Proceedings of Symposia in Pure Mathematics
Volume **40** (1983), Part 1

# RATIONAL CURVES WITH CUSPS

DAVID EISENBUD[1]

My object in these notes is to state and indicate the proof of a transversality result from my recent joint work with Joe Harris. The notes follow closely the verbal account I gave in Arcata.

Throughout, I will speak only of projective varieties over **C**.

Before coming to the result, I would like to indicate its context.

Suppose one wishes to study an invariant of smooth curves of a given genus $g$. If the invariant is discrete and is defined in some reasonably algebraic way, it will generally be constant on some open subset of any family of such curves. Thus, since the moduli space $\mathfrak{M}_g$ of curves of genus $g$ is irreducible, it will be meaningful to speak of the value of the invariant on a "general curve" of genus $g$ —that is, the value of the invariant taken on an open dense subset of $\mathfrak{M}_g$. Of course, one will want to know the range of values of the invariant too!

Here are some examples (in which "curve" means "smooth projective curve", and $g \geqslant 3$):

(1) What is the smallest degree of an embedding of a curve of genus $g$?

(a) Any curve can be embedded as a curve of degree $g + 3$; in fact a generic line bundle $\mathfrak{L}$ of that degree will satisfy $\dim_{\mathbf{C}} H^0(\mathfrak{L}) = 4$, and the ratios of 4 independent sections will embed the curve in $\mathbf{P}^3$. Hyperelliptic curves do not embed with lower degree than this.

(b) Some curves can be embedded with quite low degree, the extreme cases being the plane curves, which embed with degree

$$d = \frac{3 + \sqrt{8g + 1}}{2}.$$

This is possible, of course, only for $g$ of the form $(d - 1)(d - 2)/2$.

---

1980 *Mathematics Subject Classification*. Primary 14H10, 14M15; Secondary 14E25, 14C20.

[1]I am grateful to the NSF for partial support of this work, and to the organizers of the Arcata conference for the stimulating atmosphere that they created.

(c) The general curve of genus $g$ can be embedded with degree equal to the least integer $\geqslant \frac{3}{4}g + 3$, but not lower. This is an unpublished result of Harris; he tells me that his original proof is unattractively complicated. (A rather simple proof follows by the method below.)

(2) [B–N]: For any curve $C$ of genus $g$, the line bundles of degree $d$ on $C$ are parametrized by the abelian variety $\mathrm{Jac}^d(C)$. For an open dense set of such line bundles $\mathcal{L}$ we have $\dim_{\mathbf{C}} H^0(\mathcal{L}) = \max(0, d - g + 1)$. But many line bundles of natural interest are *special* in the sense that they do not belong to this set, and so for $r > d - g$ it is interesting to ask about the set

$$W_d^r = \left\{\mathcal{L} \in \mathrm{Jac}^d C \;\middle|\; \dim_{\mathbf{C}} H^0(\mathcal{L}) \geqslant r + 1\right\}.$$

In fact, $W_d^r$ is a subvariety, so perhaps the most naive question to ask is about the codimension of (the components of) $W_d^r$. If $\mathcal{L}$ has a section, $s$, with zero locus $D$, say, then the exact sequence $0 \to \mathcal{O}_C \xrightarrow{s} \mathcal{L} \to \mathcal{O}_D \otimes \mathcal{L}^{-1} \to 0$ gives rise to the sequence:

$$\begin{array}{ccccccccc} 0 & \to & H^0(\mathcal{O}_C) & \to & H^0(\mathcal{L}) & \to & H^0(\mathcal{O}_D \otimes \mathcal{L}^{-1}) & \to & H^1(\mathcal{O}_C) \\ & & \| & & & & \| & & \| \\ & & \mathbf{C} & & & & \mathbf{C}^d & \xrightarrow[\phi_{\mathcal{L},s}]{} & \mathbf{C}^g \end{array}$$

From this we see that $\mathcal{L} \in W_d^r$ if and only if the $(d - r + 1)$-order minors of the matrix $\phi_{\mathcal{L},s}$ vanish. Since the entries of the $d \times g$ matrix $\phi_{\mathcal{L},s}$ can locally be expressed as analytic functions of $(\mathcal{L}, s) \in \mathrm{Jac}^d(C) \times H^0\mathcal{L}$, it follows that every component of $W_d^r$ has codimension $\leqslant r(g - d + r)$. This is the observation made by Brill-Noether.

Now Brill-Noether seem to have assumed that $W_d^r$ would really be nonempty, and thus of dimension $\geqslant g - r(g - d + r)$, whenever that bound is $\geqslant 0$; this was finally proven only recently by [**K, Kl** and **K-L**].

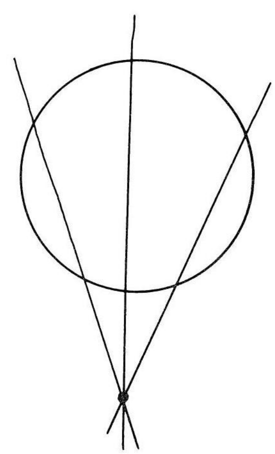
But what is the value of $\dim W_d^r$ for a general curve of genus $g$? Since the $(d - r + 1)$-order minors of a generic $d \times g$ matrix generate an ideal of codimension equal to $r(g - d + r)$, a naive guess, supported by many examples, is that, generically, $W_d^r$ has dimension equal to $g - r(g - d + r)$. This was finally proved, as outlined below, by Griffiths and Harris [**G-H-2**]; the method we will describe below gives a simplified proof.

In both cases outlined above, the statement about general curves is easily reduced, by the semicontinuity of the invariant, to the statement that there *exists one* (sufficiently general) curve for which the statement is true. This is very easy to see in the case of the Brill-Noether problem, while, in the question of embeddings, it follows from Brill-Noether that the general curve, satisfying Brill-Noether, cannot embed with smaller degree, and it remains to exhibit some curve, general enough to satisfy Brill-Noether, which does embed with the given degree.

Alas! Every smooth curve that one knows turns out to be special, and the above reduction does not immediately help except for very low genus.

Castelnuovo [**C**], in the context of Brill-Noether, had an idea in 1889 as to how to proceed; the idea seems to have been made precise by Severi [**S**] (1921) and taken up again recently by Kleiman [**Kl**]. It is: use rational curves, making them behave like curves of high genus by introducing singularities, as the "general curves". A rational curve with $g$ double points (which has "arithmetic genus" $g$) behaves from the point of view of line bundles and embeddings very much like a smooth curve of genus $g$ (for example, the Jacobian is still $g$ dimensional, though no longer compact) but it is easier to construct and control. In particular, to prove things like the results of (1) and (2) for general curves, it is enough to construct singular rational curves with the corresponding properties.

To see how one might construct the necessary curves, let us look at a simple example, and try to produce a rational curve $C$ with 3 nodes (arithmetic genus 3) for which $W_2^1$ is empty, as predicted by Brill-Noether. Suppose $\mathcal{L} \in W_2^1$, and let $|D|$ be the corresponding linear series. Pulling back $\mathcal{L}$ and $D$ to $\mathbf{P}^1$, the normalization of $C$, we see that it is represented by a linear series cut out on the nonsingular conic in $\mathbf{P}^2$ by lines; if $C$ is made from $\mathbf{P}^1$ by identifying points $\lambda_i$ with $\mu_i$ for $i = 1, 2, 3$, then the statement that the linear series comes from a series on $C$ is that the lines of the series contain $\mu_i$ iff they contain $\lambda_i$, for each $i$. Now a linear series of lines of dimension 1 in $\mathbf{P}^2$ *must* be the series of all lines through some point $x$ of $\mathbf{P}_2$, so our condition that $\mathcal{L}$ comes from $C$ becomes the condition that the 3 lines $\overline{\lambda_1\mu_1}, \overline{\lambda_2\mu_2}$ and $\overline{\lambda_3\mu_3}$ pass through a common point, as in Figure 1.

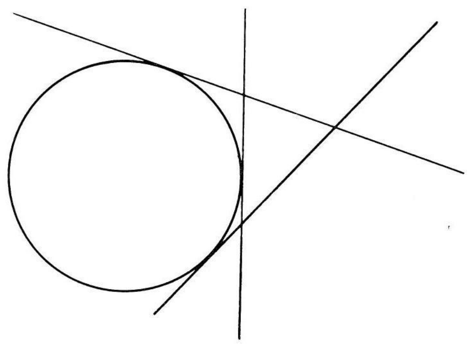


Now it is clear that for certain special choices of $\lambda_1, \lambda_2, \lambda_3, \mu_1, \mu_2$ and $\mu_3$ as above the three indicated chords will have a common point, as in Figure 1, but that for a general $\lambda_i$ and $\mu_i$ there will be no common point. Thus the general curve of (arithmetic) genus 3 has $W_2^1 = \varnothing$, and we are done, in this special case.

The above argument is easy to generalize into the (dimensional part of the) "Castelnuovo-Severi-Kleiman conjecture"; the conjecture was ultimately proven by Griffiths and Harris, giving a proof of the Brill-Noether statement on general curves by showing that a rational curve with $g$ generally placed nodes behaves as a general curve from the point of view of Brill-Noether. The proof is quite delicate, precisely because one must take the nodes to be general (as we saw above); and general sets of nodes are hard to make explicit, just as general smooth curves are hard to find.

With this in mind, it came as a kind of revelation to Harris and me that if one takes rational curves with ordinary cusps instead of rational curves with nodes, then the curves behave as general curves (say from the Brill-Noether point of view) for *all* positions of the cusps, *not* just general positions.

To see this in the example given above ($W_2^1 = \varnothing$ for a general curve of genus 3) we must, by an argument analogous to that just given, simply note that no three distinct tangents to a conic $\tilde{C}$ in $\mathbf{P}_2$ can contain a common point (see Figure 2).



This is perhaps obvious, but a rigorous proof can be made as follows: If the three tangents passed through a point $x$, then projection from $x$ would give a 2-1 map (or, if $x$ lay on the conic, a 1-1 map) of the conic onto the line, $\mathbf{P}^1$, ramified at $\geqslant 3$ points (or $\geqslant 1$ points if $x \in \tilde{C}$). But for any ramified covering map $C \to \mathbf{P}^1$ of degree $d$ we have, by an easy and standard topological argument, the Riemann-Hurwitz formula,

Euler characteristic of $\tilde{C}$
$\|$
$2 - 2g$
$\|$
$2$
$= d$ (Euler characteristic $\mathbf{P}_1$) $-$ (the number of ramification points),
$\|$ $\|$
$2$ $2$

a contradiction in either case.

The transversality result we have in mind is a generalization of these remarks. It has the following ingredients:

(1) The conic is replaced by the *rational normal curve* $C$ of degree $n$ in $\mathbf{P}^n$; it may be given parametrically by $\mathbf{P}^1 \ni (s, t) \to (s^n, s^{n-1}t, \ldots, t^n) \in \mathbf{P}^n$.

(2) The tangent line to the conic at a point $p$ is replaced by the *osculating flag* $\mathcal{F}(C, p) = \langle \mathcal{F}^0(C, p) \subset \cdots \subset \mathcal{F}^{n-1}(C, p)\rangle$ to $C$ at a point $p$: the osculating $k$-space $\mathcal{F}^k(C, p)$ to $C$ at $p$ may be defined as the intersection of all hyperplanes meeting $C \geq k + 1$ times at $p$ (it is easy to see, as in elementary differential geometry, that this is a $k$-plane!).

(3) The set of points lying on the tangent line to the conic at a point is replaced by an arbitrary *Schubert cycle*, defined with reference to $\mathcal{F}(C, p)$, in the Grassmannian of $k$ planes $(0 \leq k \leq n - 1)$ in $\mathbf{P}^n$:

$$\sigma_{a_0,\ldots,a_k}(p) := \big\{k\text{-planes } V \subset \mathbf{P}^n \mid \dim\big(V \cap \mathcal{F}^{n-k-a_i+i}(C, p)\big) \geq i \text{ for } i = 0, 1, \ldots, k\big\}.$$

Here we may assume $n - k \geq a_0 \geq \cdots \geq a_k \geq 0$. The facts we will use about the Schubert cycles are:

(*) dim Grass($k$-planes in $\mathbf{P}^n$) $= (n - k)(k + 1)$,

(*) Codim $\sigma_{a_0,\ldots,a_k}(p) = \Sigma_0^k a_i$,

(*) $\sigma_{n-k-1,0,\ldots,0}(p)$ is the set of $k$-planes meeting the tangent line to $C$ at $p$.

We may now state our transversality result.

THEOREM. *Let $C$ be a rational normal curve in $\mathbf{P}^n$, let $k$ be an integer, $0 \leq k \leq n - 1$, let $p_1, \ldots, p_g$ be points of $C$, and for $i = 1, \ldots, g$, let $\Sigma_i$ be any Schubert cycle $\sigma_{a_0^{(i)},\ldots,a_k^{(i)}}(p_i)$ defined with reference to the osculating flag at $p_i$.*

*If the points $p_1, \ldots, p_g$ are distinct, then either*

$$\operatorname{codim} \bigcap_{i=1}^{q} \Sigma_i = \sum_{i=1}^{g} \operatorname{codim} \Sigma_i,$$

*or $\bigcap_{i=1}^{q} \Sigma = \varnothing$, this being the case precisely when the product of the corresponding cohomology classes is* 0.

One could hope that for general choices of the $p_i$, the Schubert cycles $\Sigma_i$ will intersect just as transversely as do the corresponding Schubert cycles defined with reference to general flags. Harris and I have proved a somewhat weaker version of this, which will appear in **[E-H]**.

Those familiar with the Castelnuovo-Severi-Kleiman conjecture will recognize that its dimension statement follows at once from the theorem above; the result of Harris on embeddings of general curves can also be deduced relatively easily. One further application deserves mention, perhaps, since it was the original motivation behind our work. It is a formula for the dimension of the space of rational curves of degree $n$ in $\mathbf{P}^r$ with assigned cusps which plays a central role in **[E-VdV]**. First we need another definition, which also plays a key role in our proof of the theorem.

If $C$ is any smooth curve, $p \in C$ a point, and $\phi: C \to \mathbf{P}^r$ a map, then in suitable local coordinates near $p$, $\phi$ may be written as

$$\phi: t \to \left(1, t^{1+\alpha_1^\phi(p)} + \text{higher}, t^{2+\alpha_1^\phi(p)+\alpha_2^\phi(p)} + \text{higher}, \cdots\right).$$

We set $\operatorname{Cusp}\phi = \Sigma_{p\in C}\alpha_1^\phi(p)\cdot p$, the *divisor of cusps* of $\phi$, and we say that $\Sigma_{p\in C}\alpha_1^\phi(p)$ is the number of cusps of $\phi$.

COROLLARY. *The space of maps* $\phi$: $\mathbf{P}^1 \to \mathbf{P}^r$ *whose image is a nondegenerate curve of degree $n$ with* cusp $\phi$ = *a fixed effective $D$ divisor of degree $d$ has dimension*

$$(r+1)(n+1) - 1 - rd$$

*or is empty; it is empty, for example, when $d > (r+1)(n-r)/r$ and it is not empty, for example, if $d \leqslant (r+1)(n-r)/r$ and $D$ has no multiple points. In particular, the dimension of the space of nondegenerate rational curves of degree $n$ in* $\mathbf{P}^r$ *having $\geqslant d$ cusps is*

$$(r+1)(n+1) - 4 - (r-1)d \quad \text{if } d \leqslant \frac{r+1}{r}(n-r),$$

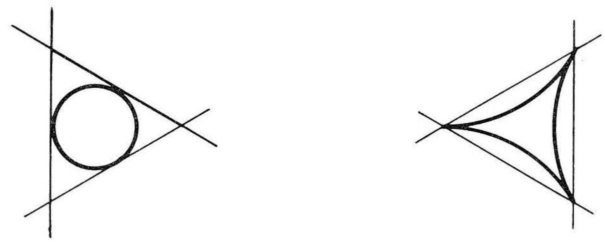
*and the space is empty otherwise.*

PROOF. The second statement follows from the first by allowing the cusps to move about (gaining $d$ dimensions) and factoring out the action of Aut($\mathbf{P}^1$). As for the first statement, it follows from the theorem at once when one realizes that in order for a projection of the rational normal curve into $\mathbf{P}^r$ to acquire an $\alpha$-fold cusp at $p$, the projection center, which has dimension $n - r - 1$, must lie in

$$\sigma_{\underbrace{r,r,\ldots,r}_{\alpha},0,\ldots,0},$$

which has codimension $\alpha r$. $\square$

Brieskorn has pointed out to me that the image, under the quadratic transformation $(x_0, x_1, x_2) \to (x_1x_2, x_0x_2, x_0x_1)$ of $\mathbf{P}^2$, of a circle tangent to each of the 3 coordinate lines $x_i = 0$ is a quartic with the maximal number $3 = \frac{3}{2}(4-2)$ of cusps (see Figure 3).



To prove the theorem, we need a generalization of the Riemann-Hurewitz formula. Given a point $p$ on a smooth curve $C$, and a linear series $L$ of divisors on $C$, we define integers $\alpha_i^L(p)$ by the formulae

$$i + \alpha_0^L(p) + \cdots + \alpha_i^L(p) = \max\{s \mid sp \text{ imposes} \leqslant i \text{ conditions on elements of } L\}$$

for $i = 0, \ldots, \dim L$. It is easy to see that $\alpha_0^L(p)$ is the multiplicity to which $p$ occurs as a fixed point of $L$, while if $\phi$ is the map $C \to \mathbf{P}^{\dim L}$ associated to $L$, then $\alpha_i^L(p) = \alpha_i^\phi(p)$ for $i \geqslant 1$.

We set $\alpha_i^L = \sum_{p \in C} \alpha_i^L(p)$.

*Generalized Riemann-Hurwitz formula.* If $L$ is a linear system of degree $n$ and dimension $r$ then

$$(r+1)n - \binom{r+1}{2}(2-2g) = \sum_{i=0}^{r} (r-i+1)\alpha_i^L.$$

This becomes the usual Riemann-Hurwitz formula if we apply it to the linear series corresponding to a map $C \to \mathbf{P}^1$. If we apply it to the complete series of canonical divisors on $C$, it becomes the well-known formula counting the Weierstrass points of $C$. It may easily be deduced (after reducing to the fixed-point free case $\alpha_0^L = 0$) by taking a certain linear combination of "general Plücker formulas" as on p. 270 of **[G-H-1]**, and is well known in this form.

PROOF OF THE THEOREM. For any point $p \in C$, the Schubert cycle $\sigma_{1,0,\ldots,0}(p)$, is a hyperplane section of the Grassmannian variety (in its Plücker embedding). To compute the codimension of $\bigcap_1^g \Sigma_i$, it is thus enough to show that the intersection of $\bigcap_1^g \Sigma_i$ with a certain number of varieties of the form $\sigma_{1,0,\ldots,0}(p_j)$ is empty. But this is itself a special case of the theorem! Thus it suffices to show that if $X = \bigcap_i^g \Sigma_i \neq \varnothing$, then

$$\sum_{i=1}^{g} \operatorname{codim} \Sigma_i = \sum_{i,j} a_j(i) \leqslant (n-k)(k+1),$$

the right-hand side being the dimension of the Grassmannian of $k$-planes in $\mathbf{P}^n$.

Suppose that $V \in X$, and consider the linear series $L$ cut out on $C$ by the hyperplanes containing $V$. From the generalized Riemann-Hurwitz formula for $L$, which has degree $n$ and dimension $n-k-1$ we get

$$(n-k)(k+1) = \sum_{i=0}^{n-k-1} (n-k-i)\alpha_i^L.$$

Working with one point $p_i$ at a time, it now suffices to show that if $p \in C$ and $V \in \sigma_{a_0,\ldots,a_k}(p)$, then the linear series of hyperplanes containing $V$ satisfies

$$\sum_0^k a_1 \leqslant \sum_0^{n-k-1} (n-k-i)\alpha_i^L(p).$$

If we set $a_i = 0$ for $i > k$, then the above inequality may be deduced at once if, setting $\alpha_i^L(p) = \alpha_i$, we prove that

$$a_{\alpha_0 + \cdots + \alpha_i} < n-k-i \quad \text{for } i = 0, \ldots, n-k-1,$$

since the sequence of $a_i$ is dominated by the sequence

$$\underbrace{n-k, \ldots, n-k}_{\alpha_0}, \ \underbrace{n-k-1, \ldots, n-k-1}_{\alpha_1}, \ldots.$$

Now fix $i$ in the above range. By the definition of the $\alpha_i$ there is no linear space of dimension $(n-k-1)-i$ of hyperplanes containing $V$ and meeting $C \geqslant i + \alpha_0 + \cdots + \alpha_i + 1$ times at $p$. In other words, the join of $V$ and $\mathcal{F}^{i+\alpha_0+\cdots+\alpha_i}(C, p)$ has dimension $> k + i$, whence

$$\dim\left(V \cap \mathcal{F}^{i+\alpha_0+\cdots+\alpha_i}\right) < \alpha_0 + \cdots + \alpha_i.$$

On the other hand, since $V \in \sigma_{a_0,\ldots,a_k}(p)$ we have

$$\dim\left(V \cap \mathcal{F}^{n-k-a_{\alpha_0+\cdots+\alpha_i}+(\alpha_0+\cdots+\alpha_i)}\right) \geqslant \alpha_0 + \cdots + \alpha_i,$$

so

$$n - k - a_{\alpha_0+\cdots+\alpha_i} + \alpha_0 + \cdots + \alpha_i > i + \alpha_0 + \cdots + \alpha_i,$$

that is, $a_{\alpha_0+\cdots+\alpha_i} < n - k - i$, as required. This completes the proof.

*Note.* After this paper was written, I learned that a dual form of the theorem, with a proof dual to that given here, was discovered by A. Iarrobino in about 1973, but never published by him. Harris and I rediscovered the result while trying to compute the dimensions of spaces of cuspidal rational curves, as in the first corollary.

## References

**[B-N]** A. Brill and M. Noether, *Über die algebraischen Functionen und ihre Anwendungen in der Geometrie*, Math. Ann. **7** (1874), 269–310.

**[C]** G. Castelnuovo, *Numero delle involutione rationali giacenti sopra una curva di dato genere*, Rend. Acad. Lincea Ser. 4 **5** (1889).

**[E-H]** D. Eisenbud and J. Harris, *General linear series and cuspidal rational curves* (in preparation).

**[E-VdV]** D. Eisenbud and A. Van de Ven, *On the variety of smooth rational curves with given degree and normal bundle*, Invent. Math. (to appear).

**[G-H-1]** P. Griffiths and J. Harris, *Principles of algebraic geometry*, Wiley, New York, 1978.

**[G-H-2]** ______, *On the variety of special linear systems on a general algebraic curve*, Duke Math. J. **47** (1980), 233–272.

**[K]** G. Kempf, *Schubert methods with an application to algebraic curves*, Publ. Math. Centrum, Amsterdam, 1971.

**[Kl]** S. Kleiman, *r-special subschemes and an argument of Severi's*, Adv. in Math. **22** (1976), 1–23.

**[K-L]** S. Kleiman and D. Laksov, *On the existence of special divisors*, Amer. J. Math. **94** (1972), 431–436.

**[S]** F. Severi, *Vorlesungen über algebraische Geometrie*, Teubner, Leipzig, 1921.

Brandeis University