Curves Of Almost Maximal Genus

by

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Chapter 3

CASTELNUOVO THEORY

3.a. The classical case

In this chapter we will discuss the "Cartesian" approach to projective curves, starting with the classical theorems of Castelnuovo in this section, and proceeding in sections 3.b and 3.c to two recent theorems on the subject extending those of Castelnuovo.

As indicated, this theory differs markedly in both technique and application from Brill-Noether theory. On the technique side, we will be primarily concerned with the projective geometry of curves C in $\mathbb{P}^r$ — that is, their extrinsic properties, such as their Hilbert functions, or the configurations of their hyperplane sections, rather than intrinsic ones. On the side of applications, as we shall see this approach is most effective in analyzing what may be called "extremal" curves: we will first establish a bound $\pi(d,r)$ on the genus of a (reduced, irreducible, non-degenerate) curve of degree d in $\mathbb{P}^r$; then study the geometry of curves whose genus is "close" to $\pi(d,r)$.

In analyzing curves in projective space, we will focus primarily on their Hilbert functions. In general, the Hilbert function h_X of a subscheme $X \subset \mathbb{P}^r$ is defined by letting $h_X(n)$ be the rank of the n^{th} graded piece of the homogeneous coordinate ring $S_X = \mathbb{C}[X_0,\ldots,X_r]/I_X$ of X ; informally, $h_X(n)$ is the number of conditions imposed by X on hypersurfaces of degree n . The Hilbert function h_X should be contrasted with the Hilbert polynomial $p_X(n) = \chi(\mathcal{O}_X(n))$, which equals h_X for large values of n — a phenomenon equivalent to the fact that for large n , i) $h^0(X,\mathcal{O}_X(n)) = \chi(\mathcal{O}_X(n))$ (i.e., $h^i(X,\mathcal{O}_X(n)) = 0$ for $i > 0$) and ii) hypersurfaces of degree n cut out a complete linear system on X (i.e., $h^1(\mathbb{P}^r,\mathcal{I}_X(n)) = 0$) . In between these two is what may be called the "abstract Hilbert function" $\tilde{h}_X$ of X , defined by

$$\tilde{h}_X(n) = h^0(X,\mathcal{O}(n)) .$$

Inasmuch as the ring

$$\tilde{S}_X = \bigoplus_{n=0}^{\infty} H^0(X,\mathcal{O}(n))$$

is (via the obvious inclusion $S_X \hookrightarrow \tilde{S}_X$) the normalization of S_X in case X is smooth and irreducible, $\tilde{h}_X$ is somewhat better-behaved than h_X , but it is still not in general a polynomial. To illustrate, if $X \subset \mathbb{P}^r$ is reduced and irreducible and of degree d and dimension k we know that

$$\tilde{h}_X(n) = p_X(n)$$

(i.e., $h^i(X,\mathcal{O}_X(n)) = 0$ for $i > 0$) for all $n > [\frac{d-1}{r-k}]$ (cf. [19]); but the best result possible along these lines for the Hilbert function is the theorem of [18] in case $\dim X = 1$:

$$h_X(n) = p_X(n) \qquad \forall n \geq d - r + 1 ;$$

no such sharp result is known in case $\dim X \geq 2$. (Although one may conjecture that $h_X(n) = p_X(n)$ for $n \geq \deg(X) - \operatorname{codim}(X)$ in general; if true, this is sharp.)

To put the above somewhat differently, our difficulty in describing the Hilbert function of a sheme $X \subset \mathbb{P}^r$ is twofold: in effect, we must first describe $\tilde{h}_X(n) = h^0(X,\mathcal{O}(n))$, and then the rank of the restriction map

$$H^0(\mathbb{P}^r,\mathcal{O}(n)) \to H^0(X,\mathcal{O}(n)) .$$

In effect, our approach to the solution of the problem is to reduce to a case where only one of the difficulties presents itself: namely, the case where X is simply a collection of points. We do this by looking at successive hyperplane sections of X , and relating their Hilbert functions to that of X . The basic comparison is the

LEMMA (3.1). *If* $X \subset \mathbb{P}^r$ *is any scheme,* X' *a hyperplane section of* X *(not containing any component of* X *), then*

$$h_X(n) - h_X(n-1) \geq h_{X'}(n) .$$

PROOF. This is quite straightforward: if we denote by $E_n \subset |\mathcal{O}_X(n)|$ the linear series cut on X by hypersurfaces of degree n , then the linear system $E'_n \subset E_n$ of divisors containing the divisor $X' \in |\mathcal{O}_X(1)|$ has codimension exactly $h_{X'}(n)$ in E_n , and contains the subseries $X' + E_{n-1}$; thus

$$\begin{aligned} h_X(n) - h_X(n-1) &= \dim E_n - \dim E_{n-1} \\ &\geq h_{X'}(n) . \end{aligned}$$

One important note here: according to the proof of the Lemma, equality holds in its statement (for any given n) if and only if $E'_n = X' + E_{n-1}$. Since the series E_n is complete for large n , to say that equality holds for *all* n is thus to say that E_n is complete for all n ; i.e. we have

REMARK (3.1.1). *For any* X *, the following are equivalent:*

i) *Equality holds for all* n *in Lemma* (3.1);

ii) *The linear series cut on* X *by hypersurfaces of degree* n *is complete, for all* n ;

iii) $h_X = \tilde{h}_X$;

iv) $S_X = \tilde{S}_X$;

v) $h^1(\mathbb{P}^r, \mathcal{I}_X(n)) = 0$ *for all* n ;

vi) *Every polynomial in* $\mathbb{P}^{r-1}$ *vanishing on* X' *is the restriction of a polynomial in* $\mathbb{P}^r$ *vanishing on* X .

In case X is one-dimensional, this is also equivalent to the condition, " X is arithmetically Cohen-Macaulay".

COROLLARY (3.2). *Let* $C \subset \mathbb{P}^r$ *be a curve of degree* d *and genus* g *and* Γ *a hyperplane section of* C *, not containing any component of* C *. Then*

$$g(C) \leq \sum_{n=1}^{\infty} d - h_\Gamma(n) .$$

PROOF. For large N , we have

$$Nd - g + 1 = h_C(N) \geq 1 + \sum_{n=1}^{N} h_\Gamma(n)$$

and the inequality follows.

We will often be dealing with the successive differences of Hilbert functions; we therefore introduce the notation

$$h'_X(n) = h_X(n) - h_X(n-1) .$$

In these terms, we can rephrase the above as

COROLLARY (3.3). *Let* C *be a curve in* $\mathbb{P}^r$ *of genus* g , Γ *a hyperplane section of* C *not containing a component of* C . *Then*

$$g \le \sum_{n=1}^{\infty} (n-1)h_\Gamma'(n) .$$

PROOF. Since $\sum_{n=0}^{\infty} h_\Gamma'(n) = d$, we have

$$g \le \sum_{n=1}^{\infty} d - h_\Gamma(n) = \sum_{n=1}^{\infty} (d - \sum_{i=0}^{n} h_\Gamma'(i))$$

$$= \sum_{n=1}^{\infty} \sum_{j=n+1}^{\infty} h_\Gamma'(j) = \sum_{n=1}^{\infty} (n-1)h_\Gamma'(n) .$$

At this point, our attention turns to the Hilbert functions of configurations Γ of points in space. To do much, however, we have to use the hypothesis that Γ is a hyperplane section of a curve $C \subset \mathbb{P}^r$; and to make some assumptions about C as well. For the remainder of this section, then, C *will denote a reduced, irreducible and non-degenerate curve in projective space* $\mathbb{P}^r_{\mathbb{C}}$ (note that we do *not* assume C is smooth). Finally, we will take Γ to be a *general* hyperplane section of C ; in particular, this means that Γ is reduced, i.e., consists of $d = \deg(C)$ distinct points.

This said, the key idea here is the *uniform position principle* (cf. [20]): that the points (and subsets) of Γ are indistinguishable from one another. More precisely, if $C \subset \mathbb{P}^r$ and we let $\mathcal{U} \subset \mathbb{P}^{r*}$ be the open set of hyperplanes intersecting C in d distinct points and $\Sigma \subset \mathcal{U} \times C$ the (d-sheeted) covering

$$\Sigma = \{(H;p) : p \in H \cap C\}$$

of C , then the Galois/monodromy group of Σ is the full symmetric group on d letters[1]. (Equivalently, the variety

$$\Sigma^{(d)} = \{(H;p_1,\ldots,p_d) : p_i \text{ distinct} \in H \cap C\} \subset \mathcal{U} \times C \times C \times \ldots \times C$$

is connected.) One immediate consequence of this is the

LEMMA (3.4). *Let* Γ *be a general hyperplane section of a reduced and irreducible curve* $C \subset \mathbb{P}^r$ *of degree* d *. Then for any subset* Γ' *of* d' *points of* Γ *and any* n *,*

$$h_{\Gamma'}(n) = \min(d', h_\Gamma(n)) .$$

Equivalently, one can say that if Γ' fails to impose independent conditions on hypersurfaces of degree n $(h_{\Gamma'}(n) < d')$, then every hypersurface of degree n containing Γ' contains Γ , $(h_{\Gamma'}(n) = h_\Gamma(n))$. In this form Lemma (3.4) follows immediately from the uniform position principle, which implies that *all* subsets of d' points of Γ impose the same number of conditions on $|\mathcal{O}(n)|$. A configuration Γ which satisfies the conclusion of Lemma (3.4) is said to have the *uniform position property*; a proof that the general hyperplane section of a reduced, irreducible curve has the U.P.P. is in [20].

One immediate consequence of Lemma (3.4) is the

COROLLARY (3.5). *Let* Γ *be as in Lemma* (3.4). *Then for any integers* m , n *, one has* $h_\Gamma(m+n) \geq \min(d, h_\Gamma(m)+h_\Gamma(n)-1)$.

PROOF. Assuming $d \geq h_\Gamma(m) + h_\Gamma(n)$, let Γ' (resp. Γ'') be a subset of $h_\Gamma(m)$ (resp. $h_\Gamma(n)$) points of Γ ; Γ' and Γ'' should have exactly one point p in common. By hypothesis, then, there exists a polynomial of degree m (resp. n) vanishing on all the points of Γ' (resp. Γ'') *except* p , and not

[1] This is known only in characteristic 0 , whence our assumption $k = \mathbb{C}$.

at p ; thus there exists a polynomial of degree $n+m$ vanishing at all the points of $\Gamma' \cup \Gamma'' - \{p\}$, but not at p . The case $d < h_\Gamma(n) + h_\Gamma(m)$

A special case of Corollary (3.5) is

COROLLARY (3.6). *Let* C *be a reduced, irreducible and non-degenerate curve of degree* d *in* $\mathbb{P}^r$ *, and* Γ *its general hyperplane section. Then for each* $n \geq 1$ *, either* $h_\Gamma(n) = d$ *or* $h_\Gamma'(n) \geq r-1$.

Here, of course, we are just observing that if C is non-degenerate, so is Γ , so that $h_\Gamma(1) = r$, and applying the previous Corollary. As trivial as this sounds, when combined with Corollary (3.3), however, it yields the famous

THEOREM (3.7) (Castelnuovo's Theorem, Part I). *With* C *as in Lemma (3.6), set* $m = [\frac{d-1}{r-1}]$ *and write* $d = m(r-1) + \varepsilon + 1$ *(so that* $0 \leq \varepsilon \leq r-2$ *). Then the genus* g *of* C *satisfies*

$$g \leq \pi(d,r) = \binom{m}{2}(r-1) + m\varepsilon .$$

Moreover, $g = \pi(d,r)$ *if and only if* C *is arithmetically Cohen-Macaulay and the Hilbert functions of* C *and* Γ *are given by*

$h_\Gamma'(0) = 1$	$h_\Gamma(0) = 1$	$h_C(0) = 1$
$h_\Gamma'(1) = r-1$	$h_\Gamma(1) = r$	$h_C(1) = r+1$
$h_\Gamma'(2) = r-1$	$h_\Gamma(2) = 2r-1$	$h_C(2) = 3r$
$h_\Gamma'(3) = r-1$	$h_\Gamma(3) = 3r-2$	$h_C(3) = 6r-2$
$\vdots$	$\vdots$	$\vdots$
$h_\Gamma'(m) = r-1$	$h_\Gamma(m) = d-\varepsilon$	
$h_\Gamma'(m+1) = \varepsilon$	$h_\Gamma(m+1) = d$.	

PROOF. It is clear that the Hilbert function given above maximizes the expression $\sum(n-1)h'(n)$ of Corollary (3.3), subject to the constraints imposed by Corollary (3.6); the remainder of the statement follows from Remark (3.1.1).

To restate the first part of this theorem, we have

COROLLARY (3.8). *For* $g > \pi(d,r)$,

$$I_{d,g,r} = \emptyset .$$

Note incidentally that $\pi(d,r)$ is roughly a quadratic polynomial in d ; specifically, for any r ,

(3.8.1) $$\pi(d,r) = \frac{d^2}{2(r-1)} + O(d) .$$

Having said this, we now turn our attention to questions of curves C of genus exactly $\pi(d,r)$: do they exist, and if so what do they look like? Again, it turns out that the key is to look at the general hyperplane section Γ of such a curve C . In particular, we may ask whether any configuration of points exists with the minimal Hilbert function specified in (3.7): The answer is that it does; specifically, if $D \subset \mathbb{P}^{r-1}$ is a *rational normal curve* — that is, an irreducible, non-degenerate curve of degree $r - 1$ (the smallest possible degree of such a curve) and, thereby, genus 0 , then whenever $d \geq \ell(r-1) + 1$, a hypersurface of degree ℓ in $\mathbb{P}^{r-1}$ will contain Γ if and only if it contains D , and will contain D if and only if it contains $\ell(r-1) + 1$ points of D ; thus

$$h_\Gamma(\ell) = h_D(\ell) = \ell(r-1) + 1 .$$

The fundamental result now is that these are in fact the only such configurations of points, i.e., LEMMA (3.9) (Castelnuovo's Lemma). *Let* $\Gamma \subset \mathbb{P}^{r-1}$ *be any configuration of* $d \geq 2r+1$ *points in linear general position, and suppose that* $h_\Gamma(2) = 2r-1$ *(equivalently, that* $h'_\Gamma(2) = r-1$ *). Then* Γ *lies on a rational normal curve* $D \subset \mathbb{P}^{r-1}$.

PROOF. This is a special case of Proposition (3.19) below; we will not give a separate proof here.

Once we have Castelnuovo's Lemma, everything falls into place readily. If Γ lies on a rational normal curve D , then since $d = \deg \Gamma > 2(r-1)$, it follows that every quadric containing Γ contains D ; since D is the intersection of the quadrics containing it, it follows that D is the intersection of the quadrics in $\mathbb{P}^{r-1}$ containing Γ ; and since C satisfies the conditions of Remark (3.1.1), it follows in turn that D is the intersection of the quadrics in $\mathbb{P}^r$ containing C with the hyperplane containing Γ . All in all, then, we see that *the intersection of the quadrics containing* C *is a surface* S *whose general hyperplane section is* D .

What kind of surface is S ? Happily, it is of a very simple type, as shown in

PROPOSITION (3.10). *Let* $X \subset \mathbb{P}^r$ *be a reduced, irreducible and non-degenerate variety of dimension* k . *Then* $\deg X \geq r-k+1$; *and if* $\deg X = r-k+1$, *then* X *is either*

i) *a quadric hypersurface (in case* $k = r-1$ *);*

ii) *a cone over the Veronese surface in* $\mathbb{P}^5$ *(only in case* $k = r-3$ *);*

or

iii) *a rational normal scroll.*

A rational normal scroll of dimension k may be characterized either as

i) the variety swept out by the $(k-1)$-planes spanned by corresponding points on k rational normal curves $C_i \subset \mathbb{P}^{a_i}$ in complementary linear subspaces $\mathbb{P}^{a_i} \subset \mathbb{P}^r$ $(\sum a_i = r - k + 1)$;

ii) the image of a projective bundle $\mathbb{P}(\mathcal{O}(-a_1) \oplus \ldots \oplus \mathcal{O}(-a_k))$ over $\mathbb{P}^1$, under the map to projective space given by sections of its (dual) tautological bundle $\mathcal{O}_{\mathbb{P}}(1)$ $(\sum a_i = r - k + 1 ;\ a_i \geq 0)$; or

iii) an irreducible determinantal variety of the form

$$\operatorname{rank}\begin{pmatrix} \ell_{0,0} & \cdots & \ell_{r-k,0} \\ \ell_{0,1} & \cdots & \ell_{r-k,1} \end{pmatrix},$$

where the ℓ_{ij} are linear functionals on $\mathbb{P}^r$.

Scrolls are fascinating varieties in their own right; unfortunately, to give them anything like their due would take us far too long. For a proof of (3.10) (due, apparently, to Bertini) and a discussion of scrolls in a setting related to the present one, see [19] or [24]; other discussions may be found in [2] and [3].

In any event, once we know that a curve C of maximal genus $\pi(d,r)$ must lie on a surface scroll S, its analysis is straightforward. (We consider here only the case of smooth scrolls.) To begin with, the group of divisor classes on S is free of rank 2, having as generators the classes H and R of the hyperplane section and line of the ruling respectively; the intersection pairing is clearly

$$H \cdot H = r - 1 , \qquad H \cdot R = 1 , \qquad R \cdot R = 0 .$$

Applying the adjunction formula to the curves H and R, moreover, we find that the canonical class of S

$$K_S \sim -2H + (r-3)R \ .$$

We see from this that if C is a curve with class $\alpha H + \beta R$ on S, the degree d and genus g of C are given by

$$d = \alpha(r-1) + \beta \ ,$$

$$g = \frac{\alpha(\alpha-1)}{2}(r-1) + (\alpha-1)(\beta-1) \ .$$

One computes that the genus equals $\pi(d,r)$ exactly when $-(r-2) \leq \beta \leq 1$. Thus, we have

THEOREM (3.11) (Castelnuovo's Theorem, Part II). *If* $r \neq 5$, *a curve* $C \subset \mathbb{P}^r$ *of degree* $d \geq 2r+1$ *whose genus equals* $\pi(d,r)$ *lies on a surface scroll, and if that scroll is smooth has class either*

$$C \sim (m+1)H - (r-2-\varepsilon)R \ ,$$

or (in case $\varepsilon = 0$ *) possibly*

$$C \sim mH + R \ .$$

If $r = 5$ *and* d *is even there is the further possibility that* C *lies on the Veronese surface.*

We will not go into any of the many corollaries of this theorem (some will be discussed following the proof of Theorem (3.15), which generalizes Castelnuovo's Theorem), except to indicate what it says about $I_{d,g,r}$. To see this, we observe first that *the space of rational normal surface scrolls* $S \subset \mathbb{P}^r$ *is irreducible of dimension* $(r+3)(r-1) - 3$. This follows readily from the determinantal description of scrolls above: the space of $2 \times (r-1)$ matrices of linear forms on $\mathbb{P}^r$ is a vector space of dimension $2(r-1)(r+1)$; and we may multiply such a matrix either on the right by an $(r-1) \times (r-1)$ or on the left by a 2×2 matrix of scalars without affecting the associated scroll, so that we have in fact

$$2(r-1)(r+1) - (r-1)^2 - 4 + 1$$

degrees of freedom left. It is a nice exercise to see that the scroll determines the matrix up to such multiplications. Secondly, since (as one can see without too much difficulty) any smooth curve on a rational surface scroll other than a section is a non-special divisor, by our previous calculations and the Riemann-Roch formula for surfaces, if C is a curve of class $C \sim \alpha H + \beta R$ on S with $\alpha \geq 2$, the complete linear system $|\mathcal{O}_S(C)|$ has dimension

$$\frac{\alpha(\alpha+1)}{2}(r-1) + (\alpha+1)(\beta+1) - 1 .$$

Since, finally, a curve $C \subset \mathbb{P}^r$ of maximal genus and degree $d \geq 2r+1$ lies on a unique scroll S, we may conclude:

COROLLARY (3.12). *Let* $\pi = \pi(d,r)$. *Then* $I_{d,g,r} = \emptyset$ *for* $g > \pi$; $I_{d,\pi,r} = I'_{d,\pi,r}$ *; and we have for* $d \geq 2r+1$ *:*

i) *If* $r \neq 5$ *and* $\varepsilon \neq 0$, $I_{d,\pi,r}$ *is irreducible of dimension*

$$\dim I_{d,\pi,r} = \left(\frac{m(m+1)}{2} + r + 2\right)(r-1) + (m+2)(\varepsilon+2) - 4 ;$$

ii) *If* $r \neq 3$, *and* $\varepsilon = 0$, $I_{d,\pi,r}$ *has exactly two irreducible components* $I^1_{d,\pi,r}$ *and* $I^2_{d,\pi,r}$. *The first has dimension given by the formula above, while*

$$\dim I^2_{d,\pi,r} = \left(\frac{m(m+1)}{2} + r + 3\right)(r-1) + 2m - 2 ;$$

iii) *In case* $r = 5$, *the situation is as above, except that in case* $d = 2k$ *is even there is an additional component of* $I_{d,\pi,5}$ *of dimension*

$27 + \frac{1}{2}k(k+3)$, *consisting of curves on Veronese surfaces.*

Note that in case ii) the dimension of $I^2_{d,\pi,r}$ is $(r-3)$ greater than that of $I^1_{d,\pi,r}$; while in case $r = 5$ and d is even, the new component has dimension 3 less than the old one in case $d \equiv 2\ (4)$, 2 less in case $d \equiv 0\ (4)$. Note also that in case the rational normal scroll S degenerates to a cone S_0 over a rational normal curve, the two classes $mH + R$ and $(m+1)H - (r-2)R$ become indistinguishable (both are simply residual to $(r-2)$ lines of the ruling of S in a complete intersection of S_0 with a hypersurface of degree $m + 1$). Indeed, it is a nice exercise to check that there exist smooth curves C on such cones S_0 whose Hilbert points lie on both components of $I_{d,\pi,r}$, furnishing the only explicit example known to the authors of a smooth curve lying on more than one component of the Hilbert scheme (or, for that matter, of a smooth curve $C \subset \mathbb{P}^r$ whose local deformation space is reducible).

3.b. The next step

As beautiful and powerful as Castelnuovo's line of argument is in the second part of his theorem, it may seem somewhat disappointing that it applies to curves of only one genus π for each d and r . Of course, we cannot hope that this approach is universally effective: as perhaps indicated by some of the open questions at the end of chapter 2, to try and describe the Hilbert functions of projective curves, and then to use this information to give explicit descriptions of the families of such curves is, in general, far beyond our means. Nonetheless, it does not seem unreasonable to ask whether such an analysis might be effectively applied to curves "near" extremal ones — that is, of relatively high genus; and this is what we shall undertake in the remainder of this chapter.

To do this, recall that the proof of the first part of Castelnuovo's theorem is based essentially on two inequalities: the inequality

$$h_\Gamma'(n) \geq r - 1$$

for n such that $h_\Gamma(n) < d$; which implies that

$$h_\Gamma(n) \geq \min(d, n(r-1)+1)$$

expressed in (3.6); and the inequality

$$h_C'(n) \geq h_\Gamma(n)$$

expressed in (3.1). Of course, if C is extremal, both these inequalities are equalities, and it is from these facts that we derive the conclusions of Part II of Castelnuovo's theorem. What we want to ask now, is, how may equality *fail* to hold in these inequalities?

To take the second first, the failure of equality to hold in (3.1) is in general very mysterious. It is measured by the Hartshorne-Rao module

$$M(C) = \bigoplus_n H^1(\mathbb{P}^r, \mathcal{I}_C(n)) \;;$$

but despite the progress made by Hartshorne [23], Rao [37], and others, we are still far away from being able to say, for example, what may be the Hilbert function of the Hartshorne-Rao module of an irreducible, reduced curve of degree d and genus g in $\mathbb{P}^r$.

It turns out to be much more rewarding to look at (3.6). The question here is, in its broadest form,

What may be the Hilbert function of a general hyperplane section of a reduced, irreducible curve?

One problem in answering this is that no one has any idea in general when a given configuration of points $\Gamma \subset P^{r-1}$ is the hyperplane section of an irreducible curve $C \subset P^r$; so that it may be more practical to ask, simply,

What may be the Hilbert function of a collection $\Gamma \subset P^{r-1}$ of points with the uniform position property?

(It would be very interesting to know if indeed these two questions have the same answer.)

This problem, as stated, seems still well beyond our reach in general. Nonetheless, quite a bit can be said, especially about Hilbert functions close to the extremal one. To see how this works, consider the case of a curve $C \subset P^3$, of degree $d = 6k + 1$ (for convenience), with general hyperplane section Γ . If C is extremal, then Γ lies on a conic curve, and has Hilbert function

$$(*)\qquad \begin{array}{ll} h'_\Gamma(0) = 1 & h_\Gamma(0) = 1 \\ h'_\Gamma(1) = 2 & h_\Gamma(1) = 3 \\ h'_\Gamma(2) = 2 & h_\Gamma(2) = 5 \\ \vdots & \vdots \\ h'_\Gamma(3k) = 2 & h_\Gamma(3k) = 6k+1 \\ h'_\Gamma(3k+1) = 0 & h_\Gamma(3k+1) = 6k+1 \\ \vdots & \vdots \end{array}$$

Again, the fact that Γ lies on a conic determines the whole Hilbert function and is itself determined by the value $h_\Gamma(2) = 5$. Now we ask: suppose $h_\Gamma(2) > 5$ — or, equivalently, suppose Γ does not lie on a conic? If, for example, Γ lies on a cubic curve D (which, by the uniform position property, must be irreducible if $d > 7$ and Γ lies on no conic), then in fact h_Γ looks quite different: since by Bezout any curve of degree $2k$ or less containing contains D, we find that

$$
(**)\qquad
\begin{array}{ll}
h'_\Gamma(0) = 1 & h_\Gamma(0) = 1 \\
h'_\Gamma(1) = 2 & h_\Gamma(1) = 3 \\
h'_\Gamma(2) = 3 & h_\Gamma(2) = 6 \\
h'_\Gamma(3) = 3 & h_\Gamma(3) = 9 \\
\vdots & \vdots \\
h'_\Gamma(2k-1) = 3 & h_\Gamma(2k-1) = 6k-3 \\
h'_\Gamma(2k) = 3 & h_\Gamma(2k) = 6k \\
h'_\Gamma(2k+1) = 1 & h_\Gamma(2k+1) = 6k+1 \\
h'_\Gamma(2k+2) = 0 & h_\Gamma(2k+2) = 6k+1\ .
\end{array}
$$

If, on the other hand, Γ lies on no cubic we have

$$
\begin{aligned}
h_\Gamma(1) &= 3 \\
h_\Gamma(2) &= 6 \\
h_\Gamma(3) &= 10
\end{aligned}
$$

and by applying the basic inequality (3.5), we see that

$$
\begin{aligned}
h_\Gamma(4) &\geq 12 \\
h_\Gamma(5) &\geq 15 \\
h_\Gamma(6) &\geq 19 \\
&\vdots
\end{aligned}
$$

i.e., h_Γ is larger than in case Γ lies on a cubic. The conclusion is striking: if the Hilbert function of Γ is greater than (*), it must be greater than or equal to (**) . Plugging the formula (**) for $h^{\prime}_{\Gamma}$ into the expression in (3.3) (and doing the same for other possible congruences of d mod 3), we arrive at a classical theorem of Halphen:

THEOREM (3.13). *Let* $C \subset \mathbb{P}^3$ *be a reduced and irreducible curve of degree* d *and genus* g *not lying on a quadric surface. Then*

$$g \le \pi_1(d,3) = \begin{cases} \dfrac{d^2}{6} - \dfrac{d}{2} + 1 , & d \equiv 0 \ (3) \\[2ex] \dfrac{d^2}{6} - \dfrac{d}{2} + \dfrac{1}{3} , & d \not\equiv 0 \ (3) \ . \end{cases}$$

PROOF. We have established the inequality in the statement of the theorem under the hypothesis that the general hyperplane section Γ of C does not lie on a conic. If C is linearly normal (i.e., $h^1(\mathbb{P}^3,\mathcal{I}_C(1)) = 0$), by the exact sequence

$$0 \to \mathcal{I}_{C,\mathbb{P}^3}(1) \to \mathcal{I}_{C,\mathbb{P}^3}(2) \to \mathcal{I}_{\Gamma,\mathbb{P}^2}(2) \to 0$$

this is equivalent to the condition that C not lie on a quadric. If, on the other hand, C is not linearly normal — i.e., C is the biregular projection of a non-degenerate curve in $\mathbb{P}^4$ — then $g \le \pi(d,4)$, which is less than the bound given.

COROLLARY (3.14). *Suppose* $\pi(d,3) \ge g > \pi_1(d,3)$. *Then* $\mathcal{I}_{d,g,3} \ne \emptyset$ *if and only if there exist integers* $a \ge b \ge 0$ *such that* $g = (a-1)(b-1)$ *and* $d = a + b$ *; and if there do exist such integers, then* $\mathcal{I}_{d,g,3}$ *is irreducible of dimension* $ab + a + b + 9$.

Note that this provides us with our first examples of d , g and r with $g \le \pi(d,r)$ such that $\mathcal{I}_{d,g,r} = \emptyset$. In fact, these are the only examples in $\mathbb{P}^3$: the theorem of Gruson-Peskine [17] states that $\mathcal{I}_{d,g,3} \ne \emptyset$ if and only if $g \le \pi_1(d,r)$ or $g \le \pi(d,r)$ and there exist integers $a,b \ge 0$ such that $d = a + b$, $g = (a-1)(b-1)$.

Halphen's theorem points the way clearly toward generalizations for curves in $\mathbb{P}^r$. The idea is simply to ask, what is the smallest possible Hilbert function of a collection of points in uniform position other than the one of (3.7), and can we describe geometrically those configurations Γ that achieve it? The answer is in a form directly analogous to Halphen's for $\Gamma \subset \mathbb{P}^2$: as we will prove below, the lowest possible Hilbert function after (3.7) is achieved by collections of points lying on *elliptic normal curves* $D \subset \mathbb{P}^{r-1}$, where by *elliptic normal curve* we mean an irreducible, non-degenerate curve in $\mathbb{P}^{r-1}$ which is of degree r, genus 1 and linearly normal (any two of the last three conditions implies the third). The Hilbert function of such a collection of points is

$$
(3.14.1)\qquad
\begin{array}{ll}
h'_\Gamma(0) = 1 & h_\Gamma(0) = 1 \\
h'_\Gamma(1) = r - 1 & h_\Gamma(1) = r \\
h'_\Gamma(2) = r & h_\Gamma(2) = 2r \\
 & h_\Gamma(3) = 3r \\
\vdots & \vdots \\
h'_\Gamma(m_1) = r & h_\Gamma(m_1) = m_1 r = d - \varepsilon_1 - 1 \\
h'_\Gamma(m_1+1) = \varepsilon_1 + 1 & h_\Gamma(m_1+1) = d \\
h'_\Gamma(m_1+2) = 0 & h_\Gamma(m_1+2) = d \\
\vdots & \vdots
\end{array}
$$

where

$$m_1 = [\frac{d-1}{r}]$$

and

$$\varepsilon_1 = d - 1 - m_1 r ,$$

except in case $\varepsilon_1 = r - 1$, where we may have $h_\Gamma^!(m_1+1) = r - 1$, $h_\Gamma^!(m_1+2) = 1$ if Γ is the complete intersection of an elliptic normal curve with a hypersurface of degree m_1 . (It is a good exercise to show that the Hilbert function of a configuration of points lying on a rational curve of degree r in P^{r-1} is close to, but strictly greater than, this.) Plugging this into (3.3), we arrive at the statement of

THEOREM (3.15). *For any* d *and* $r \geq 4$, *set*

$$m_1 = [\frac{d-1}{r}] , \qquad \varepsilon_1 = d - m_1 r - 1 ,$$

$$\mu_1 = \begin{cases} 1 & \text{if } \varepsilon_1 = r - 1 \\ 0 & \text{if } \varepsilon_1 \neq r - 1 , \end{cases}$$

$$\pi_1(d,r) = \binom{m_1}{2} r + m_1(\varepsilon_1+1) + \mu_1 ;$$

let m , ε *and* π *be as in the statement of* (3.7). *Then, for any irreducible, reduced and non-degenerate curve* $C \subset P^r$ *of degree* d *and genus* g ,

i) *if* $g > \pi_1(d,r)$ *and* $d \geq 2r + 1$, *then* C *lies on a surface of degree* $r - 1$; *and*

ii) *if* $g = \pi_1(d,r)$ *and* $d \geq 2r + 3$, *then* C *lies on a surface of degree* r *or less.*

Before launching into the proof of this theorem, let us list some of the consequences.

COROLLARY (3.16). *Suppose* $r \neq 3$ *or* 5, *and* $\pi(d,r) \geq g > \pi_1(d,r)$. *Then* $I_{d,g,r} = I'_{d,g,r}$ *and* $I_{d,g,r} \neq \emptyset$ *if and only if there exist integers* $\alpha > 0$ *and* β *such that*

$$(*)\qquad \begin{cases} d = \alpha(r-1) + \beta \\ g = \binom{\alpha}{2}(r-1) + (\alpha-1)(\beta-1) . \end{cases}$$

Moreover, $I_{d,g,r}$ *has exactly one irreducible component for each such pair* (α,β), *of dimension*

$$\left(\frac{\alpha(\alpha+1)}{2} + r + 3\right)(r-1) + (\alpha+1)(\beta+1) - 4 .$$

PROOF. The only thing that needs to be checked is that for any α, β satisfying $(*)$, there exist smooth curves with class $\alpha H + \beta R$ on some scroll in $\mathbb{P}^r$. But now by relatively elementary considerations, cf. [24]

$\exists$ smooth, irreducible curves

$$C \sim \alpha H + \beta R \quad \text{on some scroll in } \mathbb{P}^r$$

$\iff$ $\exists$ reduced, irreducible curves

$$C \sim \alpha H + \beta R \quad \text{on some scroll in } \mathbb{P}^r$$

$\iff$ $\alpha[\frac{n-1}{2}] + \beta \geq 0$.

Thus, we simply have to show that any α, β satisfying $(*)$ satisfy this further condition. In case r is odd, this is easy: $[\frac{r-1}{2}] = (\frac{r-1}{2})$, and we have

$$0 \leq g = (\alpha-1)[\alpha(\tfrac{r-1}{2}) + \beta - 1] .$$

Suppose now that $r = 2k$ is even, and

$$\beta = -\alpha(k-1) - \beta' , \qquad \beta' > 0 .$$

Then

$$g = (\alpha-1)[\alpha(\tfrac{n-1}{2}) + \beta - 1]$$

$$= (\alpha-1)(\tfrac{\alpha}{2} - \beta' - 2)$$

$$= \frac{(\alpha-1)(\alpha-4)}{2} - \beta'(\alpha-1) .$$

But we also have

$$d = \alpha(r-1) + \beta = \alpha k - \beta'$$

and so

$$g \geq \pi_1(d,r) \geq \frac{(d-r)^2}{2r}$$

$$= \frac{((\alpha-2)k - \beta')^2}{4k}$$

$$= \frac{k}{4}(\alpha-2)^2 - \frac{\beta'(\alpha-2)}{2} + \frac{\beta'^2}{4k}$$

and since $k \geq 2$, this is a contradiction.

NOTE. In case $r = 5$, we simply have to add a component of $I_{d,g,5}$ corresponding to curves on the Veronese, whenever $d = 2k$, $g = \frac{(k-1)(k-2)}{2}$.

COROLLARY (3.17). *If* $C \subset P^r$ *is a reduced, irreducible and non-degenerate curve of degree* d *and genus* g *and* C *is arithmetically Cohen-Macaulay, then*

$$g > \pi_1(d,r) \Rightarrow g = \pi(d,r) .$$

REMARK. This reflects the fact that all curves in the genus range $\pi_1 < g \leq \pi$ have hyperplane sections with the same (minimal) postulation; the difference in their genera is accounted for entirely by their failure to be arithmetically Cohen-Macaulay.

Part of the force of part ii) of Theorem (3.15) stems from the fact that surfaces of degree r in $\mathbb{P}^r$ are relatively few. In fact, any such surface must be either i) the projection of a rational normal scroll from $\mathbb{P}^{r+1}$; ii) a cone over a curve of degree r ; or iii) a del Pezzo surface (which we may here define to be a rational surface with elliptic normal hyperplane section)[1]. Moreover, since $\pi_1(d,r) \geq \pi(d,r+1) + [\frac{d}{r}]$, a curve C of genus $g = \pi_1(d,r)$ cannot lie on the projection of a scroll unless it has a double line and it is a classical theorem that *there are no del Pezzo surfaces in* $\mathbb{P}^r$ *for* $r \geq 10$ except for those with a double line. Thus, for $r \geq 10$ a curve of genus $\pi_1(d,r)$ must lie either on a scroll, a cone over a curve of degree r , or on a del Pezzo with a double line; and we may see that

COROLLARY (3.18). *Suppose* $r \geq 10$, $d \geq 2r + 3$, *and* $g = \pi_1(d,r)$. *Then there is one component* $I^1_{\alpha\beta}$ *of* $I_{d,g,r}$ *for every pair of integers* α , β *satisfying* (*) ; *the general member of* $I^1_{\alpha\beta}$ *is a smooth curve lying on the projection of a scroll. There is one component* I^2 *of* $I_{d,g,r}$ *whose general member lies on a cone, and has an ordinary* (ε_1+1)*-fold point at the vertex of that cone. Finally, in case* $\varepsilon = r - 3$, $r - 2$, $r - 1$ *or* 0 , *there is a further component* I^3 *of* $I_{d,g,r}$ *whose general member lies on a del Pezzo with a double line, and has* m_1 , m_1 , $m_1 + 1$ *and* m_1 *ordinary nodes, respectively.*

Note in particular that the general member of I^2 (in case $\varepsilon \neq 0,1$) or I^3 cannot be smoothed in $\mathbb{P}^r$, even though its singularities are abstractly smoothable.

[1] Note that a surface $S \subset \mathbb{P}^r$ of degree r whose general hyperplane section is a singular curve of genus 1 — in our terms, a del Pezzo surface with a double line — is also the projection of a scroll in $\mathbb{P}^{r+1}$. In the present setting, it makes sense to think of it as a del Pezzo, since e.g. $\omega_S = \mathcal{O}_S(-1)$.

Finally, we remark that in contrast to (3.8.1),

$$\pi_1(d,r) = \frac{d^2}{2r} + O(d)$$

for any r.

Let us turn now to the proof of Theorem (3.15). This consists of essentially two steps, the first of which is the following proposition (so as not to carry around too many -1's needlessly, we express this in terms of points in $\mathbb{P}^n$).

PROPOSITION (3.19). *Let* $\Gamma \subset \mathbb{P}^n$ *be a collection of* $d \geq 2n + 1 + 2m$ *points in uniform position, spanning* $\mathbb{P}^n$. *If* $h_\Gamma(2) \leq 2n + m$, *then* Γ *lies on an* m*-dimensional rational normal scroll.*

PROOF. The proof will consist of a claim, a construction, and four more claims.

CLAIM A. *Let* $\Lambda = \overline{p_1,\ldots,p_{n-1}} \cong \mathbb{P}^{n-2}$. *Then there are* $n - m$ *linearly independent quadrics in* $\mathbb{P}^n$ *containing* $\Gamma \cup \Lambda$.

PROOF. There are exactly $2n + 1$ independent quadrics in $\mathbb{P}^n$ containing Λ, and any such quadric contains Γ as well $\Longleftrightarrow$ it contains $p_n,\ldots,p_{2n+m}$; so there are at least $2n + 1 - (n+m+1) = n - m$ quadrics containing $\Gamma \cup \Lambda$.

CONSTRUCTION. Let L_o be the hyperplane spanned by $p_1,\ldots,p_n$, M_o the hyperplane spanned by $p_1,\ldots,p_{n-1},p_{n+1}$ [1]. Let $Q_1,\ldots,Q_{n-m}$ be independent quadrics containing $\Gamma \cup \Lambda$. Then since $L_o \cap M_o = \Lambda$, we can write

$$Q_i = \begin{vmatrix} L_o & L_i \\ M_o & M_i \end{vmatrix},$$

[1] We will abuse notation here and use the same letter to denote a polynomial and the hypersurface it determines.

where L_i , M_i are linear forms on P^n . Now, let $X \subset P^n$ be the locus

$$X = \{\mathrm{rank}\begin{pmatrix} L_o & \dots & L_{n-m} \\ M_o & \dots & M_{n-m} \end{pmatrix} \leq 1\} .$$

CLAIM B. *For any* $(\beta_o,\beta_1) \neq (0,0)$, *the linear forms* $\{\beta_o L_i + \beta_1 M_i\}_{i=0,\dots,n-m}$ *are independent.*

PROOF. Since $Q_1,\dots,Q_{n-m}$ are independent, for any $\alpha_1,\dots,\alpha_{n-m} \neq (0,\dots,0)$ the quadric

$$Q_\alpha = \sum \alpha_i Q_i = \begin{vmatrix} L_o & \sum_{i=1}^{n-m} \alpha_i L_i \\ M_o & \sum_{i=1}^{n-m} \alpha_i M_i \end{vmatrix}$$

is non-zero, and of course Q_α contains Γ . If

$$\sum_{\alpha=1}^{n-m} \alpha_i(\beta_o L_i + \beta_1 M_i) = \alpha_o(\beta_o L_o + \beta_1 M_o)$$

for some α_o , β_o , β_1 , however, Q_α would have rank 2 , and no reducible quadric can contain Γ .

CLAIM C. *No linear combination of* $L_1,\dots,L_{n-m}$ *contains* Λ .

PROOF. Since $L_o(p_n) = 0$ and $M_o(p_n) = 0$, every L_i must contain p_n . If some linear combination of $L_1,\dots,L_{n-m}$ contained Λ , it would have to be equal to $\overline{\Lambda,p_n} = \overline{p_1,\dots,p_n} = L_o$, contradicting the independence of $L_o,\dots,L_{n-m}$.

NOTE. By Claim C , the points $p_1,\dots,p_{n-1}$ impose $n-m$ independent conditions on $\{L_1,\dots,L_{n-m}\}$. We may then assume $p_1,\dots,p_{n-m}$ do, and choose the L_i so that

$$L_i \ni p_1, \ldots, \hat{p}_i, \ldots, p_{n-m} .$$

CLAIM D. X *is a rational normal scroll of dimension* m .

PROOF. This follows from Claim B : a priori

$$X = \bigcup_{[\beta_o, \beta_1] \in \mathbb{P}^1} \Gamma_\beta$$

where

$$\Gamma_\beta = \bigcap_{i=0}^{n-m} (\beta_o L_i + \beta_1 M_i = 0)$$

and by Claim B, for every β , Γ_β is an $(m-1)$-plane. Thus X is irreducible of dimension m ; that it is a scroll follows.

CLAIM E. X *contains* Γ .

PROOF. Clearly X contains $p_n, \ldots, p_d \in \Gamma$: by general position, for $\alpha \geq r$ we have

$$(L_o(p_\alpha), M_o(p_\alpha)) \neq (0,0)$$

and hence

$$\begin{vmatrix} L_o(p_\alpha) & L_i(p_\alpha) \\ M_o(p_\alpha) & M_i(p_\alpha) \end{vmatrix} = 0 \ \forall\, i \Rightarrow \begin{vmatrix} L_i(p_\alpha) & L_j(p_\alpha) \\ M_i(p_\alpha) & M_j(p_\alpha) \end{vmatrix} = 0 \ \forall\, i,j .$$

To see that X contains $p_1, \ldots, p_{n-1}$ as well, note that by the choice of L_i made following Claim C , the quadric

$$Q_i = \begin{vmatrix} L_i & L_j \\ M_i & M_j \end{vmatrix}$$

contains $p_1,\ldots,\hat{p}_i,\ldots,\hat{p}_j,\ldots,p_{n-m}$ as well as $p_n,\ldots,p_d$. This is a total of

$$n - m - 2 + d - n + 1 = d - m - 1 \geq 2n + m$$

points. But $h_\Gamma(2) = 2n + m$, and so by the hypothesis of uniform position, we may conclude that Q_{ij} contains Γ ; and the claim — and the proof of Proposition (3.19) — follow.

Note that in case $m = 1$, this is just the classical Castelnuovo Lemma (3.9). The case we are concerned with at present, however, is the case $m = 2$: as indicated in the discussion preceding (3.15), we want to argue that if $h_\Gamma(2) = 2n + 2$, then Γ lies on a curve of degree $n + 1$. Proposition (3.19) assures us that such a Γ lies on a surface scroll; the second part of the proof of (3.15) is thus the

PROPOSITION (3.20). *Let* $X \subset \mathbb{P}^n$ *be a rational normal surface scroll,* $n \geq 3$, *and* $\Gamma \subset X$ *a collection of* $d \geq 2n + 5$ *points in uniform position. Then if* $h_\Gamma(2) = 2n + 2$, Γ *lies on an elliptic normal curve.*

PROOF. The proof consists of a rather protracted analysis of the linear system cut on X by the quadrics in $\mathbb{P}^r$ containing Γ , with the ultimate goal of establishing that this linear series has a fixed component. To begin with, we need the

LEMMA (3.21). *Let* $\mathcal{E}$ *be a linear system of projective dimension* $s \geq 2$ *on a smooth surface* S , *having* e *distinct base points and no fixed components; let* $E \in \mathcal{E}$ *be a general member of this series and* g *its genus.*

i) *If* E *is reducible, we have*

$$e \leq \tfrac{1}{4}(E \cdot E) .$$

ii) *If* E *is irreducible, then either*

$$\text{a)}\quad e \leq (E \cdot E) - g - s + 1$$

and

$$e \leq (E \cdot E) - 2g - 1\ ;$$

or

$$\text{b)}\quad e \leq (E \cdot E) - 2s + 2\ .$$

PROOF. First of all, if E has $m \geq 2$ components, then since every component of E must pass through every base point of $\mathcal{E}$ we have $(E \cdot E) \geq m^2 e$ and i) follows.

Assume now that E is irreducible. Make a sequence of blow-ups of S at base points of $\mathcal{E}$ that are singular points of E to obtain a surface

$$\tilde{S} \xrightarrow{\eta} S$$

such that the general element $\tilde{E}$ of the proper transform $\tilde{\mathcal{E}}$ of $\mathcal{E}$ is smooth, and consider the linear system cut on $\tilde{E}$ by $\tilde{\mathcal{E}}$. This has dimension $s - 1$ and degree at most $(\tilde{E} \cdot \tilde{E}) - \tilde{e}$, where $\tilde{e}$ is the number of distinct base points of $\tilde{\mathcal{E}}$. Since $\tilde{E}$ is smooth and irreducible, if the genus of $\tilde{E}$ is $\tilde{g}$, *either*

a) $\deg \tilde{\mathcal{E}}|_{\tilde{E}} > 2\tilde{g}$, so

$$(\tilde{E} \cdot \tilde{E}) - \tilde{e} \geq \deg \tilde{\mathcal{E}}|_{\tilde{E}} \geq 2\tilde{g} + 1$$

and by Riemann-Roch

$$s - 1 \leq (\tilde{E} \cdot \tilde{E}) - \tilde{e} - \tilde{g}\ ;$$

or

b) $\deg \tilde{\mathcal{E}}|_{\tilde{E}} \leq 2\tilde{g}$, in which case by Clifford's theorem,

$$(\tilde{E}\cdot\tilde{E}) - \tilde{e} \geq 2(s-1) \; .$$

But now every time we blow up an m-fold base point of the linear series E and take the proper transform, $(E\cdot E)$ decreases by m^2, the genus decreases by $m(m-1)/2$, and e decreases by at most 1, thus

$$(\tilde{E}\cdot\tilde{E}) - \tilde{e} \leq (E\cdot E) - e$$

$$(\tilde{E}\cdot\tilde{E}) - 2\tilde{g} - \tilde{e} \leq (E\cdot E) - 2g - e$$

and

$$(\tilde{E}\cdot\tilde{E}) - \tilde{g} - \tilde{e} \leq (E\cdot E) - g - e$$

and the lemma follows.

Applying Lemma (3.21) to a rational normal surface scroll X in $\mathbb{P}^n$, we find that

If $E \subset |\alpha H + \beta R|$ is a linear system of projective dimension s, having e distinct base points and no fixed components, then *either*

i) $e \leq \frac{1}{4}(\alpha^2(n-1) + 2\alpha\beta)$;

iia) $e \leq \frac{\alpha(\alpha+1)}{2}(r-1) + (\alpha+1)(\beta+1) - s - 1$

and

$$e \leq \alpha(n-1) + 2\alpha + 2\beta - 3 \; ;$$

or

iii) $e \leq \alpha^2(n-1) + 2\alpha\beta - 2s + 2$.

Now, with $\Gamma \subset X$ as in the statement of Proposition (3.20)[1], let $\mathcal{D} \subset |2H|$ be the linear series cut on X by the quadrics in $\mathbb{P}^n$ containing Γ. Since $h_X(2) = 3n$ (cf. Proposition (3.23) below), $\mathcal{D}$ has projective dimension

$$\dim \mathcal{D} = h_X(2) - h_\Gamma(2) - 1 = n - 3 .$$

Let D_o be the fixed part of $\mathcal{D}$, $\mathcal{E}$ the variable part, and say

$$\mathcal{E} \sim \alpha H + \beta R$$

$$D_o \sim (2-\alpha)H - \beta R .$$

(Note that since $\mathcal{E}$ and D_o are effective, $0 \leq \alpha \leq 2$.) We will show that $\alpha = 0$, $\beta = n - 3$, and $\Gamma \subset D_o$. To do this, let e be the number of points of Γ that are base points of $\mathcal{E}$, and f the number of points of Γ lying on D_o . We now proceed by cases.

CASE I. $\alpha = 2$. In this case, note first that $\beta \leq 0$, and since $D_o \sim -\beta R$ consists of $-\beta$ lines, by uniform position $f \leq -2\beta$. Applying Lemma (3.21) to $\mathcal{E}$, we see that either

i) $e \leq \frac{1}{4}(4(n-1) + 4\beta)$

$\quad = n - 1 + \beta$;

ii) $e \leq 2(n-1) + 2\beta - 1$;

or

iii) $e \leq 4(n-1) + 4\beta - 2(n-3) + 2$

$\quad = 2n + 4 + 4\beta$

and applying Riemann-Roch to $\mathcal{E}$,

[1] In case X is singular (i.e., a cone) this argument should take place on its desingularization.

$$\frac{\alpha(\alpha+1)}{2}(n-1) + (\alpha+1)(\beta+1) - 1 \geq n - 3$$

$$\Rightarrow \beta \geq \frac{-2n+1}{3} .$$

Now in Case I.i) we have

$$\begin{aligned} d \leq e + f &\leq n - 1 + \beta - 2\beta \\ &\leq n - 1 + \frac{2n-1}{3} \\ &< 2n ; \end{aligned}$$

in Case I.ii) we have

$$\begin{aligned} d \leq e + f &\leq 2(n-1) + 2\beta - 1 - 2\beta \\ &= 2n - 3 \end{aligned}$$

and in Case I.iii)

$$\begin{aligned} d \leq e + f &\leq 2n + 4 + 4\beta - 2\beta \\ &\leq 2n + 4 , \end{aligned}$$

in any case a contradiction since $d \geq 2n + 5$.

CASE II. $\alpha = 1$, $\beta \geq 0$. Note that here $D_o \sim H - \beta R$ lies in a linear subspace of dimension 2 at most $n - \beta - 1 < n$, so $f \leq n - \beta$. By sub-cases, then, we have by (3.21) either

i) $e \leq \frac{1}{4}(n-1+2\beta)$

ii) $e \leq (n-1) + 2(\beta+1) - (n-3) - 1$
$= 2\beta + 3$

or

iii) $e \le (n-1) + 2\beta - 2(n-3) + 2$

$= 2\beta - n + 7$.

Noting that since $\deg D_0 = n - 1 - \beta \ge 0$,

$$\beta \le n - 1 ,$$

we have in Case II.i)

$$d \le e + f \le \frac{n-1}{4} + \frac{\beta}{2} + n - \beta \le 2n ;$$

in Case II.ii)

$$\begin{aligned} d \le e + f &\le 2\beta + 3 + n - \beta \\ &= \beta + n + 3 \\ &\le 2n + 2 \end{aligned}$$

and in Case II.iii)

$$\begin{aligned} d \le e + f &\le 2\beta - n + 7 + n - \beta \\ &\le \beta + 7 \\ &\le n + 6 < 2n + 5 \end{aligned}$$

so again we have a contradiction.

CASE III. $\alpha = 1$, $\beta < 0$. Here we note first that by Riemann-Roch applied to E ,

$$\frac{\alpha(\alpha+1)}{2}(n-1) + (\alpha+1)(\beta+1) - 1 \ge n - 3$$

$$\Rightarrow \beta \ge -1$$

so we need only deal with the case $\alpha = 1$, $\beta = -1$. Note moreover that in this case E is smooth, since if it had a multiple point p, E would have as fixed component the line through p; so we need only consider case ii) of (3.21). This states, in our present circumstances, that E *has at most one base point*, and correspondingly that D_o must contain the remaining $d - 1$ points of Γ. But $D_o \sim H + F$ is a rational normal curve in $\mathbb{P}^r$ or a degeneration of one and so imposes only $2n + 1$ conditions on quadrics; it follows by uniform position that $\Gamma \subset D_o$ can impose only $2n + 1$ conditions on quadrics, a contradiction.

CASE IV. $\alpha = 0$, $\beta \geq 0$. Here E consists simply of β variable lines of the scroll X; since $\dim E = n - 3$, $\beta \geq n - 3$, and since E has no fixed components, it can likewise have no base points. Thus

$$\Gamma \subset D_o \sim 2H - \beta R , \qquad \beta \geq n - 3 .$$

Now if $\beta \geq n - 2$, then

$$h^0(X, \mathcal{O}(2H - D_o)) \geq n - 1$$

and since X is arithmetically Cohen-Macaulay, this says there are $n - 1$ or more quadrics containing D_o — and hence Γ —, independent modulo those containing X. Thus Γ lies on at least $\binom{n-1}{2} + n - 1 > \binom{n+2}{2} - (2n+2)$ independent quadrics, contradicting the assumption that Γ imposes $2n + 2$ conditions on quadrics.

We conclude therefore that $D_o \sim 2H - (n-3)R$. We have

$$\deg D_o = 2(n-1) - (n-3) = n + 1$$

and applying the adjunction formula, we see that the genus of D_o is 1; so we will be done once we show that D_o is irreducible. Now, a component D_o' of D_o lying in a hyperplane can contain at most

$$\dim \overline{D_o'} + 1 \le \deg D_o' + 1 \le 2 \deg D_o'$$

points of Γ ; since $d \ge 2n + 5 > 2 \deg D_o$, we conclude that D_o must have a non-degenerate irreducible component D_o' . If this component is D_o , we are done; if not, since $\deg D_o = r + 1$, D_o' must be a rational normal curve and the remaining component of D_o a line L . But then L can contain at most 2 points of Γ , leaving at least $2r + 3$ on D_o' ; by uniform position, then, Γ would have to lie on the rational normal curve D_o' and impose $2n + 1$ conditions on quadrics. Thus D_o is an irreducible elliptic normal curve. Q.E.D. for Proposition (3.20).

From Propositions (3.19) and (3.20), the proof of Theorem (3.15) easily follows. Suppose now that $C \subset \mathbb{P}^r$ is a reduced, irreducible and non-degenerate curve of degree d and genus g , and that $\Gamma \subset \mathbb{P}^{r-1}$ is a general hyperplane section of C . We ask what $h_\Gamma(2)$ may be. To begin with, if $h_\Gamma(2) \ge 2r + 1$, we have, applying (3.5),

$$h_\Gamma(0) = 1$$
$$h_\Gamma(1) = r$$
$$h_\Gamma(2) \ge 2r + 1$$
$$h_\Gamma(3) \ge 3r$$
$$h_\Gamma(4) \ge 4r + 1$$
$$h_\Gamma(5) \ge 5r$$
$$\vdots$$
$$h_\Gamma(2k) = \min(d, 2kr+1)$$
$$h_\Gamma(2k+1) = \min(d, (2k+1)r)$$
$$\vdots$$

and since this is greater than the Hilbert function of a collection of points on an elliptic normal curve, (3.14.1) we may apply Corollary (3.2) or (3.3) to obtain $g < \pi_1(d,r)$.

If, on the other hand, $h_\Gamma(2) = 2r$, then applying Propositions (3.19) and (3.20) we may conclude that Γ lies on an elliptic normal curve. Its Hilbert function is then as described in (3.14.1) and we conclude that $g \leq \pi_1(d,r)$.

Thus, if $g > \pi_1(d,r)$, it follows that $h_\Gamma(2) = 2r - 1$, and hence that Γ lies on a rational normal curve. As before, since $\pi_1(d,r) > \pi(d,r+1)$, C must be linearly normal, so that every quadric containing Γ is the restriction of a quadric containing C ; and it follows that C lies on a surface of degree $r - 1$.

Similarly, for part ii) of the theorem, if $g = \pi_1(d,r)$ we conclude that $h_\Gamma(2) = 2r$ or $2r - 1$, and then that C lies on a surface of degree r or $r - 1$. Q.E.D.

It is interesting to observe the necessity of the condition $d \geq 2r + 3$ in part ii) of (3.15): if $d = 2r + 2$, $\pi_1(d,r) = r + 4$, and if C is any curve with a g^1_4 $|E|$, the residual linear series $|K(-E)|$ on C is a g^r_{2r+2} . This will in general be very ample, and the image of C under the associated map to $\mathbb{P}^r$ should not lie on any surface of degree r or $r - 1$. (Compare this with the case $g = \pi(d,r) = r + 5$: here the residual series to a g^r_{2r+2} is a g^2_6 , and if $g > 10$ (that is, $r > 5$) a curve with a g^2_6 is either hyperelliptic, elliptic-hyperelliptic or trigonal[1]. In the first two cases the residual g^r_{2r+2} is not very ample; so C must be trigonal — and indeed the divisors of the g^1_3 on C span lines in $\mathbb{P}^r$ which sweep out the scroll containing C .)

[1] cf. [2], [13].

Note that if $d = 2r + 3$ and $g = \pi_1(d,r) = r + 6$, the residual series to the g^r_{2r+3} is a g^2_7 . If $g \le 15$ — that is, $r \le 9$ — then C may be birational to a plane septic $\overline{C} \subset \mathbb{P}^2$, and the linear series g^r_{2r+3} is then cut out on C by plane cubics through the singularities of $\overline{C} \subset \mathbb{P}^2$; these plane cubics simultaneously give a rational map of the plane to $\mathbb{P}^r$, whose image is a del Pezzo. When $r > 9$, however, there are no del Pezzos, and there are likewise no curves of degree $2r + 3$ and *geometric* genus $r + 6$ in $\mathbb{P}^r$. By the theorem, $I_{2r+3,r+6,r}$ consists generically of curves lying on elliptic normal cones, and having an ordinary spatial triple point at the vertex of the cone. These are then curves of geometric genus $r + 4$; and indeed it is not hard to see that they are exactly the projections of elliptic-hyperelliptic canonical curves $\tilde{C}$ of genus $r + 4$ from 3 general points of $\tilde{C}$.

3.c. Results for curves of large degree

The statement and proof of Theorem (3.15) suggest further extensions of the same idea. If indeed it is the case that configurations Γ of points in uniform position with "small" Hilbert functions lie on curves D of low degree, then their Hilbert functions should reflect those of the curves D . Just as there is a large gap between the smallest possible Hilbert function (3.7) of such a configuration Γ and the next smallest (3.14.1), then, there should be gaps between the Hilbert functions of configurations lying on curves of degree r , degree $r + 1$, and so on. This in turn should give rise to a sequence of functions $\pi_\alpha(d,r)$ with (as we shall see)

$$\pi_\alpha(d,r) = \frac{d^2}{2r-2+2\alpha} + O(r)$$

for $\alpha = 0,\ldots,r-1$, such that a curve $C \subset \mathbb{P}^r$ whose genus g exceeds $\pi_\alpha(d,r)$

must lie on a surface of degree $r - 2 + \alpha$ or less. The problem is, we do not know that there cannot be a configuration of points with Hilbert function in this range, that simply lies on no curve D of small degree at all. One circumstance in which we *can* eliminate this possibility, however, is if the degree d of Γ is extremely large: then we know, for example, that the intersection of the quadrics containing Γ is positive-dimensional, and this is at least a start.

To state the theorem we will prove, we need some notation. First, for d and r given, and any α, set

$$m_\alpha = \left[\frac{d-1}{r-1+\alpha}\right]$$

and

$$\varepsilon_\alpha = d - 1 - m_\alpha(r-1+\alpha) .$$

If $\alpha \le r - 2$, set

$$\mu_\alpha = \max\left(0, \left[\frac{\alpha-r+2+\varepsilon_\alpha}{2}\right]\right)$$

and

$$\pi_\alpha(d,r) = \binom{m_\alpha}{2}(r-1+\alpha) + m_\alpha(\varepsilon_\alpha+\alpha) + \mu_\alpha ;$$

if $\alpha = r - 1$ set

$$\mu_{r-1} = \left[\frac{2+\varepsilon_{r-1}}{2}\right] + \begin{cases} 1 & \text{if } \varepsilon_{r-1} = 2r - 3 \\ 0 & \text{if } \varepsilon_{r-1} \neq 2r - 3 \end{cases}$$

and

$$\pi_{r-1}(d,r) = \binom{m_{r-1}}{2}(2r-2) + m_{r-1}(\varepsilon_{r-1}+r) - 1$$

$$+ \mu_{r-1} + \begin{cases} 1 & \text{if } \varepsilon_{r-1} = 2r-3 \\ 0 & \text{if } \varepsilon_{r-1} \neq 2r-3 . \end{cases}$$

Then we have the

THEOREM (3.22). *There exists a function* $d_o = d_o(r)$ *such that the following holds: If* $C \subset P^r$ *is a reduced, irreducible and non-degenerate curve of degree* $d \geq d_o$ *and genus* g *, and*

$$g > \pi_\alpha(d,r) ,$$

for some $\alpha \leq r-1$ *, then* C *lies on a surface of degree less than or equal to* $r+\alpha-2$ *in* P^r *. In particular, if* $g > \pi_{r-1}(d,r)$ *, then* C *lies on a birationally ruled surface. Moreover, we may take*

$$d_o = \begin{cases} 36r , & r \leq 6 \\ 288 , & r = 7 \\ 2^{r+1} , & r \geq 8 . \end{cases}$$

PROOF. The proof of this theorem consists essentially of a long argument to the effect that, if d is large and the Hilbert scheme of Γ small, then Γ must lie on a curve of low degree. To start with, the following proposition will be quite useful:

PROPOSITION (3.23). *Let* $X \subset P^n$ *be a reduced, irreducible and non-degenerate subscheme of degree* d *, dimension* k *, and codimension* $c = n-k$ *. Then*

i) $h_X(\ell) \geq \binom{\ell+k-1}{k}c + \binom{\ell+k}{k}$;

ii) *for* $\alpha \leq c$ *, if* $d \geq c+\alpha+1$ *, then*

$$h_X(\ell) < \binom{\ell+k-1}{k}c + \binom{\ell+k}{k} + \binom{\ell+k-2}{k}\alpha ;$$

and

iii) *for* $c + 1 \leq \alpha \leq 2c$, *if* $d > 2^{\alpha-c}(2\alpha-\alpha+1)$ (*or* $d > 2c + 1$, *if* $\alpha = c + 1$) *and* X *is an irreducible component of the intersection of the quadrics containing* X , *then*

$$h_X(\ell) \geq \binom{\ell-k+1}{k}c + \binom{\ell-k}{k} + \binom{\ell+k-2}{k}\alpha .$$

PROOF. Parts i) and ii) follow from the basic inequality of Lemma (3.1), applied k times, together with the inequality

$$h_\Gamma(2) \geq \min(d,2c+1)$$

for a collection Γ of d points in uniform position in P^c , and Lemma (3.4). As for part iii) , it follows similarly from the inequality

$$h_\Gamma(2) \geq c + \alpha + 1$$

on the intersection Γ of X with k general hyperplanes. This inequality is in turn a consequence of Lemma (3.19): if indeed we had $h_\Gamma(2) \leq c + \alpha$, Γ would lie on a rational normal scroll of dimension $\alpha - c$ and degree $2c - \alpha + 1$ and then, since $\deg \Gamma = d > 2^{\alpha-c}(2c- +1)$, the intersection of the quadrics containing Γ would have to have a positive-dimensional component containing all or part of Γ ; this in turn contradicts the hypothesis that X is a component of an intersection of quadrics. Q.E.D.

We now proceed with the proof of (3.22). As indicated, the key point here is an analysis of the Hilbert function of Γ , described in the following lemma. In order to state it, we introduce some new functions h_α (depending on d and r) , which will be the minimal Hilbert functions of configurations lying on curves of degree $r - 1 + \alpha$: we set, for $0 \le \alpha \le r - 2$,

$$h_\alpha(\ell) = \begin{cases} \ell(r-1+\alpha) - \alpha + 1 , & 1 \le \ell \le m_\alpha \\ d - \mu_\alpha , & \ell = m_\alpha + 1 \\ d , & \ell \ge m_\alpha + 2 \end{cases}$$

and for $\alpha = r - 1$, we set

$$h_{r-1}(\ell) = \begin{cases} r , & \ell = 1 \\ \ell(2r-2) - r + 1 , & 2 \le \ell \le m_{r-1} \\ d - \mu_{r-1} , & \ell = m_{r-1} + 1 \\ d - 1 , \text{ if } \ell = m_{r-1} + 2 , \ \varepsilon = 2r - 3 & \\ d , \text{ if } \ell \ge m_{r-1} + 3 ; \text{ or } \ell \ge m_{r-1} + 2 , \ \varepsilon \ne 2r - 3 . & \end{cases}$$

Our lemma is then

LEMMA (3.24). *Let* C *be as in the statement of* (3.22), $\Gamma \subset \mathbb{P}^{r+1}$ *a general hyperplane section. Then*

i) *For* $0 \le \alpha \le r - 2$, *if* Γ *lies on an irreducible curve* $D \subset \mathbb{P}^{r-1}$ *of degree* $e = r - 1 + \alpha$, *then*

$$h_\Gamma \ge h_\alpha ;$$

ii) *If* Γ *lies on an irreducible curve* $D \subset \mathbb{P}^r$ *of degree* $e = 2r - 2$, *then*

$$h_\Gamma \ge h_{r-1} ;$$

iii) *Finally, if* Γ *lies on no curve of degree* $2r - 2$ *or less, then*

$$h_\Gamma \ge h_{r-1}$$

and

$$h_\Gamma \neq h_{r-1} .$$

PROOF. The first two parts are relatively straightforward. Start with i) : to begin with, for $\ell \leq m_\alpha$ we have $\ell e < d$, so that every hypersurface of degree ℓ containing Γ contains D ; by Proposition (3.23), then,

$$h_\Gamma(\ell) = h_D(\ell) \geq \ell(r-1+\alpha) - \alpha + 1 .$$

Next, let β be the genus of D ; by Castelnuovo's Theorem, $\beta \leq \alpha$. Since

$$d \geq (r-1+\alpha)(\alpha+1)$$

we have

$$m_\alpha + 1 \geq \frac{d}{r-1+\alpha} \geq \alpha + 1 = e - (r-1) + 1 ,$$

so that by the main theorem of [18] the linear series cut out on D by hypersurfaces of degree $m_\alpha + 1$ is complete; since this series is non-special we have

$$\begin{aligned} h_D(m_\alpha+1) &= h^0(D,\mathcal{O}_D(m_\alpha+1)) \\ &= (m_\alpha+1)(r-1+\alpha) - \beta + 1 . \end{aligned}$$

At the same time, the hypersurfaces of degree $m_\alpha + 1$ containing Γ cut out on D the complete linear series $|\mathcal{O}_D(m_\alpha+1)(-\Gamma)|$ [1] ; so that

$$h_\Gamma(m_\alpha+1) = h_D(m_\alpha+1) - h^0(D,\mathcal{O}_D(m_\alpha+1)(-\Gamma)) .$$

Since

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[1] Note that since we may assume $d > e^2$, the curve D is unique; by the uniform position principle, then, Γ meets $D_{sing} \iff \Gamma \subset D_{sing}$; it follows that $\Gamma \subset D - D_{sing}$.

$$\begin{aligned}\deg \mathcal{O}_D(m_\alpha+1)(-\Gamma) &= (m_\alpha+1)(r-1+\alpha) - d \\ &= (m_\alpha+1)(r-1+\alpha) - (m_\alpha(r-1+\alpha) + 1 + \varepsilon_\alpha) \\ &= r + \alpha - 2 - \varepsilon_\alpha ,\end{aligned}$$

by Clifford's theorem we have

$$h^o(D,\mathcal{O}_D(m_\alpha+1)(-\Gamma)) \le \begin{cases} [\frac{r+\alpha-2-\varepsilon_\alpha}{2}] + 1 , & \text{if } r + \alpha - 2 - \varepsilon_\alpha \le 2\beta \\ r + \alpha - \varepsilon_\alpha - 1 - \beta , & \text{if } r + \alpha - 2 - \varepsilon_\alpha \ge 2\beta \end{cases}$$

We conclude, then, that

$$\begin{aligned} h_\Gamma(m_\alpha+1) &\ge \begin{cases} d - [\frac{\alpha-r+2+\varepsilon_\alpha}{2}] , & \text{if } r + \alpha - 2 - \varepsilon_\alpha \le 2\beta \\ d , & \text{if } r + \alpha - 2 - \varepsilon_\alpha \ge 2\beta \end{cases} \\ &\ge d - \mu_\alpha . \end{aligned}$$

Finally, since $\mu_\alpha < r - 1$, this in turn implies that

$$h_\Gamma(m_\alpha+2) \ge \min(d,h_\Gamma(m_\alpha+1) + r - 1) = d$$

and we are done with Part i) of the lemma.

The same argument, with slightly altered numbers $(h_D(\ell) \ge (2\ell-1)(r-1)$; $\beta \le r)$ yields Part ii) of the lemma as well[1], except in case $\ell = m_\alpha + 2$. Here, if $\varepsilon = 2r - 3$, we may have $h_\Gamma(m_\alpha+1) = d - \mu_{r-1} = d - r$ and $h_\Gamma(m_\alpha+2) = d - 1$ (this will occur exactly when D is a canonical curve and a complete intersection of D with a hypersurface of degree $m_\alpha + 1$); if $\varepsilon \neq 2r - 1$, we have $h_\Gamma(m_\alpha+2) = d$ as before. It would be interesting to know

[1] We do need to use the fact that $\beta = r \Rightarrow D$ nonhyperelliptic.

whether, as seems likely, i) and ii) hold even without the hypothesis $d \geq d_o$.

The remaining part of the lemma is, as we indicated at the outset, the nasty one: the hypotheses give no direct indication what subvarieties of P^{r-1} may contain Γ . Before embarking on the analysis of this case, it will be helpful to bunch together some of the extreme cases into the following

SUBLEMMA (3.25). *If* Γ *lies on an irreducible curve* $D \subset P^r$ *of degree* e *with*

$$2r - 2 < e < \frac{d}{4} ,$$

then $h_\Gamma \geq h_{r-1}$, $h_\Gamma \neq h_{r-1}$.

PROOF. We have to consider a number of cases:

Case 1. $e \geq 3r$. In this case, we have

$$h_\Gamma(1) = h_D(1) = r$$
$$h_\Gamma(2) = h_D(2) \geq 3r - 3$$
$$h_\Gamma(3) = h_D(3) \geq 6r - 8$$
$$h_\Gamma(4) = h_D(4) \geq 9r - 10$$

where the equalities $h_\Gamma(i) = h_D(i)$ follow from the assumption $e < \frac{d}{4}$, and the inequalities follow from Castelnuovo's analysis applied to D . Applying (3.5), we have then

$$\begin{aligned} h_\Gamma(4n) &\geq \min(d, n \cdot h_\Gamma(4) - n + 1) \\ &\geq \min(d, 9n(r-1) + 1) \end{aligned}$$

and similarly,

$$h_\Gamma(4n+1) \geq \min(d,(9n+1)(r-1) + 1)$$

$$h_\Gamma(4n+2) \geq \min(d,(9n+3)(r-1))$$

$$h_\Gamma(4n+3) \geq \min(d,(9n+6)(r-1) - 2)$$

and the inequality $h_\Gamma \geq h_{r-1}$, $h_\Gamma \neq h_{r-1}$ follows.

Case 2. $3r - 5 \leq e \leq 3r - 1$. Here we argue similarly: inasmuch as we may assume $d > 6(3r - 1)$, we have $h_\Gamma(i) = h_\Gamma(D)$ for $i = 1,\ldots,6$, and hence by Castelnuovo

$$h_\Gamma(1) \geq r$$

$$h_\Gamma(2) \geq 3r - 3$$

$$h_\Gamma(3) \geq 6r - 8$$

$$h_\Gamma(4) \geq 9r - 13$$

$$h_\Gamma(5) \geq 12r - 18$$

$$h_\Gamma(6) \geq 15r - 23 .$$

Applying (3.5), then,

$$h_\Gamma(6n) \geq \min(d,15n(r-1) - 9n + 1)$$

$$h_\Gamma(6n+1) \geq \min(d,(15n+1)(r-1) - 9n + 1)$$

$$h_\Gamma(6n+2) \geq \min(d,(15n+3)(r-1) - 9n)$$

$$h_\Gamma(6n+3) \geq \min(d,(15n+6)(r-1) - 9n - 2)$$

$$h_\Gamma(6n+4) \geq \min(d,(15n+9)(r-1) - 9n - 4)$$

$$h_\Gamma(6n+5) \geq \min(d,(15n+12)(r-1) - 9n - 6)$$

and the inequality $h_\Gamma \geq h_{r-1}$ follows since $r - 1 \geq 3$.

Case 3. $2r - 2 < e \leq 3r - 6$. In this case, write $e = 2r - 2 + \beta$ and set $m = [\frac{d-1}{e}]$, $\varepsilon = d - 1 - me$. We have then for $\ell \leq m$,

$$h_\Gamma(\ell) = h_D(\ell) \geq \ell(2r-2+\beta) - (2\beta+r-1)$$
$$= (2\ell-1)(r-1) + (\ell-2)\beta$$
$$\geq h_{r-1}(\ell)$$

by Castelnuovo. On the other hand, since

$$d > (3r-6)(2r-3) + 1$$

we have

$$m - 1 \geq r + \beta = e - r + 2$$

so that by the theorem of [18] the linear series cut on D by hypersurfaces of degree $\ell \geq m - 1$ is complete. In particular, since

$$\deg \mathcal{O}_D(m+3)(-\Gamma) = (m+3)(2r-2+\beta) - d$$
$$= 3(2r-2+\beta) - 1 - \varepsilon$$
$$> 2r - 2 + 4\beta$$
$$\geq 2p_a(D) - 2$$

the line bundle $\mathcal{O}_D(m+3)(-\Gamma)$ is non-special, and hence as before

$$h_\Gamma(\ell) = d \geq h_{r-1}(\ell) \quad \text{for} \quad \ell \geq m + 3 .$$

We estimate $h_\Gamma(\ell)$ in the remaining cases $\ell = m + 1$, $m + 2$ similarly. In either case, if $\mathcal{O}_D(\ell)(-\Gamma)$ is non-special, we have $h_\Gamma(\ell) = d \geq h_{r-1}(\ell)$; otherwise, by Clifford's theorem we have

$$h_\Gamma(m+1) = h_D(m+1) - h^0(D,\mathcal{O}_D(m+1)(-\Gamma))$$

$$\geq (m+1)(2r-2+\beta) - (r-1+2\beta) - (\frac{2r+\beta-3-\varepsilon}{2} + 1)$$

$$= 2m(r-1) + (m-\frac{3}{2})\beta + \frac{\varepsilon-1}{2}$$

$$\geq h_{r-1}(m+1) + ((m-\frac{3}{2})\beta - r + \frac{1}{2})$$

$$\geq h_{r-1}(m+1) \ ,$$

since, as remarked before, $m \geq r + \beta \geq r + 1$. Likewise, if $\mathcal{O}_D(m+2)(-\Gamma)$ is special, then

$$\begin{aligned} \deg \mathcal{O}_D(m+2)(-\Gamma) &= 2(2r-2+\beta) - 1 + \varepsilon \\ &\leq 2p_a(D) - 2 \\ &= 2r - 2 + 4\beta \end{aligned}$$

so

$$\varepsilon \geq 2r - 2\beta - 3$$

and therefore

$$h_\Gamma(m+2) = h_D(m+2) - h^0(D,\mathcal{O}_D(m+2)(-\Gamma))$$

$$\geq (m+2)(2r-2+\beta) - (r-1+2\beta) - (\frac{2(2r-2+\beta)-1-\varepsilon}{2} + 1)$$

$$\geq (2m+1)(r-1) + (m-1)\beta + \frac{\varepsilon-1}{2}$$

$$\geq (2m+2)(r-1) + (m-2)\beta - 1$$

$$= h_{r-1}(m+2) + ((m+2)\beta - r)$$

$$\geq h_{r-1}(m+2) \ .$$

Since $m \geq r+3$. Q.E.D. for Sublemma (3.25).

We return now to the proof of the remaining case iii) of Lemma (3.24). To complete the proof, we will assume that Γ does not lie on any curve of degree less than or equal to $2r-2$, and show that $h_\Gamma \geq h_{r-1}$; by the sublemma, if we can show that Γ lies on a curve of degree $e < \frac{d}{4}$ we will be done.

To start with, consider the intersection B_2 of the quadrics containing Γ . Since $d > 2^{r-1}$, we know by Fulton's Bezout Theorem [9] that B_2 must contain a positive-dimensional component A_2 containing one or more of the points of Γ ; by uniform position (and the fact that $\deg A_2 \leq 2^{r-1} < d$) it follows that A_2 must contain all of Γ .

If A_2 is a curve, then since $\deg A_2 \leq 2^{r-2} < \frac{d}{4}$, we are done; thus we may assume that $\dim A_2 \geq 2$. On the other hand, if $\dim A_2 \geq 4$, then by Proposition (3.23),

$$h_\Gamma(2) \geq h_{A_2}(2) \geq 5r-10 \geq 4r-3$$

when $r \geq 7$; applying (3.5), then,

$$h_\Gamma(2n) \geq \min(d, 4n(r-1)+1) \geq h_{r-1}(2n)$$
$$h_\Gamma(2n+1) \geq \min(d, (4n+1)(r-1)+1) \geq h_{r-1}(2n+1) .$$

Leaving aside for the moment the cases

(3.26) $\qquad r = 5 , \quad h_\Gamma(2) = 15 , \quad A_2 = \mathbb{P}^4$

and

(3.27) $\qquad r = 6 , \quad h_\Gamma(2) = 20 , \quad A_2 =$ quadric hypersurface we may assume that

$$\dim A_2 \leq 3 \ .$$

We consider the cases I. $\dim A_2 = 3$ and II. $\dim A_2 = 2$ separately. First, in case $\dim A_2 = 3$, if degree $A_2 \geq r$, then by (3.23)

$$h_\Gamma(2) \geq h_{A_2}(2) \geq 4r - 3$$

and we may conclude $h_\Gamma \geq h_{r-1}$ as above. On the other hand, if $\deg A_2 \leq r - 1$, then assuming that $d > 27(r-1)$, we see that the intersection with A_2 of the cubics containing Γ must contain a positive-dimensional irreducible component containing some, and hence as before all, of Γ. Now, if A_3 is a curve, $\deg A_3 \leq 9(r-1) < \frac{d}{4}$, and by the sublemma we are done; if A_3 is a surface, then since by hypothesis A_3 is not cut out by quadrics we must have

$$\deg A_3 \geq r - 1 \ ,$$

hence

$$h_{A_3}(3) \geq 6r - 5$$

by (3.23), and thus

$$h_\Gamma(2) \geq h_{A_2}(2) \geq 4r - 6$$
$$h_\Gamma(3) \geq h_{A_3}(3) \geq 6r - 5 \ .$$

Applying (3.5), then,

$$h_\Gamma(3n) \geq \min(d, 6n(r-1) + 1)$$
$$h_\Gamma(3n+1) \geq \min(d, (6n+1)(r-1) + 1)$$
$$h_\Gamma(3n+2) \geq \min(d, (6n+4)(r-1) - 2)$$

so $h_\Gamma \geq h_{r-1}$.

If, on the other hand, $\dim A_3 = 3$ — i.e., $A_3 = A_2$ — then in addition to $h_\Gamma(2) \geq 4r - 2$ we have by (3.23)

$$h_\Gamma(3) \geq h_{A_3}(3) \geq 10r - 20 \geq 6r - 5$$

and as before we are done.

It remains to consider case II, $\dim A_2 = 2$. Here again there are several possibilities, at first. To begin with, if $\deg A_2 > 8(r-5)$, then by (3.23) we have

$$h_\Gamma(2) \geq h_{A_2}(2) \geq 4r - 3$$

and as we have seen already, this implies $h_\Gamma \geq h_{r-1}$. If, however, $r \leq \deg A_2 \leq 8(r-5)$, then since $d > 72(r-5)$, the intersection of A_2 with the cubics containing Γ must contain a positive-dimensional component containing some, and hence all, of Γ . If A_3 is a curve, then

$$\deg A_3 \leq 24(r-5) < \frac{d}{4}$$

and we are done; while if $A_2 = A_3$, then

$$h_\Gamma(2) \geq h_{A_2}(2) \geq 3r - 2$$

and

$$h_\Gamma(3) \geq h_{A_2}(3) \geq 6r - 5$$

and it follows by (3.5) that

$$h_\Gamma(3n) \geq \min(d, 6n(r-1)+1) \geq h_{r-1}(3n)$$
$$h_\Gamma(3n+1) \geq \min(d, (6n+1)(r-1)+1) \geq h_{r-1}(3n+1)$$
$$h_\Gamma(3n+2) \geq \min(d, (6n+3)(r-1)+1) \geq h_{r-1}(3n+2) .$$

Finally, in case $\deg A_2 = r - 2$, since $d > 16(r-2)$ we see that the intersection of the quartics containing Γ contains a component A_4 containing Γ . As before, if A_4 is a curve, then since

$$\deg A_4 \leq 4(r-2) < \frac{d}{4}$$

we are done; if $A_4 = A_2$, on the other hand, then

$$h_\Gamma(2) \geq h_{A_2}(2) = 3r - 3$$
$$h_\Gamma(3) \geq h_{A_2}(3) = 6r - 8$$
$$h_\Gamma(4) \geq h_{A_2}(4) = 10r - 20$$

and hence

$$h_\Gamma(4n) \geq \min(d, 10n(r-1) - 11n + 1)$$
$$h_\Gamma(4n+1) \geq \min(d, (10n+1)(r-1) - 11n + 1)$$
$$h_\Gamma(4n+2) \geq \min(d, (10n+3)(r-1) - 11n)$$
$$h_\Gamma(4n+3) \geq \min(d, (10n+6)(r-1) - 11n - 2)$$

and so $h_\Gamma \geq h_{r-1}$.

To complete (at last!) the proof of Lemma (3.24), we consider the exceptional cases (3.26) and (3.27). First, in case $r = 5$ and $A_2 = \mathbb{P}^4$ — that is, Γ lies on no quadrics at all — since $d > 81$ we see that the intersection of the cubics containing Γ must include a component A_3 containing Γ . If A_3 is a curve of degree 9 or less, by the sublemma we are done; if on the other hand A_3 is a curve of degree 10 or more, or higher-dimensional, then letting X be a hyperplane section of A_3 we have

$$h_{A_3}(3) \geq h_{A_3}(2) + h_X(3) = 15 + h_X(3) \geq 25$$

by (3.23). Thus

$$h_\Gamma(2) = 15 = 4r - 5$$

$$h_\Gamma(3) \geq h_{A_3}(3) \geq 25 = 6r - 5$$

and we are done.

In case 3.27 — that is, $r = 5$, and Γ lies on just one quadric — a similar argument applies: since $d > 2 \cdot 81 = 162$, the intersection of the cubics through Γ has a component A_3 containing Γ . If A_3 is a curve of degree 10 or less, we are done; if not, letting X be a hyperplane section of A_3 ,

$$h_{A_3}(3) \geq h_{A_3}(2) + h_X(3) = 20 + h_X(3) \geq 31$$

and so

$$h_\Gamma(2) = 20 = 4r - 4$$

$$h_\Gamma(3) \geq 31 = 6r - 5$$

and as we have seen, this implies $h_\Gamma \geq h_{r-1}$. Q.E.D. for Lemma (3.24).

PROOF of Theorem (3.22). Our main theorem, happily, now follows readily. Suppose that $C \subset \mathbb{P}^r$ is as in the statement of the theorem, and that

$$g > \pi_\alpha(d,r) .$$

It follows from (3.2) that for some ℓ we must have

$$h_\Gamma(\ell) < h_\alpha(\ell)$$

and hence that the general hyperplane section Γ of C lies on a curve of degree e less than or equal to $r + \alpha - 2$. Every hypersurface in $\mathbb{P}^r$ of degree m_α containing C contains the union of these curves; and from (3.23) and the fact that

$$h_C(m_\alpha) \leq d \cdot m_\alpha + 1$$

it follows that the intersection of these hypersurfaces has dimension at most 2. Thus, the union of the curves of degree e containing the hyperplane sections of C is a surface of degree $e \leq r + \alpha - 2$ containing C. Q.E.D. for Lemma (3.24).

3.d. Remarks on the two main theorems, and a conjecture

The first remark to make is that the functions $\pi_\alpha(d,r)$ introduced above are sharp, in the sense that there do exist curves $C \subset \mathbb{P}^r$ of degree d and genus $\pi_\alpha(d,r)$ lying on surfaces S of degree $r - 1 + \alpha$ in $\mathbb{P}^r$ (and, correspondingly, not lying on any surfaces of smaller degree). Indeed, from the proof of Theorem (3.22) (actually, just from parts i) and ii) of Lemma (3.24), and Lemma (3.4)) one can very nearly describe all such pairs (C,S). For one thing, if (C,S) is such a pair, Γ and D the general hyperplane sections of C and S respectively, then we must have $h_\Gamma = h_\alpha$, and in particular this means that D must be a linearly normal curve of genus α in $\mathbb{P}^{r-1}$, and surfaces $S \subset \mathbb{P}^r$ with such hyperplane sections can be studied very effectively cf. Hartshorne [22] and Horowitz [24]. The next condition on (C,S) is that the linear system $|\mathcal{O}_D(m_\alpha+1)(-\Gamma)|$ have the dimension indicated in the proof of Lemma (3.24). Now, if the degree d is divisible by $r - 1 + \alpha$, this says simply that Γ is a complete intersection of a hypersurface with D, and says nothing about D itself; but in other cases the equality $h_\Gamma(m_\alpha+1) = h_\alpha(m_\alpha+1)$ may mean by Clifford that D is hyperelliptic and $|\mathcal{O}_D(m_\alpha+1)(-\Gamma)|$ a multiple of the hyperelliptic g^1_2. This in turn allows one to analyze S further: surfaces with hyperelliptic hyperplane section have been studied by Sommese [39], Van de Ven [40] and, most recently, Ein [6]. Whatever the results of such an analysis, however, it is easy to see that examples abound of pairs (C,S); one has only to take S the cone over a hyperelliptic curve of genus α and degree $r - 1 + \alpha$ in $\mathbb{P}^{r-1}$, and C residual to a suitable collection of lines (corresponding to a multiple of the g^1_2) in a complete intersection of S with a hypersurface of degree $m_\alpha + 1$. (Note that this will in general yield singular curves C; indeed, if one restricted oneself to smooth curves, it seems one *could* improve on $\pi_\alpha(d,r)$ in many cases.)

Our second remark is that, while $\pi_\alpha(d,r)$ may be correct, the value of d_o given in the remark following the statement of (3.22) certainly is not. Indeed, part i) of Theorem (3.15) simply says that for $\alpha = 1$, we may take $d_o = 2r + 1$; and there is a good bit of evidence that we can take d_o to be smaller for other values of α as well. In fact, we may make the following

MAIN CONJECTURE. *If* $C \subset \mathbb{P}^r$ *is reduced, irreducible and non-degenerate, of degree* d *and genus* g, *and for some* $\alpha \le r - 1$,

$$g > \pi_\beta(d,r), \quad \text{for } \beta \ge \alpha \text{ [1]}$$

then C *lies on a surface* $S \subset \mathbb{P}^r$ *of degree* $r - 2 + \alpha$ *or less; in particular, if* $g > \pi_{r-1}(d,r)$, *then* C *lies on a birationally ruled surface.*

---

[1] Note that for $d \ge 4r - 3$, $\pi_\alpha(d,r)$ is monotone decreasing in α, so that this condition is simply $g > \pi_\alpha$.

Note that there is, in a sense, a lower bound on the degree d built into the statement of this conjecture: by the definition of $\pi_\alpha(d,r)$,

$$\pi_\alpha(d,r) = \pi_{\alpha-1}(d,r) \quad \text{for} \quad d < 2r - 1 + 2\alpha$$

so that the conjecture would be logically unaltered if the additional hypothesis

$$d \geq 2r - 1 + 2\alpha$$

were placed on d.

The main conjecture above is certainly the best possible. For example, for most α and r it is possible to construct a curve $C \subset \mathbb{P}^r$ of degree

$$d = 2r - 2 + 2\alpha$$

and genus g equal to exactly $\pi_\alpha(d,r)$ (note again that this d is the maximal one such that $g = \pi_\alpha(d,r)$ does not imply $g > \pi_{\alpha+1}(d,r)$); such that C apparently lies on no surface of low degree: for example, take C to be any curve of genus $\pi_\alpha(d,r) = r - 2 + 3\alpha$ with a g_4^1, that is, a pencil $|D|$ of degree 4. Then (generically)

$$h^0(C,\mathcal{O}(\alpha D)) = \alpha + 1$$

and by Riemann-Roch,

$$\deg K_C(-\alpha D) = 2r - 2 + 2\alpha$$

and

$$h^0(C,K_C(-\alpha D)) = r + 1 ;$$

generically, then, $|K_C(-\alpha D)|$ will embed C in $\mathbb{P}^r$ as just such a curve.

At the same time, we can check a number of cases of the conjecture along similar lines. For example, take the first, or "minimal", case of the conjecture: that of a curve $C \subset \mathbb{P}^r$ of degree

$$d = 2r - 1 + 2\alpha$$

and genus

$$g = \pi_\alpha(d,r) + 1 = \pi_{\alpha-1}(d,r) = r + 3\alpha .$$

If C is such a curve, we have

$$\deg K_C(-1) = 4\alpha - 1$$

and by Riemann-Roch

$$h^o(C,K_C(-1)) = \alpha + 1 ,$$

i.e., $|\mathcal{O}_C(1)|$ is residual to a $g^\alpha_{4\alpha-1}$, which gives a map $\phi : C \to \mathbb{P}^\alpha$ of degree $\leq 4\alpha - 1$.

What can we say about C on the basis of the map ϕ? First, ϕ may be 2-to-1 onto a curve $D \subset \mathbb{P}^\alpha$ of degree $\leq 2\alpha - 1$, (and genus correspondingly $\leq \alpha - 1$). In this case, the lines in $\mathbb{P}^r$ joining pairs of points in the fibers of $\phi : C \to D$ sweep out a ruled surface containing $C \subset \mathbb{P}^r$. Second, ϕ may be 3 - 1 onto its image, a curve $D \subset \mathbb{P}^\alpha$ of degree $\leq \frac{4\alpha-1}{3}$ and genus $\leq \frac{\alpha-1}{3}$. One may then check by Riemann-Roch that the fibers of $\phi : C \to D$ are triples of points of C collinear in $\mathbb{P}^r$; and the lines they span again sweep out a ruled surface in $\mathbb{P}^r$ containing C. Note that ϕ *cannot* be 4-to-1 onto its image (as it would in fact be in the case we considered above, $d = 2r - 2 + 2\alpha$ and $g = \pi_\alpha$). Finally, ϕ may be birational onto its image. But then, of course, Castelnuovo's bound applies: in particular, if

$$g = r + 3\alpha > \pi_0(4\alpha-1,\alpha) = 6\alpha - 6$$

then ϕ *cannot* be birational, one of the first two cases must obtain, and so C must lie on a ruled surface. Indeed, if we assume that our conjecture holds for curves of degree $4\alpha - 1$ in P^α, then $C \subset P^\alpha$ must lie on a ruled surface whenever

$$g = r + 3\alpha > \pi_{\alpha-1}(4\alpha-1,\alpha) = 4\alpha + 2 ;$$

and this is almost always the case.

Other cases in which d is relatively low may be worked out similarly. At the same time, Theorem (3.22) says that the conjecture holds whenever d is sufficiently large; so it is the middle range that remains mysterious.

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