dimensional linear system corresponding to it – that is, an $r+1$-dimensional vector space $V \subset H^0(\mathscr{L})$ of global sections of $\mathscr{L}$. For each point $p \in C$, let R_p be the set of orders of vanishing of sections in V at p:

$$R_p = \{\mathrm{ord}_p \sigma\}_{\sigma \in V}.$$

It is easy to see that R_p must consist of exactly $r+1$ distinct non-negative integers, so we define the ramification sequence of V at p to be the sequence of integers $0 \leqq \alpha_0^V(p) \leqq \alpha_1^V(p) \leqq \ldots \leqq \alpha_r^V(p)$ determined by the formula

$$R_p = \{i + \alpha_i^V(p)\}_{i=0,\ldots,r}.$$

When no confusion about V can result we will write $\alpha_i(p)$. A point p is called an *inflection point* for the linear series V if $\alpha_i^V(p) \neq 0$ for some i; (such a point is also sometimes called a *generalized Weierstrass point* for the linear system V). We define the *weight* $w(V, p)$ (or simply $w(p)$) of the point p with respect to V by

$$w(p) = \sum_i \alpha_i^V(p),$$

and call an inflection point *ordinary* if it has weight 1.

It is easy to check that a linear series V can have only finitely many inflection points, so we may set

$$\alpha_j^V = \sum_{p \in C} \alpha_j^V(p).$$

Again, we will usually drop the superscript V.

Proposition 1.1. *For any linear series V on C we have*

$$(r+1)d + \binom{r+1}{2}(2g-2) = \sum_{j=0}^{r} \alpha_j.$$

Proof. Taking Taylor expansions up to order r of the sections in V we get a map from V to the bundle of principal parts of order r of $\mathscr{L}$ (r-jets of sections of $\mathscr{L}$):

$$V \otimes \mathcal{O} \xrightarrow{\alpha} \mathscr{P}^r(\mathscr{L}).$$

Thus taking exterior powers we get a section

$$\mathcal{O} \xrightarrow{\sigma} \bigwedge^{r+1} \mathscr{P}^r(\mathscr{L}).$$

We will derive the desired formula by computing the number of zeros of this section in two ways.

First, let K be the canonical bundle of C. From the exact sequences

$$0 \to \mathscr{L} \otimes K^m \to \mathscr{P}^m(\mathscr{L}) \to \mathscr{P}^{m-1}(\mathscr{L}) \to 0 \qquad (m \geqq 1)$$

we derive

$$\bigwedge^{r+1} \mathscr{P}^r(\mathscr{L}) \cong (\mathscr{L} \otimes K^r) \otimes (\mathscr{L} \otimes K^{r-1}) \otimes \ldots \otimes \mathscr{L}$$

$$= \mathscr{L}^{r+1} \otimes K^{\binom{r+1}{2}}.$$

Since this bundle has degree $(r+1)d + \binom{r+1}{2}(2g-2)$, the section σ has $(r+1)d + \binom{r+1}{2}(2g-2)$ zeros.

We will now complete the argument by showing that, at any point p, σ has a zero of order $\sum_{j=0}^{r} \alpha_j(p)$, and thus σ has $\sum_{j=0}^{r} \alpha_j$ zeros in all.

To this end, let t be a local coordinate at p and choose a basis $\sigma_0, \ldots, \sigma_r$ of V with

$$\sigma_j = t^{j+\alpha_j(p)} + \text{higher order}.$$

In terms of a coordinate t at p, α is given by a matrix whose $(i,j)^{\text{th}}$ entry is $\frac{1}{i!}\frac{d^i}{dt^i}\sigma_j$. The order of vanishing of σ at p is the smallest order of vanishing of any linear combination of $(r+1) \times (r+1)$ minors of this matrix. Inspecting the matrix of leading terms of α:

$$\begin{pmatrix} t^{\alpha_0(p)} & t^{1+\alpha_1(p)} & t^{2+\alpha_2(p)} & \ldots \\ \alpha_0(p)\, t^{\alpha_0(p)-1} & (1+\alpha_1(p))\, t^{\alpha_1(p)} & (2+\alpha_2(p))\, t^{1+\alpha_2(p)} & \ldots \\ \binom{\alpha_0(p)}{2} t^{\alpha_0(p)-2} & \binom{1+\alpha_1(p)}{2} t^{\alpha_1(p)-1} & \binom{2+\alpha_2(p)}{2} t^{\alpha_2(p)} & \ldots \\ \vdots & \vdots & \vdots & \end{pmatrix}$$

we see at once that this order is $\geqq \sum_{j=0}^{r} \alpha_j(p)$. It is equal to $\sum_{j=0}^{r} \alpha_j(p)$ if and only if the matrix of coefficients in the first $r+1$ columns above has non-zero determinant. Thus the next lemma completes the proof of Proposition 1.1 by computing the determinant:

Lemma 1.2. *The* $(r+1)\times(r+1)$ *matrix whose* $(i,j)^{\text{th}}$ *entry is the binomial coefficient*

$$\binom{X_j}{i} \qquad (0\leqq i,j\leqq r)$$

has determinant $\left(\prod_{i=0}^{r}\frac{1}{i!}\right)\prod_{0\leqq i<j\leqq r}(X_j-X_i)$, *which is in particular nonzero if the* X_j *are distinct integers.*

Proof. We may treat the X_j as independent variables. Multiplying the i^{th} row by $i!$ for each i and performing an obvious row operation transforms the determinant into the well known van der Monde determinant

$$\det\begin{pmatrix} 1 & 1 & 1 & \dots \\ X_0 & X_1 & X_2 & \dots \\ X_0^2 & X_1^2 & X_2^2 & \dots \\ \vdots & & & \end{pmatrix} = \prod_{0\leqq i<j\leqq r}(X_j-X_i).$$

Remarks. 1) The number α_0 is the "number of base points" of the linear system V, while the numbers $\alpha_i-\alpha_{i-1}$ are the so-called (higher) ramification indices of C. In particular, $\alpha_1-\alpha_0$ is the "number of singularities" of the map to projective r-space associated to V.

2) The formula of Proposition 1.1 reduces to the Hurwitz formula $2-2g=d(\chi(\mathbb{P}^1))-R$ (where R is the total ramification of a map $C\to\mathbb{P}^1$ of degree d, and $\chi(\mathbb{P}^1)=2$ is the topological Euler characteristic of $\mathbb{P}^1$) in case $r=1$ and $\alpha_0=0$, so that V corresponds to a degree d map $C\to\mathbb{P}^1$.

3) The formula of Proposition 1.1 reduces to the usual formula for the number of Weierstrass points of a curve of genus g if $\mathscr{L}=K$, $V=H^0(K)$.

2. Dimensional transversality

Let C be a smooth complete complex curve of genus g and let $\mathscr{L}$ be a line bundle of degree d on C with $\dim H^0(\mathscr{L})=n+1$. Let p be a point of C, and write $\alpha_i^{\mathscr{L}}(p)$ for $\alpha_i^{H^0(\mathscr{L})}(p)$, the i^{th} ramification index of the complete linear series associated to $\mathscr{L}$ at p. (The case of primary interest to us is the case $\alpha_i^{\mathscr{L}}(p)=0$ for all i).

We will interpret the ramification indices of a linear series V of dimension r associated to $\mathscr{L}$ in terms of Schubert conditions. First we define dual flags in $\mathbb{P}(H^0(\mathscr{L}))$ and $\mathbb{P}(H^0(\mathscr{L})^\vee)$, depending on p, as follows: Set

$$\mathscr{G}(p):\ \mathscr{G}^0(p)\subset\ldots\subset\mathscr{G}^n(p)=\mathbb{P}(H^0(\mathscr{L})^\vee)$$

by letting $\mathscr{G}^i(p)$ be the set of (lines through) sections of $\mathscr{L}$ which vanish to order $\geqq\alpha_{n-i}^{\mathscr{L}}+(n-i)$ at p. The dual flag

$$\mathscr{F}(p):\ \mathscr{F}^0(p)\subset\ldots\subset\mathscr{F}^n(p)=\mathbb{P}(H^0(\mathscr{L}))$$

may be defined by setting

$$\mathscr{F}^i(p) = \mathbb{P}(\operatorname{Image} H^0(\mathscr{L}) \to \mathscr{P}^{\alpha^{\mathscr{L}}+i}(\mathscr{L})_p).$$

From the interpretation of $\mathscr{P}^m(\mathscr{L})_p$ as the set of truncated Taylor expansions at p of sections of $\mathscr{L}$, it is immediate that

$$\mathscr{F}^i(p) = \mathscr{G}^{n-i-1}(p)^{\perp},$$

so the flags $\mathscr{F}$ and $\mathscr{G}$ are indeed dual.

The space $\mathscr{F}^i(p)$ is the osculating i-plane to the image of C in $\mathbb{P}(H^0(\mathscr{L}))$ associated to the natural 1-quotient of $\mathcal{O}_C \otimes H^0(\mathscr{L})$ contained in $\mathscr{L}$; this is obvious at points where all $\alpha_i^{\mathscr{L}}(p) = 0$ (unramified points), and this is the only case in which we will use it. In the general case, the proof reduces to Lemma 1.2.

We may regard a linear series V of dimension r, associated to $\mathscr{L}$ as an r-space $\mathbb{P}(V^{\vee})$ in $\mathbb{P}(H^0(\mathscr{L})^{\vee})$; we may also regard it as giving a rational map $C \to \mathbb{P}(V)$, which is the composition of the morphism $C \to \mathbb{P}(H^0(\mathscr{L}))$ defined above and projection from a center $\mathbb{P}(V)^{\perp} \cong \mathbb{P}^{n-r-1}$ represented by the $n-r$ quotient $H^0(\mathscr{L}) \to H^0(\mathscr{L})/V$.

The next Proposition expresses the ramification sequence of V in terms of the Schubert cycles, defined in terms of $\mathscr{G}(p)$ and $\mathscr{F}(p)$, in which $\mathbb{P}(V^{\vee})$ or $\mathbb{P}(V)^{\perp}$ lie:

Proposition 2.1. *A linear series V of dimension r, associated to $\mathscr{L}$ satisfies $\mathbb{P}(V^{\vee}) \in \sigma_a(\mathscr{G}(p))$ (respectively, the projection center $\mathbb{P}(V)^{\perp}$ associated to V in $\mathbb{P}(H^0(\mathscr{L}))$ satisfies $\mathbb{P}(V)^{\perp} \in \sigma_b(\mathscr{F}(p))$ where b_i is the number of j with $a_j \geqq i+1$) if and only if*

$$\alpha_{r-i}^V(p) \geqq \alpha_{a_i+r-i}^{\mathscr{L}}(p) + a_i \quad \text{for } i = 0, \ldots, r.$$

Proof. The $(i+1)$-dimensional subspace of V of sections vanishing to highest possible order at p contains sections vanishing to order $\alpha_{r-i}^V(p) + r - i$ and higher. Thus, for any a_i we have

$$\dim \mathbb{P}(V^{\vee}) \cap \mathscr{G}^{n-r-a_i+i}(p) \geqq i$$

if and only if

$$\alpha_{r-i}^V(p) + r - i \geqq \alpha_{a_i+r-i}^{\mathscr{L}}(p) + a_i + r - i,$$

that is,

$$\alpha_{r-i}^V(p) \geqq \alpha_{a_i+r-i}^{\mathscr{L}}(p) + a_i,$$

as required. □

Combining this with Proposition 1.1 we obtain:

Corollary 2.2. *Suppose that $p_1, \ldots, p_m \in C$ are distinct points and that $a^j = (a_0^j, a_1^j, \ldots, a_r^j)$ $(j = 1, \ldots, m)$ are Schubert indices. Let $\sigma_{a^j}(\mathscr{G}(p_j))$ be the corresponding Schubert cycles in the Grassmannian of r-planes in $\mathbb{P}(H^0(\mathscr{L})^{\vee})$. If*

$$\sum_{i,j} (a_i^j + \alpha_{a_i{}^j+r-i}^{\mathscr{L}}(p)) > (r+1)d + \binom{r+1}{2}(2g-2)$$

then $\bigcap_i \sigma_{a^i}(\mathscr{G}(p_i)) = \emptyset$.

This result becomes sharp and particularly significant when applied to the curve $C=\mathbb{P}^1$. Of course then $\mathscr{L}=\mathcal{O}_{\mathbb{P}^1}(d)$, so $\mathbb{P}(H^0(\mathscr{L}))=\mathbb{P}^d$. It follows from Proposition 1.1 that $\alpha_i^{\mathscr{L}}=0$ for $i=0,\ldots,d$ in this case. Also, the number $(r+1)d+\binom{r+1}{2}(2g-2)$, on the right hand side of the inequality in Corollary 2.2 becomes $(r+1)(d-r)$, the dimension of the Grassmannian of r-planes in $\mathbb{P}^d$, while the left-hand side is the sum of the codimensions of the Schubert cycles $\sigma_{a^i}(\mathscr{G}(p_i))$. The flag $\mathscr{F}(p)$ is, in this context, the flag of osculating spaces ot the rational normal curve C of degree d in $\mathbb{P}^d$ at a point $p\in C$. We get:

Theorem 2.3. *Let $p_1,\ldots,p_m$ be distinct points on the rational normal curve C of degree d in $\mathbb{P}^d$ (respectively, on $\mathbb{P}^1$), and let $\mathscr{F}(p_i)$ be the flag of osculating spaces to C at p_i (respectively, let $\mathscr{G}(p_i)$ be the flag in $\mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^1}(d))^\vee)$ defined by order of vanishing at p_i). If, for each i, τ_i is any Schubert variety of r-planes defined in terms of $\mathscr{F}(p_i)$ (respectively $\mathscr{G}(p_i)$), then $\tau_1,\ldots,\tau_m$ are dimensionally transverse; that is, every component of $\bigcap_1^m \tau_i$ has codimension equal to $\sum_1^m \operatorname{codim}\tau_i$ in the Grassmannian of r-planes, and, in particular, $\bigcap_1^m \tau_i=\emptyset$ if and only if the product of the classes of the τ_i in the intersection ring of the Grassmannian is* 0.

Proof. The versions with $\mathscr{F}$ and $\mathscr{G}$ are dual; we prove the one with $\mathscr{G}$. By Corollary 2.2, $\bigcap_1^m \tau_i=\emptyset$ if

$$\sum_1^m \operatorname{codim}\tau_i>(d-r)(r+1)=\dim \operatorname{Grass}(r,d).$$

Suppose that $\bigcap_1^m \tau_i\neq\emptyset$, and set $c=\sum_1^m \operatorname{codim}\tau_i$, $k=(d-r)(r+1)-c$. The dimension of each component of $\bigcap_1^m \tau_i$ is automatically $\geqq k$. On the other hand, if $q_{m+1},\ldots,q_{m+k+1}$ are $k+1$ more points of C, and for $i=m+1,\ldots,m+k+1$ we let τ_i be the Schubert variety of r-planes meeting $\mathscr{G}^{d-r-1}(q_i)$, then each of the new τ_i is a hyperplane section of the Grassmannian variety of r-planes in its Plücker embedding. Thus if $\bigcap_1^m \tau_i$ has a component of dimension $>k$ we would have $\bigcap_1^{m+k+1} \tau_i\neq\emptyset$, contradicting the first remark.

It remains to note that, since $\bigcap_1^m \tau_i$ has the correct codimension (or is empty), the cohomology class which is the product of the fundamental classes of the τ_i may be written as a positive linear combination of the classes of the components of $\bigcap_1^m \tau_i$; thus the intersection is $\emptyset$ if and only if the product is 0.

Remark. The question of when a product is 0 in the intersection ring of a Grassmann variety, as in the theorem, can be settled combinatorially in concrete cases by using the "Littlewood-Richardson rule"; see, for example, Stanley [1976] or Littlewood [1950]. A special case of interest is treated in Sect. 3.

A general characterization seems complex, but it is easy to prove using the Pieri formula that a product of *special* Schubert cycles is 0 if and only if the sum of the codimensions of its factors is greater than the dimension of the ambient Grassmann variety.

3. The dimension of the family of rational curves with fixed degree and assigned cusps in a projective space

As a first application of Theorem 2.3 we compute the dimensions of spaces of rational curves with assigned cusps; this was in fact the original motivation for our work.

Let $\bar{V}_{r,d}$ be the $(r+1)(d+1)-1$-dimensional projective space of $r+1$-tuples of degree d forms in 2 variables, and let $V_{r,d}$ be the open set of those $r+1$-tuples which define morphisms $\mathbb{P}^1 \to \mathbb{P}^r$ that are birational onto their images. We call $V_{r,d}$ the variety of parametrized rational curves of degree d in $\mathbb{P}^r$. The "variety of rational curves of degree d in $\mathbb{P}^r$" is then the quotient of $V_{r,d}$ by the obvious action of $\operatorname{Aut}\mathbb{P}^1$. Since $\operatorname{Aut}\mathbb{P}^1$ acts without fixed points, and $V_{r,d}$ admits an invariant open affine cover, the quotient is again a quasi-projective variety.

Associated to a (rational) map ϕ from any smooth curve C to $\mathbb{P}^r$ there is a line bundle $\mathcal{L}$ on C, a linear series $V \subset H^0(\mathcal{L})$ of dimension r, and a vector space basis ($r+1$ vectors) for V. For each point p on C we define the ramification sequence of ϕ at p to be the ramification sequence of the associated linear series.

We say that a parametrized rational curve $f\colon \mathbb{P}^1 \to \mathbb{P}^r$ has, at a point $p \in \mathbb{P}^1$, *an ordinary cusp of order* c, if the ramification sequence of the linear series corresponding to f is $(0, c, c, \ldots, c)$; we say that it has *at least a cusp of order* c at p if this ramification sequence is $(0, a_1, \ldots, a_r)$ with $a_1 \geqq c$.

Theorem 3.1. *Let $p_1, \ldots, p_m \in \mathbb{P}^1$ be distinct points, and let $c_1, \ldots, c_m$ be positive integers. The variety X of parametrized non-degenerate rational curves of degree d in $\mathbb{P}^r$ having cusps of orders at least c_i at p_i is empty if and only if, in the Grassmannian of $d-r-1$-planes in $\mathbb{P}^d$, the product of the cohomology classes of the Schubert varieties*

$$\prod_{i=1}^{m} \sigma_{\underbrace{r, r, \ldots, r}_{c_i}}$$

is 0. *If X is not empty, then every component of X has dimension exactly $(d+1)(r+1)-1-r\sum_1^m c_i$. Further, the open subvariety of X consisting of birationally parametrized curves having ordinary cusps of order c_i at p_i and only ordinary inflection points elsewhere is dense.*

In particular, there is a non-degenerate rational curve of degree d in $\mathbb{P}^r$ with m cusps, counted with multiplicity, if and only if

$$m \leqq \frac{r+1}{r} d - r - 1.$$

Factoring out the action of $\mathrm{Aut}\,\mathbb{P}^1$, and letting the points $p_1, \ldots, p_m$ vary, we immediately obtain:

Corollary 3.2. *The space of rational curves of degree d in $\mathbb{P}^r$ having at least cusps of order c_i $(i=1, \ldots, m)$ is empty if and only if the cohomology product described in the theorem is 0. If it is not empty, then every component has dimension exactly*

$$(d+1)(r+1)-r\sum c_i+m-4.$$

The dimensions given above behave oddly in the following sense. Each additional cusp of order 1 "imposes" only $r-1$ conditions on a rational curve in $\mathbb{P}^r$. But the family of curves having the maximal number of cusps possible has dimension

$$(r+1)(d+1)-\left(\left[\frac{r+1}{r}d\right]-r-1\right)(r-1)-4 \geqq (r+1)^2+d-r-4,$$

which may be arbitrarily large compared to $r-1$.

Of course $(r+1)^2-1$ dimensions of this "come from" the automorphisms of $\mathbb{P}^r$. The reason for the remaining "discrepancy" is that the result of Theorem 1.5 is independent of the positions of the points $p_1, \ldots, p_m$.

In view of the results above, it would be interesting to know exactly when a product of cohomology classes of Schubert varieties of the form

$$\tau_{c_i} = \sigma_{\underbrace{rr, \ldots, r}_{c_i}} \subset G(d-r-1, d)$$

is 0. A complete answer is given in principle by the Littlewood-Richardson rule (see Littlewood [1950] and Stanley [1976]). The combinatorial complexity of this result leads to curiosities concerning the existence and non-existence of curves with given types of cusps, but we have not been able to make a simple, general formulation. The following facts, however, may be deduced rather easily, and suffice for many purposes:

$$\prod_1^m \tau_{c_i} = 0 \quad \text{if } r\sum_1^m c_i \geqq (r+1)(d-r) \qquad *)$$

(this is because $(r+1)(d-r)$ is the dimension of the Grassmannian). If $c_i=1$ for all i, then this vanishing condition is necessary and sufficient. But in general it is not, and one has for example:

$$\prod_1^m \tau_{c_i} = 0 \quad \text{if } \sum_1^{\min(r,m)} c_i \geqq d-r. \qquad **)$$

Proof of Theorem 3.1. From Theorem 2.3 we see that each component of the space L of linear series of dimension r in $\mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^1}(d))^\vee)$ whose ramification satisfies

$$\alpha_1(p_i) \geqq c_i \quad (i=1, \ldots, m)$$

has dimension

$$\rho = (d-r)(r+1) - r\sum_1^m c_i.$$

It also follows from Theorem 2.3 that the series in a dense open subset of L have ramification sequences

$$(0, c_i, c_i, \ldots, c_i)$$

at p_i $(i=1, \ldots, m)$ and are unramified elsewhere except for further points where the ramification sequence is $(0, \ldots, 0, 1)$, since the locus in L of series failing to satisfy these conditions is contained in a subvariety of dimension $<\rho$ in L.

If we fix coordinates on $\mathbb{P}^r$, then a rational map $\mathbb{P}^1 \to \mathbb{P}^r$ corresponds to a linear series $\mathbb{P}(V^\vee)$ with a basis for V, chosen up to scalar multiplication; thus the dimension of the space of rational maps with ramification as above is

$$[(r+1)^2-1]+(d-r)(r+1)-r\sum_1^m c_i.$$

$$=(d+1)(r+1)-1-r\sum_1^m c_i.$$

Since the generic point of the space of series has $\alpha_0(p)=0$ for every p, we see that the general rational map with ramification as above is regular.

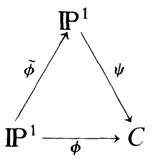
The following Lemma will complete the proof by showing that a map with only ramification as above ts birational.

Lemma 3.3. *If a regular map ϕ: $\mathbb{P}^1 \to \mathbb{P}^r$ is not birational, then for some point $p \in \mathbb{P}^1$ the ramification sequence of ϕ at p is strictly increasing; that is,*

$$0<\alpha_1^\phi(p)<\alpha_2^\phi(p)<\ldots<\alpha_r^\phi(p).$$

Thus if $r>1$ then ϕ has a cusp and additional ramification at p.

Proof. Write $C=\phi(\mathbb{P}^1)$, and let ψ: $\mathbb{P}^1 \to C$ be the normalization. There is a factorization



with $\tilde{\phi}$ not birational. Of course $\tilde{\phi}$ must have ramification points. Suppose $p \in \mathbb{P}^1$ is a γ-fold ramification point of $\tilde{\phi}$ (that is, $\tilde{\phi}$ has ramification sequence $(0, \gamma)$ at p), with $\gamma \geqq 1$. If the ramification sequence of ψ at $\tilde{\phi}(p)$ is $(0, \beta_1, \beta_2, \ldots, \beta_r)$, then computing in local coordinates we see that ϕ has ramification sequence

$$(0, \ldots, \beta_i(\gamma+1)+i\gamma, \ldots).$$

Since $\gamma \geqq 1$ and $0 \leqq \beta_1 \leqq \ldots \leqq \beta_r$ we see that this sequence is strictly increasing as desired. □

4. Linear series on curves with universal singularities

We now wish to examine the scheme of linear series on a cuspidal rational curve. A single such linear series can easily be associated with a linear series on $\mathbb{P}^1$ satisfying certain Schubert conditions. The first goal of this section is to make this into an isomorphism of schemes. (This has been a sore point of the theory; see for example Griffiths and Harris [1978] where $G^r_d(C)$ is shown to be reduced, for a general nodal rational C, *without* explicitly introducing its scheme-theoretic structure!) Because it costs nothing and includes the nodal case, we work more generally with singularities whose conductors are their maximal ideals; we call these "universal singularities."

The section closes with some results that are necessary for the reduction of our main Theorem (5.1) to the cuspidal rational case.

Let C be any projective curve. We write $G^r_d(C)$ for the variety of linear series of degree d and dimension r on C; see Arbarello et al. [1984] for a general discussion. The points of $G^r_d(C)$ correspond to pairs $(\mathscr{L}, V)$, where $\mathscr{L}$ is a line bundle of degree d on C and V is an $r+1$-dimensional vector space of global sections of $\mathscr{L}$. (If we write J for the jacobian of C, π: $C\times J\to J$ for the projection, $\mathscr{L}_n$ for a Poincaré bundle of degree n over $C\times J$, with $n\gg 0$, D for a divisor of degree $n-d$ on C, ρ: $\mathrm{Gr}(r+1, \pi_*\mathscr{L}_n)\to J$ for the canonical projection for the Grassmannian bundle associated to $\pi_*\mathscr{L}_n$, S for the universal subbundle on $\mathrm{Gr}(r+1, \pi_*\mathscr{L}_n)$, and ϕ: $S\to\rho^*\pi_*(\mathscr{L}_n\otimes\mathscr{O}_{D\times J})$ for the restriction to S of the pullback by ρ of the map $\pi_*\mathscr{L}_n\to\pi_*(\mathscr{L}_n\otimes\mathscr{O}_{D\times J})$ induced by the canonical map $\mathscr{L}_n\to\mathscr{L}_n\otimes\mathscr{O}_{D\times J}$, then $G^r_d(C)$ may be realized as the zero-locus of ϕ.) There is a universal such pair defined over $C\times G^r_d(C)$, and if $T\xrightarrow{\pi}D$ is a flat family of projective curves, then the varieties $G^r_d(\pi^{-1}(t))$ fit together into a family $G^r_d(T/D)$ over D. If $X\subset C$ is a finite set, we write $G^r_d(C, X)$ for the open subset of $G^r_d(C)$ consisting of linear series with no base points in X.

Now suppose that C is singular, and let π: $\tilde{C}\to C$ be the normalization. (For the applications below, one could assume that the singularities of C are all ordinary nodes and cusps.) For any line bundle $\mathscr{L}$ on C there is a natural inclusion

$$\bar{\pi}\colon H^0(\mathscr{L})\to H^0(\pi_*\pi^*\mathscr{L})=H^0(\pi^*\mathscr{L}),$$

and this gives rise to a natural map

$$\pi^*\colon G^r_d(C)\to G^r_d(\tilde{C}).$$

If F is a generator of $\mathscr{L}$ locally at each point of the set $P=\mathrm{Sing}\, C$ of singular points of C, and $V\subset H^0(\mathscr{L})$, then identifying V with its image in $\mathscr{O}_{C,P}$ we have

$$V\subset\mathscr{O}_{C,P}F.$$

Identifying F with

$$1\otimes F\in\mathscr{O}_{\tilde{C},\pi^{-1}P}\otimes\mathscr{O}_{C,P}\mathscr{L}_p=(\pi^*\mathscr{L})_{\pi^{-1}P},$$

we get

$$\pi^*V\subset\mathscr{O}_{\tilde{C},\pi^{-1}P}F.$$

Of course F is determined up to a unit of $\mathscr{O}_{C,P}$, and is locally a generator of $\pi^*\mathscr{L}$ at each point of $\pi^{-1}P$.

Conversely, let us fix a line bundle $\tilde{\mathscr{L}}$ on $\tilde{C}$ and consider the fiber $G_d^r(C)_{\tilde{\mathscr{L}}}$ over $\tilde{\mathscr{L}}$ of

$$G_d^r(C) \xrightarrow{\pi^*} G_d^r(\tilde{C}) \longrightarrow \operatorname{Pic}^d \tilde{C}.$$

Given $(\tilde{\mathscr{L}}, \tilde{V}) \in G_d^r(\tilde{C})$, and given an F which generates $\tilde{\mathscr{L}}$ locally at each point of $\pi^{-1}P$ and such that

$$\tilde{V} \subset \mathcal{O}_{C,P} F,$$

there is a unique "minimal" $(\mathscr{L}, V) \in G_d^r(C)$ such that

$$\tilde{\mathscr{L}} = \pi^* \mathscr{L},$$

$$F \in \mathscr{L}_P \subset \tilde{\mathscr{L}}_{\pi^{-1}P},$$

and

$$\pi^* V = \tilde{V};$$

indeed, we simply take $\mathscr{L}$ to be locally generated by F at points of P, and to coincide with $\pi_* \tilde{\mathscr{L}}$ off P. Of course $(\mathscr{L}, V)$ is unchanged if F is altered by a unit of $\mathcal{O}_{C,P}$.

In case $(\tilde{\mathscr{L}}, \tilde{V}) \in G_d^r(\tilde{C}, \pi^{-1}P)$, the element F will be determined by $\tilde{V}$ up to a unit of $\mathcal{O}_{C,P}$ by the condition $\tilde{V} \subset \mathcal{O}_{C,P} F$, and will lie in $\tilde{V}$; for then $\tilde{V}$ generates $\tilde{\mathscr{L}}_{\pi^{-1}P}$, so there is an $F' \in \tilde{V}$ with $\tilde{\mathscr{L}}_{\pi^{-1}P} = \mathcal{O}_{\tilde{C}, \pi^{-1}P} F'$, and we get $F' = \alpha F$, for some $\alpha \in \mathcal{O}_{C,P}$, necessarily a unit. Thus we may describe $G_d^r(C, P)$ as the locally closed subscheme of $G_d^r(\tilde{C}, \pi^{-1}P)$ consisting of linear series $(\tilde{\mathscr{L}}, \tilde{V})$ such that the $\mathcal{O}_{C,P}$-module $\mathcal{O}_{C,P} V \subset (\pi_* \tilde{\mathscr{L}})_P$ can be generated by 1 element.

To better characterize the linear series on $\tilde{C}$ coming from C as above, note first that if $\mathcal{O}_{C,P} \tilde{V}$ is generated by one element, then the subspace of $\tilde{V}$ of sections vanishing on each scheme-theoretic fiber $\pi^{-1}(p)$, for $p \in \operatorname{Sing} C = P$, must be of codimension 1 in V. In general, this necessary condition is not sufficient – the scheme theoretic fiber $\pi^{-1}(p) \subset \tilde{C}$ carries too little information about the singularity of C at p for there to be any hope. However, there is a natural class of singularities which are determined by their scheme-theoretic fibers, which includes ordinary double points and cusps, and for which the necessary condition above becomes sufficient.

We will say that a curve C with normalization $\pi\colon \tilde{C} \to C$ has a *universal singularity* at $p \in C$ if $\mathfrak{m}_{C,p} \mathcal{O}_{\tilde{C}, \pi^{-1}(p)} \subset \mathcal{O}_{C,p}$; that is, if the *conductor* at p contains the maximal ideal of $\mathcal{O}_{C,p}$.

The terminology is explained by the fact that if $\tilde{C} \xrightarrow{\pi'} C'$ is any morphism from C to another curve defined in a neighborhood of $\pi^{-1}p$, and if, for some $p' \in C'$ we have $\pi'^{-1}(p') \supset \pi^{-1}(p)$, and if C has a universal singularity at p, then there is a commutative diagram of morphisms

$$\begin{array}{ccc} & \tilde{C} & \\ {\scriptstyle\pi}\swarrow & & \searrow{\scriptstyle\pi'} \\ C & \xrightarrow[\phi]{} & C' \end{array}$$

where ϕ is defined in a neighborhood of p.

Universal singularities of curves are precisely those defined by "modules" in the sense of Serre [1959]. Given a smooth curve C and a collection of positive divisors $D_1, D_2, \ldots, D_m$ on C with disjoint support, we may construct the unique curve with universal singularities with normalization $\tilde{C}$, having singular fibers $D_1, \ldots, D_m$, as follows: Let the underlying set of C be obtained from that of $\tilde{C}$ by identifying the points in the support of D_i to a single point p_i, for $i=1, \ldots, m$, and define $\pi: \tilde{C} \to C$ to be the obvious set-theoretic projection. For each open set $U \subset C$ define $\mathcal{O}_C(U)$ to be the set of functions $f \in \mathcal{O}_{\tilde{C}}(\pi^{-1} U)$ such that if $p_i \in U$ then there is a constant c such that $f-c$ vanishes on D_i, considered as a subscheme of $\tilde{C}$.

The *multiplicity* of a singularity $p \in C$ is just the degree of the divisor $\pi^{-1}(p) \subset \tilde{C}$. The universal singularities of multiplicity 2 are precisely the ordinary double points and ordinary cusps.

We can now formalize and complete the discussion above:

Theorem 4.1. *Let $\pi: \tilde{C} \to C$ be the normalization of a curve C, and let $\tilde{\mathcal{L}}$ be a line bundle on $\tilde{C}$. For each $p \in C$ let $V(p) \subset H^0(\tilde{\mathcal{L}})$ be the set of sections vanishing on the subscheme $\pi^{-1}(p) \subset \tilde{C}$. Let $\tau(p)$ be the Schubert variety in* $\mathrm{Gr}(r+1, H^0(\tilde{\mathcal{L}}))$ *of $r+1$-planes meeting $V(p)$ in a space of dimension $\geqq r$, and let $\tau_0(p)$ be the open subset consisting of those $r+1$-planes in $\tau(p)$ whose elements do not all vanish at any point in the support of $\pi^{-1} p$.*

The natural map $G_d^r(C)_{\tilde{\mathcal{L}}} \to G_d^r(\tilde{C})_{\tilde{\mathcal{L}}} = \mathrm{Gr}(r+1, H^0(\tilde{\mathcal{L}}))$ induces maps

$$G_d^r(C)_{\tilde{\mathcal{L}}} \xrightarrow{\phi} \bigcap_{p \in \mathrm{Sing}\, C} \tau(p)$$

and

$$G_d^r(C, \mathrm{Sing}\, C)_{\tilde{\mathcal{L}}} \xrightarrow{\phi_0} \bigcap_{p \in \mathrm{Sing}\, C} \tau_0(p).$$

If C has only universal singularities, then this last map is an isomorphism.

Proof. The existence of the given maps is clear from the discussion above; only the isomorphism requires further proof. To construct a map $X = \bigcap_{p \in \mathrm{Sing} C} \tau_0(p) \to G_d^r(C, \mathrm{Sing}\, C)_{\tilde{\mathcal{L}}}$ it is enough to define a family over X of linear series $(V, \mathcal{L})$ of dimension r on C, having no base points in $\mathrm{Sing}\, C$, with $\tilde{\mathcal{L}} = \pi^* \mathcal{L}$. We will show that if $C \in X$, then $\mathcal{O}_{C, \mathrm{Sing} C} V \subset \pi_* \tilde{\mathcal{L}}$ is a cyclic module, and thus we may define a line bundle $\mathcal{L} \subset \pi_* \tilde{\mathcal{L}}$ by $\mathcal{L} = \mathcal{O}_C V$ in a neighborhood of $\mathrm{Sing}\, C$, and $\mathcal{L} = \pi_* \tilde{\mathcal{L}}$ off $\mathrm{Sing}\, C$. Clearly, $(\mathcal{L}, V)$ is a linear series, and $\pi^* \mathcal{L} = \tilde{\mathcal{L}}$ simply because V generates $\tilde{\mathcal{L}}$ locally near $\pi^{-1} \mathrm{Sing}\, C$. This defines a family of linear series over X as required.

It thus remains to show that $\mathcal{O}_{C, \mathrm{Sing} C} V \subset \pi_* \tilde{\mathcal{L}}$ is cyclic; of course we may do this for one point $p \in \mathrm{Sing}\, C$ at a time. By hypothesis $V' = V \cap V(p)$ is exactly r-dimensional. Choose any element $v_0 \in V - V'$. By hypothesis, v_0 does not vanish at any point in $\pi^{-1} \mathrm{Sing}\, C$, so $\tilde{\mathcal{L}}_{\pi^{-1} p} = \mathcal{O}_{\tilde{C}, \pi^{-1} p} v_0$. By definition,

$$\begin{aligned} V(p) &\subset m_{C,p} \tilde{\mathcal{L}}_{\pi^{-1} p} \\ &= m_{C,p} \mathcal{O}_{\tilde{C}, \pi^{-1} p} v_0 \\ &\subseteq \mathcal{O}_{C,p} v_0 \quad \text{(by the universality of the singularities of } C) \end{aligned}$$

so V', and thus all of V, is contained in $\mathcal{O}_{C,p} v_0$. This concludes the proof. □

Remark. All the objects defined in the theorem form nice families as $\tilde{\mathcal{L}}$ varies, so we may identify at least the portion of $G^r_d(C, \operatorname{Sing} C)$ lying over $W^s_d(\tilde{C}) - W^{s+1}_d(\tilde{C})$ for $s \geqq r$ with a certain intersection of Schubert varieties in the Grassmannian of sub-bundles of rank $r+1$ inside the universal s-bundle on $W^s_d(\tilde{C}) - W^{s+1}_d(\tilde{C})$. But this is not needed for our applications.

We will make use of the above idea in case $\tilde{C} = \mathbb{P}^1$, and the singularities of C are all ordinary cusps. In this case $\tilde{\mathcal{L}} = \mathcal{O}_{\mathbb{P}^r}(d)$, $\pi^{-1}(p)$ is a single point for each $p \in \operatorname{Sing} C$, and $V(p)$ is the (codimension 2) subspace of $H^0(\tilde{\mathcal{L}}) = H^0(\mathcal{O}_{\mathbb{P}^1}(d))$ consisting of forms vanishing to order $\geqq 2$ at $\pi^{-1} p$.

It will be useful to dualize, using the canonical isomorphism

$$\operatorname{Gr}(r+1, H^0(\mathcal{O}_{\mathbb{P}^1}(d))) \cong \operatorname{Gr}(d-r-1, \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^1}(d))).$$

The rational normal curve $\tilde{C}$ of degree d is naturally embedded in $\mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^1}(d)))$ as the family of 1-quotients $H^0(\mathcal{O}_{\mathbb{P}^1}(d)) \otimes \mathcal{O}_{\mathbb{P}^1} \to \mathcal{O}_{\mathbb{P}^1}(d)$ on $\mathbb{P}^1$, and the space dual to the space of forms vanishing to order $\geqq 2$ at a point of $\mathbb{P}^1$ is the tangent line to $\tilde{C}$ in $\mathbb{P}^1$. We thus have

Corollary 4.2. *Let $\tilde{C} \subset \mathbb{P}^d$ be the rational normal curve of degree d, and $p_1, \ldots, p_g \in \tilde{C}$ be distinct points. Let C be the curve with normalization $\tilde{C}$ having ordinary cusps at $p_1, \ldots, p_g$ and no other singularities. If $\sigma_r(p_i)_0$ denotes the open set in the Schubert variety of $d-r-1$ planes in $\mathbb{P}^d$ meeting the tangent lines to $\tilde{C}$ at p_i, consisting of those planes not containing p_i, then there is a natural isomorphism*

$$G^r_d(C, \operatorname{Sing} C) \cong \bigcap_{i=1}^{g} \sigma_r(p_i)_0,$$

in which a plane Λ in the intersection of the $\sigma_r(p_i)$ corresponds to the linear series cut out on C by all hyperplanes of $\mathbb{P}^d$ containing Λ.

As is well known, the Picard variety of a singular curve may be compactified by allowing torsion-free sheaves of rank one in addition to invertible sheaves, as for example in the paper of D'Souza [1979]. It is therefore useful to be able to classify the torsion free sheaves of rank one, at least locally. For curves with universal singularities, this is quite easy (D'Souza [1979] (2.6)).

Using this compactification of the Picard variety we may define a compactifications $\bar{W}^r_d(C)$ of $W^r_d(C)$ and $\bar{G}^r_d(C)$ of $G^r_d(C)$.

Given points $p_1, \ldots, p_s$ on a curve C, we say that $\pi\colon C' \to C$ is the partial normalization of C at $p_1, \ldots, p_s$ if π is an isomorphism off $p_1, \ldots, p_s$, and $\pi^{-1}(p_i)$ is smooth for $i = 1, \ldots, s$.

Corollary 4.4. *If C is a curve with ordinary nodes and cusps at points $p_1, \ldots, p_s$, and C' is the partial normalization at $p_1, \ldots, p_s$, then the locally closed subset of $\bar{G}^r_d(C)$ of linear series $(\mathcal{L}, V)$ with $\mathcal{L}$ not locally free precisely at the points $p_1, \ldots, p_s$ is isomorphic to $G^r_{d-s}(C')$.*

Proof. A torsion free, rank 1 sheaf $\mathcal{L}$ on C failing to be free precisely at $p_1, \ldots, p_s$ is a line-bundle on C'.

Let u be a meromorphic section of $\mathscr{L}$ which is regular at p_i and generates $\mathscr{L}$ as an $\mathcal{O}_{C',\pi^{-1}p_i}$-module for $i=1, \ldots, s$. Since $\mathcal{O}_{C',\pi^{-1}p}/\mathcal{O}_{C,p}$ is one-dimensional, the divisor of zeros of u on C is the divisor of zeros on C' plus $\sum_1^s p_i$, so the degree of $\mathscr{L}$ on C' is the degree of $\mathscr{L}$ on C minus s. $\square$

Putting this together with Theorem 2.3, Corollary 4.2, and a standard argument bounding dimensions of components of $\bar{G}^r_d$ from below we get the main result of this section, which is a strong form of the Brill-Noether equality for cuspidal rational curves:

Theorem 4.5. *If C is a g-cuspidal rational curve, then every component of $\bar{G}^r_d(C)$ has dimension*

$$(r+1)(d-r)-rg=g-(r+1)(g+r-d),$$

and $G^r_d(C, \operatorname{Sing} C)$ is dense in $\bar{G}^r_d(C)$.

Remark. It follows from this that the corresponding result holds for g-nodal rational curves if the nodes are in general position, a result proved by Griffiths and Harris [1980]. Recall that a curve C is *locally planar* if, for each $p\in C$ we have $\dim \mathfrak{m}_{C,p}/\mathfrak{m}^2_{C,p}\leqq 2$, or, equivalently (Altman and Kleiman [1979a]), C can be embedded in a smooth surface. The following Lemma is well-known.

Lemma 4.6. *If C is a projective curve of arithmetic genus g which is locally planar then every component of $\bar{G}^r_d\, C$ has dimension at least $(r+1)(d-r)-rg=g-(r+1)(g-d+r)$.*

Proof. Let D be a Cartier divisor on C of large degree e. Let $\operatorname{Pic}^{=d+e}(C)$ be the compactified Picard scheme, π_1, π_2 the projections indicated:

$$\begin{array}{ccc} & C\times \operatorname{Pic}^{=d+e}(C) & \\ \swarrow & & \searrow \\ C & & \operatorname{Pic}^{=d+e}(C), \end{array}$$

and $\mathscr{L}_{d+e}$ a Poincare sheaf on $C\times \operatorname{Pic}^{=d+e}(C)$. If $\mathscr{L}$ is a torsionfree sheaf of rank one on C then $H^0(\mathscr{L})$ is the kernel of the restriction map $H^0(\mathscr{L}(D))\to H^0(\mathscr{L}(D)|_D)$, and $\mathscr{L}(D)$ has degree $d+e$.

Thus $\bar{G}^r_d(C)$ is the locus in the Grassmannian bundle of $(r+1)$-subbundles V of the rank $d+e-g+1$ bundle $\pi_{2*}\mathscr{L}_{d+e}$ defined locally by the $(r+1)e$ equations which say that the composite map

$$V\to \pi_{2*}\mathscr{L}_{d+e}\to \pi_{2*}(\mathscr{L}_{d+e|\pi_1^{-1}(D)})$$

is 0.

It follows that the components of $\bar{G}^r_d(C)$ all have dimension at least $g+(r+1)((d+e-g+1)-(r+1))-(r+1)e=g-(r+1)(g-d+r)$. $\square$

5. Linear series and projective embedding of general curves

Our main theorem is:

Theorem 5.1. *If C is a general smooth curve of genus g or a general g-cuspidal rational curve, then:*

1) *$G_d^r(C)$ has dimension*

$$(r+1)(d-r)-rg=g-(r+1)(g-d+r)$$

and is reduced.

2) *Let $G \subset G_d^r$ be the set of linear series $\mathscr{L}$ such that the associated map $\phi_{\mathscr{L}}: C \to \mathbb{P}^r$ is a regular embedding. The set G is open, and if $r \geqq 3$, it is dense. In the cuspidal case the complement of G has codimension $\geqq r-2$ in the set of linear series with no base points at the cusps of C.*

3) *The set of $\mathscr{L} \subset G_d^r$ with only ordinary ramification is open and dense.*

Part 1) is of course the "Brill-Noether-Castelnuovo statement", proved for general smooth curves by Griffiths and Harris [1980]. From the dimensional part of statement 1) together with Lemma 4.6, it follows that for a general smooth or cuspidal curve C the natural projection map $G_d^r(C) \to W_d^r(C)$ is birational. Thus we obtain:

Corollary (Theorem 1). *With C as above and $r \geqq 3$, a general line bundle $\mathscr{L}$ in $W_d^r(C)$ has $\dim H^0(\mathscr{L})=r+1$, and $\mathscr{L}$ is very ample.*

Again using Brill-Noether, we get:

Corollary 5.2. *A general curve C of genus g can be embedded in $\mathbb{P}^3$ as a non degenerate curve of degree d if $d \geqq \frac{3}{4}g+3$. If $g \neq 0$, 1, or 3 then C admits no embedding in any projective space as a curve of lower degree.*

Proof of Corollary 5.2. By the Brill-Noether inequality, $G_d^r(C)$ is non-empty iff

$$d \geqq \frac{r}{r+1} g + r.$$

The existence of a non degenerate embedding in $\mathbb{P}^3$ with $d \geqq \frac{3}{4}g+3$ now follows from the theorem, while the non-existence in case of smaller degree follows because a general curve of genus $\neq 0$, 1 or 3 cannot be embedded in $\mathbb{P}^2$. □

Corollary 5.3. *On a general curve C, every Weierstrass point is normal* (*i.e. has gap values* $1, 2, \ldots, g-1, g+1$). *Further, no complete linear series on C has a ramification point of weight greater than $g+1$.*

Proof of Corollary 5.3. The first statement follows from Theorem 5.1 (the canonical series $|K|$ on C is the sole, and hence a general, point of G_{2g-2}^{g-1}) together with the observation that the gap sequence of a Weierstrass point $p \in C$ is exactly the sequence $\{1+i+\alpha_i^K(p)\}_{i=0,\ldots,g-1}$, where $\{\alpha_i^K(p)\}$ is the ramification sequence of the complete canonical series at p.

In the second statement, observe first that if $|D|$ is any complete series, $p \in C$ any point, and m the greatest integer such that $D - mp$ is linearly equivalent to an effective divisor, the weight of p with respect to $|D|$ satisfies $w(p, |mp|) \geqq w(p, |D|)$. Second, we observe that if $n \geqq m$, then

$$w(p, |np|) \geqq w(p, |mp|),$$

with equality for $m \geqq 2g-1$. Finally, since for $m = 2g-1$ the sequence $\{m - i - \alpha_i^{|mp|}\}_{i=g-1,\ldots,0}$ is the non-gap sequence of p,

$$w(p, |mp|) = w(p, |K|) + g$$

and applying the first half of the Corollary we conclude that for $n \gg 0$

$$\begin{aligned} w(p, |D|) &\leqq w(p, |np|) \\ &= g+1. \quad \square \end{aligned}$$

Remark. The result for mappings to $\mathbb{P}^1$ and $\mathbb{P}^2$ which corresponds to Corollary 5.2 is fairly easily accessible by means of degeneration to nodal curves; it is the following:

Proposition 5.4. *Let C be a general smooth curve and let G be an irreducible component of G_d^r. If $\mathscr{L} \in G$ is sufficiently general, then the associated map $\phi_{\mathscr{L}}$ satisfies*

i) *If $r \geqq 1$ then $\phi_{\mathscr{L}}$ has no non-ordinary ramification.*
ii) *If $r \geqq 2$ then $\phi_{\mathscr{L}}(C)$ is birational, and has no cusps.*
iii) *If $r \geqq 2$ then $\phi_{\mathscr{L}}(C)$ has no triple points.*

Remarks on the Proof of Proposition 5.4

Parts i) and ii) follow easily, by a degeneration argument, from Theorem 3.1 and Lemma 3.3 which give the corresponding facts for g-cuspidal curves. We do not know whether iii) holds for general g-cuspidal rational curves, but it can be deduced from the corresponding fact for g-nodal curves: One checks easily that the general rational plane curve of any degree d has only ordinary nodes as singularities, and one then makes a family in which g of the nodes are smoothed to get an immersion of the correct type. To show that one can do this with a general point of any component of W_d^r on a general curve, one uses the irreducibility of the space of rational plane curves of degree d. (Note that, in any case, W_d^r is irreducible as soon as $\dim W_d^r \geqq 1$, by a result of Fulton and Lazarsfeld [1981].) $\square$

Maria-Grazia Ascenzi has shown with an example of a singular rational 12-cuspidal curve of degree 10 in $\mathbb{P}^3$ that part 2 of Theorem 5.1 fails for some special curves.

Our techniques prove, at any rate, that G is dense if $r \geqq 6$. It is easy to reduce this question to the case $\dim G_d^r = 0$ (that is $(r+1)(d-r) = rg$), and it is easy to show by considering the degree of the dual curve that any g-cuspidal rational curve not satisfying Theorem 5.1 with $r=3$ must have $g \geqq 12$.

On the other hand, the reducedness in part 1) of Theorem 5.1 actually fails for some cuspidal rational curves; an example is given in Sect. 9. Also, Proposition 5.4, iii) actually fails for some cuspidal curves, as the following example, also found by Ascenzi, shows. For *general* cuspidal rational curves the question remains open.

Let $\hat{C}$ be the Lissajous figure

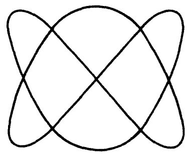


Fig. 4

defined parametrically by

$$x = \sin(2t)$$

$$y = \sin(3t).$$

The curve $\hat{C}$ is an algebraic curve of degree 6 with no cusps and 12 ordinary flexes (2 are visible; 8 more in the affine plane have complex coordinates; and a further pair coincide and share the line at infinity as a common tangent). It also has 2 (visible) tritangents. Thus the dual curve C has degree 10, and has 12 cusps, the maximum number possible for a rational plane curve of degree 10. Further, on a 12-cuspidal rational curve, $\dim G_{10}^2 = 0$, so the immersion of C given automatically represents a "general" point of G_{10}^2 on the 12-cuspidal curve corresponding to C. But C has 2 triple points and even one quadruple point!

This example is, in a sense, minimal: by considering the degree of the dual curve one sees at once that Proposition 5.4, iii) is valid for any g-cuspidal rational curve with $g < 12$.

The reader is referred to the papers of Kleiman [1976], Griffiths and Harris [1980], Gieseker [1982], and Eisenbud and Harris [1983] for further information.

In the remainder of this section we will reduce the proof of Theorem 5.1 to the cuspidal case. This reduction follows fairly well-known lines except for part 3), and we have correspondingly omitted some details, but we have tried to give a readable outline of the argument. As for the cuspidal case, part 3) of Theorem 5.1 has already been proved, as Theorem 3.1. The dimension-theoretic portion of part 1) has also been established as Theorems 2.3 and Corollary 4.2.

The cuspidal case of part 2) of Theorem 5.1 will be proved in Sects. 6 and 7, and part 1) will be treated in Sect. 8.

Reduction of Theorem 5.1 to the cuspidal case

Let C_0 be a g-cuspidal rational curve for which Theorem 5.1 holds. An ordinary cusp can locally be smoothed in a flat family, for example in the family over $D = \operatorname{Spec} \mathbb{C}[[t]]$ which is given by $y^2 + x^3 = t$. Write 0 for the closed point and η for the generic point of D. Since C_0 is 1-dimensional, it follows from general theory (for example: Winters [1972] and Rim [1972], Corollary 4.13) that there is a proper flat family of curves

$$\begin{array}{ccc} C_0 & \longrightarrow & \mathscr{C} \\ \downarrow & & \downarrow \\ \{0\} & \in & D, \end{array}$$

whose geometric generic fiber $C_{\bar{\eta}}$ is a smooth curve of genus g over the algebraic closure of the field $\mathbb{C}((t))$, such that, formally in a neighborhood of the cusp points in C_0, C is isomorphic to $\operatorname{Spec} \mathbb{C}[[x, y, t]]/(y^2 + x^3 - t)$. Indeed, an explicit family of this sort is given analytically in the appendix to this section. We will prove Theorem 5.1 for the smooth curve $C_{\bar{\eta}}$. Since the algebraic closure of $\mathbb{C}((t))$ is isomorphic to $\mathbb{C}$, and since all the assertions of Theorem 5.1 are open, as will become clear below, this will suffice.

We will write C_λ for the fiber of $\mathscr{C}$ over $\lambda \in D$. The varieties $G_d^r(C_\lambda)$ fit together in a family $\mathscr{G}_d^r \to D$, so that the fiber of $\mathscr{G}_d^r$ over λ is $G_d^r(C_\lambda)$. The problem is that $G_d^r(C_0)$ is not proper, so that in principle a "general linear series" on C_η might not specialize to a linear series on C_0 (that is, the specialization of the associated line bundle might be a torsion-free sheaf, not locally free.) The key point is that this catastrophe does not occur:

Proposition 5.5. *Let $\mathscr{C} \to D$ be a flat family of curves whose special fiber C_0 is a rational curve whose singularities are ordinary cusps, and whose general fiber is smooth. If $\mathscr{G}_d^r$ is the associated family of linear series, then every component of $\mathscr{G}_d^r$ contains a component of $G_d^r(C_0)$.*

Proof sketch. We may compactify $\mathscr{G}_d^r$ to a family $\bar{\mathscr{G}}_d^r$ whose fiber over $\lambda \in D$ is $\bar{G}_d^r(C_\lambda)$, the variety of $r+1$-dimensional spaces of sections of torsion-free rank 1 sheaves on C_λ (of course $\bar{G}_d^r(C_\eta) = G_d^r(C_\eta)$, since C_η is smooth); see Altman and Kleiman [1979] and [1980] for details. Now the family $\bar{\mathscr{G}}_d^r$ is proper, so every component of $\bar{\mathscr{G}}_d^r$ meets $\bar{G}_d^r(C_0)$. By Theorem 4.5, $G_d^r(C_0)$ is dense in $\bar{G}_d^r(C_0)$. Since, further, every component of $\mathscr{G}_d^r$ has relative dimension at least equal to the dimension of the components of $G_d^r(C_0)$, the proposition follows.

With this in hand, the reduction of part 1) of Theorem 5.1 is immediate. For the reduction of part 2, we use the following result, whose proof was clarified for us by M. Artin.

Proposition 5.6. *If*

$$\begin{array}{ccc} X & \xrightarrow{f} & Z \\ {\scriptstyle p}\downarrow & & \\ Y & & \end{array}$$

is a diagram of separated Noetherian schemes, with p proper, then the set $Y_0 = \{y \in Y \mid$ *the restriction of f to* $p^{-1}(y)$ *is a closed embedding*$\}$ *is open in Y.*

Proof. We may assume that Y is irreducible, Y_0 is non-empty, and, since a finite union of closed sets is closed, that every component of X maps onto Y. Set $F=(p,f)\colon X \to Y \times Z$. Since p is proper and X and Z are separated, F is proper. By removing from Y the closed set which is the image by p of the union of the positive-dimensional fibers of F, we may harmlessly suppose that F is quasi-finite, and thus finite.

Let B be the set of points in $F(X) \subset Y \times Z$ over which F is not an embedding. B is closed because it is the support of the coherent sheaf of $\mathcal{O}_{Y\times Z}$-modules

$$\operatorname{coker}(\mathcal{O}_{Y\times Z} \to F_* \mathcal{O}_X).$$

Clearly, $Y_0 \supset Y - p(F^{-1}(B))$, and in fact they are equal, since if $(y,z) \in Y_0 \times Z$ then

$$f^*\colon \mathcal{O}_{Z,z} \to (f_*(\mathcal{O}_X/p^* \mathfrak{m}_{Y,y}))_z$$

is onto, so, since F is finite,

$$F^*\colon \mathcal{O}_{Y\times Z, y\times z} = \mathcal{O}_{Y,y} \otimes \mathcal{O}_{Z,z} \to (F_* \mathcal{O}_X)_{y\times z}$$

is onto. This concludes the proof of Proposition 5.6. □

The openness of the set G in part 2 of the Theorem follows by applying Propositions 5.5 and 5.6 with $X = C \times G^r_d(C)$, $p\colon C \times G^r_d(C) \to G^r_d(C)$ the projection, and $f\colon C \times G^r_d(C) \to \mathbb{P}^r$ the map $(p, \mathcal{L}) \to \phi_{\mathcal{L}}(p)$. The reduction of part 2 to the cuspidal case follows in a similar way, using Proposition 5.5.

The reduction of part 3) is more difficult. It will be convenient to say that a linear series $L = (\mathcal{L}, V)$ on a cuspidal curve C_0 has ordinary ramification (or, equivalently, ordinary inflection) at a cusp point x_0 if, pulled back to the normalization $\tilde{C}_0$, L has ramification sequence $0, 1, 1, \ldots, 1$ at the preimage of x_0.

We must prove

Proposition 5.7. *Let* $(\mathcal{L}, V)$ *be a relative linear series of* $\mathcal{C}$ *over D; that is, let* $\mathcal{L}$ *be a line-bundle on* $\mathcal{C}$, *flat over D, and let V be a rank* $r+1$ *direct summand of the direct image of* $\mathcal{L}$ *on D. If the restriction* $(\mathcal{L}_{\bar\eta}, V_{\bar\eta})$ *of* $(\mathcal{L}, V)$ *to the geometric general fiber* $C_{\bar\eta}$ *of* $\mathcal{C}$ *possesses a non-ordinary inflection point, then the special fiber* $V(\mathcal{L}_0, V_0)$ *possesses a non-ordinary inflection point in* C_0.

Proof. Since the algebraic closure of $\mathbb{C}((t))$ is obtained by adjoining all the roots of t, we may replace t by $t^{1/m}$, for some m, and suppose that the non-ordinary inflection point $x \in C_{\bar{\eta}}$ is defined over $\mathbb{C}((t))$ itself; the price we pay is that, formally in the neighborhood of a cusp of C_0, the family will now be given by $\mathbb{C}[\![x, y, t]\!]/(y^2+x^3-t^m)$; making a further base change we may assume that m is divisible by 6, say $m=6n$, which will be convenient.

Since C is proper over D, the closure of x in $\mathscr{C}$ will consist of x and a point $x_0 \in C_0$, the "limit of x as $t \to 0$". We will show that x_0 is not an ordinary inflection point of $(\mathscr{L}_0, V_0)$ on C_0.

If x_0 is a smooth point of C_0 there is no problem; it is easy to see, for example, that weight is upper semi-continuous on families of germs of smooth curves.

If x_0 is a cusp, we will show that after trivializing $\mathscr{L}$ near x_0 so that V may be regarded as a vector space of functions, there is a nonzero element $f \in V$ vanishing to order $\geqq r+2$ at x_0, in the sense that

$$\dim_C(\mathcal{O}_{C_0,x_0}/(f)) \geqq r+2.$$

Writing $\tilde{\mathcal{O}}$ for the normalization of $\mathcal{O}_{C_0,x_0}$, we have

$$\dim_{\mathbb{C}}(\mathcal{O}_{C_0,x_0}/f) = \dim_{\mathbb{C}}(\tilde{\mathcal{O}}/f\tilde{\mathcal{O}})$$

(this is a general fact about 1-dimensional local domains), so this implies that f vanishes to order $\geqq r+2$ at x_0 on the normalization of C_0, whence x_0 is not an ordinary inflection point, as claimed.

Let $\alpha_0, \ldots, \alpha_r$ be the ramification sequence of $(\mathscr{L}, V)$, at x. If $\alpha_r \geqq 2$, then there is a function $f \in V$, not identically 0 on C_0, which vanishes to order $\geqq r+2$ at x; thus

$$r+2 \leqq \dim_{\mathbb{C}((t))} \mathcal{O}_{C_\eta,x}/(f) \leqq \dim_{\mathbb{C}} \mathcal{O}_{C_0,x_0}/(f)$$

as required.

If, on the other hand, $\alpha_r \leqq 1$, then since x is not an ordinary ramification point, we must have $\alpha_{r-1} = \alpha_r = 1$. This case is more difficult because the "obvious" vanishing behavior of functions in V_η is indeed imitated by the vanishing behavior of functions near an *ordinary* cusp point.

The condition $\alpha_{r-1} = \alpha_r = 1$ implies that there is a 2-dimensional subspace of V_η of functions which vanish to order $\geqq r$ at x. Let f, g be a basis, in V, for this subspace, chosen so that f and g are independent in V_0. We may trivialize $\mathscr{L}$ in a neighborhood of x_0, and regard f and g as functions on $\mathscr{C}$.

Given any (formal) section σ of $\mathscr{C}/D$ through x_0, that is, any ideal I in $R = \mathbb{C}[[x, y, t]]/y^2+x^3-t^{6n}$ such that the composition

$$\mathbb{C}[[t]] \to R \to R/I$$

is an isomorphism, there are power series u, $v \in \mathbb{C}[[t]]$ such that $uf+vg \in I$ and not both of u and v are divisible by t.

Let us write y for the general point, in C_η, of the section corresponding to I. If $y \neq x$, it is obvious that $uf+vg$ vanishes to order $\geqq r+1$ at x_0. We will next show that for suitable I, $rx+y$ is not trivial in the ideal class group of R (which in this case is just the Picard group of the puctured spectrum of R). This implies that any function which, like $uf+vg$, vanishes on the divisor $rx+y$ must vanish also on some other divisors of "the formal germ of" $\mathscr{C}$ through x_0, and therefore must vanish to order $\geqq r+2$ at x_0, as required.

It will now suffice to exhibit two formal sections which represent different classes in the ideal class group of R; writing y_1 and y_2 for their general points, it follows that $rx+y_1$ and $rx+y_2$ cannot both be trivial.

We claim that

$$I_1=(x, y-t^{3n}),$$

$$I_2=(y, x-t^{2n}),$$

which correspond to sections as above, represent distinct ideal classes.

To prove this it suffices to show that $I_1^{-1} I_2$ is not isomorphic to R or to an ideal which is primary to the maximal ideal $\mathfrak{m}$ of R.

First, we have $I_1^{-1}=\left(R+\left(\frac{y+t^{3n}}{x}\right)R\right)$, as a fractional ideal; indeed, one checks at once that the product of this ideal with I_1 is (x, y, t^{3n}), which is $\mathfrak{m}$-primary, proving at least that the given equality holds in the ideal class group, which suffices for our purpose. Multiplying, one gets

$$\begin{aligned} I_1^{-1} I_2 &= \frac{1}{x}(xy, x(x-t^{2n}), y(y+t^{3n}), (y+t^{3n})(x-t^{2n})) \\ &\cong (xy, x(x-t^{2n}), y(y+t^{3n}), (y+t^{3n})(x-t^{2n})). \end{aligned}$$

Using the relation $y^2+x^3=t^{6n}$ this may be written as $J=(xy, x(x-t^{2n}), (y+t^{3n})(x-t^{2n}))$. To show that J is not isomorphic to an $\mathfrak{m}$-primary ideal, it suffices, (since depth $\mathfrak{m}=2$) to show that J is not the product of an $\mathfrak{m}$-primary ideal and a principal ideal, and for this it suffices to show that J is neither $\mathfrak{m}$-primary nor contained in a proper principal ideal, (h), say.

Now J is not $\mathfrak{m}$-primary because it is contained in the height-one ideal $(x, y+t^{3n})R$.

Finally, to prove that J is not contained in a proper principal ideal, we exploit the quasi-homogeneity of R. If we assign weights $2n$, $3n$ and 1 to the variables x, y and t, respectively, then the defining equation $y^2+x^3-t^{6n}$ has weight $6n$. Since J is quasi homogeneous, the element h, if it exists, may be taken quasi-homogeneous, and the elements of J of weight $<6n$ would admit h as a divisor in the ring of formal power series $C[[x, y, t]]$. But $xy, x(x-t^{2n}), (y+t^{3n})(x-t^{2n})$, which are elements of J of weight $<6n$, have no common divisor in the power series ring, so we are done. This concludes the proof of Proposition 5.7, and with it the proof of Theorem 5.1, 3). ∎

The last stage of the argument above may be illuminated by looking at the resolution of the singularity of $y^2+x^3=t^{6n}$. The resolution is best computed (as Jayant Shah kindly pointed out to us) by resolving $y^2+x^3=t^6$ – a single quasi-homogeneous blow-up suffices – and then making the base-change $t=t^n$ and resolving the A_{n-1}-singularity which results. The exceptional divisor of this resolution consists of an elliptic curve E (of self-intersection -1 in S), which appears in the first blowup, and a chain of rational curves $E_1, \ldots, E_{n-1}$ of self-intersection -2, which appears as the resolution of the A_{n-1}-singularity. Moreover, the fiber of the composed map

$$S \to \mathscr{C} \to D$$

over $t=0$ consists exactly of these curves, each with multiplicity 1, together with the curve $\tilde{C}_0$ attached at the opposite end of the chain of E_i's from E:

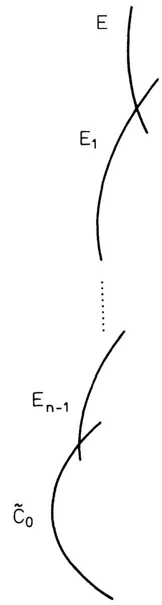


Fig. 5

Because of the multiplicity 1 statement, through any smooth point of this fiber there passes a section of S over D – for example, the sections passing through E are

$$\Gamma_{a,b} = (x - a t^{2n}, y - b t^{3n})$$

where $a^3+b^2=1$, and $\Gamma_{a,b}$ meets $E=\{(u,v) \mid v^2+u^3=1\}$ in the point (a,b).

In fact, it is not hard to see from this picture that the collection of sections $\{\Gamma_{a,b}\}$ actually injects into the divisor class group computed above: if f were a function on a neighborhood of the exceptional divisor, with divisor

$$(f) = \Gamma_{a,b} - \Gamma_{a',b'} + eE + \sum a_i A_i$$

then restricting to E we get

$$0=\Gamma_{a,b}\cdot E-\Gamma_{a',b'}\cdot E+e(E\cdot E)+a_1(A_1\cdot E)$$

so by degree count, $e=a_1$; and since $(E\cdot E)=-(A_1\cdot E)$, we have

$$\Gamma_{a,b}\cdot E=\Gamma_{a',b'}\cdot E;$$

But $(a,b)\neq(a',b')$, so that $\Gamma_{a,b}$ and $\Gamma_{a',b'}$ meet E at distinct points, and this is impossible.

Appendix to Sect. 5: An explicit smoothing of a g-cuspidal curve

The construction below is adapted from an unpublished portion of an early version of Griffiths and Harris [1980].

We will take the open unit disk

$$\Delta=\{t\in\mathbb{C}\,|\,|t|<1\}$$

as base-space.

Given points $p_i\in\mathbb{P}^1$ $(i=1,\ldots,g)$, where the cusps are to go, let Δ_i be a small open disk centered at p_i and let

$$M=\left(\mathbb{P}^1-\bigcup_1^g\Delta_i\right)\times\Delta.$$

For each i, let A_i be the annulus $\{s\in\mathbb{C}\,|\,1\leqq|s|<2\}$. The boundary of M has a neighborhood which may be identified with the disjoint union

$$\bigcup_1^g A_i\times\Delta.$$

Now let

$$B_i=\{(x,y,t)\in\mathbb{C}^2\times\Delta\,|\,y^2+x^3=t,|x|<4\},$$

and let $\phi_i\colon A_i\times\Delta\to B_i$ be the map

$$\phi_i(s,t)=(s^2,s^3\sqrt{(t/s^6)-1},t),$$

where $\sqrt{(t/s^6)-1}$ is single-valued since $|t/s^6|<1$ on $A_i\times\Delta$.

The map ϕ_i maps $A_i\times\Delta$ bi-holomorphically onto its image, which is defined in B_i by the inequality $|x|>1$; this may be seen, for example, from the inverse

$$B_i=(x,y,t)\to(y/x\sqrt{(t/x^3)-1},t).$$

Note that the map $\phi_i\colon A_i\times\Delta\to B_i$ is compatible with projection to Δ. Thus if we glue B_i to M via $\phi_i\,(i=1,\ldots,g)$ we obtain the desired family $\tilde{M}\to\Delta$; over $t=0$, each B_i has fiber a disc with a cusp at 0 so the fiber $\tilde{M}_0$ is the cuspidal rational curve with cusps at the p_i, while over a point $t\neq0$ in Δ the fiber of each B_i is a torus with a disc removed (the Seifert surface of the trefoil knot!) so the fiber $\tilde{M}_t$ is a smooth curve of genus g.

6. The embedding theorem for cuspidal curves

In this section we will undertake and almost complete the proof of the cuspidal case of part 2 of Theorem 5.1, which says:

Theorem 6.1. *Let C be a general cuspidal curve. If $G \subset G^r_d(C, \operatorname{Sing} C)$ is the open set of linear series $\mathscr{L}$ without base points at the cusps of C such that the associated map $\phi_{\mathscr{L}}\colon C \to \mathbb{P}^r$ is a regular embedding, then the codimension of $G^r_d(C, \operatorname{Sing} C) - G$ in $G^r_d(C, \operatorname{Sing} C)$ is $\geqq r-2$ everywhere.*

Proof. We note that the set in question is open by Proposition 5.6.

Let $\tilde{C}$ be the rational normal curve in $\mathbb{P}^d$. By Theorem 4.1, we may identify $G^r_d(C, \operatorname{Sing} C)$ with the set τ of $d-r-1$-planes of $\mathbb{P}^d$ which meet the tangent lines to $\tilde{C}$ at the g points $p_1, \ldots, p_g$ corresponding to the cusps of C, but do not contain any of the points p_i. Thus it is enough to show that the set of $d-r-1$-planes $\Lambda \in \tau$ violating one of the following conditions has codimension at least $r-2$:

i) Λ meets no tangent to $\tilde{C}$ except those through $p_1, \ldots, p_g$, and meets the osculating 3-planes at $p_1, \ldots, p_g$, only in points.

ii) Λ meets no proper secant to $\tilde{C}$.

The set of $\Lambda \in \tau$ meeting a particular, further tangent line to $\tilde{C}$ has codimension r in τ, and the set of $\Lambda \in \tau$ meeting the osculating 3-plane at some p_i has codimension $r-1$, so the set of $\Lambda \in \tau$ violating i) has codimension (at least) $r-1$, independently of r and the positions of $p_1, \ldots, p_g$.

Before turning to the complement of the set described in ii), we establish some notation:

If $q_1, q_2 \in \tilde{C}$, we write $\sigma(q_1, q_2)$ for the Schubert variety of $d-r-1$-planes in $\mathbb{P}^d$ meeting the (secant) line through q_1, q_2. If $q_1 = q_2 = q$, we replace the secant line by the tangent line to $\tilde{C}$ at q, and write $\sigma(q, q)$ or $\sigma(2q)$ for the corresponding Schubert variety. We write $\tilde{C}^{(k)\prime}$ for the open set of the k^{th} symmetric power $\tilde{C}^{(k)}$ consisting of k-tuples of distinct points.

We will analyze the "incidence correspondence"

$$\Gamma \subset \tilde{C}^{(g)\prime} \times \tilde{C}^{(2)} \times \operatorname{Gr}(d-r-1, d)$$

defined as the closure in $\tilde{C}^{(g)\prime} \times \tilde{C}^{(2)} \times \operatorname{Gr}(d-r-1, d)$ of

$$\{(p_1, \ldots, p_g), (q_1, q_2), \Lambda) \in \tilde{C}^{(g)\prime} \times \tilde{C}^{(2)} \times \operatorname{Gr}(d-r-1, d) \mid \Lambda \text{ does not contain any } p_i \text{ and } \Lambda \in \sigma(p_1, p_1) \cap \ldots \cap \sigma(p_g, p_g) \cap \sigma(q_1, q_2)\}$$

we write π_1, π_2 for the natural projections onto the factors indicated:

$$\begin{array}{ccc} & \Gamma & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ \tilde{C}^{(g)\prime} & & C^{(2)}. \end{array}$$

The fiber of π_1 over a point $(p_1, \ldots, p_g)$ maps onto the set of planes violating ii), so writing ρ for $(r+1)(d-r)-rg$, the dimension of $\sigma(p_1, p_2) \cap \ldots \cap \sigma(p_g, p_g)$, it suffices to show that if π_1 is onto, then the general fiber of π_1 has dimension $\leqq \rho-(r-2)$. We will do this by analyzing the fibers of π_2 over points in the diagonal of $\tilde{C}^{(2)}$, since we can control these using Theorem 3.2.

Let Γ_0 be any irreducible component of Γ. Of course, for general $((p_1,\dots,p_g),\Lambda,(q_1,q_2))\in\Gamma_0$ the plane Λ will not contain any p_i. If $\pi_1(\Gamma_0)\neq\tilde{C}^{(g)\prime}$, then the general fiber of π_1 in Γ_0 is empty, so it is enough to prove that if $\pi_1(\Gamma_0)=\tilde{C}^{(g)\prime}$ then the general fiber of π_1 in Γ_0 has dimension $\leqq\rho-(r-2)$, or equivalently that $\dim\Gamma_0\leqq g+\rho-(r-2)$. Since every component of every fiber of Γ_0 has dimension $\geqq(\dim\Gamma_0)-g$, it suffices to show that some component of some fiber of π_1 in Γ_0 has dimension $\leqq\rho-(r-2)$.

Let us fix temporarily a component Ω of a fiber $\pi_1^{-1}(x)\cap\Gamma_0$ of π_1 in Γ_0. Since π_2 is proper on $\pi_1^{-1}(x)$, the set $\pi_2(\Omega)$ is closed in $\tilde{C}^{(2)}$. The situation now depends on the form of $\pi_2(\Omega)$.

If $\pi_2(\Omega)=\tilde{C}^{(2)}$, then we can find a point of the form $(x,(q,q),\Lambda)$ in Ω with $q\neq p_i$ for $i=1,\dots,g$. Since $\pi_2^{-1}(q,q)\cap\Omega$ consists of planes meeting the tangent line to $\tilde{C}$ at q as well as the tangent lines at $p_1,\dots,p_g$, it has dimension $\rho-r$, thus the general fiber of π_2 in Ω has dimension $\leqq\rho-r$, and Ω has dimension $\leqq\rho-r+2$ as required.

If $\pi_2(\Omega)$ is a curve in $\tilde{C}^{(2)}\cong\mathbb{P}^2$, then it must meet the diagonal $\Delta=\{(q,q)\in\tilde{C}^{(2)}\}$ (which corresponds to a non-singular conic in $\tilde{C}^{(2)}\cong\mathbb{P}^2$), so again Ω contains a point of the form $(x,(q,q),\Lambda)$. If q does not occur among the p_i, we prove by the previous argument that $\dim\Omega\leqq\rho-r+1$. If it does occur, say $q=p_g$, then we note that, since any $\Lambda\in\pi_2^{-1}(q,q)\cap\Omega$ is the limit of planes meeting both the tangent line to $\tilde{C}$ at a point p approaching p_g *and* the secant line through a pair of points that approach p_g, but not containing p, Λ must meet the osculating 3-plane to $\tilde{C}$ at p_g in a line. Thus $\Lambda\in\sigma_{r,r-1,0,\dots}(\mathscr{F}(p_g))$, and by Theorem 2.3, $\dim\pi_2^{-1}(p_g,p_g)\cap\Omega\leqq\rho-(r-1)$. Thus the general fiber of π_2 in Ω (over a point in $\pi_2(\Omega)$) has dimension $\leqq\rho-r+1$, and so $\dim\Omega\leqq\rho-r+2$ as required.

If $\pi_2(\Omega)$ is a point of Δ then the argument above shows that $\dim\Omega\leqq\rho-r+1$, which is more than enough. But if $\pi_2(\Omega)$ is a point not on Δ, the above argument fails.

We may now assume that $\pi_2(\pi_1^{-1}(x)\cap\Gamma_0)$ consists of finitely many points, none in Δ, for every $x\in C^{(g)\prime}$. To treat this case, we will allow two of the component points of $x=(p_1,\dots,p_g)$ to come together.

First, we show that the image $\pi_2(\Gamma_0)$ is all of $C^{(2)\prime}$: Every automorphism of $\mathbb{P}^1\cong\tilde{C}$ extends to an automorphism of $\mathbb{P}^d$ preserving $\tilde{C}$, and thus it extends to an automorphism of Γ. Since $\operatorname{Aut}\mathbb{P}^1$ is connected, the action preserves the component Γ_0. Since $\operatorname{Aut}\mathbb{P}^1$ acts transitively on $\tilde{C}^{(2)\prime}$, we see that $\pi_2(\Gamma_0)$ is $\tilde{C}^{(2)\prime}$, as asserted.

Let $l\subset\mathbb{P}^g\cong\tilde{C}^{(g)}$ be a general line, and let $\Gamma_1\subset l\times G(d-r-1,\mathbb{P}^d)\times\tilde{C}^{(2)}$ be the closure of $\Gamma_0\cap\pi_1^{-1}(l)$. We regard π_1 and π_2 as defined on Γ_1. Since $\dim\Gamma_1=\dim\Gamma_0-g+1$, it will suffice to show that for some $x\in l$ and some component Ω of $\pi_1^{-1}(x)\cap\Gamma_1$, we have $\dim\Omega\leqq\rho-r+2$.

Since $\pi_2(\Gamma_0)=\tilde{C}^{(2)\prime}$ and l is general, we see that with the above hypotheses $\pi_2(\Gamma_1)$ is a closed 1-dimensional subset of $\tilde{C}^{(2)}$, and thus $\pi_2(\Gamma_1)$ meets Δ; that is, there is a point $x_0\in l$ and q in $\tilde{C}^{(2)}$ such that Γ_1 contains a point of the form

$(x_0, \Lambda, (q,q))$. Further, by our assumptions, $x_0 \notin \tilde{C}^{(g)\prime}$, so if we write $x_0 = (p_1, \ldots, p_g) \in \tilde{C}^{(g)}$, not all the p_i are distinct. Since l was a general line, only 2 p_i can coincide. We may thus suppose that $p_{g-1} = p_g$ but $p_1, \ldots, p_{g-1}$ are distinct.

The image of the "combined" projection

$$(\pi_1, \pi_2)\colon\ \Gamma_1 \to l \times \tilde{C}^{(2)}$$

is the closure of $(\pi_1, \pi_2)(\Gamma_1 \cap \Gamma_0)$. But $(\pi_1, \pi_2)(\Gamma_1 \cap \Gamma_0)$ is finite over l by our assumptions, so $(\pi_1, \pi_2)(\Gamma_1)$ is also finite over l. Thus

$$(\pi_1, \pi_2)^{-1}(x_0, (q,q)) \cap \Gamma_1$$

contains at least one component of $\pi_1^{-1}(x_0) \cap \Gamma_1$, and it suffices to prove that

$$\dim[(\pi_1, \pi_2)^{-1}(x_0, (q,q)) \cap \Gamma_1] \leqq \rho - (r-2).$$

For this we make use of Proposition 7.3, a relatively simple special case of one of the main results of Sect. 8, which says that as 2 points p', $p'' \in \tilde{C}$ approach a common limit p, the limit of $\sigma_r(2p') \cap \sigma_r(2p'')$ is the set of planes in $\mathrm{Gr}(d-r-1, \mathbb{P}^d)$ which either contain p or meet the osculating 2-plane to $\tilde{C}$ at p in at least a line; that is, in the notation of Sect. 2,

$$\lim_{\substack{p', p'' \to p \\ p', p'' \in \tilde{C}}} \sigma_r(p') \cap \sigma_r(p'') = \sigma_{r,r}\mathscr{F}(p) \cup \sigma_{r+1,r-1}\mathscr{F}(p).$$

We may apply this because x_0 is the limit of points in $l \cap C^{(g)\prime}$, so $\pi_1^{-1}(x_0) \cap \Gamma_1$ is contained in a limit of the form

$$\lim_{p', p'' \to p_g} \sigma_r(p') \cap \sigma_r(p'').$$

Using the above we compute

$$\begin{aligned}(\pi_1, \pi_2)^{-1}(x_0(q,q)) \subset\ & \sigma_r(2p_1) \cap \ldots \cap \sigma_r(2p_{g-2}) \\ & \cap [\sigma_{r,r}\mathscr{F}(p_g) \cup \sigma_{r+1,r-1}\mathscr{F}(p_g)] \\ & \cap \sigma_r(2q).\end{aligned}$$

If q does not occur among the p_i, we are done by Theorem 2.3, since the right hand side has dimension $\rho - r$. If q does occur among the p_i, but $q \neq p_g$, say $q = p_{g-2}$, then using Proposition 7.3 again we may replace the above containment by

$$\begin{aligned}(\pi_1, \pi_2)^{-1}(x_0, (q,q)) \subset\ & \sigma_r(2p_1) \cap \ldots \cap \sigma_r(2p_{g-3}) \\ & \cap [\sigma_{r,r}(\mathscr{F}(p_g)) \cup \sigma_{r+1,r-1}(\mathscr{F}(p_g))] \\ & \cap [\sigma_{r,r}(\mathscr{F}(p_{g-2})) \cup \sigma_{r+1,r-1}(\mathscr{F}(p_{g-2}))]\end{aligned}$$

and finish as before.

Finally, if $q = p_g$, we note that any $\Lambda \in (\pi_1, \pi_2)^{-1}(x_0, (q,q))$ is the limit of $d-r-1$-planes Λ' such that Λ' meets the tangent lines T', T'' to $\tilde{C}$ at distinct points p', p'' near p_g and some secant line S through distinct points q', q'' near p_g, but Λ' does not contain p' or p''.

Now elementary considerations show that any 3 points $r_1 \in T'$, $r_2 \in T''$ and $r_3 \in S$, all distinct from p and p', must be linearly independent, so we see that $d - r - 1 = \dim \Lambda' \geqq 2$ and that, passing to the limit, Λ contains at least a 2-plane in common with the osculating 5-plane to $\tilde{C}$ at p_g. Thus, with Proposition 7.3, we get

$$(\pi_1, \pi_2)^{-1}(x_0, (q, q)) \subset \sigma_r(2p_1) \cap \ldots \cap \sigma_r(2p_{g-2})$$
$$\cap [\sigma_{r,r,r-2}(\mathscr{F}(p_g)) \cup \sigma_{r+1,r-1,r-2}(\mathscr{F}(p_g))]$$

which has dimension $\rho - (r-2)$, as required.

This completes the proof of Theorem 6.1, and hence of part 2) of Theorem 5.1, except for the proof of Proposition 7.3.

7. Degeneration of Schubert intersections; some special cases

To conclude the proof of Theorem 5.1, 1) we must establish that for generic points $p_1, \ldots, p_g$ on the rational normal curve $\tilde{C}$ of degree d in $\mathbb{P}^d$ the intersection of Schubert cycles

$$\sigma_r(2p_1) \cap \ldots \cap \sigma_r(2p_g)$$

is reduced. We will do this by looking at the limit of this variety as one of the p_i approaches another, using an induction, in Sect. 9. The necessary theorem describing the limit will be proved in Sect. 8. In this section we prove a simple special case of the degeneration theorem which was used in the proof of Theorem 6.1 above, and give some illustrative material on the possible degeneration of the varieties of lines in $\mathbb{P}_3$ meeting two given, variable lines. The material in this section will not be referred to in the following sections, and, since Proposition 7.1 is in any case a special case of the main result of the next section, this section may be skipped without destroying the logical continuity of the paper.

We now establish some notation which differs slightly from what we use elsewhere:

Let $C \subset \mathbb{P}^d$ be the rational normal curve in $\mathbb{P}^d$, let r be an integer with $1 \leqq r \leqq d-1$, and set $k = d - r - 1$. Inside the Grassmann variety $\mathrm{Gr}(k, \mathbb{P}^d)$ we consider the following Schubert varieties, associated to a point $p \in C$:

$\sigma_{r+1}(p) = \{k\text{-planes containing } p\}$.

$\sigma_r(2p) = \{k\text{-planes meeting the tangent line to } C \text{ at } p\}$

$\sigma_{r+1,r-1}(p, 4p) = \{k$-planes containing p and meeting the osculating 3-plane to C at p in at least a line$\}$

$\sigma_{r,r-1}(2p, 4p) = \{$k-planes meeting the tangent line to C at p and meeting the osculating 3-plane to p at C in at least a line$\}$

$\sigma_{r,r}(2p, 3p) = \{k$-planes meeting the osculating 2-plane to C at p in at least a line (and thus meeting the tangent line to C at p)$\}$.

In the Chow ring (or, over $\mathbb{C}$, the homology intersection ring) of $\mathrm{Gr}(k, \mathbb{P}^d)$ the classes of the varieties defined above are independent of p, and we may

drop p in referring to them. A simple computation (Pieri's formula; see for example Griffiths and Harris [1978] p. 203) gives $\sigma_r \cdot \sigma_r = \sigma_{r+1,r-1} + \sigma_{r,r}$.
Correspondingly, we have:

Proposition 7.1. *With notation as above, we have*

$$\lim_{\substack{p \to p_0 \\ \text{along } C}} (\sigma_r(2p) \cap \sigma_r(2p_0)) = \sigma_{r+1,r-1}(p_0, 4p_0) \cup \sigma_{r,r}(2p_0, 3p_0).$$

Proof. We may normalize so that C is given parametrically by

$$t \to (1, t, t^2, \ldots, t^d) = x(t),$$

and we take p_0 to the point $(1, 0, \ldots, 0) = x(0)$. We suppose that $L(t)$ is a curve in the Grassmannian $\mathrm{Gr}(k, \mathbb{P}^d)$ of k-planes such that for every $t \neq 0$, $L(t)$ meets the tangent lines to C at $x(0)$ and at $x(t)$, and we wish to prove that $L(0)$ either contains $x(0)$ or meets the osculating 2-plane to C at $x(0)$ in a line.

If $L(0)$ does not contain $x(0)$ then we may take the point of intersection of $L(t)$ with the tangent to C at $x(0)$ to have coordinates of the form

$$y(t) = (a(t), 1, 0, \ldots, 0).$$

Similarly, since the tangent line to C at $x(t)$ is spanned by $x(t)$ and $(0, 1, 2t, 3t^2, \ldots)$, we may take its intersection with $L(t)$ to have the form

$$z(t) = (b(t), 1 + t\,b(t), 2t + t^2\,b(t), \ldots).$$

Now the line spanned by $y(t)$ and $z(t)$ also contains

$$w(t) = (b(t) - a(t), t\,b(t), 2t + t^2\,b(t), \ldots).$$

If $b(0) \neq a(0)$, then the limit point $w(0)$ is $(1, 0, 0, \ldots)$; this together with $y(0)$ spans the tangent line to C at 0, and we are done.

If, on the other hand, $b(0) = a(0)$, then we may re-write the vector $w(t)$ as $t \cdot w_1(t)$, with w_1 regular at 0, and we see that

$$w_1(0) = (b'(0) - a'(0), b(0), 2, 0, 0, \ldots).$$

Thus $y(0)$ and $w_1(0)$ span a line common to $L(0)$ and the osculating 2-plane to C at 0, and we are done. □

This result should be contrasted with what happens when a secant through points $q_1 + q_2$ approaches the tangent line to p at C, a situation treated in Griffiths and Harris [1980]: Writing $\sigma_r(q_1, q_2)$ for the variety of k-planes meeting the secant through q_1, q_2, we have only the weak result

$$\lim_{q_1, q_2 \to p} \sigma_r(2p) \cap \sigma_r(q_1, q_2) \subset \sigma_{r,r-1}(2p, 4p) \qquad *)$$

which follows because any $\Lambda \in \sigma_r(2p) \cap \sigma_r(q_1, q_2)$ shares a line at least with the 3-space V_{q_1,q_2} spanned by the tangent line to C at p and the secant line through (q_1, q_2), and V_{q_1,q_2} approaches the osculating 3-plane to C at p as q_1, $q_2 \to p$. Clearly, exactly this much can be proved for the Schubert cycles associated to an arbitrary deformation of a line. The dependence of the limit *) on the way in which q_1 and q_2 approach p, and the nature of the limit, is illustrated by the following concrete examples:

Example 1. Let $d=3$, $k=1$, so that $r=1$ and we are considering lines in $\mathbb{P}^3$. Let C be the rational normal curve, and let Q be a smooth quadric on which C lies. The lines $\mathscr{L}(t)$ of one of the rulings of Q are secants to C which, for two values of $t \in \mathbb{P}^1$, say $t=0$ and ∞, are tangent to C; let $p=\mathscr{L}(0)\cap C$. If we take $q_1(t)$, $q_2(t)$ to be the points at which $\mathscr{L}(t)$ meets C, then

$$\lim_{t\to 0} \sigma_1(2p)\cap\sigma_1(q_1(t), q_2(t))$$

is the union of the set of lines meeting the tangent line $\mathscr{L}(0)$ and contained in the tangent plane to Q at the point of their intersection with $\mathscr{L}(0)$; this is a sort of "twisted" version of $\sigma_{11}(2p, 3p)$ since instead of being contained in a fixed plane the lines in the intersection meeting $\mathscr{L}(0)$ but not containing p are required to lie in a variable plane.

To see that this limit really depends on the way in which q_1, q_2 tend to p, note that $\mathscr{L}(0)\cup C$ is the complete intersection of Q with another quadric Q', and thus there is a whole pencil, at least, of different degenerations $q_1(t)$, $q_2(t)\to p$ found as above. The limits in this case are really different; since $Q\cap Q' = \mathscr{L}(0)\cup C$ as schemes, Q and Q' do *not* share tangent planes at a general point of $L(0)$.

Example 2. More generally, let $t\to\mathscr{L}(t)$ be any curve in the Grassmannian $G = \mathrm{Gr}(1,\mathbb{P}^3)$ of lines in $\mathbb{P}^3$, and let $\sigma_1(\mathscr{L}(t))$ be the variety of lines meeting $\mathscr{L}(t)$. We wish to analyze

$$\lim_{t\to 0} \sigma_1(\mathscr{L}(0))\cap\sigma_1(\mathscr{L}(t)).$$

We will regard G as embedded, as usual, in $\mathbb{P}^5$, as the quadric

$$p_{01}p_{23}-p_{02}p_{13}+p_{03}p_{12}.$$

Recall that the bilinear form associated to this quadric induces a duality on $\mathbb{P}^5$ in which a k-dimensional linear subspace $X\subset\mathbb{P}^5$ corresponds, via annihilation $X\to X^\perp$, to a $4-k$ dimensional subspace. A subspace is isotropic with respect to this form precisely when it is contained in G.

With these identifications one sees easily that if M is any line, regarded as a point in G, then $\sigma_1(M)$ is the intersection of the Grassmannian with $\{M\}^\perp$, (which is the tangent hyperplane to the Grassmannian at M). Thus

$$\sigma_1(\mathscr{L}(0))\cap\sigma_1(\mathscr{L}(t))$$

is the intersection with G of the annihilator of the chord to $\mathscr{L}$ through $\mathscr{L}(0)$ and $\mathscr{L}(t)$, and

$$\lim_{t\to 0} \sigma_1(\mathscr{L}(0))\cap\sigma_1(\mathscr{L}(t))$$

is the intersection of G and the 3-space in which is the annihilator of the tangent line T to $\mathscr{L}$ at $\mathscr{L}(0)$. (This makes sense and is correct even if $\mathscr{L}$ is singular at $\mathscr{L}(0)$).

Thus the following elementary result gives a complete picture of the limit:

Proposition 7.2. *Let T be any line in $\mathbb{P}^5$, $T^{\perp}\cap G$ is reduced and irreducible if and only if $T\not\subset G$. Further*

1) *If $T\subset G$ and $\mathscr{L}\subset\mathbb{P}^3$ is any point of T, then there is a point $p\in\mathbb{P}^3$ on $\mathscr{L}$ and a plane $V\subset\mathbb{P}^3$ containing $\mathscr{L}$ such that $T^{\perp}$ is the union of the plane in G of lines through p and the plane in G of lines in V.*

2) *If $T\not\subset G$ but T is tangent to G at a point $\mathscr{L}$, then $T^{\perp}\cap G$ is a cone over a nonsingular (plane) conic. Further, there is a $1-1$ correspondence between nonsingular conics $D\subset T^{\perp}\cap G$ and nonsingular conics $E\subset G$, tangent to T, by the relation $D\to(D^{\perp}\cap G)=E$, $E\to(E^{\perp}\cap G)=D$. The lines in $\mathbb{P}^3$ corresponding to points on such pairs of conics in G form the two rulings of a nonsingular quadric $Q\subset\mathbb{P}^3$, and $T^{\perp}\cap G$ may be described as the set of lines in $\mathbb{P}^3$ which meet $\mathscr{L}$ and are tangent to Q.*

Proof. First, if $T^{\perp}\cap G$ is reducible or non reduced, then it is the union of two (possibly coincident) planes P_1 and P_2. Since $P_i=P_i^{\perp}$ we get

$$\begin{aligned} T\subset T^{\perp\perp}\subset(T^{\perp}\cap G)^{\perp}=(P_1\cup P_2)^{\perp}&=P_1^{\perp}\cap P_2^{\perp}\\ &=P_1\cap P_2; \end{aligned}$$

thus in particular $T\subset G$. On the other hand, every line in G may be represented as the intersection of the (distinct) uniquely determined planes P_1, P_2 in G, one consisting of all lines through a certain point and the other consisting of all lines in a certain plane. Thus $T^{\perp}=(P_1\cap P_2)^{\perp}=P_1^{\perp}\cup P_2^{\perp}=P_1\cup P_2$. This proves the first statement, and also part 1).

For part 2) note first that $T^{\perp}\subset\mathscr{L}^{\perp}$, the tangent plane to G at $\mathscr{L}$, so $T^{\perp}\cap G$ must be singular at $\mathscr{L}$. Since $T^{\perp}\cap G$ is an irreducible, reduced, singular quadric in $\mathbb{P}^3$, it is the cone over a nonsingular conic D and the vertex must of course be $\mathscr{L}$. Since D has degree 2, it follows that the union of the lines in $\mathbb{P}^3$ corresponding to points of D is a quadric Q. If this quadric were *singular*, or were a plane counted twice, then the 2-plane spanned by D would be contained in G, and thus in $T^{\perp}\cap G$, so $T^{\perp}\cap G$ would be reducible. Thus Q is nonsingular.

Now let D be any nonsingular conic in G, and let P be the plane spanned by D. If $P\not\subset G$, then as before we see that D corresponds to the rulings of a nonsingular quadric Q. If we set $E=D^{\perp}\cap G=P^{\perp}\cap G$, then $P^{\perp}\not\subset G$ and E, as the intersection of the two-plane $P^{\perp}$ with G is a conic. Since every line corresponding to a point of E meets every line corresponding to a point of D, we see that E is the nonsingular conic representing the other ruling of Q.

Returning to the cone of a nonsingular conic D on the cone $T^{\perp}\cap G$, we see at once that $E=D^{\perp}\cap G$ is a nonsingular conic through $\mathscr{L}$ and lying on the plane $D^{\perp}$, which contains T. Since E is nonsingular, $D^{\perp}$ meets the tangent hyperplane to G only in the line T, which must therefore be the tangent line to E.

Conversely, given a nonsingular conic E in G passing through $\mathscr{L}$ and tangent to T, the plane spanned by E, since it is not contained in G, meets G in E

and thus meets the tangent plane to G only in T; that is, $\mathscr{L}^\perp \not\supset E$. Thus $E^\perp \not\ni \mathscr{L}$, and since $E^\perp$ is a plane contained in $T^\perp$, $E^\perp$ meets $T^\perp \cap G$ in a nonsingular conic D, as claimed.

Since $T^\perp \subset \mathscr{L}^\perp$, all the lines corresponding to the points of $T^\perp \cap G$ meet $\mathscr{L}$. Thus it remains to show that they are all tangent to a nonsingular quadric $Q \subset \mathbb{P}^3$ associated with D and E as above. We will do this by showing that any point of $T^\perp \cap G$ corresponding to a line in $\mathbb{P}^3$ that meets Q in two distinct points, must either be $\mathscr{L}$ or lie in D. Indeed, let $\mathscr{L}'$ be such a line. If $\mathscr{L}'$ is not in D, then $\mathscr{L}'$ meets Q on two distinct lines of the ruling corresponding to D. That is, there are points $\mathscr{L}_1 \neq \mathscr{L}_2 \in D$ such that $\mathscr{L}' \in \mathscr{L}^\perp \cap \mathscr{L}^\perp \cap T^\perp$. But $\mathscr{L}_2^\perp \cap T$ is the tangent plane to $T^\perp \cap G$ at $\mathscr{L}_i$, so $(\mathscr{L}_1^\perp \cap T) \cap (\mathscr{L}_2^\perp \cap T) \cap G = \{\mathscr{L}\}$, and we are done. □

With this result in hand, we can restate the previous Proposition in a nice geometric form. Since it can easily be verified for curves other than the rational normal curve (and we will soon prove this, in far more generality), we will drop that part of the hypothesis.

Corollary 7.3. *Let C be any nondegenerate arc in $\mathbb{P}^3$. If C' is the arc of tangent lines in the Grassmannian $G \subset \mathbb{P}^5$, then the tangent line to C' at a point corresponding to $p \in C$ is the intersection of the plane of lines through p and the plane of lines contained in the osculating 2-plane to C at p.*

8. Degeneration of Schubert intersections; main results

A *non-degenerate arc* C through $x_0 \in \mathbb{P}^d$ is the image of a representation of the germ of an analytic map $(\mathbb{C}, 0) \to (\mathbb{P}^d, x_0)$ whose image lies in no hyperplane. We fix such an arc C for all of this section. If x lies on this arc we write

$$\mathscr{F}(x) = \mathscr{F}_C(x) = \{\{x\} = \mathscr{F}^0(x) \subset \ldots \subset \mathscr{F}^d(x) = \mathbb{P}^d\}$$

for the osculating flag to C at x.

Fix integers k and r with $0 \leqq k, r \leqq d$. In the Grassmann variety $\mathrm{Gr}(k, \mathbb{P}^d)$ we let σ be an arbitrary Schubert variety defined in terms of the flag $\mathscr{F}(x_0)$, and we write

$$\tau(x) = \sigma_{d-k-r}(\mathscr{F}(x)) = \{\Lambda \in \mathrm{Gr}(k, \mathbb{P}^d) \mid \Lambda \text{ meets } \mathscr{F}^r(x)\}$$

for the special Schubert variety associated to $\mathscr{F}^r(x)$.

In the Chow ring of $\mathrm{Gr}(k, \mathbb{P}^d)$ we may write

$$\sigma \cdot \tau = \sum_\rho i(\sigma, \tau; \rho)\rho,$$

where ρ ranges over all the Schubert varieties in $\mathrm{Gr}(k, \mathbb{P}^d)$. Because τ is a special Schubert variety, $i(\sigma, \tau; \rho)$ is given by Pieri's formula (stated in detail below), and is either 0 or 1.

With this notation, our main result is:

Theorem 8.1. $\lim\limits_{\substack{x \to x_0 \\ \text{along } C}} \sigma \cap \tau(x) = \bigcup\limits_{i(\sigma, \tau; \rho) \neq 0} \rho$ *as schemes.*

A simple, well-known case of the above, for $k=0$, is this: If C is a nondegenerate curve in $\mathbb{P}^d$, then the intersection of an osculating s-plane to C at a point x_0 with an osculating r-plane to C at a point x approaches the osculating $r+s-d$ plane at x_0 as $x \to x_0$ along C. Proposition 7.1 is also a special case. Of course only the behavior of $\mathscr{F}^r(x)$, as $x \to x_0$, is important. The following definitions abstract what we need to know, and allow us to prove a more general result, valid in all characteristics:

First recall that, given a system of coordinates $x_0, \ldots, x_d$ on $\mathbb{P}^d_K$, where K is any field, and an integer r between 0 and d, we may introduce Plücker coordinates p_α for the Grassmannian of r-planes in $\mathbb{P}^d$. The indices α are strictly increasing $r+1$-tuples of integers $\alpha = (\alpha_0, \ldots, \alpha_r)$ with

$$0 \leqq \alpha_0 < \alpha_1 < \ldots < \alpha_r \leqq d.$$

We partially order these indices "componentwise"; that is,

$$\alpha = (\alpha_0, \ldots, \alpha_r) \leqq \alpha' = (\alpha'_0, \ldots, \alpha'_r)$$

if $\alpha_i \leqq \alpha'_i$ for $i = 0, \ldots, r$.

We can now make the central definition: Let $\mathscr{F} = \mathscr{F}^0 \subset \mathscr{F}^1 \subset \ldots \subset \mathscr{F}^d = \mathbb{P}^d_k$ be a flag of subspaces, and let $\mathscr{F}^r(t)$ be a deformation of $\mathscr{F}^r = \mathscr{F}^r(0) \subset \mathbb{P}^d$ over $K[|t|]$, that is, a $K[|t|]$-valued point of $\mathrm{Gr}(r, \mathbb{P}^d)$.

Let $x_0, \ldots, x_d$ be coordinates such that $\mathscr{F}^i$ has equations $x_{i+1} = \ldots = x_d = 0$. Let $p_\alpha(t)$ be the associated Plücker-coordinates of the space $\mathscr{F}^r(t)$, and write $\operatorname{ord} p_\alpha(t)$ for the order of vanishing of $p_\alpha(t)$ at $t=0$. We will say that $\mathscr{F}^r(t)$ is *monotone with respect to* $\mathscr{F}$ if:

1) No $p_\alpha(t)$ is identically 0.

2) If $p_\alpha(t)$ and $p_\beta(t)$ are Plücker coordinates of $\mathscr{F}^r(t)$ with $\alpha < \beta$ then $\operatorname{ord} p_\alpha(t) < \operatorname{ord} p_\beta(t)$.

It is easy to see that this definition depends on $\mathscr{F}$ and $\mathscr{F}^r(t)$, but not on the particular coordinates chosen. Its interest for us comes primarily from the following:

Proposition 8.2. *Let C be a nondegenerate arc in $\mathbb{P}^d$, parametrized by $f\colon U \to C$, where U is an analytic neighborhood of $0 \in \mathbb{C}$. If, for $0 \leqq r \leqq d$, $\mathscr{F}^r(t)$ is the osculating r-plane to C at $f(t)$, then $\mathscr{F}^r(t)$ is monotone with respect to $\mathscr{F}(0)$.*

We postpone the proof for a moment. In the light of Proposition 8.2 we see that Theorem 8.1 follows at once from a more general result, valid in any characteristic:

Theorem 8.3. *Let $\mathscr{F} = \mathscr{F}^0 \subset \ldots \subset \mathscr{F}^d$ be a flag in $\mathbb{P}^d$, over any field, and let $\mathscr{F}^r(t)$ be a deformation of $\mathscr{F}^r = \mathscr{F}^r(0)$, monotone with respect to $\mathscr{F}$. If σ is any Schubert variety in $\mathrm{Gr}(k, \mathbb{P}^d)$, and $\tau(t)$ is the special Schubert variety of k-planes meeting $\mathscr{F}^r(t)$, then*

$$\lim_{t \to 0} \sigma \cap \tau(t) = \bigcup_{i(\sigma, \tau; \rho) \neq 0} \rho,$$

as schemes.

The limit in the theorem should be interpreted as follows: If the base field is k, the family $\sigma \cap \tau(t)$, for $t \neq 0$, can be completed in a unique way to a flat family of subschemes of the Grassmannian over the base $k[|t|]$, and we simply take the fiber over 0 as the limit.

Over the complex numbers, this can be interpreted (at least set-theoretically) as a limit of compact sets.

We now turn to the proofs of 8.2 and 8.3:

Proof of Proposition 8.2. The map f may be written, in suitable coordinates, as

$$f(t)=(1, t^{1+v_1}+\text{(higher order)}, t^{2+v_2}+\text{(higher order)}, \ldots)$$

with $0 \leqq v_1 \leqq v_2 \leqq \ldots$. The osculating i-space $\mathscr{F}^i$ to C at $f(0)=(1,0,\ldots)$ is then given by the equations

$$X_{i+1}=\ldots=X_d=0.$$

On the other hand, the osculating space $\mathscr{F}^r(t)$ is spanned, for small $t \neq 0$, by the first $r+1$ derivatives

$1, \quad t^{1+v_1}+\text{higher}, t^{2+v_2}+\text{higher}, t^{3+v_3}+\text{higher}, \ldots$

$0 \quad (1+v_1)t^{v_1}+\text{higher}, (2+v_2)t^{1+v_2}+\text{higher}, (3+v_3)t^{2+v_3}+\text{higher}, \ldots$

$0 \quad (1+v_1)v_1 t^{v_1-1}+\text{higher}, (2+v_2)(1+v_2)t^{v_2}+\text{higher}, (3+v_3)(2+v_3)t^{1+v_3} \ldots$

$\vdots \quad \ldots$

The entry in the i^{th} row and j^{th} column (counting from 0!) will be

$$(v_j+j)(v_j+j-1)\ldots(v_j+j-i+1)t^{v_j+j-i}+\text{(higher order)}.$$

Now let $p_\alpha(t)$ be a Plücker coordinate of $\mathscr{F}^r(t)$ corresponding to the columns $\alpha_0<\alpha_1<\ldots<\alpha_r$. If we write $u(z_0,\ldots,z_r)$ for the determinant of the $(r+1)\times(r+1)$ matrix whose (i,j) entry, for $0 \leqq i, j \leqq r$, is

$$z_j(z_j-1)\ldots(z_j-i+1)$$

then one checks at once that

$$p_\alpha(t)=u(v_{\alpha_0}+\alpha_0,\ldots,v_{\alpha_r}+\alpha_r)t^{\Sigma(v_{\alpha_j}+\alpha_j-j)}+\text{higher order}.$$

If $\alpha<\beta$ are both indices of Plücker coordinates, then

$$\sum(v_{\alpha_j}+\alpha_j-j)<\sum(v_{\beta_j}+\beta_j-j).$$

To prove the Proposition it now suffices to prove that, for every Plücker index α,

$$u(v_{\alpha_0}+\alpha_0,\ldots,v_{\alpha_r}+\alpha_r)\neq 0.$$

But by Lemma (1.2),

$$u(z_0,\ldots,z_r)=\prod_{0\leqq i<j\leqq r}(z_i-z_j);$$

since the sequence $v_{\alpha_0}+\alpha_0,\ldots,v_{\alpha_r}+\alpha_r$ is strictly increasing, we are done.

The proof of Theorem 8.3 will be a computation in the homogeneous coordinate ring of the Grassmannian. Before undertaking it we must review the formula for $i(\sigma, \tau; \rho)$, where τ is a "special" Schubert cycle, and the "standard basis" theorem for the homogeneous coordinate rings of the Grassmannian and Schubert varieties.

First, let $\sigma = \sigma_a$ be an arbitrary Schubert cycle of k-planes in $\mathbb{P}^d$, where

$$a = (a_0, a_1, \ldots, a_k)$$

with $d-k \geqq a_0 \geqq \ldots \geqq a_k \geqq 0$, and let τ be the "special" Schubert cycle of k-planes meeting an r-plane,

$$\tau = \sigma_{d-k-r, 0, \ldots}.$$

The intersection numbers we want are given by "Pieri's Formula" (see for example Griffiths and Harris [1978] p. 203):

$$i(\sigma_a, \tau; \sigma_b) = \begin{cases} 1 & \text{if } \operatorname{codim} \sigma_a + \operatorname{codim} \tau = \operatorname{codim} \sigma_b \text{ and } a_{i-1} \geqq b_i \geqq a_i \\ 0 & \text{otherwise.} \end{cases}$$

Here we must take $a_{-1} = d-k$ by convention. The condition $\operatorname{codim} \sigma_a + \operatorname{codim} \tau = \operatorname{codim} \sigma_b$ may, in light of the fact that $\operatorname{codim} \sigma_a = \sum a_i$, and similarly for σ_b and τ, may be rewritten

$$\sum_{i=0}^{k} b_i - a_i = d-k-r.$$

We next recall facts about the homogeneous coordinate ring of $G(k, \mathbb{P}^d)$. These were certainly known to Hodge [1943] (or even earlier to Alfred Young, for example), but we follow the formulations given by Deconcini et al. [1980] and [1982]; see also Eisenbud [1979] for an exposition. The theory works over an arbitrary ground-ring.

The homogeneous coordinate ring of $G(k, \mathbb{P}^d)$ in the usual "Plücker" embedding is generated by the Plücker coordinates p_α, where $\alpha = (\alpha_0, \ldots, \alpha_k)$, $0 \leqq \alpha_0 < \ldots < \alpha_k \leqq d$, which may be thought of as the minor of a generic $k+1 \times d+1$ matrix corresponding to the set of columns with indices $\alpha_0, \ldots, \alpha_k$. (We sometimes make the convention $\alpha_{-1} = -1$.)

A monomial $p_{\alpha^{(1)}} p_{\alpha^{(2)}} \ldots p_{\alpha^{(m)}}$ in the Plücker coordinates is called *standard* if

$$\alpha^{(1)} \leqq \alpha^{(2)} \leqq \ldots \leqq \alpha^{(m)}$$

in the usual (componentwise) partial ordering of the Plücker coordinates. The homogeneous coordinate ring A of the Grassmannian satisfies the following two axioms, which make it a

Hodge Algebra:

Hodge-1): The set of standard monomials is a free basis for A as a module over the ground ring.

Hodge-2) (Here given in a special case): If α, β are incomparable Plücker indices, then the unique expression for $p_\alpha p_\beta$ as a linear combination of standard monomials takes the form:

$$p_\alpha p_\beta = \sum_{\gamma' \geqq \alpha, \beta, \gamma} n_{\gamma,\gamma'} p_\gamma p_{\gamma'},$$

where the $n_{\gamma,\gamma'}$ are integers (depending also on α, β).

(In fact, we have $\gamma \leqq \alpha$, $\beta \leqq \gamma'$ in every term; this gives us the choice of whether to regard A as a Hodge algebra based on the poset of Plücker indices or on the opposite poset, and we have chosen the opposite poset for notational convenience. The choice made in Deconcini et al. [1982] and Eisenbud [1979] is the other one.)

We are now ready to begin work:

We will identify generators for the homogeneous ideal I of $\bigcup_{i(\sigma,\tau;\rho) \neq 0} \rho$, and show that they all occur as limits of elements of the homogeneous ideal I_t of $\sigma \cap \tau(t)$. If we write I_0 for the homogeneous ideal of $\lim_{t \to 0}(\sigma \cap \tau(t))$, which is also (by construction) the limit of the ideals I_t, then the Hilbert function of the homogeneous coordinate ring modulo I_t is constant by flatness. Since $I_0 \supset I$, this shows in particular that every component of $\sigma \cap \tau(t)$ has codimension (at least) equal to $\operatorname{codim} \sigma + \operatorname{codim} \tau(t)$. With this it follows from the Cohen-Macaulayness of σ and of $\tau(t)$ that $\sigma \cap \tau(t)$ is Cohen-Macaulay, and thus that every component has codimension equal to $\operatorname{codim} \sigma + \operatorname{codim} \tau(t)$. Since $\sum_{i(\sigma,\tau;\rho) \neq 0} \deg \rho = (\deg \sigma)(\deg \tau)$, and this is also the degree of the variety defined by I_0, it will follow, from the fact that I is reduced, that $I = I_0$, and the first formula of the theorem will be proven.

If now I_t was not reduced then, since $\tau \cap \sigma(t)$ is Cohen-Macaulay, it would have a non-reduced component, so $\lim_{t=0} \sigma \cap \tau(t)$ would also have a non-reduced component, which is not the case. This will conclude the proof.

It remains to identify the generators of I and prove that they occur as limits of elements of I_t.

First, the homogeneous ideal of a Schubert cycle σ_b $(b = (b_0, \ldots, b_k))$ of k-planes, defined in terms of the same coordinates on $\mathbb{P}^d$ as the p_γ, is generated by the set of Plücker coordinates p_γ with

$$\gamma \nleqq \beta = (d-k-b_0, d-k-b_1+1, \ldots, d-k-b_i+i, \ldots),$$

and thus the quotient ring has a basis, over the ground-field, consisting of all those standard monomials involving a factor p_γ with $\gamma \leqq \beta$.

If $\{\sigma_{b^{(i)}}\}$ is an arbitrary set of Schubert cycles defined in terms of $\mathscr{F}$, and if we write

$$\beta^{(i)} = (d-k-b_0^i, d-k-b_1^i+1, \ldots)$$

for the Plücker indices corresponding to the $b^{(i)}$, then it follows that the homogeneous ideal of

$$\bigcup_i \sigma_{b^{(i)}}$$

is spanned by the standard monomials containing factors p_γ with $\gamma \nleqq \beta^{(i)}$ for each i. (Since the γ occurring in a given standard monomial are linearly ordered, we may assume that a single one satisfies all these conditions.) Thus

$$\{p_\gamma \mid \gamma \nleqq \beta^{(i)} \text{ for each } i\}$$

generates the homogeneous ideal of $\bigcup \sigma_{b^{(i)}}$.

Returning to the setting of Theorem 8.3, note that the homogeneous ideal of $\lim_{t\to 0} \sigma_a \cap \tau(t)$ contains the homogeneous ideal of σ_a, so we will be especially interested in those generators p_γ for the ideal of

$$\bigcup_{i(\sigma_a,\tau,\rho)\neq 0} \rho$$

which satisfy

$$\gamma \leqq \alpha = (d-k-a_0, d-k-a_1+1, \ldots).$$

The following lemma identifies these:

Lemma 8.4. *Let σ_a be a Schubert cycle and let*

$$\alpha = (d-k-a_0, d-k-a_1+1, \ldots)$$

be the corresponding Plucker index.

A Plucker index β satisfies

$$\beta \nleqq (d-k-b_0, d-k-b_1+1, \ldots)$$

for every Schubert cycle σ_b with

$$i(\sigma_a, \sigma_{d-k-r,0\ldots}; \sigma_b) \neq 0$$

if and only if

$$\beta \nleqq \alpha$$

or

$$\beta \leqq \alpha \quad \text{and} \quad \sum_{i=0}^{k} \alpha_i - \max(\beta_i, \alpha_{i-1}+1)) < d-k-r.$$

Proof. The $\bar\beta$ which are $\leqq \alpha$ and are equal to some

$$(d-k-b_0, d-k-b_1+1, \ldots)$$

for which $i(\sigma_i, \tau; \sigma_b) \neq 0$ satisfy $\alpha_i \geqq \bar\beta_i \geqq \alpha_{i-1}+1$ and $\sum \alpha_i - \bar\beta_i = d-k-r$; if $\beta \leqq \bar\beta$, then clearly $\sum \alpha_i - \max(\beta_i, \alpha_{i-1}+1) \geqq d-k-r$, so if β satisfies the reverse (strict) inequality, then $\beta \nleqq \bar\beta$ for each such $\bar\beta$, as required. Of course the same is true if $\beta \nleqq \alpha$.

On the other hand if $\beta \leqq \alpha$ and

$$\sum_{i=0}^{k} (\alpha_i - \max(\beta_i, \alpha_{i-1}+1)) \geqq d-k-r,$$

then by increasing certain β_i appropriately we may construct $\bar\beta \geqq \beta$ with $\alpha_i \geqq \bar\beta_i \geqq \alpha_{i-1}+1$ and $\sum (\alpha_i - \bar\beta_i) = d-k-r$. This completes the proof of (8.4). □

It now remains to show that the generators of the ideal of $\bigcup_{i(\sigma_a,\tau;\rho)\neq 0} \rho$ which are not in the ideal of σ_a, namely those p_β with

$$\beta \leqq \alpha = (d-k-a_0, d-k-a_1+1, \ldots)$$

and

$$\sum_{i=0}^{k} (\alpha_i - \max(\beta_i, \alpha_{i-1}+1)) < d-k-r,$$

are in the ideal of $\lim_{t\to 0} \sigma_a \cap \tau(t)$, or, to put it more algebraically, are the leading coefficients of elements of the ideal I_t of $\sigma_a \cap \tau(t)$, when these are developed as power series in t. Since the ideal I_a of σ_a is contained in I_t, we may reformulate this by saying that a p_β as above must be shown to occur as the coefficient of the lowest power of t in the expansion of an element of I_t whose coefficient is not in I_a.

We will work with elements of the ideal of $\tau(t)$, which is contained in I_t. We may suppose that $\mathscr{F}^r(t)$ is given as the span of $r+1$ points depending on t, and we write $q_\gamma(t)$ for the Plücker coordinate involving columns $\gamma_0, \ldots, \gamma_r$.

A k-plane V with Plücker coordinates p_β meets $\mathscr{F}^r(t)$ if and only if the linear span of V and $\mathscr{F}^r(t)$ has dimension $\leqq r+k$, that is, if and only if the $r+k+2$-order minors of an $r+k+2 \times d+1$ matrix, whose rows give the coordinates of $r+1$ points spanning $\mathscr{F}^r(t)$ and $k+1$ points spanning V, vanish. Expanding along the $r+1$ rows corresponding to $\mathscr{F}^r(t)$, the determinant involving a set J of $r+k+2$ columns may be written, confusing the sequence $(\beta_0, \beta_1, \ldots)$ with the set $\{\beta_0, \beta_1, \ldots\}$, as

$$\sum_{\beta \cup \gamma = J} p_\beta \cdot q_\gamma(t).$$

Since $\mathscr{F}^r(t)$ is monotone, the $q_\gamma(t)$ are all nonzero and have orders strictly increasing with increasing γ; thus the order of the coefficient of p_β in the above expression decreases strictly with increasing β.

The theorem will now be proven if we show that for each $\beta \leqq \alpha$ with

$$\sum (\alpha_i - \max(\beta_i, \alpha_{i-1}+1)) < d-k-r,$$

there is a J with $r+k+2$ elements such that β is the unique maximal Plücker index satisfying

$$\beta \subset J \quad \text{and} \quad \beta \leqq \alpha.$$

Of course it will be enough to construct such a J with any larger number of elements.

If $u \leqq v$ are integers, we write $\langle u, v\rangle$ for the set of $v-u+1$ integers $u, u+1, \ldots, v$ between u and v. If K is a finite set, we write $\# K$ for its cardinality.

Let $\beta \leqq \alpha$ be a Plücker index satisfying

$$\sum_{i=0}^{k} \alpha_i - \max(\beta_i, \alpha_{i-1}+1)) < d-k-r.$$

Set

$$J'' = \bigcup_{i=0}^{k} \langle \max(\beta_i + 1, \alpha_{i-1}+1), \alpha_i \rangle$$

$$J' = J'' - (\beta \cap J'')$$

$$J = \langle 0, d\rangle - J'.$$

We will show that J satisfies the condition above.

Since $\beta \subset J$, we must show:

1) $\# J' < d-k-r$.

2) If β' is a Plücker index with $\beta' \leqq \alpha$ and $\beta' \cap J' = \emptyset$, then $\beta' \leqq \beta$.

For 1), note first that

$$\# \langle \max(\beta_i + 1, \alpha_{i-1}+1), \alpha_i \rangle = \begin{cases} \alpha_i - \max(\beta_i, \alpha_{i-1}+1) & \text{if } \beta_i > \alpha_{i-1} \\ 1 + \alpha_i - \max(\beta_i, \alpha_{i-1}+1) & \text{if } \beta_i \leqq \alpha_{i-1}. \end{cases}$$

On the other hand, if $\beta_i \leqq \alpha_{i-1}$ then $\beta_i \in (\beta \cap J'')$; since then, if k is the smallest index with $\beta_i \leqq \alpha_k$ we will have $k < i$ and so

$$\beta_i \in \langle \max(\alpha_{k-1}+1, \beta_k+1), \alpha_k \rangle.$$

(The converse is obvious but irrelevant to us). Thus

$$\begin{aligned} \# J' &= \sum \# \langle \max(\beta_i+1, \alpha_{i-1}+1), \alpha_i \rangle - \#(\beta \cap J'') \\ &= \sum_i (\alpha_i - \max(\beta_i, \alpha_{i-1}+1)) + \#\{i \mid \beta_i \leqq \alpha_{i-1}\} - \#(\beta \cap J'') \\ &\sum_i (\alpha_i - \max(\beta_i, \alpha_{i-1}+1)) < d-k-r \end{aligned}$$

the last inequality being part of the hypothesis on β.

2) We must show $\beta'_i \leqq \beta_i$ for each i. Suppose, by induction that we know $\beta'_j \leqq \beta_j$ for all $j > i$, so in particular, $\beta'_i < \beta_j$ for $j > i$. By hypothesis $\beta'_i \leqq \alpha_i$ and $\beta'_i \in J$. Since the only elements of $J \cap \langle \beta_i + 1, \alpha_i \rangle$ are the β_j for $j > i$, we see that $\beta'_i \leqq \beta_i$ as claimed. This concludes the proof. □

9. $G^r_d(C)$ is reduced for general cuspidal rational C

To prove the result in the heading, it is enough to show that if $C \subset \mathbb{P}^d$ is a rational normal curve, and for $x_i \in C$ we write $\sigma_r(x_i)$ for the variety of $d-r-1$-planes meeting the tangent line to C at x_i, then, for sufficiently general points $x_1, \ldots, x_g$ the intersection

$$\sigma_r(x_1) \cap \ldots \cap \sigma_r(x_g)$$

is reduced. We will prove this inductively, letting the x_i coalesce one by one; for the induction we need to prove slightly more, replacing $\sigma_r(x_1)$ by an arbitrary Schubert variety defined in terms of the osculating flag to C at x_1. The proof works for arbitrary non degenerate arcs, so we state the result in the generality:

Theorem 9.1. *Let C be a non-degenerate arc through x_0 in $\mathbb{P}^n_C$ and let $\sigma \subset G(k, n)$ be a Schubert variety of k-planes defined in terms of the osculating flag to C at x_0. For $x \in C$ and $0 \leqq r \leqq n$ an integer, let $\tau^r(x)$ be the Schubert variety of k-planes meeting the osculating r-space to C at x.*

If $x_1, \ldots, x_n \in C$ are a set of sufficiently general points, and $0 \leqq r_i \leqq n$ are integers, then the varieties

$$\sigma, \tau^{r_1}(x_1), \ldots, \tau^{r_m}(x_m)$$

meet transversely at every generic point of their intersection.

We will prove this theorem by showing that $\sigma \cap \bigcap_1^m \tau^{r_i}(x_i)$ has codimension equal to $\operatorname{codim} \sigma + \sum_1^m \operatorname{codim} \tau^{r_i}(x_i)$, (which implies that it is Cohen-Macaulay) and that it is reduced.

Example. The following example shows that the general position hypothesis in the theorem cannot be omitted:

Let $C \subset \mathbb{P}^3$ be the twisted cubic, and consider four distinct points $x_1, x_2, x_3, x_4 \in C$. In the Grassmannian G of lines in $\mathbb{P}^3$, embedded by the Plücker embedding in $\mathbb{P}^5$, the Schubert variety $\tau^1(x_i)$ is a hyperplane section. By Theorem 1.5, $\bigcap_1^4 \tau^1(x_i) = X$ is 0-dimensional, and since G is a quadric in $\mathbb{P}^5$, X consists of either 2 distinct reduced points or one point with multiplicity 2.

Let σ be the automorphism of $\mathbb{P}^1$ which permutes x_1, x_2, x_3 cyclically. We will show by symmetry that if x_4 is one of the two fixed points of σ, then X is non-reduced. One can show that these are the only choices of x_4 for which X is non reduced.

We may choose coordinates in $\mathbb{P}^1$ so that σ is represented on $\mathbb{A}^1 \subset \mathbb{P}^1$ by multiplication by ω, a primitive cube-root of 1, and $\{x_1, x_2, x_3\} \subset \mathbb{A}^1$ is an orbit for σ, for example $\{1, \omega, \omega^2\}$. The two fixed points of σ are 0 and ∞. Since inversion interchanges these, we may assume $x_4 = 0$. Note that the stabilizer of $\{x_1, x_2, x_3, x_4\}$ in $\operatorname{Aut} \mathbb{P}^1$ acts as the alternating group A_4 on $\{x_1, x_2, x_3, x_4\}$, which we may regard as the set of vertices of a regular tetrahedron inscribed in the Riemann Sphere:

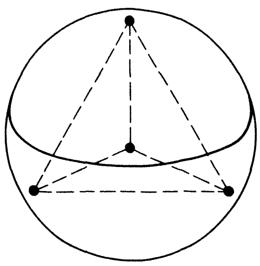


Fig. 6

We will show that, for each i, all the lines in $X = \bigcap_j \tau^1(x_j)$ meet the tangent line to C at x_i in the same point; this will show that there is only one such line, as required.

It will be convenient to represent $\mathbb{P}^3$ as the symmetric product of 3 copies of $\mathbb{P}^1$ (that is, $\mathbb{P}^3 = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1/\text{permutations}$), or, equivalently, the space of polynomials $f(x)$ of degree $\leqq 3$ modulo scalars. Thus we may represent a point of $\mathbb{P}^3$ by its preimage $(a, b, c) \in \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ that is, the polynomial $f(x) = (x-a)(x-b)(x-c)$. With this convention, we may take C to be the set

$$\begin{aligned} C &= \{(a, a, a) \mid a \in \mathbb{P}'\} \\ &= \{(x-a)^3 \mid a \in \mathbb{P}^1\} \end{aligned}$$

and the tangent line T_0 to C at 0 is then the set

$$\begin{aligned} T_0 &= \{(0, 0, a) \mid a \in \mathbb{P}^1\} \\ &= \{x^2(x-a) \mid a \in \mathbb{P}^1\}. \end{aligned}$$

Further, $\operatorname{Aut}\mathbb{P}^1$ acts as a group of (linear) automorphisms of $\mathbb{P}^3$ by $\operatorname{Aut}\mathbb{P}^1$ α: $(a, b, c) \mapsto (\alpha a, \alpha b, \alpha c)$, stabilizing C. In particular, we see that the stabilizer $A_4 \subset \operatorname{Aut}\mathbb{P}^1$ of $\{x_1, x_2, x_3, x_4\}$ acts on $\mathbb{P}^3$ so that, by symmetry, it is enough to prove that all lines in X meet T_0 in the same point.

Since there are at most two lines in X, and the element $\sigma \in \operatorname{Aut}\mathbb{P}^1$ has order 3, and stabilizes X, it must stabilize each line in X, so that such a line L must meet T_0 in a fixed point p of σ. But σ acts on T_0 by $\sigma(0, 0, a) = (0, 0, \sigma a)$, so $p = 0$ or $p = \infty$. By Theorem 2.3, L cannot meet C, so $p = \infty$ for every such line L, as required.

We now turn to the proof.

Proof of Theorem 9.1. By induction on the dimension of σ we may suppose that the intersection

$$\sigma' \cap \bigcap_1^m \tau^{r_i}(x_i)$$

is reduced, Cohen Macaulay, and of the correct codimension for any Schubert cycle σ', defined in terms of the flag $\mathscr{F}$, with $\dim \sigma' < \dim \sigma$. By Theorem 8.1

$$\lim_{x_1 \to x_0} \sigma \cap \tau^{r_1}(x_1) = \bigcup_{i(\sigma, \tau^{r_1}; \rho) \neq 0} \rho.$$

Thus

$$\lim_{x_1 \to x_0} \sigma \cap \bigcap_1^m \tau^{r_i}(x_i) = (\bigcup_{i(\sigma, \tau^{r_1} \rho) \neq 0} \rho) \cap \left(\bigcap_2^m \tau^{r_i}(x_i)\right)$$

which has the given codimension by induction on m. It follows that for small $x_1 \neq x_0$,

$$\operatorname{codim}\left(\sigma \cap \bigcap_1^m \tau^{r_i}(x_i)\right)$$

has the given value. Since σ and $\tau^{r_i}(x_i)$ are each Cohen-Macaulay, the additivity of codimension implies that $\sigma \cap \bigcap_1^m \tau(x_i)$ is, too. But this implies, in particular, that if $\sigma \cap \bigcap_1^m \tau(x_i)$ were not reduced, then it would have to be multiple along a whole, maximal dimensional component. The limiting position of the component would be a multiple component of

$$\Big(\bigcup_{i(\sigma,\tau^{r_1};\rho)\neq 0}\rho\Big)\cap\bigcap_2^m\tau^{r_i}(x_i).$$

By the induction hypothesis, such a multiple component would have to occur inside an intersection of two of the above terms, say $\rho'\cap\rho''\cap\bigcap_2^m\tau^{r_i}(x_i)$. But $\rho'\cap\rho''$ is itself a Schubert cycle, ρ''', say, defined in terms of $\mathscr{F}$, and of strictly lower dimension than ρ or ρ', so

$$\dim\rho'''\cap\bigcap_2^m\tau^{r_i}(x_i)<\dim\rho\cap\bigcap_2^m\tau^{r_i}(x_i)$$

for any ρ with $i(\sigma,\tau^{r_1};\rho)\neq 0$. This concludes the argument, and with it the proof of Theorem 5.1.

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